MODULI SPACES AND DEFORMATION THEORY, CLASS 14

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Contents

1. When is $F(k[V])$ a vector space?

Recall that if V is a finite-dimensional k-vector space, we can define the Artin ring $k[V]$. I will now always assume $\epsilon^2 = 0$, so the dual numbers are $k[\epsilon]$.

Recall that I defined the categorical product of rings $A \times_C B$. Check that $k[V \oplus W] = k[V] \times_k k[W]$.

Lemma. Suppose F is a functor (covariant, on $\mathcal{C}$) such that

$$F(k[V] \times_k k[W]) \xrightarrow{\sim} F(k[W]) \times F(k[W])$$

for finite dimensional vector spaces V and W over k. Then $F(k[V])$ and in particular $t_F = F(k[\epsilon])$, has a canonical vector space structure, such that $F(k[V]) \cong t_F \otimes V$.

I already essentially gave the proof for $V = (\epsilon)$, and the general proof is essentially the same.

Proof. $k[V]$ is a "vector space object" in $\hat{\mathcal{C}}$. In other words, for each $A \in \hat{\mathcal{C}}$, $\operatorname{Hom}(A, k[V])$ is a k-vector space. By:

$$\operatorname{Hom}(A, k[V]) \cong \operatorname{Der}_k(A, V).$$

The addition map is given by $k[V] \times_k k[V] \to k[V]$ $(x, 0), (0, x) \mapsto x$ $(x \in V)$. Scalar multiplication by a is given by the endomorphism $x \mapsto ax$ of $k[V]$.

So if F commutes with the necessary products, $F(k[V])$ gets a vector space structure.

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For the last statement, here's a sketch. Note that $\mathrm{Hom}(k[\epsilon]/\epsilon^2, k[V])$ is naturally identified with V. For any element of $\mathrm{Hom}(k[\epsilon]/\epsilon^2, k[V])$ we get a map $t_F = F(k[\epsilon]/\epsilon^2) \to F(k[V])$, hence $t_F \times V \to F(k[V])$. In fact this is $\otimes$. The desired result is true if V is one-dimensional; then use induction, as V is finite-dimensional, and $k[V] = \times_k^{\dim V} k[\epsilon]/(\epsilon^2)$. (I said something wrong in class.) ☐

Note that this isn't so hard to check. For example, deformation functors of schemes of finite type over k have this property (not even nonsingularity required).

2. Schlessinger's Criterion for existence of universal deformations and hulls (miniversal deformations)

In $\mathcal{C}$, define a *small extension* to be a surjection $A'' \to A$, so $A = A''/I$, and $m_{A''} I = 0$, and I is one-dimensional.

For the purposes of this course *only*, define a *fairly small extension* to be a surjection $A'' \to A$, so $A = A''/I$, and $m_{A''} I = 0$, without requiring that I is one-dimensional.

Note: Then for any A in $\mathcal{C}$, you can filter A into a series of fairly small extensions (by powers of the maximal ideal).

Then you can filter A into a series of small extensions (explain).

Fix our functor $F : \mathcal{C} \to \mathrm{Sets}$.

Let $A' \to A$ and $A'' \to A$ be morphisms in $\mathcal{C}$, and consider the map

(1) $$F(A' \times_A A'') \to F(A') \times_{F(A)} F(A'').$$

Note that if F is a prorepresentable functor, by $R \in \mathcal{C}$ say, then this map is

$$\mathrm{Hom}(R, A' \times_A A'') \to \mathrm{Hom}(R, A') \times_{\mathrm{Hom}(R,A)} \mathrm{Hom}(R, A'')$$

is always a bijection (explain). This is because $\times$ is a categorical product!

Schlessinger's Theorem. [Put on one board permanently!]

It has two parts, and I'll say it slowly, with translations and remarks.

(1) F has a hull iff F has properties H1–H3:

H1. (1) is a surjection whenever $A'' \to A$ is a small extension.

Translation: You can glue.

Remark: Hence equivalently whenever $A'' \to A$ is *any* surjection.

H2. (1) is a bijection when $A = k$, $A'' = k[\epsilon]/\epsilon^2$.

Translation: Uniqueness of gluing when adding $k[\epsilon]/\epsilon^2$.

Remark: Hence true when $A'' = k[V]$ by induction.

Remark: Hence the criterion of the lemma above are satisfied, so t_F is a k-vector space.

H3. $\dim_k(t_F) < \infty$.

Translation: finite-dimensional tangent space.

(2) F is pro-representable if and only if F has the additional property

H4.

$$F(A' \times_A A') \to F(A') \times_{F(A)} F(A'). \tag{2}$$

is a bijection for any small extension $A' \to A$.

Translation: bijection for gluing a small extension to itself.

That ends the statement. So we have four things to prove.

The first part is easy: if F is prorepresentable, then H1–H4 are all satisfied. Before two of the remaining 3 are quite short.

2.1. **Initial remarks.**

Before I get to them, I want to make some initial remarks.

Suppose F satisfies H1–H3.

Consider any fairly small extension $p : A' \to A$, i.e. $0 \to I \to A' \to A \to 0$, so $m_{A'} I = 0$. We have an isomorphism

$$A' \times_{A'/I} A' \xrightarrow{\sim} A' \times_k k[I]$$

induced by the map $(x, y) \mapsto (x, x_0 + y - x)$ (explain).

Now given a small extension $p : A' \to I$, By H2, we get

$$F(A' \times_A A') = F(A' \times_k k[I]) \xrightarrow{\sim} F(A') \times_{F(k)} F(k[I]) = F(A') \times (t_F \otimes I).$$

Hence we get

$$F(A') \times (t_F \otimes I) \to F(A') \times_{F(A)} F(A').$$

For each $\eta \in F(A)$, this determines a group action of $t_f \otimes I$ on $F(p)^{-1}(\eta)$, i.e. those $F(A')$'s lifting $F(A)$, assuming the set is nonempty. The fact that this is a surjection (H1) means that the action is transitive. H4 is precisely the condition that this set is a principal homogeneous space under $t_F \otimes I$. (Say more here.)

So explicitly, what this is telling us is explicitly is that if F already has a hull, then its obstruction to be representable is the existence of an automorphism of an object y in some $F(A)$, that cannot be extended to an automorphism of some object $y' \in F(A')$ for some A'.

3. Proof of Schlessinger, Part 1

I'll show that hull and H4 imply prorepresentable. Then I'll show that hull implies H1–H3. Finally, next time I'll show that H1–H3 imply hull.

Hull and H4 imply prorepresentable.

Suppose we have hull + H4. Say $(R, r \in F(A))$ is a hull. Hence get $h_R(A) \to F(A)$. We want this to be an isomorphism.

We prove this by induction on the length of A. Trivially true for $A = k$.

Consider small $p : A' \to A$, $\ker p = I$, one-dimensional.

Assume $h_R(A) \xrightarrow{\sim} F(A)$. For each $a \in F(A)$, $h_R(p)^{-1}(a)$ $F(p)^{-1}(a)$ are both principal homogeneous spaces under $t_F \otimes I$ (or empty). Since $h_R(A')$ maps *onto* $F(A')$, we have $h_R(A') \xrightarrow{\sim} F(A')$ (either both are empty, or both are principal homogeneous spaces).

Coming next day:

Hull implies H1–H3, and vice versa.