# INTRODUCTION TO ALGEBRAIC GEOMETRY, CLASS 19

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Contents



Problem sets won't be back for a while.

## 1. Recap of where we are

Integral closure in a function field. DVRs — 3 characterizations. Dedekind domains (integrally closed noetherian domain of dimension one).

**Key Technical Theorem (tricky).** Let $K$ be a finitely generated function field of dimension one over $\overline{k}$, and let $x \in K$. Then the set of discrete valuations of $K/\overline{k}$ where $v(x) < 0$ is *finite*.

The proof involved the following construction.

**Key technical construction.** Suppose $(R, \mathfrak{m})$ is a DVR of $K$, and $y \in \mathfrak{m}$. Consider $\overline{k}[y]$, the subring of $K$ generated by $y$. Since $\overline{k}$ is algebraically closed, $y$ is transcendental over $\overline{k}$, so $\overline{k}[y]$ is a polynomial ring, i.e. $y$ doesn't satisfy any relations. Hence $K$ is a finite field extension of $\overline{k}(y)$. Let $B$ be the integral closure of $\overline{k}[y]$ in $K$. Then by an earlier theorem (proved in the last few classes), $B$ is a Dedekind domain, and is also a finitely generated algebra over $\overline{k}$.

Hence $B$ corresponds to an affine variety $Y$, which maps to $\mathbb{A}^1$ with coordinate $y$; draw it! It is dimension 1 and nonsingular. Also, $y$ is a function on $Y$. □

Using this key technical construction, we got the following corollaries:

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**Corollary.** Let $v$ be a discrete valuation of the field $K/\overline{k}$, a finitely-generated function field of transcendence degree 1. Then there is a nonsingular curve $C$, with function field $K$, and a point $p \in C$, such that $v$ is the valuation induced by $p$.

**Corollary.** Given $y \in K \setminus \overline{k}$, all discrete valuations of $K/\overline{k}$ such that $v(y) \geq 0$ (i.e. that $y$ is in the corresponding DVR) are accounted for by points on the curve $Y$ above.

Then we proved an important result: on extension of morphisms to projective varieties, over nonsingular points of curves:

**Theorem.** Let $X$ be a nonsingular curve, $p$ a point of $X$, $Y$ a projective variety, and $\phi : X - p \to Y$ a morphism. Then there exists a unique morphism $\overline{\phi} : X \to Y$ extending $\phi$.

By separatedness, as $Y$ is a variety, we have uniqueness; all that is necessary is existence.

Last day, we discussed an example of a map from $\mathbb{P}^1$ to $\mathbb{P}^n$: $[x; y] \mapsto [x^3; x^2y; xy^2]$ is defined away from $[0; 1]$; it's clear how to extend. You divide by $x$. We'll carry this example along as we do the proof.

Now let's get some consequences.

## 2. Consequences of theorem on extension of morphisms to projective varieties, over nonsingular points of curves

**From here on in, all curves will be assumed to be separated.** I should have made this part of the definition of "curve".

**Theorem.** Every nonsingular (separated) curve $C$ is quasiprojective.

Anything quasiprojective is separated, so that condition is necessary.

*Proof.* Cover $C$ with finite number of (nonsingular) affines $C_i \subset \mathbb{A}^{n_i}$ (recall that $C$ is compact, so we can do this). Let $Y_i \subset \mathbb{P}^{n_i}$ be projective closure of $C_i$; we get map $C \to Y_i$ by previous result. Hence get map $\phi : C \to \prod Y_i \subset \prod \mathbb{P}^{n_i}$, which by the Segre embedding we can consider as a closed subvariety of some bigger projective space $\mathbb{P}^N$ (recall our proof that the product of projective varieties was projective!). Let $Y$ be the closure (in $\mathbb{P}^N$) of the image of $C$; then $Y$ has dimension 1. (*Exercise:* how do we know that the dimension of the closure of the image is at most the dimension of the source?)

Next, let's check that $\phi$ is an isomorphism from $C$ onto its image. Recall that to check that $\phi$ is an isomorphism, we need only construct a candidate inverse morphism, and show that that composition is the identity on the level of sets. (This is a strategy that doesn't work for schemes.)

We'll do this first for each open $U_i$ in our cover of $C$. Notice that for each $i$, the open immersion $U_i \to Y_i$ factors through $U_i \hookrightarrow C \to Y \hookrightarrow \prod Y_i \to Y_i$. So to reverse the morphism $\phi : U_i \to \phi(U_i)$, take the image $\phi(U_i)$ of $U_i$ in $Y$, project to $Y_i$, note that it is in the image of the open immersion $U_i \to Y_i$.

Hence this shows that $U_i \to Y$ is an isomorphism onto its image $\phi(U_i)$. Let $\phi_i^{-1} : \phi(U_i) \to U_i$ be the inverse morphism; this gives a rational map $\phi_i^{-1} : \phi(C) \dashrightarrow C$. All of these rational maps agree on a dense open set $\cap U_i$, so as $C$ is separated, they can be glued together to give a morphism $\phi^{-1} : \phi(C) \to C$, which is pointwise an inverse to $\phi : C \to \phi(C)$. Hence $\phi$ is an isomorphism. $\square$

Immediately we have:

**Corollary.** Hence every nonsingular (separated) curve is birational to a projective curve.

We can tweak the proof of the Theorem above to get other interesting results. Here are two:

**Proposition.** Let $C_1$ and $C_2$ be two separated nonsingular curves, and let $\alpha : C_1 \dashrightarrow C_2$ be a *birational map* between them. Then they can be glued together via $\alpha$, i.e. there is another nonsingular curve $C = U_1 \cup U_2$, with $C_i$ isomorphic to $U_i$, and the birational map $U_1 \dashrightarrow U_2$ induced by $\alpha$.

*Examples.* $\mathbb{A}^1 \dashrightarrow \mathbb{A}^1$, $y \mapsto 1/y$, get $\mathbb{P}^1$. I also gave an example where you might think you'd get something non-separated, but you don't.

*Proof.* As $C_1$ and $C_2$ are birational, they have an isomorphic open subset $V$, which we can consider as being an open subset of both, although $\alpha$ should really be in the notation as well. By the theorem, consider $C_i$ as an open subset of $Y_i \subset \mathbb{P}^{n_i}$. Consider the morphism $\phi : V \to \mathbb{P}^{n_1} \times \mathbb{P}^{n_2}$.

Note that the rational map $\phi_1 : C_1 \dashrightarrow \mathbb{P}^{n_1} \times \mathbb{P}^{n_2}$ extends to a morphism (as $C_1$ is nonsingular and $\mathbb{P}^{n_2}$ is projective), and is an isomorphism onto its image (just take $\phi_1(C_1) \hookrightarrow \mathbb{P}^{n_1} \times \mathbb{P}^{n_2} \to \mathbb{P}^{n_1}$). So in particular its image $\phi_1(C_1)$ is nonsingular. (And of course it contains $\phi(V)$.) Just take the union of the images of $C_1$ and $C_2$ in $\mathbb{P}^{n_1} \times \mathbb{P}^{n_2}$. $\square$

**Proposition.** If two nonsingular *projective* curves $C_1$ and $C_2$ are birational, then they are isomorphic.

*Proof.* Consider $C_i$ as closed subvarieties of two projective spaces. Let $\phi_i : C_i \dashrightarrow C_{3-i}$ be the two rational maps composing to give the identity. Then they both extend to morphisms, by the main result of the previous section (saying that you can extend morphisms from nonsingular curves to projective varieties). Finally, $\phi_1 \circ \phi_2$ is the identity: it is a morphism from $C_2$ to itself, that agrees with the identity on a non-empty open subset. Hence (as $C_2$ is separated), it must *be* the identity. $\square$

## 3. Theorem: Finitely generated fields over $\overline{k}$ of transcendence degree 1 correspond to nonsingular projective curves (over $\overline{k}$)

Our next topic will be proving the statement in the title. This is very powerful. For example, if you want to prove things in algebra about finitely generated fields $/\overline{k}$ of transcendence degree 1, then you can prove them using geometry.

One consequence in this vein (which will likely be an exercise near the end of the semester):

**Luroth's theorem.** Any subfield of $\overline{k}(y)$ containing $\overline{k}$ is isomorphic either to $\overline{k}(y)$ or $\overline{k}$.

Conversely, you can prove things about geometry using algebra.

*Remark.* One of the directions of the statement is easy: given a nonsingular projective curve, its function field is a finitely generated field over $\overline{k}$ of transcendence degree 1.

Recall that if two nonsingular projective curve $C_1$ and $C_2$ are birational, then they are isomorphic (the last proposition of the previous section). So in order to prove the result, we'll have to take a finitely generated field $K$ of transcendence degree 1, and produce a nonsingular projective curve with function field $K$.

*Remark.* Note that we need *finitely generated*: every nonsingular projective curve has a function field of transcendence degree 1, that is finitely generated. But there are finitely genreated fields over $\overline{k}$ of transcendence degree 1 that are *not* finitely generated, e.g. $\overline{k}(t, t^{1/2}, t^{1/3}, \dots)$.

As an immediate corollary, we have:

**Corollary.** The following 3 categories are equivalent:

(i) nonsingular projective curves, and dominant morphisms;
(ii) curves, and dominant rational maps *(in the problem set, I added quasiprojective, but that isn't necessary)*;
(iii) function fields of dimension 1 over $\overline{k}$, and $\overline{k}$-homomorphisms.

I'll leave the details to you as an exercise. This is important to understand. (Sketch what the objects and morphisms are, and how the links go.)

With these insights, you can prove things such as: any nonsingular rational curve is an open subset of $\mathbb{P}^1$.

3.1. **Proof of Theorem.** *Hartshorne's argument.* Hartshorne has an interesting construction in Section I.6, but I feel like it is somewhat convoluted. I'll give a different proof, but first I'll give Hartshorne's strategy.

Here's how he constructs the curve $C$, as a prevariety. He does it by first giving the points, then the topology, then the structure sheaf. These definitions are very short: (i) the points of $C$ correspond to the discrete valuations of $K$. (ii) the topology on $C$ is the cofinite topology. (iii) the structure sheaf is given by: for each open $U \subset C$, let $\mathcal{O}_C(U) = \cap_{p \in U} R_p$ inside $K$.

Finally, he needs to show that this really gives a variety, and that it is projective. This is tricky.

*Another argument.* Here's another argument.

First, we construct a separated nonsingular curve $C$ with function field $K$. We already know that the points of $C$ will give discrete valuations of $K$; we'll ensure that they give *all* of the discrete valuations of $K$.

From a Corollary to the Key Technical Theorem, given any discrete valuation $v$ of $K/\overline{k}$, there is a nonsingular curve $Y_1$, one of whose points corresponds to $v$. Moreover, almost all the discrete valuations will be represented by points on $Y_1$. By repeating this with the valuations that we're missing, we have a finite number of nonsingular birational curves $Y_1, \ldots, Y_n$, all with function field $K$ (and hence all birational), such that each discrete valuation is represented by some point on at least one of the $Y_i's$.

By the Corollary at the end of the previous section, we can glue the $Y_i$'s together to get a single nonsingular curve $C$, such that all of the discrete valuations of $K/\overline{k}$ correspond to points of $C$. (*Sophisticated remark.* At this point, it is still possible that each valuation is represented by more than one point of $C$. This actually can't happen though; see the "earlier lemma" below.)

Thus: *there is a nonsingular curve $C$ (with function field $K$) such that all the valuations of $K$ correspond to points of $C$ (and vice versa).*

We will be done once we prove:

**Proposition.** $C$ is projective.

*Proof.* Suppose it isn't projective. By the previous section, $C$ is quasiprojective, so embed it in some $\mathbb{P}^n$, and let $p$ be any point in its closure $\overline{C}$ but not in $C$. Using $p$, we'll construct a *new* DVR of $K/\overline{k}$, contradicting the fact that $C$ is supposed to have points corresponding to *all* the DVRs of $K/\overline{k}$.

Leo pointed out that it wouldn't matter if all the points of $\overline{C}$ not in $C$ were nonsingular, but as it is enlightening to the rest of the argument, we'll still show that:

*Step 1: $p$ can't be a nonsingular point of $\overline{C}$.* Notice that $p$ can't be a nonsingular point of $\overline{C}$, or else it would correspond to a DVR of $K$. Then it would be a distinct DVR from any of the other points on $C$: if it were the same as the DVR arising from a point $q \in C$, then take an affine $U = U_i \cap C$ containing $p$ and $q$ (where $U_i$ is a standard affine open of projective space; it's easy to find such an open missing two points in projective space). On this affine curve, we have two distinct points giving rise to the same valuation. But we had a lemma a couple of classes ago (Class 17 to be precise, restated in Class 18):

*Earlier Lemma.* Let $Y$ be a prevariety, and suppose $p$ and $q$ are two points contained in a single affine open $U$, and $\mathcal{O}_{Y,q} \subset \mathcal{O}_{Y,p}$ (as subrings of $k(Y)$). Then $p = q$.

(I also gave a geometric argument that could replace this.)

In our situation, $\mathcal{O}_{Y,q} = \mathcal{O}_{Y,p}$. Hence we've shown that $p$ can't be a nonsingular point of $\overline{C}$, so we have a picture like this (draw it).

Take any affine open $U = U_i \cap C$ as before, containing $p$, so $U$ is an affine curve. Take the normalization in its function field. (Draw a picture.) Let $\tilde{U}$ be the variety corresponding to the normalization of $A(U)$; note that we have a normalization morphism $\nu : \tilde{U} \to U$. If we could prove that $\nu$ is surjective (which we will in a moment), then we'd be essentially done. Let $\tilde{p}$ be a point mapping to $p$; it is a nonsingular point of $\tilde{U}$. It corresponds to a DVR of $K/\overline{k}$. By our *earlier lemma*, it is a different DVR from any of the points of $U$. But we could have chosen $U$ to include any given point of of $C$, so this DVR isn't the same as the DVR corresponding to any point of $C$, giving a contradiction. $\square$

Next day, we will finish the proof, by showing that we get a contradiction even if $p$ is a singular point of $\overline{C}$.

In the last few minutes, I gave an aside on normalization; in the rest of the proof, we will (prove and) use the fact that the normalization morphism is surjective. Precisely: **Commutative Algebra Lemma.** Let $U$ be an affine variety, with coordinate ring $A(U)$. Let $\tilde{U}$ be the variety corresponding to the normalization of $A(U)$ (in its function field), and let $\nu : \tilde{U} \to U$ be the *normalization morphism* corresponding to the inclusion $A(U) \hookrightarrow A(\tilde{U})$. Then $\nu$ is surjective.

This will involve Nakayama's Famous Lemma.

**Coming next:** Finishing this proof; using Nakayama's lemma. Then on to the last topic of the semester: Invertible sheaves, line bundles. The canonical sheaf, genus. Riemann-Roch Theorem: statement (no proof) and applications. Riemann-Hurwitz.