# MODULI SPACES AND DEFORMATION THEORY, CLASS 10

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Contents



## 1. Examples of the infinitesimal lifting property

Last time I introduced:

*The Infinitesimal Lifting Property.* Suppose $\operatorname{Spec} A$ is a nonsingular variety over $\overline{k}$. Given a ($\overline{k}$-algebra) homomorphism $f : A \to B$, then there is a morphism $g : A \to B'$ lifting it (draw diagram). *Keep on board.*

And then used it to motivate the definitions of formally smooth, as well as formally unramified and formally etale. (Then we could define smoothness of a morphism as formal smoothness plus quasicompactness.)

I'll prove this in a few minutes, but first let me do some examples to convince you that this is reasonable.

*Example 1.* A node is not nonsingular.

More precisely, $\operatorname{Spec} k[x,y]/xy$ is not. Let's see why. Quick check: Zariski cotangent space is 2-dimensional at origin.

*Date*: Thursday, October 12, 2000.

*Example 1a: a first attempt.*

$$\begin{array}{ccc} & & 0 \\ & & \downarrow \\ & & (\epsilon) \\ & & \downarrow \\ & & k[\epsilon]/\epsilon^2 \\ & \stackrel{x=a\epsilon, y=b\epsilon}{\nearrow} & \downarrow \\ k[x,y]/xy & \stackrel{x=y=0}{\rightarrow} & k \\ & & \downarrow \\ & & 0. \end{array}$$

No problem. Picture. There is never any problem to lifting to first order. Reason: there's a map from $k \to k[\epsilon]/\epsilon^2$. Topologists?

*Example 1 b.*

$$\begin{array}{ccc} & & 0 \\ & & \downarrow \\ & & (\epsilon^2) \\ & & \downarrow \\ & & k[\epsilon]/\epsilon^3 \\ & \stackrel{x=a\epsilon+c\epsilon^2, y=b\epsilon+d\epsilon^2}{\nearrow} & \downarrow \\ k[x,y]/xy & \stackrel{x=a\epsilon, y=b\epsilon}{\rightarrow} & k[\epsilon]/\epsilon^2 \\ & & \downarrow \\ & & 0. \end{array}$$

Say $ab \neq 0$. Then no lifrting. If $ab = 0$, then there is a lifting.

Thus we see that this isn't smooth, and moreover, we are "sensing" the formal-local shape of the node.

Also, note that if there is a lifting, there is a two-dimensional vector space of liftings.

*Example 2.* $k[t]/t^{100}$. What doesn't lift? (Draw picture.)

*Example 3.* $xy(x+y) = 0$. (Picture.) What doesn't lift? This is "smooth to third order".

## 2. Proof of the infinitesimal lifting property

We begin the proof. If you've seen this before, you may want to pay attention to where nonsingularity comes into it. By my count, it comes in twice. Also, note that you can do this over *any* base, not just $\operatorname{Spec} k$.

*Lemma.* The set of liftings is empty or forms a "$\mathrm{Hom}_A(\Omega_{A/I}, I)$-torsor" (whether or not $A$ is smooth over $k$!).

I think did this last time, but let me do it again.

*Proof.*

$$\begin{array}{ccc} & & 0 \\ & & \downarrow \\ & & I \\ & & \downarrow \\ & & B' \\ & \stackrel{g?}{\nearrow} & \downarrow \\ A & \stackrel{f}{\rightarrow} & B \\ & & \downarrow \\ & & 0. \end{array}$$

Note: $I$ is a $B$-module, hence an $A$-module.

First, suppose you have two liftings $g$, $g'$.

We get a map $\theta = g - g' : A \to I$ as $k$-modules (*not* as $A$-modules) — claerly additive, $k$-multiplicative.

Check Leibnitz: $\theta(ab) = g(a)g(b) - g'(a)g'(b)$, so

$$a\theta(b) + b\theta(a) = g(a)(g(b) - g(b')) + g'(b)(g(a) - g'(a)).$$

Fascinatingly, I'm going to write the same equation on the board while proving the product rule for my honors calulus students at 1 today.

Conversely, suppose we have one lifting $g$, and a derivation $\theta$ of $A$ into $I$. Then $g + \theta$ is another extension. Proof: $g' : A \to B'$. It is additive clearly. It is multiplicative:

$$\begin{aligned} g'(ab) - g'(a)g'(b) &= (g(ab) + \theta(ab)) - (g(a) + \theta(a))(g(b) + \theta(b)) \\ &= \theta(ab) - g(b)\theta(a) - g(a)\theta(b) = 0. \end{aligned}$$

□

Hence the choices of extension is either empty, or an "affine $\mathrm{Hom}_A(\Omega_{A/k}, I)$-space". □

Next, let's worry about existence in the case when $\mathrm{Spec}\, A$ is nonsingular. Intermediate step:

$$\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
J & & I \\
\downarrow & & \downarrow \\
P & \stackrel{h?}{\to} & B' \\
\downarrow & & \downarrow \\
A & \stackrel{f}{\to} & B \\
\downarrow & & \downarrow \\
0 & & 0
\end{array}$$

Everything on the left is a $P$-module, and everything on the right is a $B'$-module. We can definitely get some map $h : P \to B'$; just lift $x_1, \ldots, x_n$. (We have lots of choices!) This gives a map of $P$-modules from $J$ to $I$. Now everything on the right is a $P$-module too. Notice that $J^2$ maps to $I^2 = 0$. Hence we get a map $\overline{h} : J/J^2 \to I$, respecting the $P$-module structure. Both are $A$-modules.

Now $J/J^2$ is geometrically meaningful; it is the conormal sheaf for $A$.

$$0 \to J/J^2 \to \Omega_{P/K} \otimes A \to \Omega_{A/\overline{k}} \to 0$$

(Hartshorne II.8). The conormal sequence is left-exact for nonsingular varieties. Apply $\mathrm{Hom}_A(\cdot, I)$ we get

$$0 \to \mathrm{Hom}_A(\Omega_{A/\overline{k}}, I) \to \mathrm{Hom}_A(\Omega_{P/\overline{k}} \otimes A, I) \to \mathrm{Hom}_A(J/J^2, I).$$

This is also exact on the right as well, as $\Omega_{A/\overline{k}}$ is locally free. Reason: $\Omega_{A/\overline{k}}$ is projective, which means that $\mathrm{Ext}^i(\Omega_{A/\overline{k}}, I) = 0$ for all $i > 0$.

Next observation: $\mathrm{Hom}_A(\Omega_{P/\overline{k}} \otimes A, I) = \mathrm{Hom}_P(\Omega_{P/\overline{k}}, I)$. Reason: the $A$-module structure on I is the same as the $P$-module structure, as $I = I/I^2$. In general, if $P \to A \to 0$, then and $B$ is a $P$-module, and $C$ is an $A$-module, then $\mathrm{Hom}_A(B \otimes A, C) \cong \mathrm{Hom}_P(B, C)$.

Thus we have an exact sequence:

$$0 \to \mathrm{Hom}_A(\Omega_{A/\overline{k}}, I) \to \mathrm{Hom}_P(\Omega_{P/\overline{k}}, I) \to \mathrm{Hom}_A(J/J^2, I) \to 0.$$

Choose any $\theta : \mathrm{Hom}_P(\Omega_{P/\overline{k}}, I)$ that maps to $\overline{h}$. (Say strategy now.)

Then $\theta$ is a $P$-derivation $\theta : P \to I \hookrightarrow B'$, that restricts to $\overline{h} : J \to I$ (of $A$-modules, not just $P$-modules). Recall that we already had $h : P \to B'$, so $h - \theta : P \to B'$. This is a *ring homomorphism*: (i) it sends 1 to 1 (as $h(1) - \theta(1) = 1 - 0 = 1$). (ii) It is additive, as $h$ and $\theta$ are. (iii) It is multipliciative:

$$\begin{aligned}
(h(a) - \theta(a))(h(b) - \theta(b)) &= h(a)h(b) - h(a)\theta(b) - h(b)\theta(a) \\
&= h(ab) - \theta(ab)
\end{aligned}$$

by Leibnitz. (In fact, we've done this calculation before.)

It sends $J$ to 0: $j \in J$ gives

$$h(j) - \theta(j) = h(j) - \overline{h}(j) = 0.$$

Hence it is a map $h - \theta : A \to B'$. It restricts to $f : A \to B$.

That does it! $\square$

In many cases, you can quantify the "obstruction to this lifting"; here we used nonsingularity at two places, so this quantification is still a little mysterious.

Here's an application, that will come up in deforming nonsingular varieties

**Theorem.** Let $X$ be a nonsingular variety over $\overline{k}$, and let $\mathcal{F}$ be a coherent sheaf on $X$. Then there is a bijeciton between the set of infinitesimal extensions of $X$ by $\mathcal{F}$ up to isomorphism, and the group $H^1(X, \mathcal{F} \otimes \mathcal{T})$, where $\mathcal{T}$ is the tangent sheaf of $X$.

As mentioned last time, in order to do this, we do the affine case, and then patch. In the affine case, there is no higher cohomology.

**Lemma.** Suppose in addition that $X$ is affine, $X = \operatorname{Spec} A$, $\mathcal{F} = \tilde{M}$. Then any extension is isomorphic to the trivial one.

Precisely, the trivial one is the morphism $A \oplus M$ (recall the ring structure). Suppose you have some other $0 \to M \to \tilde{M} \to A \to 0$; we want to show that $\tilde{M} \cong M \oplus A$, such that the projections to $A$ agree. So this is just an algebra question. I think it's hard as an algebra question! But we use the previous question, and it becomes easy.

Consider $0 \to M \to \tilde{M} \to A \to 0$, and map $A$ isomorphically to $A$; then there is a lifting to $\tilde{M}$. (Draw it in, and note that it is a morphism of rings.) Then it is quick to check that $\tilde{M} = A \oplus M$ (as $\tilde{M}$-modules), and that the ring structures agree. $\square$

*Remark.* For future reference, note that we have some choices of the lifting, i.e. choice of expression of $\tilde{M}$ as $A \oplus M$. How many? Answer: $\operatorname{Hom}_A(\Omega_A/\overline{k}, M) = H^0(\operatorname{Spec} A, \mathcal{F} \otimes \mathcal{T}_A)$. Intuition: $H^0$ of this sheaf parametrizes "automorphisms", and $H^1$ will paremetrize deformations.

*Proof of main result.* An important cohomological remark before we start: on a separated space, we can compute cohomology using Cech, and a particular affine cover.

Cover $X$ with a finite number of affines $U_1, \ldots, U_n$.

Suppose first that we have an extension. Then on each affine, we have an identification with $A_i \oplus M_i = H^0(U_i, \mathcal{O} \oplus \mathcal{F})$. There is ambiguity here precisely of

$H^0(U_i, \mathcal{F} \otimes \mathcal{T}_X)$. These identifications don't always agree; on pairwise intersections, their difference lies in $H^0(U_i \cap U_j, \mathcal{F} \otimes \mathcal{T}_A)$. These pairwise intersections satisfy a cocycle condition. We have ambiguity up to $\prod_i H^0(U_i, \mathcal{F} \otimes \mathcal{T}_X)$. Hence we get an element of $H^1(X, \mathcal{F} \otimes \mathcal{T}_A)$.

Conversely, given an element of $H^1(X, \mathcal{F} \otimes \mathcal{T}_A)$, write it in cocycle form, and get an extension. If we change our cycle by something in $\prod_i H^0(U_i, \mathcal{F} \otimes \mathcal{T}_A)$, we get the same extension back. □