FOUNDATIONS OF ALGEBRAIC GEOMETRY PROBLEM SET 3

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This set is due at noon on Friday October 19. You can hand it in to Jarod Alper (jarod@math.stanford.edu) in the big yellow envelope outside his office, 380-J. It covers classes 5 and 6.

Please *read all of the problems*, and ask me about any statements that you are unsure of, even of the many problems you won't try. Hand in eight solutions, where each "-" problem is worth half a solution and each "+" problem is worth one-and-a-half. If you are ambitious (and have the time), go for more. Try to solve problems on a range of topics. You are encouraged to talk to each other, and to me, about the problems. Some of these problems require hints, and I'm happy to give them!

Class 5.

1. If $f : X \to Y$ is a continuous map, and $\mathcal{G}$ is a sheaf on Y, show that $f^{-1}\mathcal{G}^{\text{pre}}(U) := \varinjlim_{V \supset f(U)} \mathcal{G}(V)$ defines a presheaf on X. (Possible hint: Recall the explicit description of direct limit: sections are sections on open sets containing $f(U)$, with an equivalence relation.)

2. Show that the stalks of $f^{-1}\mathcal{G}$ are the same as the stalks of $\mathcal{G}$. More precisely, if $f(x) = y$, describe a natural isomorphism $\mathcal{G}_y \cong (f^{-1}\mathcal{G})_x$. (Possible hint: use the concrete description of the stalk, as a direct limit. Recall that stalks are preserved by sheafification.)

3-. *(easy but useful)* If U is an open subset of Y, $i : U \to Y$ is the inclusion, and $\mathcal{G}$ is a sheaf on Y, show that $i^{-1}\mathcal{G}$ is naturally isomorphic to $\mathcal{G}|_U$.

4-. *(easy but useful)* If $y \in Y$, $i : \{y\} \to Y$ is the inclusion, and $\mathcal{G}$ is a sheaf on Y, show that $i^{-1}(\mathcal{G})$ is naturally isomorphic to the stalk $\mathcal{G}_y$.

5. Show that f^{-1} is an exact functor from sheaves of abelian groups on Y to sheaves of abelian groups on X. (Hint: exactness can be checked on stalks.) The identical argument will show that f^{-1} is an exact functor from $\mathcal{O}_Y$-modules (on Y) to $f^{-1}\mathcal{O}_Y$-modules (on X), but don't bother writing that down. (Remark for experts: f^{-1} is a left-adjoint, hence right-exact by abstract nonsense. The left-exactness is true for "less categorical" reasons.)

6+. *(The construction of f^{-1} satisfies the adjoint property* If $f : X \to Y$ is a continuous map, and $\mathcal{F}$ is a sheaf on X and $\mathcal{G}$ is a sheaf on Y, describe a bijection

$$\operatorname{Mor}_X(f^{-1}\mathcal{G}, \mathcal{F}) \leftrightarrow \operatorname{Mor}_Y(\mathcal{G}, f_*\mathcal{F}).$$

Observe that your bijection is "natural" in the sense of the definition of adjoints.

Date: Sunday, October 13, 2007.

7. (a) Suppose $Z \subset Y$ is a closed subset, and $i : Z \hookrightarrow Y$ is the inclusion. If $\mathcal{F}$ is a sheaf on Z, then show that the stalk $(i_*\mathcal{F})_y$ is 0 if $y \in Z$, and $\mathcal{F}_y$ if $y \in Z$.
(b) *Important definition:* Define the *support* of a sheaf $\mathcal{F}$ of sets, denoted $\operatorname{Supp}\mathcal{F}$, as the locus where the stalks are non-empty:

$$\operatorname{Supp}\mathcal{F} := \{x \in X : \mathcal{F}_x \neq \emptyset\}.$$

(More generally, if the sheaf has value in some category, the support consists of points where the stalk is not the initial object. For sheaves of abelian groups, the support consists of points with non-zero stalks.) Suppose $\operatorname{Supp}\mathcal{F} \subset Z$ where Z is closed. Show that the natural map $\mathcal{F} \to f_* f^{-1}\mathcal{F}$ is an isomorphism. Thus a sheaf supported in a closed subset can be considered a sheaf on that closed subset.

8. Suppose $\mathcal{F}$ is a sheaf. Show that you can recover $\mathcal{F}$ from just knowing its behavior on a base.

9+. In class, we mostly prove the following theorem: *Suppose* $\{B_i\}$ *is a base on* X, *and* F *is a sheaf of sets on this base. Then there is a unique sheaf* $\mathcal{F}$ *extending* F *(with isomorphisms* $\mathcal{F}(B_i) \cong F(B_i)$ *agreeing with the restriction maps).* In the proof, I did not describe a certain inverse map $\mathcal{F}(B) \to F(B)$. Do so, and verify that it is inverse to the obvious map $F(B) \to \mathcal{F}(B)$.

10+. *(morphisms of sheaves correspond to morphisms of sheaf on a base)* Suppose $\{B_i\}$ is a base for the topology of X.
(a) Verify that a morphism of sheaves is determined by the induced morphism of sheaves on the base.
(b) Show that a morphism of sheaves on the base (i.e. such that the diagram

$$\begin{array}{ccc} F(B_i) & \longrightarrow & G(B_i) \\ \downarrow & & \downarrow \\ F(B_j) & \longrightarrow & G(B_j) \end{array}$$

commutes for all $B_j \hookrightarrow B_i$) gives a morphism of the induced sheaves.

11+. Suppose $X = \cup U_i$ is an open cover of X, and we have sheaves $\mathcal{F}_i$ on U_i along with isomorphisms $\phi_{ij} : \mathcal{F}_i|_{U_i \cap U_j} \to \mathcal{F}_j|_{U_i \cap U_j}$ that agree on triple overlaps (i.e. $\phi_{ij} \circ \phi_{jk} = \phi_{ij}$ on $U_i \cap U_j \cap U_k$). Show that these sheaves can be glued together into a unique sheaf $\mathcal{F}$ on X, such that $\mathcal{F}_i = \mathcal{F}|_{U_i}$, and the isomorphisms over $U_i \cap U_j$ are the obvious ones. (Thus we can "glue sheaves together", using limited patching information.) (You can use the ideas of this section to solve this problem, but you don't necessarily need to. Hint: As the base, take those open sets contained in *some* U_i.)

12. *(for those with a little experience with manifolds)* Prove that a continuous function of differentiable manifolds $f : X \to Y$ is differentiable if differentiable functions pull back to differentiable functions. (Hint: check this on small patches. Once you figure out what you are trying to show, you'll realize that the result is immediate.)

13. Show that a morphism of differentiable manifolds $f : X \to Y$ with $f(p) = q$ induces a morphism of stalks $f^{\#} : \mathcal{O}_{Y,q} \to \mathcal{O}_{X,p}$. Show that $f^{\#}(\mathfrak{m}_{Y,q}) \subset \mathfrak{m}_{X,p}$.

14-. *A small exercise about small schemes.* (a) Describe the set $\operatorname{Spec} k[\epsilon]/\epsilon^2$. This is called the ring of dual numbers, and will turn out to be quite useful. You should think of ϵ as a very small number, so small that its square is 0 (although it itself is not 0).
(b) Describe the set $\operatorname{Spec} k[x]_{(x)}$. (We will see this scheme again later.)

15-. Show that for primes of the form $\mathfrak{p} = (x^2 + ax + b)$ in $\mathbb{R}[x]$, the quotient $\mathbb{R}[x]/\mathfrak{p}$ is *always* isomorphic to $\mathbb{C}$.

16-. Describe the set $\mathbb{A}^1_{\mathbb{Q}}$. (This is harder to picture in a way analogous to $\mathbb{A}^1_{\mathbb{R}}$; but the rough cartoon of points on a line remains a reasonable sketch.)

Class 6.

17. Show that all the prime ideals of $\mathbb{C}[x, y]$ are of the form (0), $(f(x, y))$, or $(x - a, y - b)$.

18. Ring elements that have a power that is 0 are called *nilpotents*. If I is an ideal of nilpotents, show that $\operatorname{Spec} B/I \to \operatorname{Spec} B$ is a bijection. Thus nilpotents don't affect the underlying set.

19. *(only if you haven't already seen this fact)* Prove that the nilradical $\mathfrak{N}(A)$ is the intersection of all the primes of A.

20-. Show that if (S) is the ideal generated by S, then $V(S) = V((S))$.

21. (a) Show that $\varnothing$ and $\operatorname{Spec} A$ are both open.
(b) Show that $V(I_1) \cup V(I_2) = V(I_1 I_2)$. Hence show that the intersection of any finite number of open sets is open.
(c) (The union of any collection of open sets is open.) If I_i is a collection of ideals (as i runs over some index set), check that $\cap_i V(I_i) = V(\sum_i I_i)$.

22. If $I \subset R$ is an ideal, then define its *radical* by

$$\sqrt{I} := \{r \in R : r^n \in I \text{ for some } n \in \mathbb{Z}^{\geq 0}\}.$$

For example, the nilradical $\mathfrak{N}$ is $\sqrt{(0)}$. Show that $V(\sqrt{I}) = V(I)$. We say an *ideal is radical* if it equals its own radical.

23. *(practice with the concept)* If $I_1, \ldots, I_n$ are ideals of a ring A, show that $\sqrt{\cap_{i=1}^n I_i} = \cap_{i=1}^n \sqrt{I_i}$. (We will use this property without referring back to this exercise.)

24. *(for future use)* Show that $\sqrt{I}$ is the intersection of all the prime ideals containing I. (Hint: Use Problem 19 on an appropriate ring.)

25+. Suppose $A \to B$ is a ring homomorphism, and $\pi : \operatorname{Spec} B \to \operatorname{Spec} A$ is the induced map of sets. By showing that closed sets pull back to closed sets, show that π is a *continuous map*.

26+. Suppose that $I, S \subset B$ are an ideal and multiplicative subset respectively. Show that $\operatorname{Spec} B/I$ is naturally a *closed* subset of $\operatorname{Spec} B$. Show that the Zariski topology on $\operatorname{Spec} B/I$

(resp. $\operatorname{Spec} S^{-1}B$) is the subspace topology induced by inclusion in $\operatorname{Spec} B$. (Hint: compare closed subsets.)

27. *(useful for later)* Suppose $I \subset B$ is an ideal. Show that f vanishes on $V(I)$ if and only if $f^n \in I$ for some n.

28-. Describe the topological space $\operatorname{Spec} k[x]_{(x)}$.

29-. Show that on an irreducible topological space, any nonempty open set is dense. (The moral of this is: unlike in the classical topology, in the Zariski topology, non-empty open sets are all "very big".)

30. Show that $\operatorname{Spec} A$ is irreducible if and only if A has only one minimal prime. (Minimality is with respect to inclusion.) In particular, if A is an integral domain, then $\operatorname{Spec} A$ is irreducible.

31-. Show that the closed points of $\operatorname{Spec} A$ correspond to the maximal ideals.

32-. If $X = \operatorname{Spec} A$, show that $[\mathfrak{p}]$ is a specialization of $[\mathfrak{q}]$ if and only if $\mathfrak{q} \subset \mathfrak{p}$. Verify to your satisfaction that we have made our intuition of "containment of points" precise: it means that the one point is contained in the *closure* of another.

33. Verify that $[(y - x^2)] \in \mathbb{A}^2$ is a generic point for $V(y - x^2)$.

34. (a) Suppose $I = (wz - xy, wy - x^2, xz - y^2) \subset k[w, x, y, z]$. Show that $\operatorname{Spec} k[w, x, y, z]$ is irreducible, by showing that I is prime. (One possible approach: Show that quotient ring is a domain, by showing that it is isomorphic to the subring of $k[a, b]$ including only monomials of degree divisible by 3. There are other approaches as well, some of which we will see later. This is an example of a hard question: how do you tell if an ideal is prime?) We will later see this as the cone over the *twisted cubic curve*.
(b) Note that the ideal of part (a) may be rewritten as

$$\operatorname{rank} \begin{pmatrix} w & x & y \\ x & y & z \end{pmatrix} = 1,$$

i.e. that all determinants of 2×2 submatrices vanish. Generalize this to the ideal of rank 1 $2 \times n$ matrices. This will correspond to the cone over the *degree* n *rational normal curve*.

35. Show that any decreasing sequence of closed subsets of $\mathbb{A}^2_{\mathbb{C}} = \operatorname{Spec} \mathbb{C}[x, y]$ must eventually stabilize. Note that it can take arbitrarily long to stabilize. (The closed subsets of $\mathbb{A}^2_{\mathbb{C}}$ were described in class.)

36. Suppose $0 \to M' \to M \to M'' \to 0$, and M' and M'' satisfy the ascending chain condition for modules. Show that M does too. (The converse also holds; we won't use this, but you can show it if you wish.)

37. If A is Noetherian, show that $\operatorname{Spec} A$ is a Noetherian topological space. Show that the converse is not true. Describe a ring A such that $\operatorname{Spec} A$ is not a Noetherian topological space.

38. If A is any ring, show that the irreducible components of $\operatorname{Spec} A$ are in bijection with the minimal primes of A.

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