MODULI SPACES AND DEFORMATION THEORY, CLASS 12

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Contents

1. Caution with these notes!

I recall noticing during class that I occasionally wrote $\mathcal{C}$ where I meant to have written $\hat{\mathcal{C}}$. (That's the danger of TeX macros!) I haven't gone through to check them.

My sketch of argument that this formal construction gives you the formal neighbourhood of a point in a "nice" moduli stack isn't complete; I will complete it (in notes) some time soon.

2. Artin rings

Recall: $\mathcal{C}$ is the category of local Artin rings over k, with residue field k. In other words, the objects are $(A, \mathfrak{m})$ with residue field k, and morphisms induce isomorphism of the residue field.

Let $\hat{\mathcal{C}}$ be the category of *complete* Noetherian local k-algebras, with residue field k, for which $A/\mathfrak{m}^n$ is in $\mathcal{C}$ for all n. Notice that $\mathcal{C}$ is a full subcategory of $\mathcal{C}$.

These are precisely the local ring morphisms. However, it is best to think of these as being continuous ring morphisms (as this generalizes to give the correct concept for formal schemes).

We denoted t_A^* by $\mathfrak{m}/\mathfrak{m}^2$; the Zariski cotangent space of $\operatorname{Spec} A$.

Here are some basic facts about Artin rings.

Date: Thursday, October 19, 2000.

Algebra exercise. A morphism $B \to A$ in $\mathcal{C}$ is surjective if and only if the induced map $t_B^* \to t_A^*$ is surjective. $\mathcal{C}$ replaced by $\hat{\mathcal{C}}$.

I also defined formal smoothness (and also f. unr. and f. et.).

Definition. Suppose $G \to F$, in $\hat{\mathcal{C}}$. Then we need to check if

$$\begin{array}{ccc} \operatorname{Spf} A & \to & \operatorname{Spf} F \\ \downarrow & \overset{?}{\nearrow} & \downarrow \\ \operatorname{Spf} B & \to & \operatorname{Spf} G \end{array}$$

where $B \to A$ is a surjection in $\hat{\mathcal{C}}$. Then we say $\operatorname{Spec} F \to \operatorname{Spec} G$ is *smooth*.

Similarly, you can define *etale* (exists exactly one) and *unramified* (at most one).

Thus $\operatorname{Spf} F \to \operatorname{Spf} G$ in $\hat{\mathcal{C}}$ is smooth (etale, unr) if for all $\operatorname{Spf} A \to \operatorname{Spf} B$ in $\mathcal{C}$, where $B \to A$ is surjective,

$$\operatorname{Hom}(F, B) \to \operatorname{Hom}(F, A) \times_{\operatorname{Hom}(G,A)} \operatorname{Hom}(G, B)$$

as sets, is surjective, (bijective, injective).

3. Functors of Artin rings

We will consider *covariant functors* $F : \mathcal{C} \to \text{Sets}$ such that $F(k)$ consists of one element.

Example 1. If X is a variety over k, to $A \in \mathcal{C}$, associate

$$\begin{array}{ccc} X & \to & \tilde{X} \\ \downarrow & & \downarrow \\ \operatorname{Spec} k & \to & \operatorname{Spec} A \end{array}$$

where the right arrow is flat.

Informal definition. These are *(formal) deformations* of X/k.

We will also use this to describe deformations of other things, such as varieties with specified divisors or points, or abelian varieties with given level structure, etc.

If we have a nice moduli problem, this is *precisely* the set of morphisms $\operatorname{Spec} A \to \mathcal{M}$ that restricts to the given $\operatorname{Spec} k \to \mathcal{M}$.

Notation you shouldn't worry about:

Definition. By *couple*, we mean (A, a) where $A \in \mathcal{C}$ and $a \in F(A)$. Morphism of couples:

$$f : (A, a) \to (A', a')$$

$f : A \to A'$, $F(f)(a) = a'$.

So if your functor is a deformation functor, then deformations give couples. We're basically only interested in deformations, so I won't use the "couple" language.

If we extend F to $\mathcal{C}$ by the formula $\hat{F}(A) = \lim_{\leftarrow} F(A/\mathfrak{m}^n)$, we may speak analogously of *pro-couples* and *morphisms of procouples.* So we should also say something like *pro-deformations* to remind ourselves that we're working over a complete thing, but I'll still use the word deformation.

Example 2. For any $R \in \mathcal{C}$, set $h_R(A) = \mathrm{Hom}(R, A)$. It is a covariant functor. If F is any functor on $\mathcal{C}$, we have a canonical isomorphism $\hat{F}(R) \xrightarrow{\sim} \mathrm{Hom}(h_R, F)$. We say a procouple (R, r) for F *prorepresents* F if the morphism $h_R \to F$ induced by r is an isomorphism.

Reality check. Hence if you have a good moduli stack parametrizing some sort of object, then given one of the objects C, the deformations of C are prorepresentable!

This requires the following results.

Exercise. Suppose $f : (Y, q) \to (X, p)$ is a formally etale morphism, and the residue fields at p and q are the same. Show that this induces an isomorphism of formal neighborhoods of Y at q and X at p, i.e. for all n, $A/\mathfrak{p}^n \to B/\mathfrak{q}^n$ is an isomorphism.

Exercise. Prove the formal version of the Infinitesimal Lifting Property for Deligne-Mumford stacks. In other words, suppose $\mathcal{M} \to \mathcal{N}$ is a formally smooth (representable) morphism of Deligne-Mumford stacks. Then

$$\begin{array}{ccc} \mathrm{Spec}\, A/I & \to & \mathcal{M} \\ \downarrow & \overset{?}{\nearrow} & \downarrow \\ \mathrm{Spec}\, A & \to & \mathcal{N} \end{array}$$

where $A \in \mathcal{C}$, $I^2 = 0$. (Ditto for formally smooth, formally etale.)

As observed earlier. Show that $I^2 = 0$ can be removed from the previous exercise. (Explain why.)

Exercise. Suppose $A \to B$ is an etale morphism in $\mathcal{C}$. Then $A \to B$ is an isomorphism. (Even in $\hat{\mathcal{C}}$ by the earlier observation!)

Key: isomorphism of residue fields; note that $\mathrm{Spec}\, k \to \mathrm{Spec}\, l$ is etale whenever k is a separable extension of l.

Note: We can now define what it means for a morphism of functors to be smooth, etale, unramified:

Definition. $F \to G$ in $\hat{\mathcal{C}}$ is smooth (etale, unr) if for all $\mathrm{Spf}\, A \to \mathrm{Spf}\, B$ in $\mathcal{C}$, where $B \to A$ is surjective,

$$F(B) \to F(A) \times_{G(A)} G(B)$$

as sets, is surjective, (bijective, injective).

By definition, this is what you think it is when F and G are prorepresentable.

We can also define the tangent space to a covariant functor of Artin rings.

Definition. The tangent space to a functor F is $F(k[\epsilon]/\epsilon^2)$.

Really this is a tangent set; there's no vector space structure in general.

4. Universal deformations, versal deformations, miniversal deformations (hulls)

Know these words!

Suppose (R, r) *prorepresents* a functor (e.g. deformations of some smooth curve). This essentially a *fine moduli space* property. For any $a \in F(A)$, we get a unique $f : R \to A$ so that $f(r) = a$.

Definition. If the *deformation functor* of X/k is prorepresentable by (R, r), then (R, r) is a *universal deformation* of X.

Exercise. A universal deformation is unique up to unique isomorphism. (Say precisely.)

This is sometimes too good to ask.

Example¿ Assume you have a universal deformation. Take a trivial family over $\operatorname{Spec} S$ $(S \in \mathcal{C})$. $X \times S \to S$. Hence

$$\begin{array}{ccccc} X \times S & & \to & & X \times S \\ & \searrow & & \swarrow & \\ & & S & & \end{array}$$

has one element. Hence those elements of $\operatorname{Hom}(S, \operatorname{Aut} X)$ sending the closed point to the identity has 1 element. Hence X has no infinitesimal automorphisms.

Definition. A *versal deformation* of X is a deformation (R, r) such that if $n : (X, x) \to (Y, y)$ is a surjective homomorphism of deformations (i.e. $n : X \to Y$ is surjective), and $f : (R, r) \to (Y, y)$ is a homomorphism, then there is a g completing the following diagram:

$$\begin{array}{ccc} & & (X, x) \\ & \overset{g?}{\nearrow} & \downarrow n \\ (R, r) & \overset{f}{\to} & (Y, y) \end{array}$$

(Explain Miles Reid story.)

Note: this definition "looks like" the infinitesimal lifting property! No coincidence. This can be restated as: a versal deformation of X is a morphism of functors $h_R \to \operatorname{Def} X$ that is smooth. (Explain why.)

What are (X, x), and (Y, y)? They could be couples or pro-couples; the definition would be the same. This came up earlier. (Exercise: show this.)

In general, for any k-point of an Artin stack over k, we get a versal deformation space.

Remark. Do *universal*, note get uniqueness!

Example. (More detail later.) Deformations of a node. $xy = 0$ up to isomorphism, or formally $xy = 0$ in $\operatorname{Spf} k[x, y]/xy$. Any node has an etale neighborhood isomorphic to the first, and its formal neighborhood is isomorphic to the second. There is a "deformation space" $xy - t$ over $\operatorname{Spf} k[[t]]$.

Given any deformation of the node, we get a map to the base — but not uniquely! Here's an explicit example of why:

Exercise.

$$\begin{array}{ccc} \operatorname{Spf} k[x,y][[t]]/(xy-t) & \dashrightarrow & \operatorname{Spf} k[x,y][[u]]/(xy-u) \\ \downarrow & & \downarrow \\ \operatorname{Spf} k[[t]] & \to & \operatorname{Spf} k[[u]] \end{array}$$

for *any* map $k[[u]]$ to $k[[t]]$ sending u to a unit times t (i.e. something with valuation 1, i.e. u maps to a power series with leading term at, $a \neq 0$).

Note: in node example, $\operatorname{Spf}[x, y][[t, v]]/(xy - t)/ \operatorname{Spf} k[[t, v]]$ also works!

What makes $xy - t$ over $\operatorname{Spf} k[[t]]$ better? It is "minimal" in an explicit way.

Definition. A versal deformation (R, r) is a *miniversal* or *minimal versal* deformation or a *hull* if the induced map $t_R^* \to t_{\mathrm{Def}}$ is an isomorphism.

The *hull* of deformations to the node is (as we will see) $\operatorname{Spf} k[x, y][[t]]/ \operatorname{Spf} k[[t]]$.

Exercise. Two hulls are noncanonically isomorphic. This isn't too hard, but here's a hint: when you decode it, it involves showing that a surjective endomorphism of any noetherian ring is an isomorphism.

Exercise. A universal deformation is a miniversal deformation (hull); a miniversal deformation is a versal deformation.