MATH 113 PRACTICE FINAL EXAM

Each problem is worth 6 points. Justify your answers completely (unless otherwise noted). The problems are not listed in order of difficulty, so use your time wisely. The starred problems are intended to be more challenging; don't spend too much time on them!

1.

(a) Is $A = \begin{pmatrix} 3/5 & -4/5 \\ -4/5 & -3/5 \end{pmatrix}$ the matrix of a reflection? (Hint: how can you tell if an operator is a reflection in terms of its eigenvalues and eigenvectors?)

(b) For which real numbers k is the following matrix invertible?

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 1 \\ -1 & k & 0 & 0 & 0 \\ 0 & 0 & 1 & 3 & 9 \\ 0 & 0 & 1 & 4 & 16 \\ 0 & 0 & 1 & k & k^2 \end{pmatrix}$$

(c) Suppose $T : \mathbb{R}^3 \to \mathbb{R}^3$ is an operator corresponding to rotating by 90 degrees about the line generated by the vector $(1, 2, 3)$. Find the determinant of T, and the characteristic polynomial. (Hint: don't try to find the matrix explicitly.)

2. Show that

$$A = \begin{pmatrix} -1 & 0 \\ -4 & 1 \end{pmatrix}$$

is diagonalizable by writing it as $A = C^{-1}DC$ where C is an invertible matrix, and D is as diagonalizable matrix. Explain this formula in terms of a change of basis.

3. Show that every subspace of $\mathbb{F}^n$ is the solution set of some homogeneous system of n equations in n unknowns.

4. Suppose A is a linear operator on a 10-dimensional space V, such that $A^2 = 0$.
(a) Show that $\mathrm{Im}(A) \subset \mathrm{Ker}(A)$.
(b) Show that the rank of A is at most 5. (Hint: (a) and the rank-nullity theorem might help.)
(c) Give an example to show that the rank of A could be 5.

5. Suppose $T : V \to V$ is a linear operator on a finite-dimensional vector space. Show that T and T^* have the same characteristic polynomial, and the same minimal polynomial.

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6. Suppose $T : V \to V$ is a linear operator on a finite-dimensional *complex* vector space. If the *minimal* polynomial doesn't have repeated prime factors, show that the matrix is diagonalizable.

7. Suppose $T : V \to V$ is a linear operator on a finite-dimensional vector space. Show that there *always* exist vectors $\vec{v} \in V$ such that $\min P_{T,\vec{v}} = \min P_T$. (Hint: use the prime-power decomposition theorem.)

8*. Recall that the dot product is defined as follows: if $\vec{v} = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$ and $\vec{w} = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$, $\vec{v} \cdot \vec{w} = a_1 b_1 + \cdots + a_n b_n$. Suppose A is a matrix with column vectors $\vec{v}_1, \ldots, \vec{v}_n$ (so $|\det A|$ is the volume of the parallelepiped spanned by $\vec{v}_1, \ldots, \vec{v}_n$). Show that

$$(\det A)^2 = \det \begin{pmatrix} \vec{v}_1 \cdot \vec{v}_1 & \vec{v}_1 \cdot \vec{v}_2 & \cdots & \vec{v}_1 \cdot \vec{v}_n \\ \vec{v}_2 \cdot \vec{v}_1 & \vec{v}_2 \cdot \vec{v}_2 & \cdots & \vec{v}_2 \cdot \vec{v}_n \\ \vdots & & \ddots & \vdots \\ \vec{v}_n \cdot \vec{v}_1 & \vec{v}_n \cdot \vec{v}_2 & \cdots & \vec{v}_n \cdot \vec{v}_n \end{pmatrix}$$

(This is a useful way of computing volume.)