FOUNDATIONS OF ALGEBRAIC GEOMETRY PROBLEM SET 9

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This set covers classes 17 and 18.

Please *read all of the problems*, and ask me about any statements that you are unsure of, even of the many problems you won't try. Hand in nine solutions, where each "-" problem is worth half a solution, each "+" problem is worth one-and-a-half, and each "++" problem is worth two. *You are allowed to hand in up to three problems from previous sets that you have not done.* If you are ambitious (and have the time), go for more. Try to solve problems on a range of topics. You are encouraged to talk to each other, and to me, about the problems. Some of these problems require hints, and I'm happy to give them!

1. Show that $f : X \to Y$ is quasiseparated if and only if for any affine open $\operatorname{Spec} A$ of Y, and two affine open subsets U and V of X mapping to $\operatorname{Spec} A$, $U \cap V$ is a *finite* union of affine open sets. (Hint: compare this to the proposition showing that the intersection of two affine open sets on a separated scheme over an affine scheme is affine.)

2. *(a nonquasiseparated scheme)* Let $X = \operatorname{Spec} k[x_1, x_2, \dots]$, and let U be $X - [\mathfrak{m}]$ where $\mathfrak{m}$ is the maximal ideal $(x_1, x_2, \dots)$. Take two copies of X, glued along U. Show that the result is not quasiseparated. (This open immersion $U \hookrightarrow X$ came up earlier, as an example of a nonquasicompact open subset of an affine scheme.)

3. Prove that the condition of being quasiseparated is local on the target. (Hint: the condition of being quasicompact is local on the target; use a similar argument.)

4. Suppose $\pi : Y \to X$ is a morphism, and $s : X \to Y$ is a *section* of a morphism, i.e. $\pi \circ s$ is the identity on X. Show that s is a locally closed immersion. Show that if π is separated, then s is a closed immersion.

5. Show that a A-scheme is separated (over A) if and only if it is separated over $\mathbb{Z}$. (In particular, a complex scheme is separated over $\mathbb{C}$ if and only if it is separated over $\mathbb{Z}$, so complex geometers and arithmetic geometers can communicate about separated schemes without confusion.)

6+. *(useful exercise: The locus where two morphisms agree)* Suppose f and g are two morphisms $X \to Y$, over some scheme Z. We can now give meaning to the phrase 'the locus where f and g agree', and that in particular there is a smallest locally closed subscheme where they agree. Suppose $h : W \to X$ is some morphism (perhaps a locally closed immersion). We say that f and g agree on h if $f \circ h = g \circ h$. Show that there is a locally closed subscheme $i : V \hookrightarrow X$ such that any morphism $h : W \to X$ on which f and g agree factors

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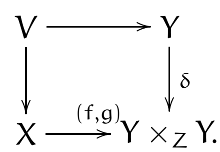
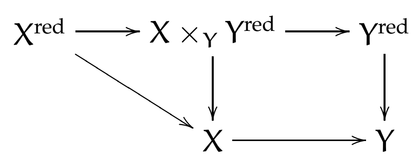
uniquely through i, i.e. there is a unique $j : W \to V$ such that $h = i \circ j$. (You may recognize this as a universal property statement.) Show further that if $V \to Z$ is separated, then $i : V \hookrightarrow X$ is a closed immersion. Hint: define V to be the following fibered product:



As δ is a locally closed immersion, $V \to X$ is too. Then if $h : W \to X$ is any scheme such that $g \circ h = f \circ h$, then h factors through V.

7. Show that the line with doubled origin X is not separated, by finding two morphisms $f_1, f_2 : W \to X$ whose domain of agreement is not a closed subscheme. (Another argument was given in an exercise, I believe last day.)

8. Suppose P is a class of morphisms such that closed immersions are in P, and P is closed under fibered product and composition. Show that if $f : X \to Y$ is in P then $f^{red} : X^{red} \to Y^{red}$ is in P. (Two examples are the classes of separated morphisms and quasiseparated morphisms.) Hint:



9-. Interpret rational functions on a separated integral scheme as rational maps to $\mathbb{A}^1_{\mathbb{Z}}$. (This is analogous to functions corresponding to morphisms to $\mathbb{A}^1_{\mathbb{Z}}$, an earlier exercise.)

10. In class, we prove that two S-morphisms $f_1, f_2 : U \to Z$ from a reduced scheme to a separated S-scheme agreeing on a dense open subset of U are the same. Give examples to show how this breaks down when we give up reducedness of the base or separatedness of the target. Here are some possibilities. For the first, consider the two maps $\operatorname{Spec} k[x,y]/(y^2, xy) \to \operatorname{Spec} k[t]$, where we take f_1 given by $t \mapsto x$ and f_2 given by $t \mapsto x + y$; f_1 and f_2 agree on the distinguished open set $D(x)$. (See Figure 1.) For the second, consider the two maps from $\operatorname{Spec} k[t]$ to the line with the doubled origin, one of which maps to the "upper half", and one of which maps to the "lower half". these to morphisms agree on the dense open set $D(f)$. (See Figure 2.)

11. Show that the graph of a rational map is independent of the choice of representative of the rational map.

12. *(important)* Show that you can compose two rational maps $f : X \dashrightarrow Y$, $g : Y \dashrightarrow Z$ if f is dominant. In particular, integral separated schemes and dominant rational maps between them form a category which is geometrically interesting.

13-. Show that dominant rational maps give morphisms of function fields in the opposite direction.

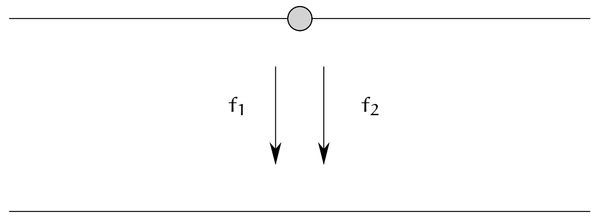


FIGURE 1. Two different maps from a nonreduced scheme agreeing on an open set

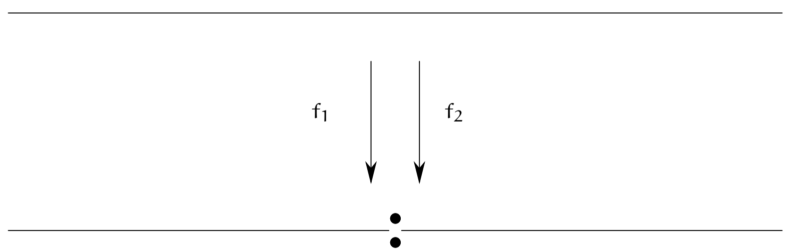


FIGURE 2. Two different maps to a nonseparated scheme agreeing on an open set

14. Let K be a finitely generated field extension of k. Show there exists an irreducible k-variety with function field K. (Hint: let $x_1, \dots, x_n$ be generators for K over k. Consider the map $k[t_1, \dots, t_n] \to K$ given by $t_i \mapsto x_i$, and show that the kernel is a prime ideal $\mathfrak{p}$, and that $k[t_1, \dots, t_n]/\mathfrak{p}$ has fraction field K. This can be interpreted geometrically: consider the map $\operatorname{Spec} K \to \operatorname{Spec} k[t_1, \dots, t_n]$ given by the ring map $t_i \mapsto x_i$, and take the closure of the image.)

15. Use our discussion in class to find a "formula" yielding all Pythagorean triples.

16. Show that the conic $x^2+y^2 = z^2$ in $\mathbb{P}^2_k$ is isomorphic to $\mathbb{P}^1_k$ for any field k of characteristic not 2. (We've done this earlier in the case where k is algebraically closed, by diagonalizing quadrics.)

17. Find all rational solutions to $y^2 = x^3+x^2$, by finding a birational map to $\mathbb{A}^1$, mimicking what worked with the conic.

18. Find a birational map from the quadric $Q = \{x^2 + y^2 = w^2 + z^2\}$ to $\mathbb{P}^2$. Use this to find all rational points on Q. (This illustrates a good way of solving Diophantine equations. You will find a dense open subset of Q that is isomorphic to a dense open subset of $\mathbb{P}^2$, where you can easily find all the rational points. There will be a closed subset of Q where the rational map is not defined, or not an isomorphism, but you can deal with this subset in an ad hoc fashion.)

19++. *(a first view of a blow-up)* Let k be an algebraically closed field. (We make this hypothesis in order to not need any fancy facts on nonsingularity.) Consider the rational map $\mathbb{A}^2_k \dashrightarrow \mathbb{P}^1_k$ given by $(x, y) \mapsto [x; y]$. I think you have shown earlier that this rational map cannot be extended over the origin. Consider the graph of the birational map, which we denote $\mathrm{Bl}_{(0,0)} \mathbb{A}^2_k$. It is a subscheme of $\mathbb{A}^2_k \times \mathbb{P}^1_k$. Show that if the coordinates on $\mathbb{A}^2$ are x, y, and the coordinates on $\mathbb{P}^1$ are u, v, this subscheme is cut out in $\mathbb{A}^2 \times \mathbb{P}^1$ by the single equation $xv = yu$. Describe the fiber of the morphism $\mathrm{Bl}_{(0,0)} \mathbb{A}^2_k \to \mathbb{P}^1_k$ over each closed point of $\mathbb{P}^1_k$. Describe the fiber of the morphism $\mathrm{Bl}_{(0,0)} \mathbb{A}^2_k \to \mathbb{A}^2_k$. Show that the fiber over $(0, 0)$ is an effective Cartier divisor (a closed subscheme that is locally principal and not a zero-divisor). It is called the *exceptional divisor*.

20. *(the Cremona transformation, a useful classical construction)* Consider the rational map $\mathbb{P}^2 \dashrightarrow \mathbb{P}^2$, given by $[x; y; z] \to [1/x; 1/y; 1/z]$. What is the the domain of definition? (It is bigger than the locus where $xyz \neq 0$!) You will observe that you can extend it over codimension 1 sets. This will again foreshadow a result we will soon prove.

21. Show that $\mathbb{A}^1_{\mathbb{C}} \to \operatorname{Spec} \mathbb{C}$ is not proper, by finding a base change that turns this into a non-closed map. (Hint: Consider $\mathbb{A}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}} \to \mathbb{P}^1_{\mathbb{C}}$.)

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