INTRODUCTION TO ALGEBRAIC GEOMETRY, CLASS 17

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Contents

Thanks for responses to diophantine e-mail. I'll respond soon en masse.

Hand in problem sets at end.

1. A little more on completions: changing coordinates

A few quick notes:

Note that the completion of $\overline{k}[x,y]$ at $\mathfrak{m} = (x,y)$ is $\overline{k}[[x,y]]$.

Note that you can make sense of the infinite sum $x_0 + x_1 + \dots$ in $\hat{R}$, the completion of a ring with respect to $\mathfrak{m}$ if $x_i \subset \mathfrak{m}^i$ (in the case of a DVR, $v(x_i) \geq i$).

Note that if $f(x) \in \overline{k}[[x]]$ and $u \in \mathfrak{m} \subset \hat{R}$, then $f(u)$ makes sense as an element of $\hat{R}$. Reason: $u^n \in \mathfrak{m}^n$.

In $\overline{k}[[x]]$, let $u = x + g_{>1}(x)$. Then this invertible, i.e. you can write $x = h(u)$. Here's how, informally. $x = u - g_{>1}(x)$; this will tell us the "degree 1" term of h. Then $x = u - g_{>1}(u - g_{>1}(x))$; this will tell us the "degree 2" term. And so on.

This result can be expressed in another way: Suppose you have an element $f \in \overline{k}[[x]]$, which I prefer to think of as a "function in a formal neighbourhood of a point on a nonsingular curve". Suppose $v(f) = 1$, i.e. $f = ax + g_{>1}(x)$, where $a \neq 0$. Then there is a change of coordinates so that $f(u) = u$, which is given by $u = ax + g_{>1}(x)$. (Note that changes of coordinates are by definition invertible.)

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In $\overline{k}[[x,y]]$, suppose $f(x,y) = f_1 + f_2 + \dots$, and $f_1 \neq 0$, or equivalently,

$$\left(\frac{\partial f}{\partial x}(0,0), \frac{\partial f}{\partial y}(0,0) \right) \neq (0,0)$$

(the Jacobian condition for nonsingularity!). Let's change coordinates to put this function in nice form.

First, make a linear change of coordinates so that $f_1 = y$. Hence $f(x,y) = y + f_{>1}$, where $f_{>1}$ has elements of degree greater than 1. We can change coordinates (invertibly, by definition of change of coordinates) again so that $f(u,v) = v$, by taking $u = x$, $v = f(x,y) = y + f_{>1}$. We need to check that this is invertible. I sketched why; in short, $y = v - f_{>1}(u, v - f_{>1}(u, \dots))$. More precisely, we can read off the first n degrees of y (in terms of u and v).

Here's the beginning of the second-last exercise on the problem set I just gave out. Suppose the characteristic of $\overline{k}$ is not 2, and $f(x,y) = f_2 + f_{>2} \in \overline{k}[[x,y]]$, with $f_2 \neq 0$. Then $f_2 = \alpha x^2 + \beta xy + \gamma y^2$. Suppose further that $\beta^2 - 4\alpha\gamma \neq 0$, i.e. f_2 factors into $(\delta_1 x + \epsilon_1 y)(\delta_2 x + \epsilon_2 y)$, where

$$\det \begin{pmatrix} \delta_1 & \epsilon_1 \\ \delta_2 & \epsilon 2 \end{pmatrix} \neq 0.$$

For convenience, make a linear change of coordinates $a = \delta_1 x + \epsilon_1 y$, $b = \delta_2 x + \epsilon_2 y$ (which is invertible by linear algebra). Then you can further change coordinates to u, v, so that $u = a + g_{>1}(a,b)$, $v = b + h_{>1}(a,b)$ and $uv = f$. (In the problem set, you will justify this step, using *Hensel's Lemma*.)

This is indeed invertible, by the same trick:

$$a = u - g_{>1}(u - g_{>1}(...), v - h_{>1}(...))$$

$$b = v - h_{>1}(u - g_{>1}(...), v - h_{>1}(...)).$$

As a consequence, the change of coordinates is

$$u = \delta_1 x + \epsilon_1 y + \text{higher}$$

$$v = \delta_2 x + \epsilon_2 y + \text{higher}$$

so long as it is "invertible to first order", i.e.

$$\det \begin{pmatrix} \delta_1 & \epsilon_1 \\ \delta_2 & \epsilon 2 \end{pmatrix} \neq 0.$$

This generalizes to more variables.

2. Integral closure and Dedekind domains

Reminder: R is integrally closed means that it is integrally closed in its function field K. In other words, the only solutions in K to $x^n + a_{n-1}x^{n-1} + \cdots + a_0 = 0$ where $a_i \in R$, lie in R.

There are two "canonical" examples.

Example. We already know $\overline{k}[t]$ has dimension one, and it's obviously a domain, and it is Noetherian by the Hilbert Basis Theorem. $\overline{k}[t]$ is integrally closed in its function field, by explicit calculation: If $f \in \overline{k}(t)$ is in the integral closure, so $f^n + a_{n-1}(t) f^{n-1} + \cdots + a_0(t) = 0$, where $a_i(t)$ are polynomials. Write f in lowest terms as $p(t)/q(t)$, where p and q are relatively prime polynomials. Then $p(t)^n + a_{n-1}(t)p(t)^{n-1}q(t) + \cdots + a_0(t)q(t)^n = 0$. Then $q(t)$ must divide $p(t)^n$, so as p and q are relatively prime, $q(t)$ must be a constant, so f was indeed a polynomial.

Indeed its localizations at the primes $(t - a)$ are DVR's, as we observed earlier.

Example. We already know that $\mathbb{Z}$ has dimension one, and it is obviously a Noetherian domain. Let's check that it is integrally closed in its function field. (Mimic the above proof.)

For more on integral closure, see next day's notes.

We'll need two more facts about integral closures.

Theorem (Integral closure is a local property). If R is integrally closed if and only if the localization of R at each of its maximal ideals is integrally closed.

(One direction is easy; the other is harder.)

Definition. Let A be an integral domain, which is a finitely generated algebra over $\overline{k}$. Let K be the quotient field of A, and let L be a finite algebraic extensions of K. Then the *integral closure of A in L* consists of those elements of L satisfying $x^n + a_{n-1}x^{n-1} + \cdots + a_0 = 0$ where $a_i \in A$.

For example, the *integral closure of A in its quotient field* is *the integral closure of A*.

It isn't hard to show that integral closure A' of A in L must be integrally closed, i.e. the only solutions to such equations in L, where now the a_i are supposed to lie in A', lie in A'.

Theorem (Finiteness of integral closure). Then A' is a finite A-module, and is also a finitely generated $\overline{k}$-algebra.

Corollary. Every prime in A has a preimage in A'. *(Appears in class 19 or 20; should be here.)*

Definition. A *Dedekind domain* is an integrally closed noetherian domain of dimension one.

Quick scheme-theoretic remark. It's basically immediate that a ring R is a Dedekind domain if and only if $\operatorname{Spec} R$ is a nonsingular dimension one scheme — just use that commutative algebra fact from last day (the 4 equivalent definitions of DVR). (That's not quite right: we need to include *Noetherian* in the adjectives in front of *scheme*. I haven't defined what that means, but you might be able to guess.)

Remark. Because a Dedekind domain R has dimension one, it means that its prime ideals are: the ideal (0), and the maximal ideals. If you localize R at a maximal ideal $\mathfrak{m}$, you get a discrete valuation ring, by the commutative algebra facts above.

Example, generalizing $\overline{k}[t]$. Let X be a nonsingular affine curve, with ring of functions R. Then R is a Noetherian domain of dimension one. It is also integrally closed: by the commutative algebra fact and the Theorem on "locality of integral closure" above, we need to check that for each maximal ideal of $\mathfrak{m}$ of R, the localization $R_{\mathfrak{m}}$ is a regular local ring. But this is the stalk of the structure sheaf at a point, so by our algebraic definition of nonsingularity, this is true.

To show you how nontrivial it is, we've shown for example that the ring $\overline{k}[x,y]/(y^2 - x^3 + x)$ is integrally closed in its function field (in characteristic different from 2). Just check that $y^2 - x^3 + x$ cuts out a nonsingular curve in $\mathbb{A}^2$, using the Jacobian condition, and you're done!

Exercise. The function field of this curve is not isomorphic to $\overline{k}(t)$. Hence this really is a different example. (Hints given.)

Theorem. Take any Dedekind domain R that is a finitely generated algebra over $\overline{k}$, and let K be its field of fractions. Let L be a finite extension of K, and let S be the integral closure of R in L. Then S is also a Dedekind domain, and is also a finitely generated algebra over $\overline{k}$.

(Perhaps keep this on the board.)

Proof. By the Theorem on finiteness of integral closure, S is a finitely generated algebra over $\overline{k}$. It is a domain, as it is a subring of a domain L. It has dimension 1, as its function field L has the same transcendence degree over $\overline{k}$ as K (as L is finite over K). And it is integrally closed. $\square$

Something more general is true: you can excise the "finitely generated algebra over $\overline{k}$". I'll give the result because it will be of interest to those of you with arithmetic interests, but as we won't use it in the course, I won't prove it. Those of you learning schemes will find it useful, though.

Theorem that we won't use. Take any Dedekind domain R, and let K be its field of fractions. Let L be a finite extension of K, and let S be the integral closure of R in L. Then S is also a Dedekind domain.

Example, generalizing $\mathbb{Z}$. Let $R = \mathbb{Z}$, so $K = \mathbb{Q}$. Let L be a finite extension of $\mathbb{Q}$, called a *number field*. Then S in this case is called the *ring of integers* in the number field. For example, if $L = \mathbb{Q}(i)$, then $S = \mathbb{Z}[i]$, the example that has come up in the problem sets.

3. Extending rational maps of nonsingular curves

Goal: Rational maps of nonsingular curves to projective varieties can be extended to morphisms.

Discuss reasons why you can't extend $\mathbb{P}^1 \dashrightarrow \mathbb{A}^1$, $\mathbb{A}^2 \dashrightarrow \mathbb{P}^1$. Perhaps mention same thing with cusp and node.

Discussed extending map from $\mathbb{P}^1$ to $\mathbb{P}^4$. Then analytically. This is l'Hopital in a vague sense: can take limits.

Lemma. Let Y be a prevariety, and suppose P and Q are two points contained in a single affine open U, and $\mathcal{O}_{Y,Q} \subset \mathcal{O}_{Y,P}$ (as subrings of $k(Y)$). Then $P = Q$.

Recall that if any two points are contained in an affine open, then Y is separated, i.e. a variety. Recall that all quasiprojective varieties and affine varieties have this property; this was even how we showed that quasiprojective varieties are in fact varieties.

Proof. We can replace Y be U, so we may as well assume Y is affine. Let $A(Y)$ be the affine coordinate ring. Then there are maximal ideals $\mathfrak{m}, \mathfrak{n} \subset A(Y)$ such that $\mathcal{O}_{Y,P} = A(Y)_{\mathfrak{m}}$ and $\mathcal{O}_{Y,Q} = A(Y)_{\mathfrak{n}}$. If $\mathcal{O}_{Y,Q} \subset \mathcal{O}_{Y,P}$, we must have $\mathfrak{m} \subset \mathfrak{n}$. But $\mathfrak{m}$ is a maximal ideal, so $\mathfrak{m} = \mathfrak{n}$, so $P = Q$ (as maximal ideals are in 1-1 correspondence with points, by the Nullstellensatz). □

Also used in Class 19 or 20. Compare also Lemma I used in proof of Key Technical Theorem.

Key technical Theorem (tricky). Let K be a function field of dimension one over $\overline{k}$, and let $x \in K$. Then the set of discrete valuations of $K/\overline{k}$ where $v(x) < 0$ is *finite*.

Here's the geometric idea. Suppose $K = \overline{k}(t)$, the function field of $\mathbb{P}^1$. $x \in K$, so x is a rational function of $\mathbb{P}^1$. Then the discrete valuations of $K/\overline{k}$ correspond to points of $\mathbb{P}^1$ (exercise). The points/valuations v_p where $v_p(x) < 0$ are precisely the points where x has poles; there are only a finite number of such points/valuations.

Coming up. I'll prove the "key technical theorem". Then I'll prove that if X is a nonsingular curve, p a point of X, Y a projective variety, and $\phi : X - p \to Y$ a morphism, then there exists a unique morhpism $\bar{\phi} : X \to Y$ extending ϕ. From there, we will prove that every nonsingular curve is quasiprojective, so every curve is birationally equivalent to a nonsingular projective curve.