INTRODUCTION TO ALGEBRAIC GEOMETRY, CLASS 5

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Contents

Where we are.

We've defined the category of affine algebraic sets over an algebraically closed field $\overline{k}$, which means we've defined what the sets are, and what the morphisms are. A fundamental result we proved is that

Theorem. Morphisms from one affine algebraic set $X \subset \mathbb{A}^m$ to another $Y \subset \mathbb{A}^n$ canonically correspond to morphisms of rings $A(Y) \to A(X)$:

$$\mathrm{Hom}_{\text{alg sets}}(X, Y) \cong \mathrm{Hom}_{\text{rings over } \overline{k}}(A(Y), A(X)).$$

1. Sheaves continued: Examples, and stalks at a point

First recall the definition of a *sheaf* F of rings (or sets, or modules, or groups) on a topological space X:

- To every open subset U of X there is associated a ring denoted $F(X)$ or $\Gamma(U, F)$, called "sections over U".
- If $U \subset V$ (both open) then there is a restriction map $\mathrm{res}_{V,U}\, F(V) \to F(U)$.
- Restriction maps "commute": if $U \subset V \subset W$ (all open), then

$$\begin{array}{ccccc} F(W) & & \overset{\mathrm{res}_{W,V}}{\rightarrow} & & F(V) \\ & \mathrm{res}_{W,U} \searrow & & \swarrow \mathrm{res}\, V, U & \\ & & F(U) & & \end{array}$$

  commutes. (The conditions so far describe a *presheaf*.)
- *Gluability.* If $U_1 \cup \cdots \cup U_n = V$ (not necessarily a finite union; I should say $\cup_{i \in I} U_i$) and you are given sections $f_i \in F(U_i)$ such that "f_i and f_j" agree on $U_i \cap U_j$", i.e.

$$\mathrm{res}_{U_i, U_i \cap U_j} f_i = \mathrm{res}_{U_j, U_i \cap U_j} f_j$$

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in $F(U_i \cap U_j)$, then there is an $f \in F(V)$ such that $f_i = \mathrm{res}_{V,U_i}$ in $F(U_i)$.

- *Identity.* If $U_1 \cup \cdots \cup U_n = V$ (again not necessarily a finite union) and you have a sections $f, g \in F(V)$ such that "f and g agree on each U_i", i.e. $\mathrm{res}_{V,U_i} f = \mathrm{res}_{V,U_j} g$ in $F(U_i)$ for all i, then $f = g$ in $F(V)$.

One can also have a presheaf of sets, or modules, or groups, etc.; the definition doesn't change.

Examples are rings of functions on a topological space, or continuous or differentiable or meromorphic or polynomial or analytic functions (assuming those adjectives make sense on the space in question).

I gave an example of a presheaf without gluability, and a presheaf without identity.

I talked a bit about $F(\emptyset)$, but I'll think more as to whether $F(\emptyset)$ should be the zero-ring.

Recovering sheaves from a base of a topology.

Reminder: Given a topology on a set X, a *base* is a subset of the open sets such that

(i) for all U open,$x \in X$, there is an element V of the base such that $x \in V \subset U$.

or equivalently

(ii) For all $U \subset X$, U can be expressed as the union of elements of the base.

E.g. In the classical topology on a manifold, open balls.

Note: you can recover a sheaf from knowing its sections over the elements of a base, and the restriction maps between them.

This may turn into an exercise.

Definition: Stalk of a (pre)sheaf of rings at a point.

Given a topological space X, a point $x \in X$, a sheaf F of rings on X. The stalk of F at x is, informally, the set of "germs of functions at x", or "functions in a neighborhood of x"; denoted F_x.

Precisely,

$$\begin{aligned} F_x &= \lim_{\overrightarrow{U}} \{F(U)\} \\ &= \{(U, f \in F(U))\}/\sim \end{aligned}$$

where $(V, f \in F(V)) \sim (W, g \in F(W))$ if there is a $U \subset V, W$ such that

$$\mathrm{res}_{V,U} f = res_{W,U} g.$$

Example. Germs of differentiable functions on a manifold.

Remark. You can just worry not about sections on *all* U containing X, but just those U in a base.

Exercise (on set given out this past Tuesday). Let F be the sheaf of differentiable real-valued functions on the unit disc $\{x^2+y^2 < 1\}$ (in the classical topology). Show that the stalk of F at the origin is a local ring (i.e. that it has one maximal ideal).

2. The structure sheaf $\mathcal{O}_X$ of an irreducible algebraic set $X \subset \mathbb{A}^n(\overline{k})$

Comment. "Irreducible" can be excised with a little work, but for simplicity of exposition we'll stick with it.

Let's fix notation. Suppose we have an irreducible algebraic set $X = V(I)$. Let $R = A(X) = \overline{k}[x_1, \dots, x_n]/I$ be the ring of functions. X is irreducible, so I is prime, so R is an integral domain and has a field of fractions.

If x is a point of X, then $\mathfrak{m}_x \subset R$, and $R_{\mathfrak{m}_x}$ is a subring of R.

- fractions where the denominator doesn't vanish at x or even in a neighborhood of x.
- "functions" in a neighborhood of x; germs of functions at x.
- local ring with maximal ideal $\mathfrak{m}_x R_{\mathfrak{m}_x}$ and quotient $\overline{k}$ (quotient $\leftrightarrow$ evaluation at x)
- "functions defined in a neighborhood of x"

Reality check example: Suppose X is the curve $y^2 = x^3 + x$, and x is the point $(0, 0)$. Then one element of $R_{\mathfrak{m}_x}$ ios

$$\frac{3x^2 + 4}{y + x + 4}.$$

The residue is 1.

Now would be a good time to give a definition of what the structure sheaf is: functions that are locally quotients of polynomials.

Definition. If U is an open subset of X (in the Zariski topology), let

$$\mathcal{O}_X(U) = \cap_{x \in U} R_{\mathfrak{m}_x} \subset K(R).$$

(Recall that $K(R)$ is the *field of fractions* of the domain R.)

Claim/definition. This is a *sheaf* (called the *structure sheaf*).

Definition. Let X be an algebraic set (not necessarily irreducible). (Otherwise, in this section, X is assumed to be irreducible.) Let $R = A(X)$. Let $f \in R$. Define the *distinguished open subset*

$$D(f) = \{x \in X | f(x) \neq 0\}.$$

Note that this is really open!

Remark. $D(f_1) \cap \cdots \cap D(f_n) = D(f_1 \dots f_n)$.

I'll give three reasons we like them, two now and one later.

Reason 1 we like distinguished open sets:

Claim. The *distinguished open subsets* form a base for the Zariski topology, i.e. each open subset is a union of distinguished open subsets.

Proof. Let U be an open subset, and let C be its complement. C is an algebraic set, so $C = V(I)$ for some ideal I. Then

$$C = \cap_{f \in I}\{x | f(x) = 0\}$$

so

$$U = \cup_{f \in I} D(f).$$

□

Reason 2 we like them:

Proposition. (Once again, X is irreducible.) $\mathcal{O}_X(D(f)) = R_f$.

Proof. First,

$$R_f = \{\frac{r}{f^n} | r \in R\} \subset \mathcal{O}_X(D(f)).$$

(explain why). Next, suppose $F \in \mathcal{O}_X(D(f)) \subset K(R)$.

Let $B = \{g \in R | gF \in R\}$; it is an ideal (explain). If $f^n \in B$, we're done.

If $x \in D(f)$, then $F \in R_{\mathfrak{m}_x}$, so there are functions g, h with $F = h/g$, $g(x) \neq 0$. Then $gF = h \in R$, so $g \in B$, so B contains an element not vanishing at x. Thus $V(B) \subset \{x | f(x) = 0\}$. By the Nullstellensatz v. 6, $f \in \sqrt{B}$. □

Substituting $f = 1$, we get:

Corollary. $\mathcal{O}_X(X) = R$.

First, note that this Proposition (and Corollary) tells you that you can recover the structure sheaf knowing only the ring of regular functions, which is the same as the the global sections over all of X.

Second, let me tell you what the Corollary says in english. Any element r of $K(R)$ that lies in each $R_{\mathfrak{m}}$ for all maximal ideals — i.e. for each x, r can be written as a/b such that b is not in that maximal ideal — must actually lie in R to begin with. There's some subtlety in there: a priori, you can't tell if you have to write r as a/b *differently* for different maximal ideals. (You'll realize later that there is something subtle going on here; you won't understand it now because you'll have implicit assumptions about certain things. See the $wx = yz$ example below.)

Corollary. The stalk of $\mathcal{O}_X$ at x (denoted $\mathcal{O}_{X,x}$) is $R_{\mathfrak{m}_x}$.

Proof. Since the distinguished open sets $D(f)$ are a base for the Zariski topology of X, we have

$$\varinjlim_{x\in U} \mathcal{O}_X(U) = \varinjlim_{x\in D(f)} \mathcal{O}_X(D(f)) = \varinjlim_{f:f(x)\neq 0} R_f = \cup_{f(x)\neq 0} R_f = R_{\mathfrak{m}_x}$$

□

Important remark. *Sections of $\mathcal{O}_X$ over U are functions (from U to $\overline{k}$).*

$\cap_{x\in U} R_{\mathfrak{m}_x} = \mathcal{O}_X(U) \to \text{Functions}(U, \overline{k})$

The morphism is via $R_{\mathfrak{m}_x} \to \text{Functions}(x, \overline{k})$ (quotient by maximal ideal).

It is injective.

They are functions that are locally quotients by polynomials.

Example. If $h \in \mathcal{O}_x(U)$ for some open $U \subset X$, it need not be true that $h = f/g$ where g doesn't vanish on U (except in certain circumstances, e.g. $U = D(f)$ for a distinguished open, see below).

Let $X = V(wx - yz) \subset \mathbb{A}^4$, and let

$$U = D(y) \cup D(w) = \{p \in X | y(p) \neq 0 \text{ or } w(p) \neq 0\}.$$

Then take $h = x/y$ on $D(y)$, and z/w on $D(w)$. (They agree on the overlap, as $x/y = z/w$.)

This is a good example to ponder at length on your own.

3. Defining affine varieties and prevarieties: a beginning

Next on the agenda: we'll define affine varieties and prevarieties, give some examples, and define morphisms between them. We ended today with definitions, that I'll repeat at the start of Tuesday's class.

Definition. An *affine variety over* $\overline{k}$ $(X, \mathcal{O}_X)$ is a topological space X plus the *structure sheaf* $\mathcal{O}_X$, a sheaf of $\overline{k}$-valued functions $\mathcal{O}_X$ on X which is isomorphic to an irreducible algebraic subset of some $\mathbb{A}^n$ plus the sheaf just defined.

Often we will just say X when the structure sheaf is clear.

Remark. We can now define an affine variety structure on $\mathbb{A}^n$.

Definition. A *prevariety over* $\overline{k}$ is a topological space X plus a sheaf $\mathcal{O}_X$ of $\overline{k}$-valued functions on X such that

1. X is connected,
2. There is a finite open covering $\{U_i\}$ of X such that for all i, $(U_i, \mathcal{O}_X|_{U_i})$ is an affine variety.

In particular, irreducible affine algebraic sets can be naturally given the structure of a variety.

Homework was returned at the end of class.

Coming in the next two lectures:

(1) Affine varieties, and prevarieties. Examples.

(2) Morphisms. Some examples of morphisms.

(3) Easier variations: Open subprevarieties. Closed subprevarieties. Rational maps. Birationality.

(4) Projective varieties.