# INTRODUCTION TO ALGEBRAIC GEOMETRY, CLASS 12

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*Problem sets back at end.*

## 1. Products of projective varieties; the Segre map

**Theorem.** The product of two projective varieties is a projective variety.

*Proof.* Since a closed subvariety of a projective variety is a projective variety, it is enough to show that $\mathbb{P}^m \times \mathbb{P}^n$ is a projective variety. So we're done, modulo the following lemma.

**Lemma.** $\mathbb{P}^m \times \mathbb{P}^n$ is a projective variety.

Describe the image of $(x_0; \dots; x_m) \times (y_0; \dots; y_n)$ in $\mathbb{P}^{(m+1)(n+1)-1}$: $z_{ij} = x_i y_j$. This is called the *Segre embedding*.

The image is in the locus of rank 1 matrices; in fact it is *precisely* the rank 1 matrices.

Make clear that you can recover the points of $\mathbb{P}^m$ and $\mathbb{P}^n$.

You have defining equations: all the $2 \times 2$ minors. Hence the image $V$ is a projective prevariety.

We've described the map on points one way. Describe in another way.

Then you have to check on patches. Not hard, but I'll omit it; you can read about it in any of the references. The details of the proof are important, but more

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important is the geometric insight into what's going on. So let's go through an example. □

*Remark (another parallel between affine and projective prevarieties).* Recall that if $X$ and $Y$ are affine, with coordinate rings $A(X)$ and $A(Y)$, then the coordinate ring of $X \times Y$ is $A(X) \otimes_{\overline{k}} A(Y)$. Something similar happens with projective prevarieties too. Suppose $X$ and $Y$ are projective, and lie in $\mathbb{P}^m$ and $\mathbb{P}^n$ respectively, so $X$ has *graded* coordinate ring $R(X) = \overline{k}[x_0, \dots, x_m]/I(X)$, where $I(X)$ is a *homogeneous* ideal, and $R(Y) = \overline{k}[y_0, \dots, y_n]/I(Y)$ similarly. Then under the Segre embedding (so $X \times Y \subset \mathbb{P}^{(m+1)(n+1)-1}$), $R(X \times Y) \cong R(X) \otimes R(Y)$ where the grading behaves well under tensor product (i.e. the tensor product of the $i$th graded piece of $R(X)$ and the $j$th graded piece of $R(Y)$ lies in the $(i+j)$th graded piece of $R(X \times Y)$.

*Exercise.* Prove that this is the case if $X = \mathbb{P}^m$ and $Y = \mathbb{P}^n$. Caution: I've shown that the product is cut out by the equations of the $2 \times 2$ minors, but I didn't show that the ideal $I(X \times Y)$ is generated by the $2 \times 2$ minors, although that's true. (Analogy: In $\mathbb{A}^2$, the $y$-axis $Y$ is cut out by the equation $x^2 = 0$, but $I(Y)$ isn't generated by $x^2$; it's generated by $x$.)

1.1. $\mathbb{P}^1 \times \mathbb{P}^1$ **and the smooth quadric surface.** When you do the numerology with $m = n = 1$, we see that $\mathbb{P}^1 \times \mathbb{P}^1$ maps into $\mathbb{P}^3$, and it is given by a single equation $wx - yz = 0$.

Draw a picture.

*Remark.* Almost all quadric surfaces look the same. Hence we know the (classical) topology of almost all quadric surfaces: they are products of 2-spheres.

1.2. **Rulings of the smooth quadric surface.** Show them the lines in the real picture. We'll see these algebraically. First of all, a line in $\mathbb{P}^3$ is the intersection of two distinct hyperplanes, e.g. $w = x = 0$. They are isomorphic to $\mathbb{P}^1$, e.g for $w = x = 0$, the isomorphism is given by $(0; 0; y; z) \leftrightarrow (y; z)$. (For hyperplanes with uglier coefficients, just change coordinates!)

Give the projection to the first $\mathbb{P}^1$, and to the second $\mathbb{P}^1$: $(w_0; x_0; y_0; z_0) \mapsto (w_0; y_0)$ or $(z_0; x_0)$ is the map to the first $\mathbb{P}^1$, and $(w_0; x_0; y_0; z_0) \mapsto (w_0; z_0)$ or $(y_0; x_0)$ is the map to the second $\mathbb{P}^1$.

The two one-parameter family of lines: first note that the Segre map is

$$(a; b) \times (c; d) \mapsto (ac; bd; ad; bc).$$

The first family of lines is: fix $(a; b) = (a_0; b_0)$, and consider: $(a_0c; b_0d; a_0d; b_0c)$; this is the line that is the intersection of $b_0w = a_0x$ and $b_0y = a_0z$. The second is similar; just switch the roles of $(a; b)$ and $(c; d)$.

## 2. Defining varieties

**Definition.** A prevariety $X$ is a *variety* if for all prevarieties $Y$ and for all morphisms $f$ and $g$ from $Y$ to $X$, the locus where they agree $\{y \in Y | f(y) = g(y)\}$ is a closed subset of $Y$.

This is often called the *separatedness* condition. Mumford calls this the *Hausdorff* axiom, because it is the analogue of the Hausdorff condition in the definition of a manifold.

Note that the line with the doubled origin is not a variety.

**Remark.** An open (resp. closed) subprevariety of a variety is a variety.

**Remark.** An affine variety is a variety. (Sorry for nasty notation!) Reason: Suppose $X$ is affine, and $Y$ is any prevariety, and $f$, $g$ are two morphisms $Y \to X$. Then the subset of $Y$ where $f(y) = g(y)$, $\{y \in Y | f(y) = g(y)\}$ is as follows.

$x_1 = x_2$ in affine $X$ iff for all regular functions $s \in A(X)$, $s(x_1) = s(x_2)$. Hence $\{y \in Y | f(y) = g(y)\}$ is the locus where all regular functions $s(f(y)) - s(g(y))$ vanish (where $s$ runs through all of $A(X)$). This is a closed set.

**Remark.** We will soon see that projective prevarieties are varieties.

Special case: if $Y = X \times X$, and $f$ and $g$ are the projections. $\Delta(X)$ is the locus where $f$ and $g$ agree. Hence if $X$ is separated, then $\Delta(X)$ is closed in $X \times X$.

This special case is all you need to check:

**Proposition (Criterion for separatedness).** A prevariety $X$ is a variety iff $\Delta(X)$ is closed in $X \times X$.

*Proof.* Neat trick, which is a recurring theme. Suppose you have two $f$ and $g$, which induce a morphism $(f, g) : Y \to X \times X$. Then

$$\{y \in Y | f(y) = g(y)\} = (f, g)^{-1}(\Delta(X)).$$

□

*Exercise.* Show that the line with the doubled origin is not separated, using this condition.

**Proposition (Another criterion for separatedness).** Let $X$ be a prevariety. Assume that for all $x, y \in X$ there is an open affine $U$ containing both $x$ and $y$. Then $X$ is a variety.

*Proof.* We use the definition. Consider two functions $f, g : Y \to X$, and let $Z$ be the locus where they agree $Z = \{y \in Y | f(y) = g(y)\}$. Let $z$ be in the closure of $Z$, and let $x_1 = f(z)$, $x_2 = g(z)$. We want to show that $Z$ is closed, so we want to show that $x_1 = x_2$. By assumption, there is an open affine $V \subset X$ containing $x_1$ and $x_2$. Let $U = f^{-1}(V) \cap g^{-1}(V)$; it is an open neighbourhood of $Z$. Then consider the "restricted morphisms" $f, g : U \to V$; now $V$ is affine (hence a variety), so

$$Z \cap U = \{y \in U | f(y) = g(y)\}$$

is closed in $U$. Thus $z \in Z \cap U$, and $Z$ is indeed closed. □

**Corollary.** Every quasiprojective prevariety $X$ (i.e. open subset of a projective prevariety) is a variety.

*Proof.* As every projective prevariety $X$ is a closed subprevariety of $\mathbb{P}^n$ (for some $n$, by definition), we just need to show that $\mathbb{P}^n$ is a variety. So given any two points $y, z \in \mathbb{P}^n$, we just need to find an affine open containing both. Consider any hyperplane $H$ not meeting $y$ or $z$; there are lots! (Instead of proving this, let me just convince you that it is obvious by example. If $y = (1; 0; 0)$ and $z = (0; 1; 1)$, take the hyperplane $x_1 - x_2 = 0$.) The complement of a hyperplane is affine (proved earlier), so we're done. □

Another nice property of varieties: the intersection of any two affine opens is another affine open. I don't foresee using this, so I won't prove it, but you can find a proof in Mumford (p. 55) or Hartshorne (Exercise II.4.4).

This isn't a criterion (as it also holds for the line with the doubled origin), but it can be strengthed a little into a criterion.

Here's a prevariety that doesn't have this property: the *plane* with the doubled origin. The intersection of the two elements of the "obvious" affine cover is the plane minus the origin, which you've shown earlier is not affine.

## 3. Rational maps

I said a few words about rational maps, the topic we'll address on Thursday.

We can reinterpret the definition of separatedness as follows. Suppose I'm thinking of a morphism $f : Y \to X$, where $X$ is a *variety*. And suppose I tell you what the morphism is on a non-empty open set $U \subset Y$, i.e. I tell you $f|_U : U \to X$. Then there is only one way for you to recover the "full" morphism $f$. Because if you have two different morphisms $f_1$ and $f_2$ extending $f$, then you have two morphisms $f_1, f_2 : Y \to X$ which agree on a dense open set (the set $U$; recall that dense means that the closure of $U$ is $Y$), and agree on a closed set (as $X$ is separated). Hence they have to agree everywhere.

Coming soon: completeness (roughly, compactness), dimension, smoothness. Then we can talk about curves.