MODULI SPACES AND DEFORMATION THEORY, CLASS 1

RAVI VAKIL

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1. Preliminaries

On the off chance that any of you don't know me: Ravi Vakil, 2–271, vakil@math.mit.edu. If you don't know me, please introduce yourself. Office hours: just drop by, or make an appointment.

Sign up: name, e-mail.

From the course catalog: Emphasis on explicit applications, especially to various moduli spaces. Topics include the Hilbert scheme; moduli spaces of curves, abelian varieties, maps; deformations of varieties (nonsingular and singular), subvarieties, maps, singularities, vector bundles and other coherent sheaves, and Hodge structures.

In short, what I'd like to do with this course is to develop explicit tools to study moduli spaces using deformation theory. I'm hoping to do this in a explicit, case-by-case way, rather than an abstract general Illusie-style approach. In order to do this, we'll first need to discuss moduli spaces, such as the Hilbert scheme and the moduli space of curves. It will be important not to shy away from "stackiness" issues, but one needn't know what a stack is in order to have a good feel for what's going on!

Prerequisites: Chapter II and some of chapter III of Hartshorne. (A full year graduate course will definitely do.) Some homological algebra. Willingness to learn things on the fly. We'll use some ideas, such as flatness and etaleness of morphisms, that you may have seen, but haven't had much experience with — in that case, this is your chance to get your hands dirty! My target audience are people who know little more than that, although I hope it will be of interest to experts, not just future experts. And I also expect this to be quite hard core, which means that you

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should actively think of examples, and work out details, and ask questions both in class and by e-mail.

Many of my examples will be related to curves — for example, the moduli space of pointed curves $\overline{\mathcal{M}}_{g,n}$, and the moduli space of stable maps $\overline{\mathcal{M}}_{g,n}(X,\beta)$. As we'll see, the fact that the objects in question are one-dimensional will make certain calculations much easier.

A related course: Joe Harris (of Harvard) and Sean Keel (of U.T. Austin) will be teaching a course on the geometry of the moduli space of pointed curves, that matches well with this one. It will be largely independent, although it may use certain results of this course. I encourage you to check it out. The first Harvard-M.I.T. algebraic geometry seminar, by Sean Kell will be in effect an advertisement for thsi course. This will cause two complications in the time and location of this course. First of all, locations. The Harvard course is currently scheduled on Tuesdays and Thursdays from 1 to 2:30. (Explain proposed solution — please be discreet.) Second, time constraints. (Explain proposed solution.) So, as of Sept. 19, something funny will be happen. This course will likely meet at Harvard that week, and the time may even change.

Possible topics to be covered: (in no particular order)

- Introductory lectures: moduli spaces.
- Functor of points; moduli functors.
- Obstruction theories.
- Schlessinger's thesis.
- Hilbert scheme?
- Deformation of nonsingular varieties. (Lots of examples!)
- Singular varieties.
- Embedded deformations.
- Deformations of maps of curves (e.g. rational curves to varieties — Jason; Gromov-Witten theory, stable maps).
- The cotangent complex in easy cases; the Nederlander complex.
- Others, depending on your interests. (Let me know!)
- Suggestion from class: stable maps

References: There's no perfect reference. Here are some good ones that I'll be following. (I may try to give out copies of all interesting articles.)

- *Moduli spaces:* Harris-Morrison. Sernesi.
- *Hilbert schemes:* Mumford's *Curves on an algebraic surface*, esp. Ch. 14. (Grothendieck's *FGA*.)
- *Stacks:* Fulton et al (doesn't exist).
- *Deformation theory:*
  - Kodaira's *Deformations of complex algebraic varieties* (also Annals papers) for analytic theory.
  - Kollár's *Rational curves on algebraic vareities.*
  - Vistoli's e-print *Deformations of local complete intersections.*

– Schlessinger's article *Functors of Artin rings*, and his thesis.
– M. Artin's *Deformations of singularities*.
– (Illusie's *Complexe Cotangent et Deformations* I and II.)
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Notes: On the webpage, a few days later. I hope.

If you need a grade: occasional exercises. If you don't need a grade, I encourage you to do them as well; this is not a spectator course!

Short-term plan: some motivation for moduli spaces. The functor of points. Understanding a scheme by understanding morphisms to it. Risky: Mumford's *Picard groups of moduli problems*: $\operatorname{Pic} \mathcal{M}_{1,1} = \mathbb{Z}/12$. I'll try to convince you what this means, and the proof will give you some idea of stackiness. (If too hard, I'll downshift.) I may stall on this until Lecture 4.

2. Motivation for moduli spaces

The easiest example of a moduli space is projective space. Characteristic 0 algebraically closed field. (What goes wrong in positive characteristic?)

Degree of discriminant is 2. $\operatorname{Sym}^2(\mathbb{P}^1) \cong \mathbb{P}^2$. Discriminant locus $b^2 - 4ac = 0$. What does it mean?

If you have *any* "family" of 2 points in $\mathbb{P}^1$ over a base B, there is a universal formula for when the two points come together; this is a locus on B.

What does this mean? You have an effective Cartier divisor on $B \times \mathbb{P}^1$, of relative degree 2, not containing any fibers This induces a morphism $B \to \mathbb{P}^2$. The pullback of the discriminant conic. Let me say it stupidly. For every family, you have a Cartier divisor, and a section of it. The vanishing of that section tells you precisely where the 2 points come together.

Note: there is a universal a, b, c on this family; these are really modular forms in the bluntest sense of the word.

In short, the discriminant is a universal divisor. Given any proper curve, you get a number.

Aside: virtual number. Also, order of meeting.

Moral: If we want to study two points in $\mathbb{P}^1$, we just need to understand $\mathbb{P}^2$, with the universal curve.

In general, if we want to study some sort of object, such as elliptic curves, or elliptic curves with level n structure, then we study instead the moduli space of these objects. Any facts we prove about the moduli space of these objects is a "universal fact" about the objects themselves.

In the points-in-$\mathbb{P}^1$ case, we found the moduli space by an ad hoc method.

2.1. **Deeper into that example.** More generally, the discriminant of a degree d polynomial in 2 variables is $2d-2$. (Good exercise.)

Here's a false proof. Take a pencil. It meets the discriminatn divisor in $d-1$ points. So the degree is $d-1$. What's wrong? (Answer me in a moment.)

In case you think that was too naive, here's another more sophisticated variant. $\pi : (\mathbb{P}^1)^d \to \mathbb{P}^d$. Finite morphism of degree $d!$. Divisors $h_1, \ldots, h_d$ on left, h on right; Δ too on right. Then $\pi^* h = \sum h_i$. Check: $\pi^* h^d = d!$.

$$\pi^* \Delta = (h_1 + h_2) + (h_1 + h_3) + \cdots + (h_{d-1} + h_d) = (d-1) \sum h_i.$$

We want to calculate the degree of Δ, i.e. $h^{d-1}\Delta$. Let's pull it back to $(\mathbb{P}^1)^d$, where we understand it better; our answer is $d!$ too big. We get $(d-1)(\sum h_i)^d = (d-1)d!$. Again, we get $d-1$ as our answer.

What's gone wrong? (Ask.)

How could you tell the multiplicity? Answer: look in an analytic neighbourhood.

Let me do it for $d = 2$. The family is $x(x+ty) = 0$, as t varies. $\Delta = b^2 - 4ac = t^2$. Double root! This is deformation theory!

What's something meeting the singular locus with multiplicity one? $x^2 - ty^2$.

In particular, you can see the multiplicity of intersection by the geometry of the universal family. All you need is to know what's going on in a *formal neighbourhood* of the central fiber.

Analogous picture, and a first look at a fundamental object. $\overline{\mathcal{M}}_{g,n}$ is the moduli space of genus g, n-pointed stable curves. I'm leaving ambiguous what I mean by "space" for now; in fact, it is best to think of it as a Deligne-Mumford stack, which is a mild extension of the idea of scheme. Next day, I'll rigorously define a family of such objects, but now I'll just tell you what a single one is. It's a connected proper nodal curve over our algebraically closed field (a node is an ordinary double point, which we defined last year in the intro to alg geom course (18.725); we saw that it formally looked like $xy = 0$), which I'll draw in this way, such that any component of geometric genus 0 has at least three "node-branches", and any component of geometric genus 1 has at least one node-branch.

Combinatorial Exercise. Show that any stable curve has genus at least 2. Show that one can remove the genus 1 condition by requiring that the stable curve have genus at least two. (Hence we get a definition that is a little shorter, but in some sense less natural.)

As we'll see later, we can rephrase the definition as: a stable curve is a proper connected nodal curve with no infinitesimal automorphisms. (Draw picture.)

Then this "space" is nonsingular and proper, and has a universal curve. Draw genus 5 picture, and have boundary divisor Δ_2. Suppose we have a family of curves over a 1-dimensional base. Then the singularity looks like $xy = t^n$; the value of n is the multiplicity of intersection is precisely n. Once again, we just need to see what's going on in a formal neighbourhood.

We'll see this example (rigorously, in more detail) later.