INTRO TO ALGEBRAIC GEOMETRY, PROBLEM SET 12

Due Monday December 13 by noon at my office. Read all of the problem set, and hand in six of them, including one from each section. You're strongly encouraged to collaborate (although write up solutions separately), and you're also strongly encouraged to ask me questions (if you're stuck, or if the question is vaguely worded, or if you want to try out an argument).

Line bundles.

1. Suppose C is a nonsingular curve, p is a point of C, and $\mathcal{L}$ is an invertible sheaf on C. Describe a natural identification of global sections of $\mathcal{L} \otimes \mathcal{O}_C(-p)$ (denoted $\mathcal{L}(-p)$ for short) with the global sections of $\mathcal{L}$ vanishing at p. If C is a projective nonsingular curve, prove that $h^0(C, \mathcal{L}) - h^0(C, \mathcal{L}(-p))$ is 0 or 1.

Differentials.

2. Suppose U is an affine variety, and $r \in A(U)$ is a regular function. Show $d(r^n) = nr^{n-1}dr$ if n is a non-negative integer. If r is an invertible regular function (i.e. $r \in A(U)^*$) show that $dr^n = nr^{n-1}dr$ for any integer n.
3. *The canonical sheaf of projective space.* For some m, $\wedge^n \Omega^1_{\mathbb{P}^n} \cong \mathcal{O}_{\mathbb{P}^n}(m)$. Find m as a function of n. (We haven't actually defined the canonical sheaf of a higher-dimensional nonsingular variety, but think of sections on each affine open with as being of the form $f dx_1 \wedge \cdots \wedge dx_n$, where f and x_i are regular functions on the open. Make whatever assumptions you need.)
4. Suppose X is a nonsingular curve, and p a point on X. Let U be an affine neighborhood of p, and let $\mathfrak{m}$ be the maximal ideal of $A(U)$ corresponding to p. Then we earlier identified the cotangent space to X at p with $\mathfrak{m}/\mathfrak{m}^2$. As Ω^1 is the "cotangent sheaf", there should be a natural surjective map (of $A(U)$-modules) from $\Omega^1(U) \to \mathfrak{m}/\mathfrak{m}^2$. Describe this map explicitly. (Hence there is a natural map from the stalk Ω^1_p to $\mathfrak{m}/\mathfrak{m}^2$.)
5. *Pullback of differentials.* Suppose $\pi : X \to Y$ is a morphism of nonsingular curves. Show that there is a natural map from global differentials on Y (i.e. global sections of Ω^1_Y) to global differentials on X. The naturality of your construction will show that there is a natural morphism of invertible sheaves on X, $\pi^{-1}\Omega^1_Y \to \Omega^1_X$. (Equivalently, there is a natural morphism of sheaves on Y, $\Omega^1_Y \to \pi_* \Omega^1_X$; but $\pi_* \Omega^1_X$ isn't an invertible sheaf, so we don't yet have the language to effectively discuss it.)
6. *Residues.* In complex analysis, a differential on a one-dimensional complex manifold with a simple pole at a point p has a naturally-defined residue at that point: in any analytic coordinate z at p (i.e. p corresponds to $z = 0$), if

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the differential is of the form $(a_{-1}/z + f(z))dz$ where $f(z)$ is analytic, then the residue is a_{-1}, and this is *independent of the choice of analytic coordinate*. Show that the same is true in algebraic geometry: suppose X is a nonsingular curve, p a point of X, and s a differential on X with simple pole at p; define the residue of s at p (and show that it is well-defined, i.e. that it doesn't depend on any choice).

Genus.

7. Suppose C is a nonsingular projective curve, and $\mathcal{L}$ an invertible sheaf on C with degree 1, and $h^0(C, \mathcal{L}) = 2$. Show that C has genus 0.
8. *Hyperelliptic curves.* Assume $\operatorname{char} \overline{k} \neq 2$. Suppose $f(x_0, x_1)$ is a polynomial of homogeneous degree n with no double zeros, where n is even. Let C_0 be the affine plane curve given by $y^2 = f(1, x_1)$, with the (generically 2-to-1) morphism $C_0 \to U_0$. Let C_1 be the affine plane curve given by $z^2 = f(x_0, 1)$, with the morphism $C_1 \to U_1$. Check that C_0 and C_1 are nonsingular. Show that you can glue together C_0 and C_1 (and the double covers) so as to give a double cover $C \to \mathbb{P}^1$. (For computational convenience, you may assume that neither $[0; 1]$ nor $[1; 0]$ are zeros of f.) What goes wrong if n is odd? Show that the genus of C is $n/2 - 1$. (This is a special case of the *Riemann-Hurwitz formula*.) This provides examples of curves of any genus. (Not for credit: What goes wrong if $\operatorname{char} \overline{k} = 2$?)
9. *Hyperelliptic curves take 2.* Again, assume $\operatorname{char} \overline{k} \neq 2$. Suppose C is a projective nonsingular curve of positive genus, and $\mathcal{L}$ is a line bundle of degree 2. Prove that $h^0(C, \mathcal{L}) \leq 2$. If equality holds, prove that $|\mathcal{L}|$ gives a morphism $C \to \mathbb{P}^1$. (It is a fact that all such morphisms are of the form described in the previous question, and vice versa.)
10. *Nonsingular plane curves.* Suppose C is a nonsingular degree d curve in $\mathbb{P}^2$. Then $\Omega^1_C \cong \mathcal{O}_C(d-3)$. Hence prove that the genus of C is $\binom{d-1}{2}$.

The Riemann-Roch formula.

11. *Genus 1 and 2 curves.* Prove that all projective nonsingular genus 1 curves are hyperelliptic in the sense of Problem 9. Prove that all projective nonsingular genus 2 curves are hyperelliptic (use $\mathcal{L} = \Omega^1$). Hence (modulo the parenthetical fact in Problem 9 you have now seen explicit descriptions of all genus 2 curves (see Problem 8).
12. *Genus 3 curves.* Suppose C is a genus 3 projective nonsingular curve. Suppose furthermore that C is *not* hyperelliptic (see above), i.e. that for each degree 2 invertible sheaf $\mathcal{L}$, $h^0(C, \mathcal{L}) < 2$. Prove that C can be expressed as a nonsingular plane quartic. (Hint: Consider $|\Omega^1_C|$.) (A mild addition to your argument will show that C can be expressed as a nonsingular plane quartic in essentially only one way.) Hence you can describe what all genus 3 curves look like.
13. *Dimension of the Picard variety.* The set of degree d line bundles on an nonsingular projective algebraic curve C of genus g can also be given the structure of an algebraic variety X. Here's a calculation that will convince you that the dimension of X should be g. Suppose $d > 2g - 2$. Calculate the

number of sections any degree d line bundle must have (or, more precisely, the dimension of the vector space of sections). Each section (up to multiple by non-zero constant) gives a degree d divisor on C, and distinct degree d divisors on C give rise to distinct sections of degree d line bundles (up to non-zero constant). Explain why the degree d divisors on C should be parametrized by a dimension d variety. You now know the dimension of the "space of degree d divisors", and you know the dimension of the "space of degree d divisors corresponding to a fixed degree d line bundle"; use this to calculate the dimension of the "space of degree d line bundles".

Now if d' is any integer, show that the space of degree d' line bundles is isomorphic to the space of degree d line bundles. (Hint: let $\mathcal{L}$ be any line bundle of degree $d - d'$; tensoring by $\mathcal{L}$ will give the isomorphism.)