FOUNDATIONS OF ALGEBRAIC GEOMETRY CLASS 1

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CONTENTS

1. WELCOME

Welcome! This is Math 216A, Foundations of Algebraic Geometry, the first of a three-quarter sequence on the topic. I'd like to tell you a little about what I intend with this course.

Algebraic geometry is a subject that somehow connects and unifies several parts of mathematics, including obviously algebra and geometry, but also number theory, and depending on your point of view many other things, including topology, string theory, etc. As a result, it can be a handy thing to know if you are in a variety of subjects, notably number theory, symplectic geometry, and certain kinds of topology. The power of the field arises from a point of view that was developed in the 1960's in Paris, by the group led by Alexandre Grothendieck. The power comes from rather heavy formal and technical machinery, in which it is easy to lose sight of the intuitive nature of the objects under consideration. This is one reason why it used to strike fear into the hearts of the uninitiated.

The rough edges have been softened over the ensuing decades, but there is an inescapable need to understand the subject on its own terms.

This class is the second version of an experiment. I hope to try several things, which are mutually incompatible. Over the year, I want to cover the foundations of the subject quite completely: the varieties and schemes, the morphisms between them, their properties, cohomology theories, and more. I would like to do this rigorously, while trying hard to keep track of the geometric intuition behind it. I'm going to try to do this without working from a text, so I'll occasionally talk myself into a corner, and then realize I'll have to go backwards and fix something earlier.

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Beginning algebraic geometry traditionally requires a lot of background. I'm going to try to assume as little as possible, ideally just commutative ring theory, and some comfort with things like prime ideals and localization. The more you know, the better, of course. But if I say things that you don't understand, please slow me down in class, and also talk to me after class. Given the amount of material that there is in the foundations of the subject, I'm afraid I'm going to move faster than I would like, which means that for you it will be like drinking from a firehose. If it helps, I'm very happy to do my part to make it easier for you, and I'm happy to talk about things outside of class. I also intend to post notes for as many classes as I can. They will usually appear before the next class, but not always.

In particular, this will not be the type of class where you can sit back and hope to pick up things casually. The only way to avoid losing yourself in a sea of definitions is to become comfortable with the ideas by playing with examples.

To this end, I intend to give problem sets, to be handed in. They aren't intended to be onerous, and if they become so, please tell me. But they *are* intended to force you to become familiar with the ideas we'll be using.

Okay, I think I've said enough to scare most of you away from coming back, so I want to emphasize that I'd like to do everything in my power to make it better, short of covering less material. The best way to get comfortable with the material is to talk to me on a regular basis about it.

Office hours: I haven't decided if it will be useful to have formal office hours rather than being available to talk after class, and also on many days by appointment.

Grader/TA: Jarod Alper, jarod@math.

Texts: Here are some books to have handy. Hartshorne's *Algebraic Geometry* has most of the material that I'll be discussing. It isn't a book that you should sit down and read, but you might find it handy to flip through for certain results. It may be at the bookstore, and is on 2-day reserve at the library. Mumford's *Red Book of Varieties and Schemes* has a good deal of the material I'll be discussing, and with a lot of motivation too. That is also on 2-day reserve in the library. The second edition is strictly worse than the 1st, because someone at Springer retyped it without understanding the math, introducing an irritating number of errors. If you would like something gentler, I would suggest Shafarevich's books on algebraic geometry. Another excellent foundational reference is Eisenbud and Harris' book *The Geometry of Schemes*, and Harris' earlier book *Algebraic Geometry* is a beautiful tour of the subject.

For background, it will be handy to have your favorite commutative algebra book around. Good examples are Eisenbud's *Commutative Algebra with a View to Algebraic Geometry*, or Atiyah and Macdonald's *Commutative Algebra*. If you'd like something with homological algebra, category theory, and abstract nonsense, I'd suggest Weibel's book *Introduction to Homological Algebra*.

Assumptions. All my rings are commutative, and with unit. I currently don't require $0 \neq 1$, so the 0-ring, with one ring, counts as a ring for me. I may regret this later. I believe in the axiom of choice, and in particular that every proper ideal in a ring is contained in a maximal ideal.

2. Why algebraic geometry?

It is hard to define algebraic geometry in its vast generality in a couple of sentences. So I'll talk around it a bit.

As a motivation, consider the study of manifolds. Real manifolds are things that locally look like bits of real n-space, and they are glued together to make interesting shapes. There is already some subtlety here — when you glue things together, you have to specify what kind of gluing is allowed. For example, if the transition functions are required to be differentiable, then you get the notion of a differentiable manifold.

A great example of a manifold is a submanifold of $\mathbb{R}^n$ (consider a picture of a torus). In fact, any compact manifold can be described in such a way. You could even make this your definition, and not worry about gluing. This is a good way to think about manifolds, but not the best way. There is something arbitrary and inessential about defining manifolds in this way. Much cleaner is the notion of an *abstract manifold*, which is the current definition used by the mathematical community.

There is an even more sophisticated way of thinking about manifolds. A differentiable manifold is obviously a topological space, but it is a little bit more. There is a very clever way of summarizing what additional information is there, basically by declaring what functions on this topological space are differentiable. The right notion is that of a sheaf, which is a simple idea, that I'll soon define for you. It is true, but non-obvious, that this ring of functions that we are declaring to be differentiable determines the differentiable manifold structure.

Very roughly, algebraic geometry, at least in its geometric guise, is the kind of geometry you can describe with polynomials. So you are allowed to talk about things like $y^2 = x^3 + x$, but not $y = \sin x$. So some of the fundamental geometric objects under consideration are things in n-space cut out by polynomials. Depending on how you define them, they are called *affine varieties* or *affine schemes*. They are the analogues of the patches on a manifold. Then you can glue these things together, using things that you can describe with polynomials, to obtain more general varieties and schemes. So then we'll have these algebraic objects, that we call varieties or schemes, and we can talk about maps between them, and things like that.

In comparison with manifold theory, we've really restricted ourselves by only letting ourselves use polynomials. But on the other hand, we have gained a huge amount too. First of all, we can now talk about things that aren't smooth (that are *singular*), and we can work with these things. Algebraic geometry provides particularly powerful tools for dealing with singular objects. (One thing we'll have to do is to define what we mean by smooth and singular!) Also, we needn't work over the real or complex numbers, so we can talk about arithmetic questions, such as: what are the rational points on $y^2 = x^3 + x^2$? (Here, we work over the field $\mathbb{Q}$.) More generally, the recipe by which we make geometric objects out of things to do with polynomials can generalize drastically, and we can make a geometric object out of rings. This ends up being surprisingly useful — all sorts of old facts in algebra can be interpreted geometrically, and indeed progress in the field of commutative algebra these days usually requires a strong geometric background.

Let me give you some examples that will show you some surprising links between geometry and number theory. To the ring of integers $\mathbb{Z}$, we will associate a smooth curve $\operatorname{Spec} \mathbb{Z}$. In fact, to the ring of integers in a number field, there is always a smooth curve, and to its orders (subrings), we have singular = non-smooth curves.

An old flavor of Diophantine question is something like this. Given an equation in two variables, $y^2 = x^3 + x^2$, how many rational solutions are there? So we're looking to solve this equation over the field $\mathbb{Q}$. Instead, let's look at the equation over the field $\mathbb{C}$. It turns out that we get a complex surface, perhaps singular, and certainly non-compact. So let me separate all the singular points, and compactify, by adding in points. The resulting thing turns out to be a compact oriented surface, so (assuming it is connected) it has a genus g, which is the number of holes it has. For example, $y^2 = x^3 + x^2$ turns out to have genus 0. Then Mordell conjectured that if the genus is at least 2, then there are at most a finite number of rational solutions. The set of complex solutions somehow tells you about the number of rational solutions! Mordell's conjecture was proved by Faltings, and earned him a Fields Medal in 1986. As an application, consider Fermat's Last Theorem. We're looking for integer solutions to $x^n + y^n = z^n$. If you think about it, we are basically looking for rational solutions to $X^n + Y^n = 1$. Well, it turns out that this has genus $\binom{n-1}{2}$ — we'll verify something close to this at some point in the future. Thus if n is at least 4, there are only a finite number of solutions. Thus Falting's Theorem implies that for each $n \geq 4$, there are only a finite number of counterexamples to Fermat's last theorem. Of course, we now know that Fermat is true — but Falting's theorem applies much more widely — for example, in more variables. The equations $x^3 + y^2 + z^{14} + xy + 17 = 0$ and $3x^{14} + x^{34}y + \cdots = 0$, assuming their complex solutions form a surface of genus at least 2, which they probably do, have only a finite number of solutions.

So here is where we are going. Algebraic geometry involves a new kind of "space", which will allow both singularities, and arithmetic interpretations. We are going to define these spaces, and define maps between them, and other geometric constructions such as vector bundles and sheaves, and before long, cohomology groups.

Motivating example: Varieties. This course will deal with the geometric notion of a *scheme*, which generalizes the earlier notion of a variety. Ideally I'd like to give you a semester's worth of a pre-course, dealing with varieties.

3. A LITTLE BIT OF CATEGORY THEORY

That which does not kill me, makes me stronger. — Nietzsche

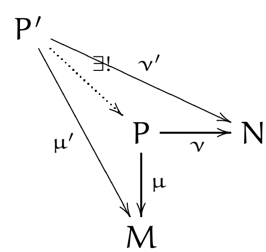
Before we get to any interesting geometry, we need to develop the language to discuss things cleanly and effectively. This is best done in the language of categories. If algebraic geometry tends to strike fear into peoples' hearts, category theory tends to induce sleep and boredom, as abstract meaningless concepts are introduced and symbols are pushed around. If I use the word topoi, you can shoot me. Here's how you should think about category theory for our purposes. There is not much to know about categories to get started; it is just a very useful language. Like all mathematical languages, category theory comes with an embedded logic, which allows us to abstract intuitions in settings we know well to far more general situations.

Our motivation is as follows. We will be creating some new mathematical objects (such as schemes, and families of sheaves), and we expect them to act like objects we have seen before. We could try to nail down precisely what we mean by "act like", and what minimal set of things we have to check in order to verify that they act the way we expect. Fortunately, we don't have to — other people have done this before us, by defining key notions, such as *abelian categories*, which behave like modules over a ring.

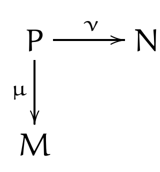
For example, we will define the notion of *product* of the geometric spaces (schemes). We could just give a definition of product, but then you should want to know why this precise definition deserves the name of "product". As a motivation, we revisit the notion of product in a situation we know well: (the category of) sets. One way to define the product of sets U and V is as the set of ordered pairs $\{(u,v) : u \in U, v \in V\}$. But someone from a different mathematical culture might reasonably define it as the set of symbols $\{[v,u] : u \in U, v \in V\}$. These notions are "obviously the same". Better: there is "an obvious bijection between the two".

This can be made precise by giving a better definition of product, in terms of a *universal property*. Given two sets M and N, a product is a set P, along with maps $\mu : P \to M$ and $\nu : P \to N$, such that for *any other set* P' *with maps* $\mu' : P' \to M$ *and* $\nu' : P' \to N$, these maps must factor *uniquely* through P:

(1)



Thus a product is a *diagram*



and not just a set P, although the maps μ and ν are often left implicit.

This definition agrees with the usual definition, with one twist: there isn't just a single product; but any two products come with a *canonical* isomorphism between them. In other words, the product is unique up to unique isomorphism. Here is why: if you have a product

$$\begin{array}{ccc} P_1 & \xrightarrow{\nu_1} & N \\ {\scriptstyle \mu_1}\downarrow & & \\ M & & \end{array}$$

and I have a product

$$\begin{array}{ccc} P_2 & \xrightarrow{\nu_2} & N \\ {\scriptstyle \mu_2}\downarrow & & \\ M & & \end{array}$$

then by the universal property of my product (letting (P_2, μ_2, ν_2) play the role of (P, μ, ν), and (P_1, μ_1, ν_1) play the role of (P', μ', ν') in (1)), there is a unique map $f : P_1 \to P_2$ making the appropriate diagram commute (i.e. $\mu_1 = \mu_2 \circ f$ and $\nu_1 = \nu_2 \circ f$). Similarly by the universal property of your product, there is a unique map $g : P_2 \to P_1$ making the appropriate diagram commute. Now consider the universal property of my product, this time letting (P_2, μ_2, ν_2) play the role of both (P, μ, ν) and (P', μ', ν') in (1). There is a unique map $h : P_2 \to P_2$ such that

$$\begin{array}{cccc} P_2 & & & \\ & \searrow^{h} & \overset{\nu_2}{\searrow} & \\ {\scriptstyle \mu_2}\searrow & & P_2 \xrightarrow[\nu_2]{} & N \\ & & {\scriptstyle \mu_2}\downarrow & \\ & & M & \end{array}$$

commutes. However, I can name two such maps: the identity map id_{P_2}, and $g \circ f$. Thus $g \circ f = \mathrm{id}_{P_2}$. Similarly, $f \circ g = \mathrm{id}_{P_1}$. Thus the maps f and g arising from the universal property are bijections. In short, there is a unique bijection between P_1 and P_2 preserving the "product structure" (the maps to M and N). This gives us the right to name any such product $M \times N$, since any two such products are canonically identified.

This definition has the advantage that it works in many circumstances, and once we define category, we will soon see that the above argument applies verbatim in any category to show that products, if they exist, are unique up to unique isomorphism. Even if you haven't seen the definition of category before, you can verify that this agrees with your notion of product in some category that you have seen before (such as the category of vector spaces, where the maps are taken to be linear maps; or the category of real manifolds, where the maps are taken to be submersions).

This is handy even in cases that you understand. For example, one way of defining the product of two manifolds M and N is to cut them both up into charts, then take products of charts, then glue them together. But if I cut up the manifolds in one way, and you cut them up in another, how do we know our resulting manifolds are the "same"? We could wave our hands, or make an annoying argument about refining covers, but instead, we should just show that they are indeed products, and hence the "same" (i.e. isomorphic).

This argument is essentialy Yoneda's lemma, which we will formalize shortly in Section 5.

4. CATEGORIES AND FUNCTORS

Let's now get our hands dirty. We begin with an informal definition of categories and functors.

4.1. Categories.

A category $\mathcal{C}$ consists of a collection of *objects*, and for each pair of objects, a set of maps, or *morphisms* (or *arrows*), between them. The collection of objects of a category $\mathcal{C}$ are often denoted $\mathrm{obj}(\mathcal{C})$, but we will usually denote the collection $\mathcal{C}$ also by $\mathcal{C}$. If $A, B \in \mathcal{C}$, then the morphisms from A to B are denoted $\mathrm{Mor}(A, B)$. A morphism is often written $f : A \to B$, and A is said to be the *source* of f, and B the *target* of f. Morphisms compose as expected: there is a composition $\mathrm{Mor}(A, B) \times \mathrm{Mor}(B, C) \to \mathrm{Mor}(A, C)$, and if $f \in \mathrm{Mor}(A, B)$ and $g \in \mathrm{Mor}(B, C)$, then their composition is denoted $g \circ f$. Composition is associative: if $f \in \mathrm{Mor}(A, B)$, $g \in \mathrm{Mor}(B, C)$, and $h \in \mathrm{Mor}(C, D)$, then $h \circ (g \circ f) = (h \circ g) \circ f$. For each object $A \in \mathcal{C}$, there is always an *identity morphism* $\mathrm{id}_A : A \to A$, such that when you (left- or right-)compose a morphism with the identity, you get the same morphism. More precisely, if $f : A \to B$ is a morphism, then $f \circ \mathrm{id}_A = f = \mathrm{id}_B \circ f$.

If we have a category, then we have a notion of *isomorphism* between two objects (if we have two morphisms $f : A \to B$ and $g : B \to A$, *both* of whose compositions are the identity on the appropriate object), and a notion of *automorphism* of an object (an isomorphism of the object with itself).

4.2. *Example.* The prototypical example to keep in mind is the category of sets, denoted **Sets**. The objects are sets, and the morphisms are maps of sets.

4.3. *Example.* Another good example is the category $\mathbf{Vec}_k$ of vector spaces over a given field k. The objects are k-vector spaces, and the morphisms are linear transformations.

4.A. UNIMPORTANT EXERCISE. A category in which each morphism is an isomorphism is called a *groupoid*. (This notion is not important in this class. The point of this exercise is to give you some practice with categories, by relating them to an object you know well.)
(a) A perverse definition of a group is: a groupoid with one element. Make sense of this.
(b) Describe a groupoid that is not a group.
(For readers with a topological background: if X is a topological space, then the fundamental groupoid is the category where the objects are points of x, and the morphisms from $x \to y$ are paths from x to y, up to homotopy. Then the automorphism group of x_0 is the (pointed) fundamental group $\pi_1(X, x_0)$. In the case where X is connected, and the $\pi_1(X)$ is not abelian, this illustrates the fact that for a connected groupoid — whose definition you can guess — the automorphism groups of the objects are all isomorphic, but not canonically isomorphic.)

4.B. EXERCISE. If A is an object in a category $\mathcal{C}$, show that the isomorphisms of A with itself $\mathrm{Isom}(A, A)$ form a group (called the *automorphism group of* A, denoted $\mathrm{Aut}(A)$). What are the automorphism groups of the objects in Examples 4.2 and 4.3? Show that two isomorphic objects have isomorphic automorphism groups.

4.4. *Example: abelian groups.* The abelian groups, along with group homomorphisms, form a category **Ab**.

4.5. *Example: modules over a ring.* If A is a ring, then the A-modules form a category $\mathbf{Mod}_A$. (This category has additional structure; it will be the prototypical example of an *abelian category*, which we'll define next day.) Taking $A = k$, we obtain Example 4.3; taking $A = \mathbb{Z}$, we obtain Example 4.4.

4.6. *Example: rings.* There is a category **Rings**, where the objects are rings, and the morphisms are morphisms of rings (which I'll assume send 1 to 1).

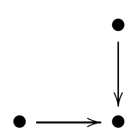
4.7. *Example: topological spaces.* The topological spaces, along with continuous maps, form a category **Top**. The isomorphisms are homeomorphisms.

4.8. *Example: partially ordered sets.* A *partially ordered set*, or *poset*, is a set $(S, \geq)$ along with a binary relation $\geq$ satisfying:

(i) $x \geq x$,
(ii) $x \geq y$ and $y \geq z$ imply $x \geq z$ (transitivity), and
(iii) if $x \geq y$ and $y \geq x$ then $x = y$.

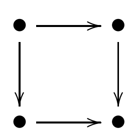
A partially ordered set $(S, \geq)$ can be interpreted as a category whose objects are the elements of S, and with a single morphism from x to y if and only if $x \geq y$ (and no morphism otherwise).

A trivial example is $(S, \geq)$ where $x \geq y$ if and only if $x = y$. Another example is



Here there are three objects. The identity morphisms are omitted for convenience, and the three non-identity morphisms are depicted. A third example is

(2)



Here the "obvious" morphisms are again omitted: the identity morphisms, and the morphism from the upper left to the lower right. Similarly, depicts a partially ordered set, where again, only the "generating morphisms" are depicted.

4.9. *Example: the category of subsets of a set, and the category of open sets in a topological space.* If X is a set, then the subsets form a partially ordered set, where the order is given by inclusion. Similarly, if X is a topological space, then the *open* sets form a partially ordered set, where the order is given by inclusion. (What is the initial object? What is the final object?)

4.10. Functors.

A *covariant functor* F from a category $\mathcal{A}$ to a category $\mathcal{B}$, denoted $F : \mathcal{A} \to \mathcal{B}$, is the following data. It is a map of objects $F : \mathrm{obj}(\mathcal{A}) \to \mathrm{obj}(\mathcal{B})$, and for each $a_1, a_2 \in \mathcal{A}$ a morphism $m : a_1 \to a_2$, $F(m)$ is a morphism from $F(A_1) \to F(A_2)$ in $\mathcal{B}$. F preserves identity morphisms: for $A \in \mathcal{A}$, $F(\mathrm{id}_A) = \mathrm{id}_{F(A)}$. F preserves composition: $F(m_1 \circ m_2) = F(m_1) \circ F(m_2)$.

If $F : \mathcal{A} \to \mathcal{B}$ and $G : \mathcal{B} \to \mathcal{C}$, then we may define a functor $G \circ F : \mathcal{A} \to \mathcal{C}$ in the obvious way. Composition of functors is associative.

4.11. *Example: a forgetful functor..* Consider the functor from the category of complex vector spaces $\mathbf{Vec}_k$ to **Sets**, that associates to each vector space its underlying set. The functor sends a linear transformation to its underlying map of sets. This is an example of a *forgetful functor*, where some additional structure is forgotten. Another example of a forgetful functor is $\mathbf{Mod}_A \to \mathbf{Ab}$ from A-modules to abelian groups, remembering only the abelian group structure of the A-module.

4.12. *Topological examples.* Examples of covariant functors include the fundamental group functor π_1, which sends a topological space with X choice of a point $x_0 \in X$ to a group $\pi_1(X, x_0)$, and the ith homology functor $\mathbf{Top} \to \mathbf{Ab}$, which sends a topological space X to its ith homology group $H_i(X, \mathbb{Z})$. The covariance corresponds to the fact that a (continuous) morphism of pointed topological spaces $f : X \to Y$ with $f(x_0) = y_0$ induces a map of fundamental groups $\pi_1(X, x_0) \to \pi_1(Y, y_0)$, and similarly for homology groups.

4.13. *Example.* Suppose A is an element of a category $\mathcal{C}$. Then there is a functor $h_A : \mathcal{C} \to$ **Sets** sending $B \in \mathcal{C}$ to $\mathrm{Mor}(A, B)$, and sending $f : B_1 \to B_2$ to $\mathrm{Mor}(A, B_1) \to \mathrm{Mor}(A, B_2)$ described by

$$[g : A \to B_1] \mapsto [f \circ g : A \to B_1 \to B_2].$$

4.14. *Example: partially ordered sets as index categories.* Partially ordered sets will often turn up as index categories. As a first example, if $\square$ is the category of (2), and $\mathcal{A}$ is a category, then a functor $\square \to \mathcal{A}$ is precisely the information of a commuting square in $\mathcal{A}$.

4.15. Definition. A *contravariant functor* is defined in the same way as a covariant functor, except the arrows switch directions: in the above language, $F(A_1 \to A_2)$ is now an arrow from $F(A_2)$ to $F(A_1)$.

It is wise to always state whether a functor is covariant or contravariant. If it is not stated, the functor is often assumed to be covariant.

4.16. *Topological example (cf. Example 4.12).* The the ith cohomology functor $H^i(\cdot, \mathbb{Z}) :$ **Top** $\to$ **Ab** is a contravariant functor.

4.17. *Example.* If $\mathbf{Vec}_k$ is the category of complex k-vector spaces, then taking duals gives a contravariant functor $^\vee : \mathbf{Vec}_k \to \mathbf{Vec}_k$. Indeed, to each linear transformation $f : V \to W$, we have a dual transformation $f^\vee : W^\vee \to V^\vee$, and $(f \circ g)^\vee = g^\vee \circ f^\vee$.

4.18. *Example.* There is a contravariant functor **Top** $\to$ **Rings** taking a topological space X to the continuous functions on X. A morphism of topological spaces $X \to Y$ (a continuous map) induces the pullback map from functions on Y to maps on X.

4.19. *Example (cf. 4.13).* Suppose A is an element of a category $\mathcal{C}$. Then there is a contravariant functor $h^A : \mathcal{C} \to$ **Sets** sending $B \in \mathcal{C}$ to $\mathrm{Mor}(B, A)$, and sending $f : B_1 \to B_2$ to $\mathrm{Mor}(B_2, A) \to \mathrm{Mor}(B_1, A)$ described by

$$[g : B_2 \to A] \mapsto [g \circ f : B_2 \to B_1 \to A].$$

This example initially looks weird and different, but the previous two examples are just special cases of this; do you see how? What is A in each case?

5. Universal properties

Given some category that we come up with, we often will have ways of producing new objects from old. In good circumstances, such a definition can be made using the notion of a *universal property*. Informally, we wish that there is an object with some property. We first show that if it exists, then it is essentially unique, or more precisely, is unique up to unique isomorphism. Then we go about constructing an example of such an object to show existence.

With a little practice, universal properties are useful in proving things quickly slickly. However, explicit constructions are often intuitively easier to work with, and sometimes also lead to short proofs.

We have seen one important example of a universal property argument already in our discussion of products. You should go back and verify that our discussion there gives a notion of product in category, and shows that products, *if they exist*, are unique up to canonical isomorphism.

5.1. Another good example of a universal property construction is the notion of a tensor product of A-modules

$$\otimes_A : \quad \mathrm{obj}(\mathbf{Mod}_A) \times \mathrm{obj}(\mathbf{Mod}_A) \longrightarrow \mathrm{obj}(\mathbf{Mod}_A)$$

$$M \times N \longmapsto M \otimes_A N$$

The subscript A is often suppressed when it is clear from context. Tensor product is often defined as follows. Suppose you have two A-modules M and N. Then elements of the tensor product $M \otimes_A N$ are of the form $m \otimes n$ ($m \in M$, $n \in N$), subject to relations $(m_1 + m_2) \otimes n = m_1 \otimes n + m_2 \otimes n$, $m \otimes (n_1 + n_2) = m \otimes n_1 + m \otimes n_2$, $a(m \otimes n) = (am) \otimes n = m \otimes (an)$ (where $a \in A$).

If A is a field k, we get the tensor product of vector spaces.

5.A. EXERCISE (IF YOU HAVEN'T SEEN TENSOR PRODUCTS BEFORE). Calculate $\mathbb{Z}/10 \otimes_{\mathbb{Z}} \mathbb{Z}/12$. (This exercise is intended to give some hands-on practice with tensor products.)

5.B. EXERCISE: RIGHT-EXACTNESS OF $\cdot \otimes_A N$. Show that $\cdot \otimes_A N$ gives a covariant functor $\mathbf{Mod}_A \to \mathbf{Mod}_A$. Show that $\cdot \otimes_A N$ is a *right-exact functor*, i.e. if

$$M' \to M \to M'' \to 0$$

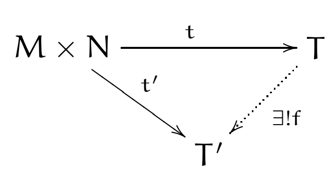
is an exact sequence of A-modules, then the induced sequence

$$M' \otimes_A N \to M \otimes_A N \to M'' \otimes_A N \to 0$$

is also exact. (For experts: is there a universal property proof?)

This is a weird definition, and really the "wrong" definition. To motivate a better one: notice that there is a natural A-bilinear map $M \times N \to M \otimes_A N$. *Any* A-bilinear map $M \times N \to C$ factors through the tensor product uniquely: $M \times N \to M \otimes_A N \to C$. (Think this through!)

We can take this as the *definition* of the tensor product as follows. It is an A-module T along with an A-bilinear map $t : M \times N \to T$, such any other such map factors through t that given any other $t' : M \times N \to T'$, there is a unique map $f : T \to T'$ such that $t' = f \circ t$.



5.C. EXERCISE. Show that $(T, t : M \times N \to T)$ is unique up to unique isomorphism. Hint: first figure out what "unique up to unique isomorphism" means for such pairs. Then follow the analogous argument for the product. (This exercise will prime you for Yoneda's Lemma.)

In short: there is an A-bilinear map $t : M \times N \to M \otimes_A N$, unique up to unique isomorphism, defined by the following universal property: for any A-bilinear map $t' : M \times N \to T'$ there is a unique $f : M \otimes_A N \to T'$ such that $t' = f \circ t$.

Note that this argument shows uniqueness *assuming existence*. We need to still show the existence of such a tensor product. This forces us to do something constructive.

5.D. EXERCISE. Show that the construction of §5.1 satisfies the universal property of tensor product.

The uniqueness of tensor product is our second example of the proof of uniqueness (up to unique isomorphism) by a *universal property*. If you have never seen this sort of argument before, then you might think you get it, but you don't, so you should think over it some more. We will be using such arguments repeatedly in the future. We'll soon formalize this way of thinking in Yoneda's Lemma.

Before getting to it, we'll give another exercise that involves universal properties.

5.2. Definition. An object of a category $\mathcal{C}$ is an *initial object* if it has precisely one map to every other object. It is a *final object* if it has precisely one map from every other object. It is a *zero-object* if it is both an initial object and a final object.

5.E. EXERCISE. Show that any two initial objects are canonically isomorphic. Show that any two final objects are canonically isomorphic.

This (partially) justifies the phrase "*the* initial object" rather than "*an* initial object", and similarly for "*the* final object" and "*the* zero object".

5.F. EXERCISE. State what the initial and final objects are in **Sets**, **Rings**, and **Top** (if they exist).

Next day: Yoneda's lemma. Limits. Maybe even some sheaves.

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FOUNDATIONS OF ALGEBRAIC GEOMETRY CLASS 2

RAVI VAKIL

CONTENTS

First, some bureaucratic details.

- We will move to **380-F** for Monday's class.
- Please sign up on this sign-up sheet. I'm going to use it to announce important things like room changes and problem sets.
- Problem sets will be due on Fridays, and I'll try to give them out at least a week in advance. The first set will be out by tomorrow, on the class website. I'll announce it by e-mail. The problems will all be from the notes, and almost all from the class.
- Jarod will be hosting problem sessions on Wednesdays from 5-6 pm, starting next week, at a location to be announced later. This is a great chance to ask him lots of questions, and to hear interesting questions from other people.

If you weren't here last day, you can see the notes on-line. The main warning is that this is going to be a hard class, and you should take it only if you really want to, and also that you should ask me lots of questions, both during class and out of class. And you should do lots of problems.

1. WHERE WE WERE

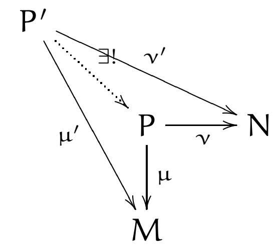
Last day, we begin by discussing some category theory. Keep in mind that our motivation in learning this is to formalize what we already know, so we can use it in new contexts. Today we should finish with category theory, and we may even begin to discuss sheaves.

The most important notion from last day was the fact that universal properties essentially determine things up to unique isomorphism.

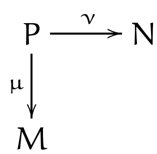
Date: Wednesday, September 26, 2007. Revised Oct. 13.

For example, in any category, the product of two objects M and N is an object P, along with maps $\mu : P \to M$ and $\nu : P \to N$, such that for *any other object* P' *with maps* $\mu' : P' \to M$ *and* $\nu' : P' \to N$, these maps must factor *uniquely* through P:

(1)

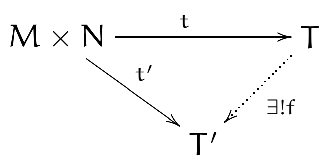


Thus a product is a *diagram*



and not just a set P, although the maps μ and ν are often left implicit.

Another good example of a universal property construction is the notion of a tensor product of A-modules. It is an A-module T along with an A-bilinear map $t : M \times N \to T$, such that any other such map factors through t: given any other $t' : M \times N \to T'$, there is a unique map $f : T \to T'$ such that $t' = f \circ t$.



I gave you the exercise of showing that $(T, t : M \times N \to T)$ (should it exist) is unique up to unique isomorphism. You should really do this, because I'm going to use universal property arguments a whole lot. If you know how to do one of these arguments, you'll know how to do them all.

I then briefly gave other examples: initial objects, final objects, and zero-objects (=initial+final).

2. Yoneda's lemma

2.1. Yoneda's Lemma.

Suppose A is an object of category $\mathcal{C}$. For any object $C \in \mathcal{C}$, we have a set of morphisms $\mathrm{Mor}(C, A)$. If we have a morphism $f : B \to C$, we get a map of sets

$$\mathrm{Mor}(C, A) \to \mathrm{Mor}(B, A), \tag{2}$$

by composition: given a map from C to A, we get a map from B to A by precomposing with f. Hence this gives a contravariant functor $h^A : \mathcal{C} \to$ **Sets**. Yoneda's Lemma states that the functor h^A determines A up to unique isomorphism. More precisely:

2.2. *Yoneda's lemma.* — Given two objects A and A', and bijections

$$i_C : \mathrm{Mor}(C, A) \to \mathrm{Mor}(C, A') \tag{3}$$

that commute with the maps (2), then the i_C must be induced from a unique isomorphism $A \to A'$.

2.A. IMPORTANT EXERCISE (THAT EVERYONE SHOULD DO ONCE IN THEIR LIFE). Prove this. (Hint: This sounds hard, but it really is not. This statement is so general that there are really only a couple of things that you could possibly try. For example, if you're hoping to find an isomorphism $A \to A'$, where will you find it? Well, you're looking for an element $\mathrm{Mor}(A, A')$. So just plug in $C = A$ to (3), and see where the identity goes. You'll quickly find the desired morphism; show that it is an isomorphism, then show that it is unique.)

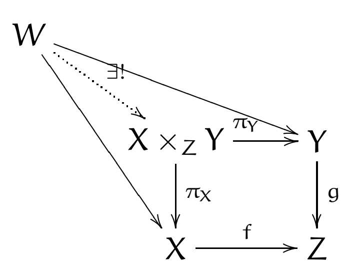
2.3. *Remark.* There is an analogous statement with the arrows reversed, where instead of maps into A, you think of maps *from* A.

2.4. *Remark: the full statement of Yoneda's Lemma.* It won't matter so much for us (so I didn't say it in class), but it is useful to know the full statement of Yoneda's Lemma. A covariant functor $F : \mathcal{A} \to \mathcal{B}$ is *faithful* if for all $A, A' \in \mathcal{A}$, the map $\mathrm{Mor}_{\mathcal{A}}(A, A') \to \mathrm{Mor}_{\mathcal{B}}(F(A), F(A'))$ is injective, and *full* if it is surjective. A functor that is full and faithful is *fully faithful*. A subcategory $i : \mathcal{A} \to \mathcal{B}$ is a *full subcategory* if i is full. If $\mathcal{C}$ is a category, consider the contravariant functor

$$h : \mathcal{C} \to \mathbf{Sets}^{\mathcal{C}}$$

where the category on the right is the "functor category" where the objects are contravariant functors $\mathcal{C} \to \mathbf{Sets}$. (What are the morphisms in this category? You will rediscover the notion of *natural transformation of functors*.) This functor h sends A to h^A. Yoneda's lemma states that this is a fully faithful functor, called the *Yoneda embedding*.

2.5. Example: Fibered products. (This notion of fibered product will be important for us later.) Suppose we have morphisms $X, Y \to Z$ (in *any* category). Then the *fibered product* is an object $X \times_Z Y$ along with morphisms to X and Y, where the two compositions $X \times_Z Y \to Z$ agree, such that given any other object W with maps to X and Y (whose compositions to Z agree), these maps factor through some unique $W \to X \times_Z Y$:



By a universal property argument, if it exists, it is unique up to unique isomorphism. (You should think this through until it is clear to you.) Thus the use of the phrase "the fibered product" (rather than "a fibered product") is reasonable, and we should reasonably be allowed to give it the name $X \times_Z Y$. We know what maps to it are: they are precisely maps to X and maps to Y that agree on maps to Z.

The right way to interpret this is first to think about what it means in the category of sets.

2.B. EXERCISE. Show that in **Sets**,

$$X \times_Z Y = \{(x \in X, y \in Y) : f(x) = g(y)\}.$$

More precisely, describe a natural isomorphism between the left and right sides. (This will help you build intuition for fibered products.)

2.C. EXERCISE. If X is a topological space, show that fibered products always exist in the category of open sets of X, by describing what a fibered product is. (Hint: it has a one-word description.)

2.D. EXERCISE. If Z is the final object in a category $\mathcal{C}$, and $X, Y \in \mathcal{C}$, then "$X \times_Z Y = X \times Y$": "the" fibered product over Z is canonically isomorphic to "the" product. (This is an exercise about unwinding the definition.)

2.E. UNIMPORTANT EXERCISE. Show that in the category **Ab** of abelian groups, the kernel K of $f : A \to B$ can be interpreted as a fibered product:

$$\begin{array}{ccc} K & \longrightarrow & A \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & B \end{array}$$

We make a definition to set up an exercise.

2.6. Definition. A morphism $f : X \to Y$ is a **monomorphism** if any two morphisms $g_1, g_2 : Z \to X$ such that $f \circ g_1 = f \circ g_2$ must satisfy $g_1 = g_2$. This a generalization of of an injection of sets. In other words, there is a unique way of filling in the dotted arrow so that the following diagram commutes.

$$\begin{array}{ccc} Z & & \\ {\scriptstyle \leq 1}\,\vdots & \searrow & \\ X & \xrightarrow{f} & Y. \end{array}$$

Intuitively, it is the categorical version of an injective map, and indeed this notion generalizes the familiar notion of injective maps of sets.

2.7. *Remark.* The notion of an **epimorphism** is "dual" to this diagramatic definition, where all the arrows are reversed. This concept will not be central for us, although it is necessary for the definition of an abelian category. Intuitively, it is the categorical version of a surjective map.

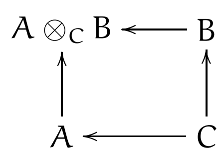
2.F. EXERCISE. Prove a morphism is a monomorphism if and only if the natural morphism $X \to X \times_Y X$ is an isomorphism. (What is this natural morphism?!) We may then take this as the definition of monomorphism. (Monomorphisms aren't very central to future discussions, although they will come up again. This exercise is just good practice.)

2.G. EXERCISE. Suppose $X \to Y$ is a monomorphism, and $W, Z \to X$ are two morphisms. Show that $W \times_X Z$ and $W \times_Y Z$ are canonically isomorphic. We will use this later when talking about fibered products. (Hint: for any object V, give a natural bijection between maps from V to the first and maps from V to the second.)

2.H. EXERCISE. Given $X \to Y \to Z$, show that there is a natural morphism $X \times_Y X \to X \times_Z X$, assuming that both fibered products exist. (This is trivial once you figure out what it is saying. The point of this exercise is to see why it is trivial.)

2.I. UNIMPORTANT EXERCISE. Define *coproduct* in a category by reversing all the arrows in the definition of product. Show that coproduct for **Sets** is disjoint union.

2.J. EXERCISE. Suppose $C \to A, B$ are two ring morphisms, so in particular A and B are C-modules. Define a ring structure $A \otimes_C B$ with multiplication given by $(a_1 \otimes b_1)(a_2 \otimes b_2) = (a_1 a_2) \otimes (b_1 b_2)$. There is a natural morphism $A \to A \otimes_C B$ given by $a \mapsto (a, 1)$. (Warning: This is not necessarily an inclusion.) Similarly, there is a natural morphism $B \to A \otimes_C B$. Show that this gives a coproduct on rings, i.e. that



satisfies the universal property of coproduct.

2.K. IMPORTANT EXERCISE FOR LATER. We continue the notation of the previous exercise. Let I be an ideal of A. Let I^e be the extension of I to $A \otimes_C B$. (These are the elements $\sum_j i_j \otimes b_j$ where $i_j \in I$, $b_j \in B$.) Show that there is a natural isomorphism

$$(A/I) \otimes_C B \cong (A \otimes_C B)/I^e.$$

(Hint: consider $I \to A \to A/I \to 0$, and use the right exactness of $\otimes_C B$.)

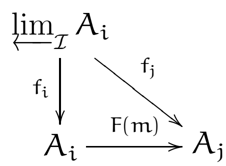
Hence the natural morphism $B \to B \otimes_C (A/I)$ is a surjection. As an application, we can compute tensor products of finitely generated k algebras over k. For example, we have a canonical isomorphism

$$k[x_1, x_2]/(x_1^2 - x_2) \otimes_k k[y_1, y_2]/(y_1^3 + y_2^3) \cong k[x_1, x_2, y_1, y_2]/(x_1^2 - x_2, y_1^3 + y_2^3).$$

Limits and colimits provide two important examples defined by universal properties. They generalize a number of familiar constructions. I'll give the definition first, and then show you why it is familiar. (For example, we'll see that the p-adics are a limit, and fractions are a colimit.)

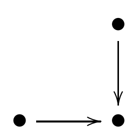
3.1. Limits. We say that a category is an *index category* (a technical condition intended only for experts) the objects form a set. An example is a partially ordered set (in which there in particular there is only one morphism between objects), and indeed all of our examples will be partially ordered sets. Suppose $\mathcal{I}$ is any index category (such as a partially ordered set), and $\mathcal{C}$ is any category. Then a functor $F : \mathcal{I} \to \mathcal{C}$ (i.e. with an object $A_i \in \mathcal{C}$ for each element $i \in \mathcal{I}$, and appropriate commuting morphisms dictated by $\mathcal{I}$) is said to be a *diagram indexed by* $\mathcal{I}$. Commuting squares can be interpreted in this way.

Then the *limit* is an object $\varprojlim_{\mathcal{I}} A_i$ of $\mathcal{C}$ along with morphisms $f_i : \varprojlim_{\mathcal{I}} A_i$ such that if $m : i \to j$ is a morphism in $\mathcal{I}$, then



commutes, and this object and maps to each A_i is universal (final) respect to this property. (The limit is sometimes called the *inverse limit* or *projective limit*.) By the usual universal property argument, if the limit exists, it is unique up to unique isomorphism.

3.2. *Examples: products.* For example, if $\mathcal{I}$ is the partially ordered set



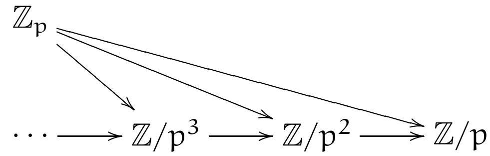
we obtain the fibered product.

If $\mathcal{I}$ is

we obtain the product.

If $\mathcal{I}$ is a set (i.e. the only morphisms are the identity maps), then the limit is called the *product* of the A_i, and is denoted $\prod_i A_i$. The special case where $\mathcal{I}$ has two elements is the example of the previous paragraph.

3.3. *Example: the* p*-adics.* The p-adic numbers, $\mathbb{Z}_p$, are often described informally (and somewhat unnaturally) as being of the form $\mathbb{Z}_p = ? + ?p + ?p^2 + ?p^3 + \cdots$. They are an example of a limit in the category of rings:



Limits do not always exist. For example, there is no limit of $\cdots \to \mathbb{Z}/p^3 \to \mathbb{Z}/p^2 \to \mathbb{Z}/p \to 0$ in the category of finite rings.

However, you can often easily check that limits exist if the elements of your category can be described as sets with additional structure, and arbitrary products exist (respecting the set structure).

3.A. EXERCISE. Show that in the category **Sets**,

$$\{(a_i)_{i \in I} \in \prod_i A_i : F(m)(a_i) = a_j \text{ for all } [m : i \to j] \in \operatorname{Mor}(\mathcal{I})\},$$

along with the projection maps to each A_i, is the limit $\varprojlim_{\mathcal{I}} A_i$.

This clearly also works in the category $\mathbf{Mod}_A$ of A-modules, and its specializations such as $\mathbf{Vec}_k$ and **Ab**.

From this point of view, $2 + 3p + 2p^2 + \cdots \in \mathbb{Z}_p$ can be understood as the sequence $(2, 2 + 3p, 2 + 3p + 2p^2, \dots)$.

3.4. Colimits. More immediately relevant for us will be the dual of the notion of inverse limit. We just flip all the arrows in that definition, and get the notion of a *direct limit*. Again, if it exists, it is unique up to unique isomorphism. (The colimit is sometimes called the direct limit or injective limit.)

A limit maps *to* all the objects in the big commutative diagram indexed by $\mathcal{I}$. A colimit has a map *from* all the objects.

Even though we have just flipped the arrows, somehow colimits behave quite differently from limits.

3.5. *Example.* The ring $5^{-\infty}\mathbb{Z}$ of rational numbers whose denominators are powers of 5 is a colimit $\varinjlim 5^{-i}\mathbb{Z}$. More precisely, $5^{\infty}\mathbb{Z}$ is the colimit of

$$\mathbb{Z} \longrightarrow 5^{-1}\mathbb{Z} \longrightarrow 5^{-2}\mathbb{Z} \longrightarrow \cdots$$

The colimit over an index *set* I is called the *coproduct*, denoted $\coprod_i A_i$, and is the dual notion to the product.

3.B. EXERCISE. (a) Interpret the statement "$\mathbb{Q} = \varinjlim \frac{1}{n}\mathbb{Z}$". (b) Interpret the union of some subsets of a given set as a colimit. (Dually, the intersection can be interpreted as a limit.)

Colimits always exist in the category of sets:

3.C. EXERCISE. Consider the set $\{(i \in \mathcal{I}, a_i \in A_i)\}$ modulo the equivalence generated by: if $m : i \to j$ is an arrow in $\mathcal{I}$, then $(i, a_i) \sim (j, F(m)(a_i))$. Show that this set, along with the obvious maps from each A_i, is the colimit.

Thus in Example 3.5, each element of the direct limit is an element of something upstairs, but you can't say in advance what it is an element of. For example, 17/125 is an element of the $5^{-3}\mathbb{Z}$ (or $5^{-4}\mathbb{Z}$, or later ones), but not $5^{-2}\mathbb{Z}$.

3.6. *Example: colimits of* A*-modules.* A variant of this construction works in a number of categories that can be interpreted as sets with additional structure (such as abelian groups, A-modules, groups, etc.). While in the case of sets, the direct limit is a quotient object of the direct sum (= disjoint union) of the A_i, in the case of A-modules (for example), the direct limit is a quotient object of the direct sum of rings. thus the direct limit is $\oplus A_i$ modulo $a_j - F(m)(a_i)$ for every $m : i \to j$ in $\mathcal{I}$.

3.D. EXERCISE. Verify that the A-module described above is indeed the colimit.

3.7. Summary. One useful thing to informally keep in mind is the following. In a category where the objects are "set-like", an element of a colimit can be thought of ("has a representative that is") an element of a single object in the diagram. And an element of a limit can be thought of as an element in each object in the diagram, that are "compatible". Even though the definitions of limit and colimit are the same, just with arrows reversed, these interpretations are quite different.

4. ADJOINTS

Here is another example of a construction closely related to universal properties. We now define adjoint functors. Two *covariant* functors $F : \mathcal{A} \to \mathcal{B}$ and $G : \mathcal{B} \to \mathcal{A}$ are *adjoint* if there is a natural bijection for all $A \in \mathcal{A}$ and $B \in \mathcal{B}$

$$\tau_{AB} : \mathrm{Mor}_{\mathcal{B}}(F(A), B) \to \mathrm{Mor}_{\mathcal{A}}(A, G(B)).$$

In this instance, let me make precise what "natural" means. For all $f : A \to A'$ in $\mathcal{A}$, we require

(4)
$$\begin{array}{ccc} \mathrm{Mor}_{\mathcal{B}}(F(A'), B) & \xrightarrow{Ff^*} & \mathrm{Mor}_{\mathcal{B}}(F(A), B) \\ \downarrow{\scriptstyle \tau} & & \downarrow{\scriptstyle \tau} \\ \mathrm{Mor}_{\mathcal{A}}(A', G(B)) & \xrightarrow{f^*} & \mathrm{Mor}_{\mathcal{A}}(A, G(B)) \end{array}$$

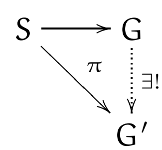
to commute, and for all $g : B \to B'$ in $\mathcal{B}$ we want a similar commutative diagram to commute. (Here f^* is the map induced by $f : A \to A'$, and Ff^* is the map induced by $Ff : L(A) \to L(A')$.)

4.A. EXERCISE. Write down what this diagram should be. (Hint: do it by extending diagram (4) above.)

You've actually seen this before, in linear algebra, when you have seen adjoint matrices. Here is another example.

4.B. EXERCISE. Suppose M, N, and P are A-modules. Describe a natural bijection $\mathrm{Mor}_A(M \otimes_A N, P) = \mathrm{Mor}_A(M, \mathrm{Mor}_A(N, P))$. (Hint: try to use the universal property.) If you wanted, you could check that $\cdot \otimes_A N$ and $\mathrm{Mor}_A(N, \cdot)$ are adjoint functors. (Checking adjointness is never any fun!)

4.1. Example: groupification. Here is another motivating example: getting an abelian group from an abelian semigroup. An abelian semigroup is just like a group, except you don't require an inverse. One example is the non-negative integers $0, 1, 2, \dots$ under addition. Another is the positive integers under multiplication $1, 2, \dots$. From an abelian semigroup, you can create an abelian group, and this could be called groupification. Here is a formalization of that notion. If S is a semigroup, then its groupification is a map of semigroups $\pi : S \to G$ such that G is a group, and any other map of semigroups from S to a *group* G' factors *uniquely* through G.



4.C. EXERCISE. Define groupification H from the category of abelian semigroups to the category of abelian groups. (One possibility of a construction: given an abelian semigroup S, the elements of its groupification $H(S)$ are (a, b), which you may think of as $a - b$, with the equivalence that $(a, b) \sim (c, d)$ if $a + d = b + c$. Describe addition in this group, and show that it satisfies the properties of an abelian group. Describe the semigroup map $S \to H(S)$.) Let F be the forgetful morphism from the category of abelian groups **Ab** to the category of abelian semigroups. Show that H is left-adjoint to F.

(Here is the general idea for experts: We have a full subcategory of a category. We want to "project" from the category to the subcategory. We have $\mathrm{Mor}_{\text{category}}(S, H) = \mathrm{Mor}_{\text{subcategory}}(G, H)$ automatically; thus we are describing the left adjoint to the forgetful functor. How the argument worked: we constructed something which was in the small category, which automatically satisfies the universal property.)

4.D. EXERCISE. Show that if a semigroup is *already* a group then groupification is the identity morphism, by the universal property.

4.E. EXERCISE. The purpose of this exercise is to give you some practice with "adjoints of forgetful functors", the means by which we get groups from semigroups, and sheaves from presheaves. Suppose A is a ring, and S is a multiplicative subset. Then $S^{-1}A$-modules are a fully faithful subcategory of the category of A-modules (meaning: the objects of the first category are a subset of the objects of the second; and the morphisms between any two objects of the second that are secretly objects of the first are just the morphisms from the first). Then $M \to S^{-1}M$ satisfies a universal property. Figure out what the universal property is, and check that it holds. In other words, describe the universal property enjoyed by $M \to S^{-1}M$, and prove that it holds.

(Here is the larger story. Let $S^{-1}A$-Mod be the category of $S^{-1}A$-modules, and A-Mod be the category of A-modules. Every $S^{-1}A$-module is an A-module, and this is an injective map, so we have a (covariant) forgetful functor $F : S^{-1}A\text{-Mod} \to A\text{-Mod}$. In fact this is a fully faithful functor: it is injective on objects, and the morphisms between any two $S^{-1}A$-modules *as A-modules* are just the same when they are considered as $S^{-1}A$-modules. Then there is a functor $G : A\text{-Mod} \to S^{-1}A\text{-Mod}$, which might reasonably be called "localization with respect to S", which is left-adjoint to the forgetful functor. Translation: If M is an A-module, and N is an $S^{-1}A$-module, then $\mathrm{Mor}(GM, N)$ (morphisms as $S^{-1}A$-modules, which is incidentally the same as morphisms as A-modules) are in natural bijection with $\mathrm{Mor}(M, FN)$ (morphisms as A-modules).)

4.2. Useful comment for experts. Here is one last useful comment intended only for people who have seen adjoints before. If (F, G) is an adjoint pair of functors, then F preserves all colimits, and G preserves all limits.

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FOUNDATIONS OF ALGEBRAIC GEOMETRY CLASS 3

RAVI VAKIL

CONTENTS

Last day: category theory in earnest. Universal properties. Limits and colimits. Adjoints.

Today: abelian categories: kernels, cokernels, and all that jazz.

Here are some additional comments on last day's material. The details of Yoneda's lemma don't matter so much; what matters most is that you understand how universal properties determine objects up to unique isomorphism.

It doesn't matter much, but limits and colimits needn't be indexed only by categories where there is at most one morphism between any two objects. I gave an example involving a G-action on a set X (where G is a finite group). The G-invariants can be interpreted as limit.

Tony Licata gave a nice argument that $\otimes$ is right-exact using a universal property argument.

1. KERNELS, COKERNELS, AND EXACT SEQUENCES: A BRIEF INTRODUCTION TO ABELIAN CATEGORIES

Since learning linear algebra, you have been familiar with the notions and behaviors of kernels, cokernels, etc. Later in your life you saw them in the category of abelian groups, and later still in the category of A-modules. Each of these notions generalizes the previous one. The notion of abelian category formalizes kernels etc.

Date: Monday, October 1, 2007. Updated November 4, 2007 to add espace étalé construction.

We now briefly introduce a few notions about abelian categories. We will soon define some new categories (certain sheaves) that will have familiar-looking behavior, reminiscent of that of modules over a ring. The notions of kernels, cokernels, images, and more will make sense, and they will behave "the way we expect" from our experience with modules. This can be made precise through the notion of an abelian category. We will see enough to motivate the definitions that we will see in general: monomorphism (and subobject), epimorphism, kernel, cokernel, and image. But we will avoid having to show that they behave "the way we expect" in a general abelian category because the examples we will see will be directly interpretable in terms of modules over rings.

Abelian categories are the right general setting in which one can do "homological algebra", in which notions of kernel, cokernel, and so on are used, and one can work with complexes and exact sequences.

Two key examples of an abelian category are the category **Ab** of abelian groups, and the category $\mathbf{Mod}_A$ of A-modules. As stated earlier, the first is a special case of the second (just take $A = \mathbb{Z}$). As we give the definitions, you should verify that $\mathbf{Mod}_A$ is an abelian category, and you should keep these examples in mind always.

We first define the notion of *additive category*. We will use it only as a stepping stone to the notion of an abelian category.

1.1. Definition. A category $\mathcal{C}$ is said to be *additive* if it satisfies the following properties.

- Ad1. For each $A, B \in \mathcal{C}$, $\mathrm{Mor}(A, B)$ is an abelian group, such that composition of morphisms distributes over addition. (You should think about what this means — it translates to two distinct statements).
- Ad2. $\mathcal{C}$ has a zero-object, denoted 0. (Recall: this is an object that is simultaneously an initial object and a final object.)
- Ad3. It has products of two objects (a product $A \times B$ for any pair of objects), and hence by induction, products of any finite number of objects.

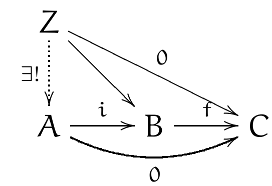
In an additive category, the morphisms are often called homomorphisms, and Mor is denoted by Hom. In fact, this notation Hom is a good indication that you're working in an additive category. A functor between additive categories preserving the additive structure of Hom, and sending the 0-object to the 0-object, is called an *additive functor*. (It is a consequence of the definition that additive functors send 0-objects to 0-objects, and preserve products.)

1.2. *Remarks.* It is a consequence of the definition of additive category that finite direct products are also finite direct sums=coproducts (the details don't matter to us). The symbol $\oplus$ is used for this notion.

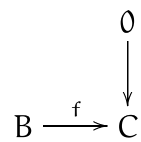
One motivation for the name 0-object is that the 0-morphism in the abelian group $\mathrm{Hom}(A, B)$ is the composition $A \to 0 \to B$.

Real (or complex) Banach spaces are an example of an additive category. The category $\mathbf{Mod}_A$ of A-modules is another example, but it has even more structure, which we now formalize as an example of an abelian category.

1.3. Definition. Let $\mathcal{C}$ be an additive category. A *kernel* of a morphism $f : B \to C$ is a map $i : A \to B$ such that $f \circ i = 0$, and that is universal with respect to this property. Diagramatically:

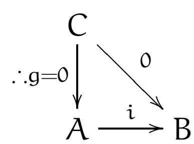


(Note that the kernel is not just an object; it is a morphism of an object to B.) Hence it is unique up to unique isomorphism by universal property nonsense. A *cokernel* is defined dually by reversing the arrows — do this yourself. Notice that the kernel of $f : B \to C$ is the limit



and similarly the cokernel is a colimit.

A morphism $i : A \to B$ in $\mathcal{C}$ is *monic* if for all $i \circ g = 0$, where the tail of g is A, implies $g = 0$. Diagramatically,



(Once we know what an abelian category is — in a few sentences — you may check that a monic morphism in an abelian category is a monomorphism.) If $i : A \to B$ is monic, then we say that A is a *subobject* of B, where the map i is implicit. Dually, there is the notion of *epi* — reverse the arrows to find out what that is. The notion of *quotient object* is defined dually to subobject.

An *abelian category* is an additive category satisfying three additional properties.

(1) Every map has a kernel and cokernel.
(2) Every monic morphism is the kernel of its cokernel.
(3) Every epi morphism is the cokernel of its kernel.

It is a non-obvious (and imprecisely stated) fact that every property you want to be true about kernels, cokernels, etc. follows from these three.

The *image* of a morphism $f : A \to B$ is defined as $\mathrm{im}(f) = \ker(\mathrm{coker}\, f)$. It is the unique factorization

$$A \xrightarrow{\text{epi}} \mathrm{im}(f) \xrightarrow{\text{monic}} B$$

It is the cokernel of the kernel, and the kernel of the cokernel. The reader may want to verify this as an exercise. It is unique up to unique isomorphism.

We will leave the foundations of abelian categories untouched. The key thing to remember is that if you understand kernels, cokernels, images and so on in the category of modules over a ring $\mathbf{Mod}_A$, you can manipulate objects in any abelian category. This is made precise by Freyd-Mitchell Embedding Theorem. However, the abelian categories we'll come across will obviously be related to modules, and our intuition will clearly carry over. For example, we'll show that sheaves of abelian groups on a topological space X form an abelian category. The interpretation in terms of "compatible germs" will connect notions of kernels, cokernels etc. of sheaves of abelian groups to the corresponding notions of abelian groups.

1.4. Complexes, exactness, and homology.

If you aren't familiar with these notions, you should definitely read this section closely!

We say

$$A \xrightarrow{f} B \xrightarrow{g} C \tag{1}$$

is a complex if $g \circ f = 0$, and is exact if $\ker g = \operatorname{im} f$. If (1) is a complex, then its homology is $\ker g / \operatorname{im} f$. We say that $\ker g$ are the *cycles*, and $\operatorname{im} f$ are the *boundaries*. Homology (resp. cohomology) is denoted by H, often with a subscript (resp. superscript), and it should be clear from the context what the subscript means (see for example the discussion below).

An exact sequence

$$A^\bullet : \qquad \cdots \longrightarrow A^{i-1} \xrightarrow{f^{i-1}} A^i \xrightarrow{f^i} A^{i+1} \xrightarrow{f^{i+1}} \cdots \tag{2}$$

can be "factored" into short exact sequences

$$0 \longrightarrow \ker f^i \longrightarrow A^i \longrightarrow \ker f^{i+1} \longrightarrow 0$$

which is helpful in proving facts about long exact sequences by reducing them to facts about short exact sequences.

More generally, if (2) is assumed only to be a complex, then it can be "factored" into short exact sequences

$$0 \longrightarrow \ker f^i \longrightarrow A^i \longrightarrow \operatorname{im} f^i \longrightarrow 0$$

$$0 \longrightarrow \operatorname{im} f^{i-1} \longrightarrow \ker f^i \longrightarrow H^i(A^\bullet) \longrightarrow 0$$

1.A. EXERCISE. Suppose

$$0 \xrightarrow{d^0} A^1 \xrightarrow{d^1} \cdots \xrightarrow{d^{n-1}} A^n \xrightarrow{d^n} 0$$

is a complex of k-vector spaces (often called $A^\bullet$ for short). Show that $\sum(-1)^i \dim A^i = \sum(-1)^i h^i(A^\bullet)$. (Recall that $h^i(A^\bullet) = \dim \ker(d^i)/\operatorname{im}(d^{i-1})$.) In particular, if $A^\bullet$ is exact, then $\sum(-1)^i \dim A^i = 0$. (If you haven't dealt much with cohomology, this will give you some practice.)

1.B. IMPORTANT EXERCISE. Suppose $\mathcal{C}$ is an abelian category. Define the category $\mathbf{Com}_{\mathcal{C}}$ as follows. The objects are infinite complexes

$$A^\bullet : \qquad \cdots \longrightarrow A^{i-1} \xrightarrow{f^{i-1}} A^i \xrightarrow{f^i} A^{i+1} \xrightarrow{f^{i+1}} \cdots$$

in $\mathcal{C}$, and the morphisms $A^\bullet \to B^\bullet$ are commuting diagrams

$$\begin{array}{ccccccccc}
A^\bullet : & \cdots \longrightarrow & A^{i-1} & \xrightarrow{f^{i-1}} & A^i & \xrightarrow{f^i} & A^{i+1} & \xrightarrow{f^{i+1}} & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
B^\bullet : & \cdots \longrightarrow & B^{i-1} & \xrightarrow{f^{i-1}} & B^i & \xrightarrow{f^i} & B^{i+1} & \xrightarrow{f^{i+1}} & \cdots
\end{array}$$

Show that $\mathbf{Com}_{\mathcal{C}}$ is an abelian category. Show that a short exact sequence of complexes

$$\begin{array}{ccccccccc}
0 : & \cdots \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
A^\bullet : & \cdots \longrightarrow & A^{i-1} & \xrightarrow{f^{i-1}} & A^i & \xrightarrow{f^i} & A^{i+1} & \xrightarrow{f^{i+1}} & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
B^\bullet : & \cdots \longrightarrow & B^{i-1} & \xrightarrow{g^{i-1}} & B^i & \xrightarrow{g^i} & B^{i+1} & \xrightarrow{g^{i+1}} & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
C^\bullet : & \cdots \longrightarrow & C^{i-1} & \xrightarrow{h^{i-1}} & C^i & \xrightarrow{h^i} & C^{i+1} & \xrightarrow{h^{i+1}} & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 : & \cdots \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \cdots
\end{array}$$

induces a *long exact sequence in cohomology*

$$\begin{array}{c}
\cdots \longrightarrow H^{i-1}(C^\bullet) \longrightarrow \\
H^i(A^\bullet) \longrightarrow H^i(B^\bullet) \longrightarrow H^i(C^\bullet) \longrightarrow \\
H^{i+1}(A^\bullet) \longrightarrow \cdots
\end{array}$$

1.5. *Exactness of functors.* If $F : \mathcal{A} \to \mathcal{B}$ is a covariant additive functor from one abelian category to another, we say that F is *right-exact* if the exactness of

$$A' \longrightarrow A \longrightarrow A'' \longrightarrow 0,$$

in $\mathcal{A}$ implies that

$$F(A') \longrightarrow F(A) \longrightarrow F(A'') \longrightarrow 0$$

is also exact. Dually, we say that F is *left-exact* if the exactness of

$$0 \longrightarrow A' \longrightarrow A \longrightarrow A'' \qquad \text{implies}$$

$$0 \longrightarrow F(A') \longrightarrow F(A) \longrightarrow F(A'') \qquad \text{is exact.}$$

A contravariant functor is *left-exact* if the exactness of

$$A' \longrightarrow A \longrightarrow A'' \longrightarrow 0 \qquad \text{implies}$$

$$0 \longrightarrow F(A'') \longrightarrow F(A) \longrightarrow F(A') \qquad \text{is exact.}$$

The reader should be able to deduce what it means for a contravariant functor to be *right-exact*.

A covariant or contravariant functor is *exact* if it is both left-exact and right-exact.

1.6. $\star$ **Interactions of adjoints, (co)limits, and (left and right) exactness.** There are some useful properties of adjoints that make certain arguments quite short. This is intended only for experts, and can be ignored by most people in the class, so this won't be said during class. We present them as three facts. Suppose $(F : \mathcal{C} \to \mathcal{D}, G : \mathcal{D} \to \mathcal{C})$ is a pair of adjoint functors.

Fact 1. F commutes with colimits, and G commutes with limits.

We prove the second statement here. The first is the same, "with the arrows reversed". We begin with a useful fact.

1.C. EXERCISE: $\mathrm{Mor}(X, \cdot)$ COMMUTES WITH LIMITS. Suppose A_i ($i \in \mathcal{I}$) is a diagram in $\mathcal{D}$ indexed by $\mathcal{I}$, and $\varprojlim A_i \to A_i$ is its limit. Then for any $X \in \mathcal{D}$, $\mathrm{Mor}(X, \varprojlim A_i) \to \mathrm{Mor}(X, A_i)$ is the limit $\varprojlim \mathrm{Mor}(X, A_i)$.

We are now ready to prove (one direction of) Fact 1.

1.7. *Proposition (right-adjoints commute with limits). — Suppose* $(F : \mathcal{C} \to \mathcal{D}, G : \mathcal{D} \to \mathcal{C})$ *is a pair of adjoint functors. If* $A = \varprojlim A_i$ *is a limit in* $\mathcal{D}$ *of a diagram indexed by* I*, then* $GA = \varprojlim GA_i$ *(with the corresponding maps* $GA \to GA_i$*) is a limit in* $\mathcal{C}$*.*

Proof. We must show that $GA \to GA_i$ satisfies the universal property of limits. Suppose we have maps $W \to GA_i$ commuting with the maps of $\mathcal{I}$. We wish to show that there exists a unique $W \to GA$ extending the $W \to GA_i$. By adjointness of F and G, we can restate this as: Suppose we have maps $FW \to A_i$ commuting with the maps of $\mathcal{I}$. We wish to show that there exists a unique $FW \to A$ extending the $FW \to A_i$. But this is precisely the universal property of the limit. □

Suppose now further that $\mathcal{C}$ and $\mathcal{D}$ are abelian categories, and F and G are additive functors. Kernels are limits and cokernels are colimits (§1.3), so we have **Fact 2.** F commutes with cokernels and G commutes with kernels.

Now suppose

$$M' \xrightarrow{f} M \longrightarrow M'' \longrightarrow 0$$

is an exact sequence in $\mathcal{C}$, so $M'' = \operatorname{coker} f$. Then by Fact 2, $FM'' = \operatorname{coker} Ff$. Thus

$$FM' \to FM \to FM'' \to 0$$

so: **Fact 3.** Left-adjoint additive functors are right-exact, and right-adjoint additive functors are left-exact. For example, the fact that $(\cdot \otimes_A N, \operatorname{Hom}_A(N, \cdot))$ are an adjoint pair (from the A-**Mod** to itself) imply that $\cdot \otimes_A N$ is right-exact (an exercise from last week) and $\operatorname{Hom}(N, \cdot)$ is left-exact.

2. Sheaves

It is perhaps suprising that geometric spaces are often best understood in terms of (nice) functions on them. For example, a differentiable manifold that is a subset of $\mathbb{R}^n$ can be studied in terms of its differentiable functions. Because geometric spaces can have few functions, a more precise version of this insight is that the structure of the space can be well understood by undestanding all functions on all open subsets of the space. This information is encoded in something called a *sheaf*. We will define *sheaves* and describe many useful facts about them. Sheaves were introduced by Leray in the 1940s. The reason for the name is from an earlier, different perspective on the definition, which we shall not discuss.

We will begin with a motivating example to convince you that the notion is not so foreign.

One reason sheaves are often considered slippery to work with is that they keep track of a huge amount of information, and there are some subtle local-to-global issues. There are also three different ways of getting a hold of them.

- in terms of open sets (the definition §4) — intuitive but in some way the least helpful
- in terms of stalks
- in terms of a base of a topology.

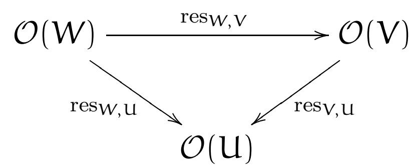
Knowing which idea to use requires experience, so it is essential to do a number of exercises on different aspects of sheaves in order to truly understand the concept.

3. Motivating example: The sheaf of differentiable functions.

We will consider differentiable functions on the topological $X = \mathbb{R}^n$, although you may consider a more general manifold X. The sheaf of differentiable functions on X is the data of all differentiable functions on all open subsets on X; we will see how to manage this data, and observe some of its properties. To each open set $U \subset X$, we have a ring of differentiable functions. We denote this ring $\mathcal{O}(U)$.

Given a differentiable function on an open set, you can restrict it to a smaller open set, obtaining a differentiable function there. In other words, if $U \subset V$ is an inclusion of open sets, we have a map $\mathrm{res}_{V,U} : \mathcal{O}(V) \to \mathcal{O}(U)$.

Take a differentiable function on a big open set, and restrict it to a medium open set, and then restrict that to a small open set. The result is the same as if you restrict the differentiable function on the big open set directly to the small open set. In other words, if $U \hookrightarrow V \hookrightarrow W$, then the following diagram commutes:



Next take two differentiable functions f_1 and f_2 on a big open set U, and an open cover of U by some U_i. Suppose that f_1 and f_2 agree on each of these U_i. Then they must have been the same function to begin with. In other words, if $\{U_i\}_{i \in I}$ is a cover of U, and $f_1, f_2 \in \mathcal{O}(U)$, and $\mathrm{res}_{U,U_i} f_1 = \mathrm{res}_{U,U_i} f_2$, then $f_1 = f_2$. Thus I can *identify* functions on an open set by looking at them on a covering by small open sets.

Finally, given the same U and cover U_i, take a differentiable function on each of the U_i — a function f_1 on U_1, a function f_2 on U_2, and so on — and they agree on the pairwise overlaps. Then they can be "glued together" to make one differentiable function on all of U. In other words, given $f_i \in \mathcal{O}(U_i)$ for all i, such that $\mathrm{res}_{U_i, U_i \cap U_j} f_i = \mathrm{res}_{U_j, U_i \cap U_j} f_j$ for all i, j, then there is some $f \in \mathcal{O}(U)$ such that $\mathrm{res}_{U,U_i} f = f_i$ for all i.

The entire example above would have worked just as well with continuous function, or smooth functions, or just functions. Thus all of these classes of "nice" functions share some common properties; we will soon formalize these properties in the notion of a sheaf.

3.1. Motivating example continued: the germ of a differentiable function. Before we do, we first point out another definition, that of the germ of a differentiable function at a point $x \in X$. Intuitively, it is a shred of a differentiable function at x. Germs are objects of the form $\{(f, \text{open } U) : x \in U, f \in \mathcal{O}(U)\}$ modulo the relation that $(f, U) \sim (g, V)$ if there is some open set $W \subset U, V$ containing x where $f|_W = g|_W$ (or in our earlier language, $\mathrm{res}_{U,W} f = \mathrm{res}_{V,W} g$). In other words, two functions that are the same in a neighborhood of x but (but may differ elsewhere) have the same germ. We call this set of germs $\mathcal{O}_x$. Notice that this forms a ring: you can add two germs, and get another germ: if you have a function f defined on U, and a function g defined on V, then $f + g$ is defined on $U \cap V$. Moreover, $f + g$ is well-defined: if f' has the same germ as f, meaning that there is some open set W containing x on which they agree, and g' has the same germ as g, meaning they agree on some open W' containing x, then $f' + g'$ is the same function as $f + g$ on $U \cap V \cap W \cap W'$.

Notice also that if $x \in U$, you get a map $\mathcal{O}(U) \to \mathcal{O}_x$. Experts may already see that this is secretly a colimit.

We can see that $\mathcal{O}_x$ is a local ring as follows. Consider those germs vanishing at x, which we denote $\mathfrak{m}_x \subset \mathcal{O}_x$. They certainly form an ideal: $\mathfrak{m}_x$ is closed under addition, and when you multiply something vanishing at x by any other function, the result also vanishes at x. Anything not in this ideal is invertible: given a germ of a function f not vanishing at x, then f is non-zero near x by continuity, so $1/f$ is defined near x. We check that this ideal is maximal by showing that the quotient map is a field:

$$0 \longrightarrow \mathfrak{m} := \text{ideal of germs vanishing at } x \longrightarrow \mathcal{O}_x \xrightarrow{f \mapsto f(x)} \mathbb{R} \longrightarrow 0$$

3.A. EXERCISE (FOR THOSE FAMILIAR WITH DIFFERENTIABLE FUNCTIONS). Show that this is the only maximal ideal of $\mathcal{O}_x$.

Note that we can interpret the value of a function at a point, or the value of a germ at a point, as an element of the local ring modulo the maximal ideal. (We will see that this doesn't work for more general sheaves, but *does* work for things behaving like sheaves of functions. This will be formalized in the notion of a *locally ringed space*, which we will see only briefly later.)

Side fact for those with more geometric experience. Notice that $\mathfrak{m}/\mathfrak{m}^2$ is a module over $\mathcal{O}_x/\mathfrak{m} \cong \mathbb{R}$, i.e. it is a real vector space. It turns out to be naturally (whatever that means) the cotangent space to the manifold at x. This insight will prove handy later, when we define tangent and cotangent spaces of schemes.

4. DEFINITION OF SHEAF AND PRESHEAF

We now formalize these notions, by defining presheaves and sheaves. Presheaves are simpler to define, and notions such as kernel and cokernel are straightforward — they are defined "open set by open set". Sheaves are more complicated to define, and some notions such as cokernel require more thought (and the notion of sheafification). But we like sheaves are useful because they are in some sense geometric; you can get information about a sheaf locally.

4.1. Definition of sheaf and presheaf on a topological space X.

To be concrete, we will define sheaves of sets. However, **Sets** can be replaced by any category, and other important examples are abelian groups **Ab**, k-vector spaces, rings, modules over a ring, and more. Sheaves (and presheaves) are often written in calligraphic font, or with an underline. The fact that $\mathcal{F}$ is a sheaf on a topological space X is often written as

$$\begin{array}{c} \mathcal{F} \\ \Big| \\ X \end{array}$$

4.2. Definition: Presheaf. A *presheaf* $\mathcal{F}$ on a topological space X is the following data.

• To each open set $U \subset X$, we have a set $\mathcal{F}(U)$ (e.g. the set of differentiable functions). (Notational warning: Several notations are in use, for various good reasons: $\mathcal{F}(U) = \Gamma(U, \mathcal{F}) = H^0(U, \mathcal{F})$. We will use them all.) The elements of $\mathcal{F}(U)$ are called *sections of* $\mathcal{F}$ *over* U.

• For each inclusion $U \hookrightarrow V$ of open sets, we have a restriction map $\mathrm{res}_{V,U} : \mathcal{F}(V) \to \mathcal{F}(U)$ (just as we did for differentiable functions).

• The map $\mathrm{res}_{U,U}$ is the identity: $\mathrm{res}_{U,U} = \mathrm{id}_{\mathcal{F}(U)}$.

• If $U \hookrightarrow V \hookrightarrow W$ are inclusions of open sets, then the restriction maps commute, i.e.

$$\begin{array}{ccc} \mathcal{F}(W) & \xrightarrow{\mathrm{res}_{W,V}} & \mathcal{F}(V) \\ & {\scriptstyle \mathrm{res}_{W,U}} \searrow \quad \swarrow {\scriptstyle \mathrm{res}_{V,U}} & \\ & \mathcal{F}(U) & \end{array}$$

commutes.

4.A. INTERESTING EXERCISE FOR CATEGORY-LOVERS: "A PRESHEAF IS THE SAME AS A CONTRAVARIANT FUNCTOR". Given any topological space X, we can get a category, called the "category of open sets" (discussed last week), where the objects are the open sets and the morphisms are inclusions. Verify that the data of a presheaf is precisely the data of a contravariant functor from the category of open sets of X to the category of sets. (This interpretation is suprisingly useful.)

4.3. Definition: Stalks and germs. We define the stalk of a sheaf at a point in two different ways. In essense, one will be hands-on, and the other will be categorical using universal properties (as a colimit).

4.4. We will define the *stalk of* $\mathcal{F}$ *at* x to be the set of *germs* of a presheaf $\mathcal{F}$ at a point x, $\mathcal{F}_x$, as in the example of §3.1. Elements are $\{(f, \text{open } U) : x \in U, f \in \mathcal{O}(U)\}$ modulo the relation that $(f, U) \sim (g, V)$ if there is some open set $W \subset U, V$ where $\mathrm{res}_{U,W} f = \mathrm{res}_{V,W}\, g$. Elements of the stalk correspond to sections over some open set containing x. Two of these sections are considered the same if they agree on some smaller open set.

4.5. A useful (and better) equivalent definition of a stalk is as a colimit of all $\mathcal{F}(U)$ over all open sets U containing x:

$$\mathcal{F}_x = \varinjlim \mathcal{F}(U).$$

(Those having thought about the category of open sets will have a warm feeling in their stomachs.) The index category is a directed set (given any two such open sets, there is a third such set contained in both), so these two definitions are the same. It would be good for you to think this through. Hence by that Remark/Exercise, we can have stalks for sheaves of sets, groups, rings, and other things for which direct limits exist for directed sets.

Elements of the stalk $\mathcal{F}_x$ are called *germs*. If $x \in U$, and $f \in \mathcal{F}(U)$, then the image of f in $\mathcal{F}_x$ is called the *germ of* f.

I repeat that it is useful to think of stalks in both ways, as colimits, and also explicitly: a germ at p has as a representative a section over an open set near p.

If $\mathcal{F}$ is a sheaf of rings, then $\mathcal{F}_x$ is a ring, and ditto for rings replaced by abelian groups (or indeed any category in which colimits exist).

(Warning: the value at a point of a section doesn't make sense.)

4.6. Definition: Sheaf. A presheaf is a *sheaf* if it satisfies two more axioms, which will use the notion of when some open sets cover another.

Identity axiom. If $\{U_i\}_{i \in I}$ is an open cover of U, and $f_1, f_2 \in \mathcal{F}(U)$, and $\mathrm{res}_{U,U_i} f_1 = \mathrm{res}_{U,U_i} f_2$, then $f_1 = f_2$.

(A presheaf satisfying the identity axiom is sometimes called a *separated presheaf*, but we will not use that notation in any essential way.)

Gluability axiom. If $\{U_i\}_{i \in I}$ is a open cover of U, then given $f_i \in \mathcal{F}(U_i)$ for all i, such that $\mathrm{res}_{U_i, U_i \cap U_j} f_i = \mathrm{res}_{U_j, U_i \cap U_j} f_j$ for all i, j, then there is some $f \in \mathcal{F}(U)$ such that $\mathrm{res}_{U,U_i} f = f_i$ for all i.

(For experts, and scholars of the empty set only: an additional axiom sometimes included is that $F(\varnothing)$ is a one-element set, and in general, for a sheaf with values in a category, $F(\varnothing)$ is required to be the final object in the category. As pointed out by Kirsten, this actually follows from the above definitions, assuming that the empty product is appropriately defined as the final object.)

Example. If U and V are disjoint, then $\mathcal{F}(U \cup V) = \mathcal{F}(U) \times \mathcal{F}(V)$. (Here we use the fact that $F(\varnothing)$ is the final object.)

The *stalk of a sheaf* at a point is just its stalk as a presheaf; the same definition applies.

Philosophical note. In mathematics, definitions often come paired: "at most one" and "at least one". In this case, identity means there is at most one way to glue, and gluability means that there is at least one way to glue.

4.B. UNIMPORTANT EXERCISE FOR CATEGORY-LOVERS. The gluability axiom may be interpreted as saying that $\mathcal{F}(\cup_{i \in I} U_i)$ is a certain limit. What is that limit?

We now give a number of examples of sheaves.

4.7. *Example.* (a) Verify that the examples of §3 are indeed sheaves (of differentiable functions, or continuous functions, or smooth functions, or functions on a manifold or $\mathbb{R}^n$).
(b) Show that real-valued continuous functions on (open sets of) a topological space X form a sheaf.

4.8. *Important Example: Restriction of a sheaf.* Suppose $\mathcal{F}$ is a sheaf on X, and $U \subset$ is an open set. Define the *restriction of $\mathcal{F}$ to* U, denoted $\mathcal{F}|_U$, to be the collection $\mathcal{F}|_U(V) = \mathcal{F}(V)$ for all $V \subset U$. Clearly this is a sheaf on U.

4.9. *Important Example: skyscraper sheaf.* Suppose X is a topological space, with $x \in X$, and S is a set. Then S_x defined by $\mathcal{F}(U) = S$ if $x \in U$ and $\mathcal{F}(U) = \{e\}$ if $x \notin U$ forms a sheaf. Here $\{e\}$ is any one-element set. (Check this if it isn't clear to you.) This is called a *skyscraper sheaf*, because the informal picture of it looks like a skyscraper at x. There is an analogous definition for sheaves of abelian groups, except $\mathcal{F}(U) = \{0\}$ if $x \in U$; and for sheaves with values in a category more generally, $\mathcal{F}(U)$ should be a final object. (Warning: the notation S_x is not ideal, as the subscript of a point will also used to denote a stalk.)

4.C. IMPORTANT EXERCISE: CONSTANT PRESHEAF AND LOCALLY CONSTANT SHEAF. (a) Let X be a topological space, and S a set with more than one element, and define $\mathcal{F}(U) = S$ for all open sets U. Show that this forms a presheaf (with the obvious restriction maps), and even satisfies the identity axiom. We denote this presheaf $\underline{S}^{\text{pre}}$. Show that this needn't form a sheaf. This is called the *constant presheaf with values in S*.
(b) Now let $\mathcal{F}(U)$ be the maps to S that are *locally constant*, i.e. for any point x in U, there is a neighborhood of x where the function is constant. Show that this is a *sheaf*. (A better description is this: endow S with the discrete topology, and let $\mathcal{F}(U)$ be the continuous maps $U \to S$. Using this description, this follows immediately from Exercise 4.E below.) We will call this the *locally constant sheaf*. This is usually called the *constant sheaf*. We denote this sheaf $\underline{S}$.

4.D. UNIMPORTANT EXERCISE: MORE EXAMPLES OF PRESHEAVES THAT ARE NOT SHEAVES. Show that the following are presheaves on $\mathbb{C}$ (with the usual topology), but not sheaves: (a) bounded functions, (b) holomorphic functions admitting a holomorphic square root.

4.E. EXERCISE. Suppose Y is a topological space. Show that "continuous maps to Y" form a sheaf of sets on X. More precisely, to each open set U of X, we associate the set of continuous maps to Y. Show that this forms a sheaf. (Example 4.7(b), with $Y = \mathbb{R}$, and Exercise 4.C(b), with $Y = S$ with the discrete topology, are both special cases.)

4.F. EXERCISE. This is a fancier example of the previous exercise.
(a) Suppose we are given a continuous map $f : Y \to X$. Show that "sections of f" form a sheaf. More precisely, to each open set U of X, associate the set of continuous maps s to Y such that $f \circ s = id|_U$. Show that this forms a sheaf. (For those who have heard of vector bundles, these are a good example.)
(b) (This exercise is for those who know what a topological group is. If you don't know what a topological group is, you might be able to guess.) Suppose that Y is a topological group. Show that maps to Y form a sheaf of *groups*. (Example 4.7(b), with $Y = \mathbb{R}$, is a special case.)

4.10. ⋆ *The espace étalé of a (pre)sheaf.* Depending on your background, you may prefer the following perspective on sheaves, which we will not discuss further. Suppose $\mathcal{F}$ is a presheaf (e.g. a sheaf) on a topological space X. Construct a topological space Y along with a continuous map to X as follows: as a set, Y is the disjoint union of all the stalks of X. This also describes a natural set map $Y \to X$. We topologize Y as follows. Each section s of $\mathcal{F}$ over an open set U determines a section of $Y \to X$ over U, sending s to each of its germs for each $x \in U$. The topology on Y is the weakest topology such that these sections are continuous. This is called the **espace étalé** of the Then the reader may wish to show that (a) if $\mathcal{F}$ is a sheaf, then the sheaf of sections of $Y \to X$ (see the previous exercise 4.F(a) can be naturally identified with the sheaf $\mathcal{F}$ itself. (b) Moreover, if $\mathcal{F}$ is a presheaf, the sheaf of sections of $Y \to X$ is the *sheafification* of $\mathcal{F}$ (to be defined later).

4.G. IMPORTANT EXERCISE: THE DIRECT IMAGE SHEAF OR PUSHFORWARD SHEAF. Suppose $f : X \to Y$ is a continuous map, and $\mathcal{F}$ is a sheaf on X. Then define $f_*\mathcal{F}$ by $f_*\mathcal{F}(V) = \mathcal{F}(f^{-1}(V))$, where V is an open subset of Y. Show that $f_*\mathcal{F}$ is a sheaf. This is called a *direct image sheaf* of *pushforward sheaf*. More precisely, $f_*\mathcal{F}$ is called the *pushforward of $\mathcal{F}$ by* f.

The skyscraper sheaf (Exercise 4.9) can be interpreted as follows as the pushforward of the constant sheaf $\underline{S}$ on a one-point space x, under the morphism $f : \{x\} \to X$.

Once we realize that sheaves form a category, we will see that the pushforward is a functor from sheaves on X to sheaves on Y.

4.H. EXERCISE (PUSHFORWARD INDUCES MAPS OF STALKS). Suppose $\mathcal{F}$ is a sheaf of sets (or rings or A-modules). If $f(x) = y$, describe the natural morphism of stalks $(f_*\mathcal{F})_y \to \mathcal{F}_x$. (You can use the explicit definition of stalk using representatives, §4.4, or the universal property, §4.5. If you prefer one way, you should try the other.)

4.11. Important Example: Ringed spaces, and $\mathcal{O}_X$-modules.. Suppose $\mathcal{O}_X$ is a sheaf of *rings* on a topological space X (i.e. a sheaf on X with values in the category of **Rings**). Then $(X, \mathcal{O}_X)$ is called a *ringed space*. The sheaf of rings is often denoted by $\mathcal{O}_X$; this is pronounced "oh-of-X". This sheaf is called the *structure sheaf* of the ringed space. We now define the notion of an $\mathcal{O}_X$-*module*. The notion is analagous to one we've seen before:

just as we have modules over a ring, we have $\mathcal{O}_X$-modules over the structure sheaf (of rings) $\mathcal{O}_X$.

There is only one possible definition that could go with this name. An $\mathcal{O}_X$-module is a sheaf of abelian groups $\mathcal{F}$ with the following additional structure. For each U, $\mathcal{F}(U)$ is a $\mathcal{O}_X(U)$-module. Furthermore, this structure should behave well with respect to restriction maps. This means the following. If $U \subset V$, then

$$\begin{array}{ccc} \mathcal{O}_X(V) \times \mathcal{F}(V) & \xrightarrow{\text{action}} & \mathcal{F}(V) \\ \downarrow \text{res}_{V,U} & & \downarrow \text{res}_{V,U} \\ \mathcal{O}_X(U) \times \mathcal{F}(U) & \xrightarrow{\text{action}} & \mathcal{F}(U) \end{array} \tag{3}$$

commutes. (You should convince yourself that I haven't forgotten anything.)

Recall that the notion of A-module generalizes the notion of abelian group, because an abelian group is the same thing as a $\mathbb{Z}$-module. Similarly, the notion of $\mathcal{O}_X$-module generalizes the notion of sheaf of abelian groups, because the latter is the same thing as a $\underline{\mathbb{Z}}$-module, where $\underline{\mathbb{Z}}$ is the locally constant sheaf with values in $\mathbb{Z}$. Hence when we are proving things about $\mathcal{O}_X$-modules, we are also proving things about sheaves of abelian groups.

4.12. *For those who know about vector bundles.* The motivating example of $\mathcal{O}_X$-modules is the sheaf of sections of a vector bundle. If X is a differentiable manifold, and $\pi : V \to X$ is a vector bundle over X, then the sheaf of differentiable sections $\phi : X \to V$ is an $\mathcal{O}_X$-module. Indeed, given a section s of π over an open subset $U \subset X$, and a function f on U, we can multiply s by f to get a new section fs of π over U. Moreover, if V is a smaller subset, then we could multiply f by s and then restrict to V, or we could restrict both f and s to V and then multiply, and we would get the same answer. That is precisely the commutativity of (3).

Next day: We know about presheaves and sheaves, so we naturally ask about morphisms between presheaves and morphisms of presheaves.

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FOUNDATIONS OF ALGEBRAIC GEOMETRY CLASS 4

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CONTENTS

Last day: abelian categories: kernels, cokernels, and all that jazz. Definition of (pre)sheaves.

A quick comment on last day's material:

When you see a left-exact functor, you should always dream that you are seeing the end of a long exact sequence. If

$$0 \to M' \to M \to M'' \to 0$$

is an exact sequence in abelian category $\mathcal{A}$, and $F : \mathcal{A} \to \mathcal{B}$ is a left-exact functor, then

$$0 \to FM' \to FM \to FM''$$

is exact, and you should always dream that it should continue in some natural way. For example, the next term should depend only on M', call it R^1FM', and if it is zero, then $FM \to FM''$ is an epimorphism. This remark holds true for left-exact and contravariant functors too. In good cases, such a continuation exists, and is incredibly useful. We'll see this when we come to cohomology.

1. MORPHISMS OF PRESHEAVES AND SHEAVES

Whenever one defines a new mathematical *object*, category theory has taught us to try to understand maps between them. We now define morphisms of presheaves, and similarly for sheaves. In other words, we will descibe the *category of presheaves* (of abelian groups, etc.) and the *category of sheaves*.

A morphism of presheaves of sets (or indeed with values in any category) $f : \mathcal{F} \to \mathcal{G}$ is the data of maps $f(U) : \mathcal{F}(U) \to \mathcal{G}(U)$ for all U behaving well with respect to restriction:

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if $U \hookrightarrow V$ then

$$\begin{array}{ccc} \mathcal{F}(V) & \xrightarrow{f(V)} & \mathcal{G}(V) \\ \downarrow{\scriptstyle \mathrm{res}_{V,U}} & & \downarrow{\scriptstyle \mathrm{res}_{V,U}} \\ \mathcal{F}(U) & \xrightarrow{f(U)} & \mathcal{G}(U) \end{array}$$

commutes. (Notice: the underlying space remains X.)

A morphism of sheaves is defined in the same way: the morphisms from a sheaf $\mathcal{F}$ to a sheaf $\mathcal{G}$ are precisely the morphisms from $\mathcal{F}$ to $\mathcal{G}$ as presheaves. (Translation: The category of sheaves on X is a full subcategory of the category of presheaves on X.)

An example of a morphism of sheaves is the map from the sheaf of differentiable functions on $\mathbb{R}$ to the sheaf of continuous functions. This is a "forgetful map": we are forgetting that these functions are differentiable, and remembering only that they are continuous.

1.1. *Side-remarks for category-lovers.* If you interpret a presheaf on X as a contravariant functor (from the category of open sets), a morphism of presheaves on X is a natural transformation of functors. We haven't defined natural transformation of functors, but you might be able to guess the definition from this remark.

1.A. EXERCISE. Suppose $f : X \to Y$ is a continuous map of topological spaces (i.e. a morphism in the category of topological spaces). Show that pushforward gives a functor from $\{$ sheaves of sets on $X \}$ to $\{$ sheaves of sets on $Y \}$. Here "sets" can be replaced by any category. (Watch out for some possible confusion: a presheaf is a functor, and presheaves form a category. It may be best to forget that presheaves form a functor for the time being.)

1.B. IMPORTANT EXERCISE AND DEFINITION: "SHEAF Hom". Suppose $\mathcal{F}$ and $\mathcal{G}$ are two sheaves of abelian groups on X. (In fact, it will suffice that $\mathcal{F}$ is a presheaf.) Let $\underline{\mathrm{Hom}}(\mathcal{F}, \mathcal{G})$ be the collection of data

$$\underline{\mathrm{Hom}}(\mathcal{F}, \mathcal{G})(U) := \mathrm{Hom}(\mathcal{F}|_U, \mathcal{G}|_U).$$

(Recall the notation $\mathcal{F}|_U$, the restriction of the sheaf to the open set U, see last day's notes.) Show that this is a sheaf. This is called the "sheaf $\underline{\mathrm{Hom}}$". Show that if $\mathcal{G}$ is a sheaf of abelian groups, then $\underline{\mathrm{Hom}}(\mathcal{F}, \mathcal{G})$ is a sheaf of abelian groups. (The same construction will obviously work for sheaves with values in any category.)

1.2. Presheaves of abelian groups or $\mathcal{O}_X$-modules form an abelian category.

We can make module-like constructions using presheaves of abelian groups on a topological space X. (In this section, all (pre)sheaves are of abelian groups.) For example, we can clearly add maps of presheaves and get another map of presheaves: if $f, g : \mathcal{F} \to \mathcal{G}$, then we define the map $f + g$ by $(f + g)(V) = f(V) + g(V)$. (There is something small to check here: that the result is indeed a map of presheaves.) In this way, presheaves of abelian groups form an additive category (recall: the morphisms between any two presheaves of abelian groups form an abelian group; there is a 0-morphism; and one can take finite products.) For exactly the same reasons, sheaves of abelian groups also form an additive category.

If $f : \mathcal{F} \to \mathcal{G}$ is a morphism of presheaves, define the presheaf kernel $\ker_{\text{pre}} f$ by $(\ker_{\text{pre}} f)(U) = \ker f(U)$.

1.C. EXERCISE. Show that $\ker_{\text{pre}} f$ is a presheaf. (Hint: if $U \hookrightarrow V$, there is a natural map $\text{res}_{V,U} : \mathcal{G}(V)/f(V)(\mathcal{F}(V)) \to \mathcal{G}(U)/f(U)(\mathcal{F}(U))$ by chasing the following diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \ker_{\text{pre}} f(V) & \longrightarrow & \mathcal{F}(V) & \longrightarrow & \mathcal{G}(V) \\ & & \downarrow \exists! & & \downarrow \text{res}_{V,U} & & \downarrow \text{res}_{V,U} \\ 0 & \longrightarrow & \ker_{\text{pre}} f(U) & \longrightarrow & \mathcal{F}(U) & \longrightarrow & \mathcal{G}(U) \end{array}$$

You should check that the restriction maps compose as desired.)

Define the presheaf cokernel $\text{coker}_{\text{pre}} f$ similarly. It is a presheaf by essentially the same argument.

1.D. EXERCISE: THE COKERNEL DESERVES ITS NAME. Show that the presheaf cokernel satisfies the universal property of cokernels in the category of presheaves.

Similarly, $\ker_{\text{pre}} f \to \mathcal{F}$ satisfies the unversal property for kernels in the category of presheaves.

It is not too tedious to verify that presheaves of abelian groups form an abelian category, and the reader is free to do so. (The key idea is that all abelian-categorical notions may be defined and verified open set by open set.) Hence we can define terms such as *subpresheaf*, *image presheaf*, *quotient presheaf*, *cokernel presheaf*, and they behave the way one expect. You construct kernels, quotients, cokernels, and images open set by open set. Homological algebra (exact sequences etc.) works, and also "works open set by open set". In particular:

1.E. EXERCISE. If $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \cdots \to \mathcal{F}_n \to 0$ is an exact sequence of presheaves of abelian groups, then $0 \to \mathcal{F}_1(U) \to \mathcal{F}_2(U) \to \cdots \to \mathcal{F}_n(U) \to 0$ is also an exact sequence for all U, and vice versa.

The above discussion carries over without any change to presheaves with values in any abelian category.

However, we are interested in more geometric objects, sheaves, where things are can be understood in terms of their local behavior, thanks to the identity and gluing axioms. We will soon see that sheaves of abelian groups also form an abelian category, but a complication will arise that will force the notion of *sheafification* on us. Sheafification will be the answer to many of our prayers. We just don't realize it yet.

Kernels work just as with presheaves:

1.F. IMPORTANT EXERCISE. Suppose $f : \mathcal{F} \to \mathcal{G}$ is a morphism of *sheaves*. Show that the presheaf kernel $\ker_{\text{pre}} f$ is in fact a sheaf. Show that it satisfies the universal property of kernels. (Hint: the second question follows immediately from the fact that $\ker_{\text{pre}} f$ satisfies the universal property in the category of *presheaves*.)

Thus if f is a morphism of sheaves, we define

$$\ker f := \ker_{\text{pre}} f.$$

The problem arises with the cokernel.

1.G. IMPORTANT EXERCISE. Let X be $\mathbb{C}$ with the classical topology, let $\underline{\mathbb{Z}}$ be the locally constant sheaf on X with group $\mathbb{Z}$, $\mathcal{O}_X$ the sheaf of holomorphic functions, and $\mathcal{F}$ the *presheaf* of functions admitting a holomorphic logarithm. (Why is $\mathcal{F}$ not a sheaf?) Consider

$$0 \longrightarrow \underline{\mathbb{Z}} \longrightarrow \mathcal{O}_X \xrightarrow{f \mapsto \exp 2\pi i f} \mathcal{F} \longrightarrow 0$$

where $\underline{\mathbb{Z}} \to \mathcal{O}_X$ is the natural inclusion. Show that this is an exact sequence *of presheaves*. Show that $\mathcal{F}$ is *not* a sheaf. (Hint: $\mathcal{F}$ does not satisfy the gluability axiom. The problem is that there are functions that don't have a logarithm that locally have a logarithm.) This will come up again in Example 2.8.

We will have to put our hopes for understanding cokernels of sheaves on hold for a while. We will first take a look at how to understand sheaves using stalks.

2. PROPERTIES DETERMINED AT THE LEVEL OF STALKS

In this section, we'll see that lots of facts about sheaves can be checked "at the level of stalks". This isn't true for presheaves, and reflects the local nature of sheaves. We will flag each case of a property determined by stalks.

2.A. IMPORTANT EXERCISE **(sections are determined by stalks)**. Prove that a section of a sheaf is determined by its germs, i.e. the natural map

$$\mathcal{F}(U) \to \prod_{x \in U} \mathcal{F}_x \tag{1}$$

is injective. (Hint # 1: you won't use the gluability axiom, so this is true for separated presheaves. Hint # 2: it is false for presheaves in general, see Exercise 2.F, so you *will* use the identity axiom.)

This exercise suggests an important question: which elements of the right side of (1) are in the image of the left side?

2.1. Important definition. We say that an element $\prod_{x \in U} s_x$ of the right side $\prod_{x \in U} \mathcal{F}_x$ of (1) consists of *compatible germs* if for all $x \in U$, there is some representative $(U_x, s'_x \in \Gamma(U_x, \mathcal{F}))$ for s_x (where $x \in U_x \subset U$) such that the germ of s'_x at all $y \in U_x$ is s_y. You'll have to think about this a little. Clearly any section s of $\mathcal{F}$ over U gives a choice of compatible germs for U — take $(U_x, s'_x) = (U, s)$.

2.B. IMPORTANT EXERCISE. Prove that any choice of compatible germs for $\mathcal{F}$ over U is the image of a section of $\mathcal{F}$ over U. (Hint: you will use gluability.)

We have thus completely described the image of (1), in a way that we will find useful.

2.2. *Remark.* This perspective is part of the motivation for the agricultural terminology "sheaf": it is the data of a bunch of stalks, bundled together appropriately.

Now we throw morphisms into the mix.

2.C. EXERCISE. Show a morphism of (pre)sheaves (of sets, or rings, or abelian groups, or $\mathcal{O}_X$-modules) induces a morphism of stalks. More precisely, if $\phi : \mathcal{F} \to \mathcal{G}$ is a morphism of (pre)sheaves on X, and $x \in X$, describe a natural map $\phi_x : \mathcal{F}_x \to \mathcal{G}_x$.

2.D. EXERCISE **(morphisms are determined by stalks)**. Show that morphisms of sheaves are determined by morphisms of stalks. Hint: consider the following diagram.

(2)
$$\begin{array}{ccc} \mathcal{F}(U) & \longrightarrow & \mathcal{G}(U) \\ \downarrow & & \downarrow \\ \prod_{x \in U} \mathcal{F}_x & \longrightarrow & \prod_{x \in U} \mathcal{G}_x \end{array}$$

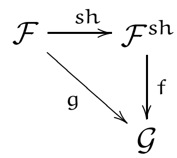
2.E. TRICKY EXERCISE **(isomorphisms are determined by stalks)**. Show that a morphism of sheaves is an isomorphism if and only if it induces an isomorphism of all stalks. (Hint: Use (2). Injectivity uses the previous exercise 2.D. Surjectivity will use gluability, and is more subtle.)

2.F. EXERCISE. (a) Show that Exercise 2.A is false for general presheaves.
(b) Show that Exercise 2.D is false for general presheaves.
(c) Show that Exercise 2.E is false for general presheaves.
(General hint for finding counterexamples of this sort: consider a 2-point space with the discrete topology, i.e. every subset is open.)

2.3. Sheafification.

Every sheaf is a presheaf (and indeed by definition sheaves on X form a full subcategory of the category of presheaves on X). Just as groupification gives a group that best approximates a semigroup, sheafification gives the sheaf that best approximates a presheaf, with an analogous universal property.

2.4. *Definition.* If $\mathcal{F}$ is a presheaf on X, then a morphism of presheaves $\mathrm{sh} : \mathcal{F} \to \mathcal{F}^{\mathrm{sh}}$ on X is a *sheafification of* $\mathcal{F}$ if $\mathcal{F}^{\mathrm{sh}}$ is a sheaf, and for any other sheaf $\mathcal{G}$, and any presheaf morphism $g : \mathcal{F} \to \mathcal{G}$, there *exists* a *unique* morphism of sheaves $f : \mathcal{F}^{\mathrm{sh}} \to \mathcal{G}$ making the diagram



commute.

2.G. EXERCISE. Show that sheafification is unique up to unique isomorphism. Show that if $\mathcal{F}$ is a sheaf, then the sheafification is $\mathcal{F} \xrightarrow{\mathrm{id}} \mathcal{F}$. (This should be second nature by now.)

2.5. Construction. We next show that any presheaf has a sheafification. Suppose $\mathcal{F}$ is a *presheaf*. Define $\mathcal{F}^{\mathrm{sh}}$ by defining $\mathcal{F}^{\mathrm{sh}}(U)$ as the set of compatible germs of the presheaf $\mathcal{F}$ over U. Explicitly:

$$\mathcal{F}^{\mathrm{sh}}(U) := \{(f_x \in \mathcal{F}_x)_{x \in U} : \forall x \in U, \exists x \in V \subset U, s \in \mathcal{F}(V) : s_y = f_y \forall y \in V\}.$$

(Those who want to worry about the empty set are welcome to.)

2.H. EASY EXERCISE. Show that $\mathcal{F}^{\mathrm{sh}}$ (using the tautological restriction maps) forms a sheaf.

2.I. EASY EXERCISE. Describe a natural map $\mathrm{sh} : \mathcal{F} \to \mathcal{F}^{\mathrm{sh}}$.

2.J. EXERCISE. Show that the map sh satisfies the universal property 2.4 of sheafification.

2.K. EXERCISE. Use the universal property to show that for any morphism of presheaves $\phi : \mathcal{F} \to \mathcal{G}$, we get a natural induced morphism of sheaves $\phi^{\mathrm{sh}} : \mathcal{F}^{\mathrm{sh}} \to \mathcal{G}^{\mathrm{sh}}$. Show that sheafification is a functor from presheaves to sheaves.

2.L. USEFUL EXERCISE FOR CATEGORY-LOVERS. Show that the sheafification functor is left-adjoint to the forgetful functor from sheaves on X to presheaves on X.

2.M. EXERCISE. Show $\mathcal{F} \to \mathcal{F}^{\mathrm{sh}}$ induces an isomorphism of stalks. (Possible hint: Use the concrete description of the stalks. Another possibility: judicious use of adjoints.)

2.6. *Unimportant remark.* Sheafification can be defined in a topological way, via the "espace étalé" construction, see Hartshorne II.1.13, and likely Serre's totemic *FAC*. This is essentially the same construction as the one given here. Another construction is described in Eisenbud-Harris.

2.7. Subsheaves and quotient sheaves.

2.N. EXERCISE. Suppose $\phi : \mathcal{F} \to \mathcal{G}$ is a morphism of sheaves (of sets) on at topological space X. Show that the following are equivalent.

(a) ϕ is a monomorphism in the category of sheaves.
(b) ϕ is injective on the level of stalks: $\phi_x : \mathcal{F}_x \to \mathcal{G}_x$ injective for all $x \in X$.
(c) ϕ is injective on the level of open sets: $\phi(U) : \mathcal{F}(U) \to \mathcal{G}(U)$ is injective for all open $U \subset X$.

(Possible hints: for (b) implies (a), recall that morphisms are determined by stalks, Exercise 2.D. For (a) implies (b), judiciously choose a skyscraper sheaf. For (a) implies (c), judiciously the "indicator sheaf" with one section over every open set contained in U, and no section over any other open set.)

If these conditions hold, we say that $\mathcal{F}$ is a *subsheaf* of $\mathcal{G}$ (where the "inclusion" ϕ is sometimes left implicit).

2.O. EXERCISE. Continuing the notation of the previous exercise, show that the following are equivalent.

(a) ϕ is a epimorphism in the category of sheaves.
(b) ϕ is surjective on the level of stalks: $\phi_x : \mathcal{F}_x \to \mathcal{G}_x$ surjective for all $x \in X$.

If these conditions hold, we say that $\mathcal{G}$ is a *quotient sheaf* of $\mathcal{F}$.

Thus **monomorphisms and epimorphisms — subsheafiness and quotient sheafiness — can be checked at the level of stalks.**

Both exercises generalize immediately to sheaves with values in any category, where "injective" is replaced by "monomorphism" and "surjective" is replaced by "epimorphism".

Notice that there was no part (c) to the previous exercise, and here is an example showing why.

2.8. *Example.* Let $X = \mathbb{C}$ with the usual (analytic) topology, and define $\mathcal{O}_X$ to be the sheaf of holomorphic functions, and $\mathcal{O}_X^*$ to be the sheaf of invertible (nowhere zero) holomorphic functions. This is a sheaf of abelian groups under multiplication. We have maps of sheaves

$$0 \longrightarrow \underline{\mathbb{Z}} \xrightarrow{\times 2\pi i} \mathcal{O}_X \xrightarrow{\exp} \mathcal{O}_X^* \longrightarrow 1 \tag{3}$$

where $\underline{\mathbb{Z}}$ is the locally constant sheaf associated to $\mathbb{Z}$. (You can figure out what the sheaves 0 and 1 mean; they are isomorphic, and are written in this way for reasons that may be clear). We will soon interpret this as an exact sequence of sheaves of abelian groups (the *exponential exact sequence*), although we don't yet have the language to do so.

2.P. EXERCISE. Show that $\mathcal{O}_X \xrightarrow{\exp} \mathcal{O}_X^*$ describes $\mathcal{O}_X^*$ as a quotient sheaf of $\mathcal{O}_X$. Show that it is not surjective on all open sets.

This is a great example to get a sense of what "surjectivity" means for sheaves. Nonzero holomorphic functions locally have logarithms, but they need not globally.

3. SHEAVES OF ABELIAN GROUPS, AND $\mathcal{O}_X$-MODULES, FORM ABELIAN CATEGORIES

We are now ready to see that sheaves of abelian groups, and their cousins, $\mathcal{O}_X$-modules, form abelian categories. In other words, we may treat them in the same way we treat vector spaces, and modules over a ring. In the process of doing this, we will see that this is much stronger than an analogy; kernels, cokernels, exactness, etc. can be understood at the level of germs (which are just abelian groups), and the compatibility of the germs will come for free.

The category of sheaves of abelian groups is clearly an additive category. In order to show that it is an abelian category, we must show that any morphism $\phi : \mathcal{F} \to \mathcal{G}$ has a kernel and a cokernel. We have already seen that ϕ has a kernel (Exercise 1.F): the presheaf kernel is a sheaf, and is a kernel.

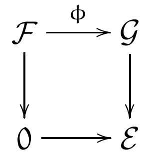
3.A. EXERCISE. Show that the stalk of the kernel is the kernel of the stalks: there is a natural isomorphism

$$(\ker(\mathcal{F} \to \mathcal{G}))_x \cong \ker(\mathcal{F}_x \to \mathcal{G}_x).$$

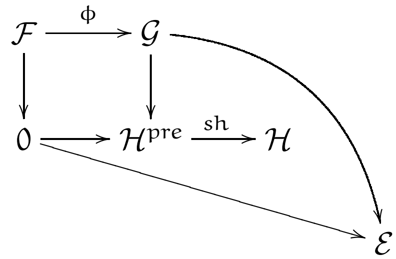
So we next address the issue of the cokernel. Now $\phi : \mathcal{F} \to \mathcal{G}$ has a cokernel in the category of presheaves; call it $\mathcal{H}^{\mathrm{pre}}$ (where the superscript is meant to remind us that this is a presheaf). Let $\mathcal{H}^{\mathrm{pre}} \xrightarrow{\mathrm{sh}} \mathcal{H}$ be its sheafification. Recall that the cokernel is defined using a universal property: it is the colimit of the diagram

$$\begin{array}{ccc} \mathcal{F} & \xrightarrow{\phi} & \mathcal{G} \\ \downarrow & & \\ 0 & & \end{array}$$

in the category of presheaves. We claim that $\mathcal{H}$ is the cokernel of ϕ in the category of sheaves, and show this by proving the universal property. Given any sheaf $\mathcal{E}$ and a commutative diagram



We construct



We show that there is a unique morphism $\mathcal{H} \to \mathcal{E}$ making the diagram commute. As $\mathcal{H}^{\text{pre}}$ is the cokernel in the category of presheaves, there is a unique morphism of presheaves $\mathcal{H}^{\text{pre}} \to \mathcal{E}$ making the diagram commute. But then by the universal property of sheafification (Defn. 2.4), there is a unique morphism of *sheaves* $\mathcal{H} \to \mathcal{E}$ making the diagram commute.

3.B. EXERCISE. Show that the stalk of the cokernel is naturally isomorphic to the cokernel of the stalk.

We have now defined the notions of kernel and cokernel, and verified that they may be checked at the level of stalks. We have also verified that the qualities of a morphism being monic or epi are also determined at the level of stalks (Exercises 2.N and 2.O). Hence sheaves of abelian groups on X form an abelian category.

We see more: all structures coming from the abelian nature of this category may be checked at the level of stalks. For example, **exactness of a sequence of sheaves may be checked at the level of stalks**. A fancy-sounding consequence: taking stalks is an exact functor from sheaves of abelian groups on X to abelian groups.

3.C. EXERCISE (LEFT-EXACTNESS OF THE GLOBAL SECTION FUNCTOR). Suppose $U \subset X$ is an open set, and $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H}$ is an exact sequence of sheaves of abelian groups. Show that

$$0 \to \mathcal{F}(U) \to \mathcal{G}(U) \to \mathcal{H}(U)$$

is exact. Give an example to show that the global section functor is not exact. (Hint: the exponential exact sequence (3).)

3.D. EXERCISE: LEFT-EXACTNESS OF PUSHFORWARD. Suppose $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H}$ is an exact sequence of sheaves of abelian groups on X. If $f : X \to Y$ is a continuous map, show that

$$0 \to f_*\mathcal{F} \to f_*\mathcal{G} \to f_*\mathcal{H}$$

is exact. (The previous exercise, dealing with the left-exactness of the global section functor can be interpreted as a special case of this, in the case where Y is a point.)

3.E. EXERCISE. Suppose $\phi : \mathcal{F} \to \mathcal{G}$ is a morphism of sheaves of abelian groups. Show that the image sheaf $\operatorname{im} \phi$ is the sheafification of the image presheaf. (You must use the definition of image in an abelian category. In fact, this gives the accepted definition of image sheaf for a morphism of sheaves of sets.)

3.F. EXERCISE. Show that if $(X, \mathcal{O}_X)$ is a ringed space, then $\mathcal{O}_X$-modules form an abelian category. (There isn't much more to check!)

We end with a useful construction using some of the ideas in this section.

3.G. IMPORTANT EXERCISE: TENSOR PRODUCTS OF $\mathcal{O}_X$-MODULES. (a) Suppose $\mathcal{O}_X$ is a sheaf of rings on X. Define (categorically) what we should mean by tensor product of two $\mathcal{O}_X$-modules. Give an explicit construction, and show that it satisfies your categorical definition. *Hint:* take the "presheaf tensor product" — which needs to be defined — and sheafify. Note: $\otimes_{\mathcal{O}_X}$ is often written $\otimes$ when the subscript is clear from the context.
(b) Show that the tensor product of stalks is the stalk of tensor product.

I then said a very little about where we are going. The last two things we'll discuss about sheaves in particular are the *inverse image sheaf* and *sheaves on a base* of a topology.

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FOUNDATIONS OF ALGEBRAIC GEOMETRY CLASS 5

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CONTENTS

Last day: morphisms of (pre)sheaves; properties determined at the level of stalks; sheaves of abelian groups on X (and $\mathcal{O}_X$-modules) form an abelian category.

1. THE INVERSE IMAGE SHEAF

We next describe a notion that is rather fundamental, but is still a bit intricate. We won't need it (at least for a long while), so this may be best left for a second reading. Suppose we have a continuous map $f : X \to Y$. If $\mathcal{F}$ is a sheaf on X, we have defined the pushforward or direct image sheaf $f_*\mathcal{F}$, which is a sheaf on Y. There is also a notion of inverse image sheaf. (We won't call it the pullback sheaf, reserving that name for a later construction, involving quasicoherent sheaves.) This is a covariant functor f^{-1} from sheaves on Y to sheaves on X. If the sheaves on Y have some additional structure (e.g. group or ring), then this structure is respected by f^{-1}.

1.1. *Definition by adjoint: elegant but abstract.* Here is a categorical definition of the inverse image: f^{-1} is left-adjoint to f_*.

This isn't really a definition; we need a construction to show that the adjoint exists. (Also, for pedants, this won't determine $f^{-1}\mathcal{F}$; it will only determine it up to unique isomorphism.) Note that we then get canonical maps $f^{-1}f_*\mathcal{F} \to \mathcal{F}$ (associated to the identity in $\operatorname{Mor}_Y(f_*\mathcal{F}, f_*\mathcal{F})$) and $\mathcal{G} \to f_*f^{-1}\mathcal{G}$ (associated to the identity in $\operatorname{Mor}_X(f^{-1}\mathcal{G}, f^{-1}\mathcal{G})$).

1.2. *Construction: concrete but ugly.* Define the temporary notation $f^{-1}\mathcal{G}^{\text{pre}}(U) = \varinjlim_{V \supset f(U)} \mathcal{G}(V)$. (Recall the explicit description of direct limit: sections are sections on open sets containing $f(U)$, with an equivalence relation.)

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1.A. EXERCISE. Show that this defines a presheaf on X.

Now define the *inverse image of* $\mathcal{G}$ by $f^{-1}\mathcal{G} := (f^{-1}\mathcal{G}^{\text{pre}})^{\text{sh}}$.

You will show that this construction satisfies the universal property in Exercise 1.F. For the exercises before that, feel free to use either the adjoint description or the construction.

1.B. EXERCISE. Show that the stalks of $f^{-1}\mathcal{G}$ are the same as the stalks of $\mathcal{G}$. More precisely, if $f(x) = y$, describe a natural isomorphism $\mathcal{G}_y \cong (f^{-1}\mathcal{G})_x$. (Possible hint: use the concrete description of the stalk, as a direct limit. Recall that stalks are preserved by sheafification.)

1.C. EXERCISE (EASY BUT USEFUL). If U is an open subset of Y, $i : U \to Y$ is the inclusion, and $\mathcal{G}$ is a sheaf on Y, show that $i^{-1}\mathcal{G}$ is naturally isomorphic to $\mathcal{G}|_U$.

1.D. EXERCISE (EASY BUT USEFUL). If $y \in Y$, $i : \{y\} \to Y$ is the inclusion, and $\mathcal{G}$ is a sheaf on Y, show that $i^{-1}(\mathcal{G})$ is naturally isomorphic to the stalk $\mathcal{G}_y$.

1.E. EXERCISE. Show that f^{-1} is an exact functor from sheaves of abelian groups on Y to sheaves of abelian groups on X. (Hint: exactness can be checked on stalks, and by Exercise 1.B, the stalks are the same.) The identical argument will show that f^{-1} is an exact functor from $\mathcal{O}_Y$-modules (on Y) to $f^{-1}\mathcal{O}_Y$-modules (on X), but don't bother writing that down. (Remark for experts: f^{-1} is a left-adjoint, hence right-exact by abstract nonsense. The left-exactness is true for "less categorical" reasons.)

1.F. IMPORTANT EXERCISE: THE CONSTRUCTION SATISFIES THE UNIVERSAL PROPERTY. If $f : X \to Y$ is a continuous map, and $\mathcal{F}$ is a sheaf on X and $\mathcal{G}$ is a sheaf on Y, describe a bijection

$$\mathrm{Mor}_X(f^{-1}\mathcal{G}, \mathcal{F}) \leftrightarrow \mathrm{Mor}_Y(\mathcal{G}, f_*\mathcal{F}).$$

Observe that your bijection is "natural" in the sense of the definition of adjoints.

1.G. EXERCISE. (a) Suppose $Z \subset Y$ is a closed subset, and $i : Z \hookrightarrow Y$ is the inclusion. If $\mathcal{F}$ is a sheaf on Z, then show that the stalk $(i_*\mathcal{F})_y$ is a one element set if $y \notin Z$, and $\mathcal{F}_y$ if $y \in Z$.
(b) *Important definition:* Define the *support* of a sheaf $\mathcal{F}$ of sets, denoted $\operatorname{Supp}\mathcal{F}$, as the locus where the stalks are not a one-element set:

$$\operatorname{Supp}\mathcal{F} := \{x \in X : |\mathcal{F}_x| \neq 1\}.$$

(More generally, if the sheaf has value in some category, the support consists of points where the stalk is not the final object. For sheaves of abelian groups, the support consists of points with non-zero stalks.) Suppose $\operatorname{Supp}\mathcal{F} \subset Z$ where Z is closed. Show that the natural map $\mathcal{F} \to i_*i^{-1}\mathcal{F}$ is an isomorphism. Thus a sheaf supported on a closed subset can be considered a sheaf on that closed subset.

Sheaves are natural things to want to think about, but hard to get one's hands on. We like the identity and gluability axioms, but they make proving things trickier than for presheaves. We have discussed how we can understand sheaves using stalks. We now introduce a second way of getting a hold of sheaves, by introducing the notion of a *sheaf on a base*.

First, let me define the notion of a *base of a topology*. Suppose we have a topological space X, i.e. we know which subsets of X are open $\{U_i\}$. Then a base of a topology is a subcollection of the open sets $\{B_j\} \subset \{U_i\}$, such that each U_i is a union of the B_j. There is one example that you have seen early in your mathematical life. Suppose $X = \mathbb{R}^n$. Then the way the usual topology is often first defined is by defining *open balls* $B_r(x) = \{y \in \mathbb{R}^n : |y - x| < r\}$, and declaring that any union of open balls is open. So the balls form a base of the usual topology. Equivalently, we often say that they *generate* the usual topology. As an application of how we use them, to check continuity of some map $f : X \to \mathbb{R}^n$, you need only think about the pullback of balls on $\mathbb{R}^n$.

Now suppose we have a sheaf $\mathcal{F}$ on X, and a base $\{B_i\}$ on X. Then consider the information $(\{\mathcal{F}(B_i)\}, \{\operatorname{res}_{B_i,B_j} : \mathcal{F}(B_i) \to \mathcal{F}(B_j)\})$, which is a subset of the information contained in the sheaf — we are only paying attention to the information involving elements of the base, not all open sets.

We can recover the entire sheaf from this information. Reason: we can determine the stalks from this information, and we can determine when germs are compatible.

2.A. EXERCISE. Make this precise.

This suggests a notion, that of a *sheaf on a base*. A sheaf of sets (rings etc.) on a base $\{B_i\}$ is the following. For each B_i in the base, we have a set $\mathcal{F}(B_i)$. If $B_i \subset B_j$, we have maps $\operatorname{res}_{ji} : \mathcal{F}(B_j) \to \mathcal{F}(B_i)$. (Things called B are always assumed to be in the base.) If $B_i \subset B_j \subset B_k$, then $\operatorname{res}_{B_k,B_i} = \operatorname{res}_{B_j,B_i} \circ \operatorname{res}_{B_k,B_j}$. So far we have defined a *presheaf on a base*.

We also require *base identity*: If $B = \cup B_i$, then if $f, g \in \mathcal{F}(B)$ such that $\operatorname{res}_{B,B_i} f = \operatorname{res}_{B,B_i} g$ for all i, then $f = g$.

We require *base gluability* too: If $B = \cup B_i$, and we have $f_i \in \mathcal{F}(B_i)$ such that f_i agrees with f_j on any basic open set in $B_i \cap B_j$ (i.e. $\operatorname{res}_{B_i,B_k} f_i = \operatorname{res}_{B_j,B_k} f_j$ for all $B_k \subset B_i \cap B_j$) then there exist $f \in \mathcal{F}(B)$ such that $\operatorname{res}_{B,B_i} = f_i$ for all i.

2.1. *Theorem. — Suppose* $\{B_i\}$ *is a base on* X*, and* F *is a sheaf of sets on this base. Then there is a unique sheaf* $\mathcal{F}$ *extending* F *(with isomorphisms* $\mathcal{F}(B_i) \cong F(B_i)$ *agreeing with the restriction maps).*

Proof. We will define $\mathcal{F}$ as the sheaf of compatible germs of F.

Define the *stalk* of F at $x \in X$ by

$$F_x = \varinjlim F(B_i)$$

where the colimit is over all B_i (in the base) containing x.

We'll say a family of germs in an open set U is compatible near x if there is a section s of F over some B_i containing x such that the germs over B_i are precisely the germs of s. More formally, define

$$\mathcal{F}(U) := \{(f_x \in F_x)_{x \in U} : \forall x \in U, \exists B \text{ with } x \subset B \subset U, s \in F(B) : s_y = f_y \forall y \in B\}$$

where each B is in our base.

This is a sheaf (for the same reasons as the sheaf of compatible germs was earlier).

I next claim that if U is in our base, the natural map $F(B) \to \mathcal{F}(B)$ is an isomorphism.

2.B. TRICKY EXERCISE. Describe the inverse map $\mathcal{F}(B) \to F(B)$, and verify that it is indeed inverse. □

Thus sheaves on X can be recovered from their "restriction to a base". This is a statement about *objects* in a category, so we should hope for a similar statement about *morphisms*.

2.C. IMPORTANT EXERCISE: MORPHISMS OF SHEAVES CORRESPOND TO MORPHISMS OF SHEAF ON A BASE. Suppose $\{B_i\}$ is a base for the topology of X.
(a) Verify that a morphism of sheaves is determined by the induced morphism of sheaves on the base.
(b) Show that a morphism of sheaves on the base (i.e. such that the diagram

$$\begin{array}{ccc} F(B_i) & \longrightarrow & G(B_i) \\ \downarrow & & \downarrow \\ F(B_j) & \longrightarrow & G(B_j) \end{array}$$

commutes for all $B_j \hookrightarrow B_i$) gives a morphism of the induced sheaves.

2.D. IMPORTANT EXERCISE. Suppose $X = \cup U_i$ is an open cover of X, and we have sheaves $\mathcal{F}_i$ on U_i along with isomorphisms $\phi_{ij} : \mathcal{F}_i|_{U_i \cap U_j} \to \mathcal{F}_j|_{U_i \cap U_j}$ that agree on triple overlaps (i.e. $\phi_{ij} \circ \phi_{jk} = \phi_{ij}$ on $U_i \cap U_j \cap U_k$). Show that these sheaves can be glued together into a unique sheaf $\mathcal{F}$ on X, such that $\mathcal{F}_i = \mathcal{F}|_{U_i}$, and the isomorphisms over $U_i \cap U_j$ are the obvious ones. (Thus we can "glue sheaves together", using limited patching information.) (You can use the ideas of this section to solve this problem, but you don't necessarily need to. Hint: As the base, take those open sets contained in *some* U_i.)

2.2. *Remark for experts.* This almost says that the "set" of sheaves forms a sheaf itself, but not quite. Making this precise leads one to the notion of a *stack*.

We are now ready to consider the notion of a *scheme*, which is the type of geometric space considered by algebraic geometry. We should first think through what we mean by "geometric space". You have likely seen the notion of a manifold, and we wish to abstract this notion so that it can be generalized to other settings, notably so that we can deal with non-smooth and arithmetic objects.

The key insight behind this generalization is the following: we can understand a geometric space (such as a manifold) well by understanding the functions on this space. More precisely, we will understand it through the sheaf of functions on the space. If we are interested in differentiable manifolds, we will consider differentiable functions; if we are interested in smooth manifolds, we will consider smooth functions and so on.

Thus we will define a scheme to be the following data

- *The set:* the points of the scheme
- *The topology:* the open sets of the scheme
- *The structure sheaf:* the sheaf of "algebraic functions" (a sheaf of rings) on the scheme.

Recall that a topological space with a sheaf of rings is called a *ringed space*.

We will try to draw pictures throughout, so our geometric intuition can guide the algebra development (and, eventually, vice versa). Pictures can help develop geometric intuition. Some readers will find the pictures very helpful, while others will find the opposite.

3.1. Example: Differentiable manifolds. As motivation, we return to our example of differentible manifolds, reinterpreting them in this light. We will be quite informal in this section. Suppose X is a manifold. It is a topological space, and has a *sheaf of differentiable functions* $\mathcal{O}_X$ (as described earlier). This gives X the structure of a ringed space. We have observed that evaluation at p gives a surjective map from the stalk to $\mathbb{R}$

$$\mathcal{O}_{X,p} \twoheadrightarrow \mathbb{R},$$

so the kernel, the (germs of) functions vanishing at p, is a maximal ideal $\mathfrak{m}_X$.

We could *define* a differentiable real manifold as a topological space X with a sheaf of rings such that there is a cover of X by open sets such that on each open set the ringed space is isomorphic to a ball around the origin in $\mathbb{R}^n$ with the sheaf of differentiable functions on that ball. With this definition, the ball is the basic patch, and a general manifold is obtained by gluing these patches together. (Admittedly, a great deal of geometry comes from how one chooses to patch the balls together!) In the algebraic setting, the basic patch is the notion of an *affine scheme*, which we will discuss soon.

Functions are determined by their values at points. This is an obvious statement, but won't be true for schemes in general. We will see an example in Exercise 4.A(a).

Morphisms of manifolds. How can we describe differentiable maps of manifolds $X \to Y$? They are certainly continuous maps — but which ones? We can pull back functions along continuous maps. Differentiable functions pull back to differentiable functions. More formally, we have a map $f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$. (The inverse image sheaf f^{-1} was defined in §1) Inverse image is left-adjoint to pushforward, so we get a map $f^\# : \mathcal{O}_Y \to f_*\mathcal{O}_X$.

Certainly given a differentiable map of manifolds, differentiable functions pullback to differentiable functions. It is less obvious that *this is a sufficient condition for a continuous function to be differentiable.*

3.A. IMPORTANT EXERCISE FOR THOSE WITH A LITTLE EXPERIENCE WITH MANIFOLDS. Prove that a continuous function of differentiable manifolds $f : X \to Y$ is differentiable if differentiable functions pull back to differentiable functions, i.e. if pullback by f gives a map $\mathcal{O}_Y \to f_*\mathcal{O}_X$. (Hint: check this on small patches. Once you figure out what you are trying to show, you'll realize that the result is immediate.)

3.B. EXERCISE. Show that a morphism of differentiable manifolds $f : X \to Y$ with $f(p) = q$ induces a morphism of stalks $f^\# : \mathcal{O}_{Y,q} \to \mathcal{O}_{X,p}$. Show that $f^\#(\mathfrak{m}_{Y,q}) \subset \mathfrak{m}_{X,p}$. In other words, if you pull back a function that vanishes at q, you get a function that vanishes at p — not a huge surprise.

Here is a little more for experts: Notice that this induces a map on tangent spaces

$$(\mathfrak{m}_{X,p}/\mathfrak{m}_{X,p}^2)^\vee \to (\mathfrak{m}_{Y,q}/\mathfrak{m}_{Y,q}^2)^\vee.$$

This is the tangent map you would geometrically expect. Again, it is interesting that the cotangent map $\mathfrak{m}_{Y,q}/\mathfrak{m}_{Y,q}^2 \to \mathfrak{m}_{X,p}/\mathfrak{m}_{X,p}^2$ is algebraically more natural than the tangent map.

Experts are now free to try to interpret other differential-geometric information using only the map of topological spaces and map of sheaves. For example: how can one check if f is a submersion? How can one check if f is an immersion? (We will see that the algebro-geometric version of these notions are *smooth morphisms* and *locally closed immersion*.)

3.2. *Side Remark.* Manifolds are covered by disks that are all isomorphic. Schemes (or even complex algebraic varieties) will not have isomorphic open sets. (We'll see an example later.) Informally, this is because in the topology on schemes, all non-empty open sets are "huge" and have more "structure".

4. THE UNDERLYING SET OF AFFINE SCHEMES

For any ring A, we are going to define something called $\operatorname{Spec} A$, the *spectrum of* A. In this section, we will define it as a set, but we will soon endow it with a topology, and later we will define a sheaf of rings on it (the structure sheaf). Such an object is called an *affine scheme*. In the future, $\operatorname{Spec} A$ will denote the set along with the topology. (Indeed, it will often implicitly include the data of the structure sheaf.) But for now, as there is no possibility of confusion, $\operatorname{Spec} A$ will just be the set.

The set $\operatorname{Spec} A$ is the set of prime ideals of A. The point of $\operatorname{Spec} A$ corresponding to the prime ideal $\mathfrak{p}$ will be denoted $[\mathfrak{p}]$.

We now give some examples. Here are some temporary definitions to help us understand these examples. Elements $a \in A$ will be called **functions** on $\operatorname{Spec} A$, and their **value** at the point $[\mathfrak{p}]$ will be $a \pmod{\mathfrak{p}}$. "An element a of the ring lying in a prime ideal $\mathfrak{p}$" translates to "a function a that is 0 at the point $[\mathfrak{p}]$" or "a function a vanishing at the point $[\mathfrak{p}]$", and we will use these phrases interchangeably. Notice that if you add or multiply two functions, you add or multiply their values at all points; this is a translation of the fact that $A \to A/\mathfrak{p}$ is a homomorphism of rings. These translations are important — make sure you are very comfortable with them!

Example 1: $\mathbb{A}^1_{\mathbb{C}} := \operatorname{Spec} \mathbb{C}[x]$. This is known as "the affine line" or "the affine line over $\mathbb{C}$". Let's find the prime ideals. As $\mathbb{C}[x]$ is an integral domain, 0 is prime. Also, $(x-a)$ is prime, where $a \in \mathbb{C}$: it is even a maximal ideal, as the quotient by this ideal is field:

$$0 \longrightarrow (x-a) \longrightarrow \mathbb{C}[x] \xrightarrow{f \mapsto f(a)} \mathbb{C} \longrightarrow 0$$

(This exact sequence should remind you of $0 \to \mathfrak{m}_x \to \mathcal{O}_x \to \mathbb{R} \to 0$ in our motivating example of manifolds.)

We now show that there are no other prime ideals. We use the fact that $\mathbb{C}[x]$ has a division algorithm, and is a unique factorization domain. Suppose $\mathfrak{p}$ is a prime ideal. If $\mathfrak{p} \neq 0$, then suppose $f(x) \in \mathfrak{p}$ is a non-zero element of smallest degree. It is not constant, as prime ideals can't contain 1. If $f(x)$ is not linear, then factor $f(x) = g(x)h(x)$, where $g(x)$ and $h(x)$ have positive degree. Then $g(x) \in \mathfrak{p}$ or $h(x) \in \mathfrak{p}$, contradicting the minimality of the degree of f. Hence there is a linear element $x - a$ of $\mathfrak{p}$. Then I claim that $\mathfrak{p} = (x-a)$. Suppose $f(x) \in \mathfrak{p}$. Then the division algorithm would give $f(x) = g(x)(x-a) + m$ where $m \in \mathbb{C}$. Then $m = f(x) - g(x)(x-a) \in \mathfrak{p}$. If $m \neq 0$, then $1 \in \mathfrak{p}$, giving a contradiction.

Thus we have a picture of $\operatorname{Spec} \mathbb{C}[x]$ (see Figure 1). There is one point for each complex number, plus one extra point. The point $[(x-a)]$ we will reasonably associate to $a \in \mathbb{C}$. Where should we picture the point $[(0)]$? Where is it? The best way of thinking about it is somewhat zen. It is somewhere on the complex line, but nowhere in particular. Because (0) is contained in all of these primes, we will somehow associate it with this line passing through all the other points. $[(0)]$ is called the "generic point" of the line; it is "generically on the line" but you can't pin it down any further than that. We'll place it far to the right for lack of anywhere better to put it. You will notice that we sketch $\mathbb{A}^1_{\mathbb{C}}$ as one-dimensional in the real sense; this is to later remind ourselves that this will be a one-dimensional space, where dimensions are defined in an algebraic (or complex-geometric) sense.

To give you some feeling for this space, let me make some statements that are currently undefined, but suggestive. The functions on $\mathbb{A}^1_{\mathbb{C}}$ are the polynomials. So $f(x) = x^2 - 3x + 1$ is a function. What is its value at $[(x-1)]$, which we think of as the point $1 \in \mathbb{C}$? Answer: $f(1)$! Or equivalently, we can evalute $f(x)$ modulo $x - 1$ — this is the same thing by the division algorithm. (What is its value at (0)? It is $f(x) \pmod{0}$, which is just $f(x)$.)

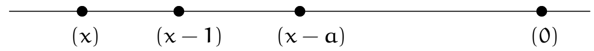

FIGURE 1. A picture of $\mathbb{A}^1_{\mathbb{C}} = \operatorname{Spec} \mathbb{C}[x]$

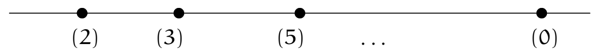

FIGURE 2. A "picture" of $\operatorname{Spec} \mathbb{Z}$, which looks suspiciously like Figure 1

Here is a more complicated example: $g(x) = (x-3)^3/(x-2)$ is a "rational function". It is defined everywhere but $x = 2$. (When we know what the structure sheaf is, we will be able to say that it is an element of the structure sheaf on the open set $\mathbb{A}^1_{\mathbb{C}} - \{2\}$.) $g(x)$ has a triple zero at 3, and a single pole at 2.

Example 2: $\mathbb{A}^1_k := \operatorname{Spec} k[x]$ *where* k *is an algebraically closed field.* This is called the affine line over k. All of our discussion in the previous example carries over without change. We will use the same picture, which is after all intended to just be a metaphor.

Example 3: $\operatorname{Spec} \mathbb{Z}$. One amazing fact is that from our perspective, this will look a lot like the affine line. This is another unique factorization domain, with a division algorithm. The prime ideals are: (0), and (p) where p is prime. Thus everything from Example 1 carries over without change, even the picture. Our picture of $\operatorname{Spec} \mathbb{Z}$ is shown in Figure 2.

Let's blithely carry over our discussion of functions on this space. 100 is a function on $\operatorname{Spec} \mathbb{Z}$. It's value at (3) is "$1 \pmod 3$". It's value at (2) is "$0 \pmod 2$", and in fact it has a double zero. $27/4$ is a rational function on $\operatorname{Spec} \mathbb{Z}$, defined away from (2). It has a double pole at (2), a triple zero at (3). Its value at (5) is

$$27 \times 4^{-1} \equiv 2 \times (-1) \equiv 3 \pmod 5.$$

Example 4: stupid examples. $\operatorname{Spec} k$ where k is any field is boring: only one point. $\operatorname{Spec} 0$, where 0 is the zero-ring, is the empty set, as 0 has no prime ideals.

4.A. A SMALL EXERCISE ABOUT SMALL SCHEMES. (a) Describe the set $\operatorname{Spec} k[\epsilon]/\epsilon^2$. This is called the ring of **dual numbers**, and will turn out to be quite useful. You should think of ϵ as a very small number, so small that its square is 0 (although it itself is not 0).
(b) Describe the set $\operatorname{Spec} k[x]_{(x)}$. (We will see this scheme again later.)

In Example 2, we restricted to the case of algebraically closed fields for a reason: things are more subtle if the field is not algebraically closed.

Example 5: $\mathbb{R}[x]$. Using the fact that $\mathbb{R}[x]$ is a unique factorization domain, we see that the primes are (0), $(x-a)$ where $a \in \mathbb{R}$, and (x^2+ax+b) where x^2+ax+b is an irreducible quadratic. The latter two are maximal ideals, i.e. their quotients are fields. For example: $\mathbb{R}[x]/(x-3) \cong \mathbb{R}$, $\mathbb{R}[x]/(x^2+1) \cong \mathbb{C}$.

4.B. UNIMPORTANT EXERCISE. Show that for the last type of prime, of the form $(x^2 + ax + b)$, the quotient is *always* isomorphic to $\mathbb{C}$.

So we have the points that we would normally expect to see on the real line, corresponding to real numbers; the generic point 0; and new points which we may interpret as *conjugate pairs* of complex numbers (the roots of the quadratic). This last type of point should be seen as more akin to the real numbers than to the generic point. You can picture $\mathbb{A}^1_{\mathbb{R}}$ as the complex plane, folded along the real axis. But the key point is that Galois-conjugate points are considered glued.

Let's explore functions on this space; consider the function $f(x) = x^3 - 1$. Its value at the point $[(x-2)]$ is $f(x) = 7$, or perhaps better, $7 \pmod{x-2}$. How about at (x^2+1)? We get

$$x^3 - 1 \equiv x - 1 \pmod{x^2+1},$$

which may be profitably interpreted as $i - 1$.

One moral of this example is that we can work over a non-algebraically closed field if we wish. It is more complicated, but we can recover much of the information we wanted.

4.C. EXERCISE. Describe the set $\mathbb{A}^1_{\mathbb{Q}}$. (This is harder to picture in a way analogous to $\mathbb{A}^1_{\mathbb{R}}$; but the rough cartoon of points on a line, as in Figure 1, remains a reasonable sketch.)

Example 6: $\mathbb{F}_p[x]$. As in the previous examples, this has a division algorithm, so the prime ideals are of the form (0) or $(f(x))$ where $f(x) \in \mathbb{F}_p[x]$ is an irreducible polynomials, which can be of any degree. Irreducible polynomials correspond to sets of Galois conjugates in $\overline{\mathbb{F}}_p$.

Note that $\operatorname{Spec} \mathbb{F}_p[x]$ has p points corresponding to the elements of $\mathbb{F}_p$, but also (infinitely) many more. This makes this space much richer than simply p points. For example, a polynomial $f(x)$ is not determined by its values at the p elements of $\mathbb{F}_p$, but it *is* determined by its values at the points of $\operatorname{Spec} \mathbb{F}_p$. (As we have mentioned before, this is not true for all schemes.)

You should think about this, even if you are a geometric person — this intuition will later turn up in geometric situations. Even if you think you are interested only in working over an algebraically closed field (such as $\mathbb{C}$), you will have non-algebraically closed fields (such as $\mathbb{C}(x)$) forced upon you.

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FOUNDATIONS OF ALGEBRAIC GEOMETRY CLASS 6

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CONTENTS

Last day: inverse image sheaf; sheaves on a base; toward schemes; the underlying set of an affine scheme.

1. MORE EXAMPLES OF THE UNDERLYING SETS OF AFFINE SCHEMES

We are in the midst of discussing the underlying set of an affine scheme. We are looking at examples and learning how to draw pictures.

Example 7: $\mathbb{A}^2_{\mathbb{C}} = \operatorname{Spec} \mathbb{C}[x, y]$. (As with Examples 1 and 2, discussion will apply with $\mathbb{C}$ replaced by *any* algebraically closed field.) Sadly, $\mathbb{C}[x, y]$ is not a Principal Ideal Domain: (x, y) is not a principal ideal. We can quickly name *some* prime ideals. One is (0), which has the same flavor as the (0) ideals in the previous examples. $(x - 2, y - 3)$ is prime, and indeed maximal, because $\mathbb{C}[x, y]/(x - 2, y - 3) \cong \mathbb{C}$, where this isomorphism is via $f(x, y) \mapsto f(2, 3)$. More generally, $(x - a, y - b)$ is prime for any $(a, b) \in \mathbb{C}^2$. Also, if $f(x, y)$ is an irreducible polynomial (e.g. $y - x^2$ or $y^2 - x^3$) then $(f(x, y))$ is prime.

1.A. EXERCISE. (Feel free to skip this exercise, as we will see a different proof of this later.) Show that we have identified all the prime ideals of $\mathbb{C}[x, y]$.

We can now attempt to draw a picture of this space. The maximal primes correspond to the old-fashioned points in $\mathbb{C}^2$: $[(x - a, y - b)]$ corresponds to $(a, b) \in \mathbb{C}^2$. We now have to visualize the "bonus points". $[(0)]$ somehow lives behind all of the old-fashioned points; it is somewhere on the plane, but nowhere in particular. So for example, it does not lie on the parabola $y = x^2$. The point $[(y - x^2)]$ lies on the parabola $y = x^2$, but nowhere in particular on it. You can see from this picture that we already want to think about "dimension". The primes $(x - a, y - b)$ are somehow of dimension 0, the primes $(f(x, y))$ are of dimension 1, and (0) is somehow of dimension 2. (All of our dimensions here are *complex* or *algebraic* dimensions. The complex plane $\mathbb{C}^2$ has real dimension 4, but

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complex dimension 2. Complex dimensions are in general half of real dimensions.) We won't define dimension precisely until later, but you should feel free to keep it in mind before then.

Note too that maximal ideals correspond to "smallest" points. Smaller ideals correspond to "bigger" points. "One prime ideal contains another" means that the points "have the opposite containment." All of this will be made precise once we have a topology. This order-reversal is a little confusing, and will remain so even once we have made the notions precise.

We now come to the obvious generalization of Example 7, affine n-space.

Example 8: $\mathbb{A}^n_{\mathbb{C}} := \operatorname{Spec} \mathbb{C}[x_1, \dots, x_n]$. (More generally, $\mathbb{A}^n_A$ is defined to be $\operatorname{Spec} A[x_1, \dots, x_n]$, where A is an arbitrary ring.)

For concreteness, let's consider $n = 3$. We now have an interesting question in algebra: What are the prime ideals of $\mathbb{C}[x, y, z]$? Analogously to before, $(x-a, y-b, z-c)$ is a prime ideal. This is a maximal ideal, with residue field $\mathbb{C}$; we think of these as "0-dimensional points". We will often write (a, b, c) for $[(x-a, y-b, z-c)]$ because of our geometric interpretation of these ideals.

There are no more maximal ideals, by Hilbert's Nullstellensatz. (This is sometimes called the "weak version" of the Nullstellensatz.) You may have already seen this result. We will prove it later (in a slightly stronger form), so we will content ourselves by stating it here.

1.1. *Hilbert's Nullstellensatz. — Suppose* $A = k[x_1, \dots, x_n]$, *where* k *is an algebraically closed field. Then the maximal ideals are precisely those of the form* $(x_1 - a_1, \dots, x_n - a_n)$, *where* $a_i \in k$.

There are other prime ideals too. We have (0), which is corresponds to a "3-dimensional point". We have $(f(x, y, z))$, where f is irreducible. To this we associate the hypersurface $f = 0$, so this is "2-dimensional" in nature. But we have not found them all! One clue: we have prime ideals of "dimension" 0, 2, and 3 — we are missing "dimension 1". Here is one such prime ideal: (x, y). We picture this as the locus where $x = y = 0$, which is the z-axis. This is a prime ideal, as the corresponding quotient $\mathbb{C}[x, y, z]/(x, y) \cong \mathbb{C}[z]$ is an integral domain (and should be interpreted as the functions on the z-axis). There are lots of one-dimensional primes, and it is not possible to classify them in a reasonable way. It will turn out that they correspond to things that we think of as irreducible curves: the natural answer to this algebraic question is geometric.

1.2. Important fact: Maps of rings induce maps of spectra (as sets). We now make an observation that will later grow up to be morphisms of schemes. If $\phi : B \to A$ is a map of rings, and $\mathfrak{p}$ is a prime ideal of A, then $\phi^{-1}(\mathfrak{p})$ is a prime ideal of B (check this!). Hence a map of rings $\phi : B \to A$ induces a map of sets $\operatorname{Spec} A \to \operatorname{Spec} B$ "in the opposite direction". This gives a contravariant functor from the category of rings to the category of sets: the composition of two maps of rings induces the composition of the corresponding maps of spectra.

We now describe two important cases of this: maps of rings inducing *inclusions* of sets. There are two particularly useful ways of producing new rings from a ring A. One is by taking the quotient by an ideal I. The other is by localizing at a multiplicative set. We'll see how Spec behaves with respect to these operations. In both cases, the new ring has a Spec that is a subset of Spec of the old ring.

First important example (quotients): $\operatorname{Spec} B/I$ **in terms of** $\operatorname{Spec} B$**.** As a motivating example, consider $\operatorname{Spec} B/I$ where $B = \mathbb{C}[x, y]$, $I = (xy)$. We have a picture of $\operatorname{Spec} B$, which is the complex plane, with some mysterious extra "higher-dimensional points". It is an important fact that the primes of B/I are in bijection with the primes of B containing I. (If you do not know why this is true, you should prove it yourself.) Thus we can picture $\operatorname{Spec} B/I$ as a subset of $\operatorname{Spec} B$. We have the "0-dimensional points" $(a, 0)$ and $(0, b)$. We also have two "1-dimensional points" (x) and (y).

We get a bit more: the inclusion structure on the primes of B/I corresponds to the inclusion structure on the primes containing I. More precisely, if $J_1 \subset J_2$ in B/I, and K_i is the ideal of B corresopnding to J_i, then $K_1 \subset K_2$. (Again, prove this yourself if you have not seen it before.)

So the minimal primes of $\mathbb{C}[x, y]/(xy)$ are the "biggest" points we see, and there are two of them: (x) and (y). Thus we have the intuition that will later be made precise: the minimal primes of A correspond to the "components" of $\operatorname{Spec} A$.

As an important motivational special case, you now have a picture of **"complex affine varieties"**. Suppose A is a finitely generated $\mathbb{C}$-algebra, generated by x_1, ..., x_n, with relations $f_1(x_1, \dots, x_n) = \cdots = f_r(x_1, \dots, x_n) = 0$. Then this description in terms of generators and relations naturally gives us an interpretation of $\operatorname{Spec} A$ as a subset of $\mathbb{A}^n_{\mathbb{C}}$, which we think of as "old-fashioned points" (n-tuples of complex numbers) along with some "bonus" points. To see which subsets of the old-fashioned points are in $\operatorname{Spec} A$, we simply solve the equations $f_1 = \cdots = f_r = 0$. For example, $\operatorname{Spec} \mathbb{C}[x, y, z]/(x^2 + y^2 - z^2)$ may be pictured as shown in Figure 1. (Admittedly this is just a "sketch of the $\mathbb{R}$-points", but we will still find it helpful later.) This entire picture carries over (along with the Nullstellensatz) with $\mathbb{C}$ replaced by any algebraically closed field. Indeed, the picture of Figure 1 can be said to represent $k[x, y, z]/(x^2+y^2-z^2)$ for most algebraically closed fields k (although it is misleading in characteristic 2, because of the coincidence $x^2 + y^2 - z^2 = (x + y + z)^2$).

1.B. EXERCISE. Ring elements that have a power that is 0 are called *nilpotents*. If I is an ideal of nilpotents, show that $\operatorname{Spec} B/I \to \operatorname{Spec} B$ is a bijection. Thus nilpotents don't affect the underlying set. (We will soon see in Exercise 2.H that they won't affect the topology either — the difference will be in the structure sheaf.)

Second important example (localization): $\operatorname{Spec} S^{-1}B$ **in terms of** $\operatorname{Spec} B$, where S is a multiplicative subset of B. There are two particularly important flavors of multiplicative subsets. The first is $B \setminus \mathfrak{p}$, where $\mathfrak{p}$ is a prime ideal. This localization $S^{-1}B$ is denoted $B_{\mathfrak{p}}$.

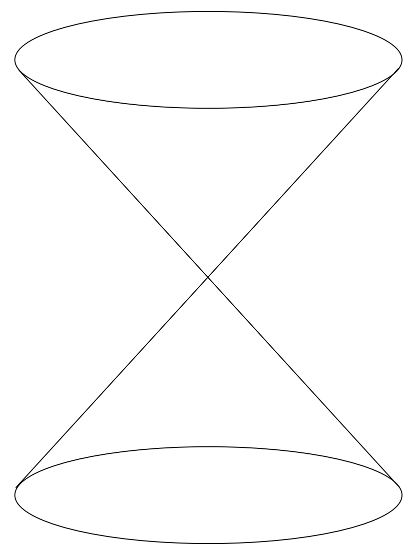

FIGURE 1. A "picture" of $\operatorname{Spec}\mathbb{C}[x,y,z]/(x^2+y^2-z^2)$

A motivating example is $B = \mathbb{C}[x,y]$, $S = B - (x,y)$. The second is $\{1, f, f^2, \dots\}$, where $f \in B$. This localization is denoted B_f. (Notational warning: If $\mathfrak{p}$ is a prime ideal, then $B_\mathfrak{p}$ means you're allowed to divide by elements not in $\mathfrak{p}$. However, if $f \in B$, B_f means you're allowed to divide by f. This can be confusing. For example, if (f) is a prime ideal, then $B_f \neq B_{(f)}$.) A motivating example is $B = \mathbb{C}[x,y]$, $f = x$.

1.3. *Essential algebra fact (to review and know).* The map $\operatorname{Spec} S^{-1}B \to \operatorname{Spec} B$ gives an order-preserving bijection of the primes of $S^{-1}B$ with the primes of B that *don't meet* the multiplicative set S.

So if $S = B - \mathfrak{p}$ where $\mathfrak{p}$ is a prime ideal, the primes of $S^{-1}B$ are just the primes of B *contained in* $\mathfrak{p}$. If $S = \{1, f, f^2, \dots\}$, the primes of $S^{-1}B$ are just those primes not containing f (the points where "f doesn't vanish" — draw a picture of $\operatorname{Spec}\mathbb{C}[x]_{x^2-x}$ to see how this works).

1.4. *Warning.* sometimes localization is first introduced in the special case where B is an integral domain. In this example, $B \hookrightarrow B_f$, but this isn't true when one inverts zero-divisors. (A **zero-divisor** of a ring B is an element a such that there is a non-zero element b with $ab = 0$. The other elements of B are called **non-zero-divisors**.) One definition of localization is as follows. The elements of $S^{-1}B$ are of the form a/s where $r \in B$ and $s \in S$, and $(a_1/s_1) \times (a_2/s_2) = (a_1 a_2 / s_1 s_2)$, and $(a_1/s_1) + (a_2/s_2) = (a_1 s_2 + s_1 a_2)/(s_1 s_2)$. We say that $a_1/s_1 = a_2/s_2$ if *for some* $s \in S$ $s(a_1 s_2 - a_2 s_1) = 0$. So for example, $B[1/0] \cong 0$.

1.5. Important comment: functions are not determined by their values at points. We are developing machinery that will let us bring our geometric intuition to algebra. There is one point where your intuition will be false, so you should know now, and adjust your intuition appropriately. Suppose we have a function (ring element) vanishing at all points. Then it is not necessarily the zero function! The translation of this question is: is the intersection of all prime ideals necessarily just 0? The answer is no, as is shown by the example of the ring of dual numbers $k[\epsilon]/\epsilon^2$: $\epsilon \neq 0$, but $\epsilon^2 = 0$. (We saw this scheme in an exercise in class 5.) Any function whose power is zero certainly lies in the intersection of all prime ideals. The converse is also true: the intersection of all the prime ideals consists of functions for which some power is zero, otherwise known as the nilradical $\mathfrak{N}$. (You should check that the nilpotents indeed form an ideal. For example, the sum of two nilpotents is always nilpotent.)

1.6. Theorem. *The nilradical $\mathfrak{N}(A)$ is the intersection of all the primes of A.*

1.C. EXERCISE. If you don't know this theorem, then look it up, or even better, prove it yourself. (Hint: one direction is easy. The other will require knowing that any proper ideal of A is contained in a maximal ideal, which requires the axiom of choice.)

In particular, although it is upsetting that functions are not determined by their values at points, we have precisely specified what the failure of this intuition is: two functions have the same values at points if and only if they differ by a nilpotent. And if there are no non-zero nilpotents — if $\mathfrak{N} = 0$ — then functions *are* determined by their values at points.

2. THE ZARISKI TOPOLOGY: THE UNDERLYING TOPOLOGICAL SPACE OF AN AFFINE SCHEME

We next introduce the *Zariski topology* on the spectrum of a ring. At first it seems like an odd definition, but in retrospect it is reasonable. For example, consider $\mathbb{A}^2_{\mathbb{C}} = \operatorname{Spec} \mathbb{C}[x, y]$, the complex plane (with a few extra points). In algebraic geometry, we will only be allowed to consider algebraic functions, i.e. polynomials in x and y. The locus where a polynomial vanishes should reasonably be a closed set, and the Zariski topology is defined by saying that the only sets we should consider closed should be these sets, and other sets forced to be closed by these. In other words, it is the coarsest topology where these sets are closed.

In particular, although topologies are often described using open subsets, it will more convenient for us to define this topology in terms of closed subsets. If S is a subset of a ring A, define the V**anishing set** of S by

$$V(S) := \{[\mathfrak{p}] \in \operatorname{Spec} A : S \subset \mathfrak{p}\}.$$

It is the set of points on which all elements of S are zero. (It should now be second nature to equate "vanishing at a point" with "contained in a prime".) We declare that these (and no other) are the closed subsets.

For example, consider $V(xy, xz) \subset \mathbb{A}^3 = \operatorname{Spec} \mathbb{C}[x, y, z]$. Which points are contained in this locus? We think of this as solving $xy = yz = 0$. Of the "old-fashioned" points (interpreted as ordered triples of complex numbers, thanks to the Hilbert's Nullstellensatz 1.1), we have the points where $y = 0$ or $x = z = 0$: the xz-plane and the y-axis respectively. Of the "new" points, we have the generic point of the xz-plane (also known as the point $[(y)]$), and the generic point of the y-axis (also known as the point $[(x,z)]$). You might imagine that we also have a number of "one-dimensional" points contained in the xz-plane.

2.A. EASIER EXERCISE. Check that the x-axis is contained in this set.

Let's return to the general situation. The following exercise lets us restrict attention to vanishing sets of *ideals*.

2.B. EASIER EXERCISE. Show that if (S) is the ideal generated by S, then $V(S) = V((S))$.

We define the Zariski topology by declaring that $V(S)$ is closed for all S. Let's check that this is a topology. We have to check that the empty set and the total space are open; the union of an arbitrary collection of open sets are open; and the intersection of two open sets are open.

2.C. EXERCISE. (a) Show that $\varnothing$ and $\operatorname{Spec} A$ are both open.
(b) Show that $V(I_1) \cup V(I_2) = V(I_1 I_2)$. Hence show that the intersection of any finite number of open sets is open.
(c) *(The union of any collection of open sets is open.)* If I_i is a collection of ideals (as i runs over some index set), check that $\cap_i V(I_i) = V(\sum_i I_i)$.

2.1. Properties of "vanishing set" function $V(\cdot)$. The function $V(\cdot)$ is obviously inclusion-reversing: If $S_1 \subset S_2$, then $V(S_2) \subset V(S_1)$. Warning: We could have equality in the second inclusion without equality in the first, as the next exercise shows.

2.D. EXERCISE/DEFINITION. If $I \subset R$ is an ideal, then define its **radical** by

$$\sqrt{I} := \{r \in R : r^n \in I \text{ for some } n \in \mathbb{Z}^{\geq 0}\}.$$

For example, the nilradical $\mathfrak{N}$ (§1.5) is $\sqrt{(0)}$. Show that $V(\sqrt{I}) = V(I)$. We say an **ideal is radical** if it equals its own radical.

Here are two useful consequences. As $(I \cap J)^2 \subset IJ \subset I \cap J$, we have that $V(IJ) = V(I \cap J)$ $(= V(I) \cup V(J)$ by Exercise 2.C(b)). Also, combining this with Exercise 2.B, we see $V(S) = V((S)) = V(\sqrt{(S)})$.

2.E. EXERCISE (PRACTICE WITH THE CONCEPT). If $I_1, \dots, I_n$ are ideals of a ring A, show that $\sqrt{\cap_{i=1}^n I_i} = \cap_{i=1}^n \sqrt{I_i}$. (We will use this property without referring back to this exercise.)

2.F. EXERCISE FOR FUTURE USE. Show that $\sqrt{I}$ is the intersection of all the prime ideals containing I. (Hint: Use Theorem 1.6 on an appropriate ring.)

2.2. Examples. Let's see how this meshes with our examples from the previous section.

Recall that $\mathbb{A}^1_{\mathbb{C}}$, as a set, was just the "old-fashioned" points (corresponding to maximal ideals, in bijection with $a \in \mathbb{C}$), and one "new" point (0). The Zariski topology on $\mathbb{A}^1_{\mathbb{C}}$ is not that exciting: the open sets are the empty set, and $\mathbb{A}^1_{\mathbb{C}}$ minus a finite number of maximal ideals. (It "almost" has the cofinite topology. Notice that the open sets are determined by their intersections with the "old-fashioned points". The "new" point (0) comes along for the ride, which is a good sign that it is harmless. Ignoring the "new" point, observe that the topology on $\mathbb{A}^1_{\mathbb{C}}$ is a coarser topology than the analytic topology.)

The case $\operatorname{Spec} \mathbb{Z}$ is similar. The topology is "almost" the cofinite topology in the same way. The open sets are the empty set, and $\operatorname{Spec} \mathbb{Z}$ minus a finite number of "ordinary" ((p) where p is prime) primes.

2.3. *Closed subsets of* $\mathbb{A}^2_{\mathbb{C}}$. The case $\mathbb{A}^2_{\mathbb{C}}$ is more interesting. You should think through where the "one-dimensional primes" fit into the picture. In Exercise 1.A, we identified all the primes of $\mathbb{C}[x, y]$ (i.e. the points of $\mathbb{A}^2_{\mathbb{C}}$) as the maximal ideals $(x-a, y-b)$ $(a, b \in \mathbb{C})$, the "one-dimensional points" $(f(x, y))$ $(f(x, y)$ irreducible), and the "two-dimensional point" (0).

Then the closed subsets are of the following form:

(a) the entire space, and
(b) a finite number (possibly zero) of "curves" (each of which is the closure of a "one-dimensional point") and a finite number (possibly zero) of closed points.

2.4. Important fact: Maps of rings induce continuous maps of topological spaces. We saw in §1.2 that a map of rings $\phi : B \to A$ induces a map of sets $\pi : \operatorname{Spec} A \to \operatorname{Spec} B$.

2.G. IMPORTANT EXERCISE. By showing that closed sets pull back to closed sets, show that π is a *continuous map*.

Not all continuous maps arise in this way. Consider for example the continuous map on $\mathbb{A}^1_{\mathbb{C}}$ that is the identity except 0 and 1 (i.e. $[(x)]$ and $[(x-1)]$ are swapped); there is no polynomial that can manage this.

In §1.2, we saw that $\operatorname{Spec} B/I$ and $\operatorname{Spec} S^{-1}B$ are naturally *subsets* of $\operatorname{Spec} B$. It is natural to ask if the Zariski topology behaves well with respect to these inclusions, and indeed it does.

2.H. IMPORTANT EXERCISE. Suppose that $I, S \subset B$ are an ideal and multiplicative subset respectively. Show that $\operatorname{Spec} B/I$ is naturally a *closed* subset of $\operatorname{Spec} B$. Show that the Zariski topology on $\operatorname{Spec} B/I$ (resp. $\operatorname{Spec} S^{-1}B$) is the subspace topology induced by inclusion in $\operatorname{Spec} B$. (Hint: compare closed subsets.)

In particular, if $I \subset \mathfrak{N}$ is an ideal of nilpotents, the bijection $\operatorname{Spec} B/I \to \operatorname{Spec} B$ (Exercise 1.B) is a homeomorphism. Thus nilpotents don't affect the topological space. (The difference will be in the structure sheaf.)

2.I. USEFUL EXERCISE FOR LATER. Suppose $I \subset B$ is an ideal. Show that f vanishes on $V(I)$ if and only if $f^n \in I$ for some n.

2.J. EXERCISE. Describe the topological space $\operatorname{Spec} k[x]_{(x)}$.

3. TOPOLOGICAL DEFINITIONS

We now describe various properties that it will be useful to have names for.

A topological space is said to be **irreducible** if it is not the union of two proper closed subsets. In other words, X is irreducible if whenever $X = Y \cup Z$ with Y and Z closed, we have $Y = X$ or $Z = X$.

3.A. EASY EXERCISE. Show that on an irreducible topological space, any nonempty open set is dense. (The moral of this is: unlike in the classical topology, in the Zariski topology, non-empty open sets are all "very big".)

3.B. EXERCISE. Show that $\operatorname{Spec} A$ is irreducible if and only if A has only one minimal prime. (Minimality is with respect to inclusion.) In particular, if A is an integral domain, then $\operatorname{Spec} A$ is irreducible.

A point of a topological space $x \in X$ is said to be **closed** if $\{x\}$ is a closed subset. In the old-fashioned topology on $\mathbb{C}^n$, all points are closed.

3.C. EXERCISE. Show that the closed points of $\operatorname{Spec} A$ correspond to the maximal ideals.

Thus Hilbert's Nullstellensatz lets us associate the closed points of $\mathbb{A}^n_{\mathbb{C}}$ with n-tuples of complex numbers. Hence from now on we will say "closed point" instead of "old-fashioned point" and "non-closed point" instead of "bonus" or "new-fangled" point when discussing subsets of $\mathbb{A}^n_{\mathbb{C}}$.

Given two points x, y of a topological space X, we say that x is a **specialization** of y, and y is a **generization** of x, if $x \in \overline{\{y\}}$. This now makes precise our hand-waving about "one point contained another". It is of course nonsense for a point to contain another. But it is not nonsense to say that the closure of a point contains another. For example, in $\mathbb{A}^2_{\mathbb{C}} = \operatorname{Spec} \mathbb{C}[x, y]$, $[(y - x^2)]$ is a generization of $(2, 4) = [(x - 2, y - 4)]$, and $(2, 4)$ is a specialization of $[(y - x^2)]$.

3.D. EXERCISE. If $X = \operatorname{Spec} A$, show that $[\mathfrak{p}]$ is a specialization of $[\mathfrak{q}]$ if and only if $\mathfrak{q} \subset \mathfrak{p}$. Verify to your satisfaction that we have made our intuition of "containment of points" precise: it means that the one point is contained in the *closure* of another.

We say that a point $x \in X$ is a **generic point** for a closed subset K if $\overline{\{x\}} = K$.

3.E. EXERCISE. Verify that $[(y - x^2)] \in \mathbb{A}^2$ is a generic point for $V(y - x^2)$.

We will soon see that there is a natural bijection between points of $\operatorname{Spec} A$ and irreducible closed subsets of $\operatorname{Spec} A$. You know enough to show this now, although we'll wait until we have developed some convenient terminology.

3.F. LESS IMPORTANT EXERCISE. (a) Suppose $I = (wz - xy, wy - x^2, xz - y^2) \subset k[w, x, y, z]$. Show that $\operatorname{Spec} k[w, x, y, z]/I$ is irreducible, by showing that I is prime. (One possible approach: Show that the quotient ring is a domain, by showing that it is isomorphic to the subring of $k[a, b]$ including only monomials of degree divisible by 3. There are other approaches as well, some of which we will see later. This is an example of a hard question: how do you tell if an ideal is prime?) We will later see this as the cone over the *twisted cubic curve*.
(b) Note that the ideal of part (a) may be rewritten as

$$\operatorname{rank} \begin{pmatrix} w & x & y \\ x & y & z \end{pmatrix} = 1,$$

i.e. that all determinants of 2×2 submatrices vanish. Generalize this to the ideal of rank 1 $2 \times n$ matrices. This notion will correspond to the cone over the *degree* n *rational normal curve*.

3.1. Noetherian conditions.

In the examples we have considered, the spaces have naturally broken up into some obvious pieces. Let's make that a bit more precise.

A topological space X is called **Noetherian** if it satisfies the **descending chain condition** for closed subsets: any sequence $Z_1 \supseteq Z_2 \supseteq \cdots \supseteq Z_n \supseteq \cdots$ of closed subsets eventually stabilizes: there is an r such that $Z_r = Z_{r+1} = \cdots$.

The following exercise may be enlightening.

3.G. EXERCISE. Show that any decreasing sequence of closed subsets of $\mathbb{A}^2_{\mathbb{C}} = \operatorname{Spec} \mathbb{C}[x, y]$ must eventually stabilize. Note that it can take arbitrarily long to stabilize. (The closed subsets of $\mathbb{A}^2_{\mathbb{C}}$ were described in §2.3.)

3.2. It turns out that all of the spectra we have considered have this property, but that isn't true of the spectra of all rings. The key characteristic all of our examples have had in common is that the rings were *Noetherian*. Recall that a ring is **Noetherian** if every *ascending* sequence $I_1 \subset I_2 \subset \cdots$ of ideals eventually stabilizes: there is an r such that $I_r = I_{r+1} = \cdots$. (This is called the **ascending chain condition** on ideals.)

Here are some quick facts about Noetherian rings. You should be able to prove them all.

- Fields are Noetherian. $\mathbb{Z}$ is Noetherian.
- If A is Noetherian, and I is any ideal, then A/I is Noetherian.
- If A is Noetherian, and S is any multiplicative set, then $S^{-1}A$ is Noetherian.
- In a Noetherian ring, any ideal is finitely generated.
- Any submodule of a finitely generated module over a Noetherian ring is finitely generated. (Hint: prove it for A^n, and use the next exercise.)

3.H. EXERCISE. Suppose $0 \to M' \to M \to M'' \to 0$, and M' and M'' satisfy the ascending chain condition for modules. Show that M does too. (The converse also holds; we won't use this, but you can show it if you wish.)

The next fact is non-trivial.

3.3. *The Hilbert basis theorem. — If A is Noetherian, then so is $A[x]$.*

Using these results, then any polynomial ring over any field, or over the integers, is Noetherian — and also any quotient or localization thereof. Hence for example any finitely-generated algebra over k or $\mathbb{Z}$, or any localization thereof is Noetherian. Most "nice" rings are Noetherian, but not all rings are Noetherian, e.g. $k[x_1, x_2, \ldots]$ because $\mathfrak{m} = (x_1, x_2, \ldots,)$ is not finitely generated.

3.I. EXERCISE. If A is Noetherian, show that $\operatorname{Spec} A$ is a Noetherian topological space.

3.J. LESS IMPORTANT EXERCISE. Show that the converse is not true: if $\operatorname{Spec} A$ is a Noetherian topological space, A need not be Noetherian. Describe a ring A such that $\operatorname{Spec} A$ is not a Noetherian topological space.

I discussed how the finiteness of the game of Chomp is a consequence of the Hilbert basis theorem.

If X is a topological space, and Z is an irreducible closed subset not contained in any larger irreducible closed subset, Z is said to be an *irreducible component* of X. (I drew a picture.)

3.K. EXERCISE. If A is any ring, show that the irreducible components of $\operatorname{Spec} A$ are in bijection with the minimal primes of A.

For example, the only minimal prime of $k[x, y]$ is (0). What are the minimal primes of $k[x, y]/(xy)$?

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FOUNDATIONS OF ALGEBRAIC GEOMETRY CLASSES 7 AND 8

RAVI VAKIL

CONTENTS

1. DECOMPOSITION INTO IRREDUCIBLE COMPONENTS, AND NOETHERIAN INDUCTION

At the end of last day, we defined *irreducible component*: If X is a topological space, and Z is an irreducible closed subset not contained in any larger irreducible closed subset, Z is said to be an **irreducible component** of X. We think of these as the "pieces of X" (see Figure 1).

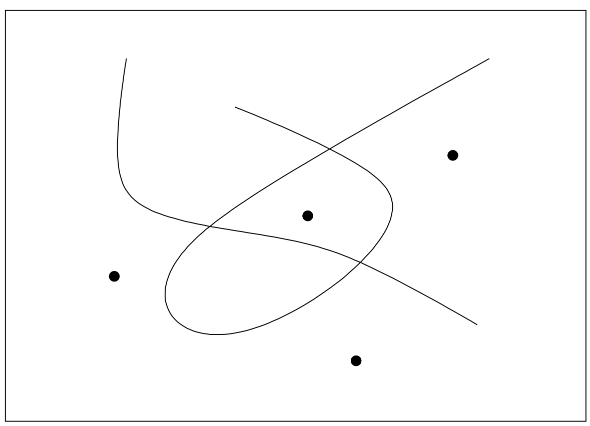


FIGURE 1. This closed subset of $\mathbb{A}^2$ has six irreducible components

We saw the exercise: If A is any ring, show that the irreducible components of $\operatorname{Spec} A$ are in bijection with the minimal primes of A.

Date: Monday, October 15, 2007 and Wednesday, October 17, 2007. New exercise added November 15, 2007.

For example, the only minimal prime of $k[x, y]$ is (0). What are the minimal primes of $k[x, y]/(xy)$?

1.1. *Proposition. — Suppose X is a Noetherian topological space. Then every non-empty closed subset Z can be expressed uniquely as a finite union $Z = Z_1 \cup \cdots \cup Z_n$ of irreducible closed subsets, none contained in any other.*

Translation: any non-empty closed subset Z has of a finite number of pieces.

As a corollary, this implies that a Noetherian ring A has only finitely many minimal primes.

Proof. The following technique is often called *Noetherian induction*, for reasons that will become clear. Justin prefers the phrase "Noetherian descent".

Consider the collection of closed subsets of X that *cannot* be expressed as a finite union of irreducible closed subsets. We will show that it is empty. Otherwise, let Y_1 be one such. If it properly contains another such, then choose one, and call it Y_2. If this one contains another such, then choose one, and call it Y_3, and so on. By the descending chain condition, this must eventually stop, and we must have some Y_r that cannot be written as a finite union of irreducible closed subsets, but every closed subset contained in it can be so written. But then Y_r is not itself irreducible, so we can write $Y_r = Y' \cup Y''$ where Y' and Y'' are both proper closed subsets. Both of these by hypothesis can be written as the union of a finite number of irreducible subsets, and hence so can Y_r, yielding a contradiction. Thus each closed subset can be written as a finite union of irreducible closed subsets. We can assume that none of these irreducible closed subsets contain any others, by discarding some of them.

We now show uniqueness. Suppose

$$Z = Z_1 \cup Z_2 \cup \cdots \cup Z_r = Z'_1 \cup Z'_2 \cup \cdots \cup Z'_s$$

are two such representations. Then $Z'_1 \subset Z_1 \cup Z_2 \cup \cdots \cup Z_r$, so $Z'_1 = (Z_1 \cap Z'_1) \cup \cdots \cup (Z_r \cap Z'_1)$. Now Z'_1 is irreducible, so one of these is Z'_1 itself, say (without loss of generality) $Z_1 \cap Z'_1$. Thus $Z'_1 \subset Z_1$. Similarly, $Z_1 \subset Z'_a$ for some a; but because $Z'_1 \subset Z_1 \subset Z'_a$, and Z'_1 is contained in no other Z'_i, we must have $a = 1$, and $Z'_1 = Z_1$. Thus each element of the list of Z's is in the list of Z''s, and vice versa, so they must be the same list. □

2. The function $I(\cdot)$, taking subsets of $\operatorname{Spec} A$ to ideals of A

We now introduce a notion that is in some sense "inverse" to the vanishing set function $V(\cdot)$. Given a subset $S \subset \operatorname{Spec} A$, $I(S)$ is the set of functions vanishing on S.

We make three quick observations:

- $I(S)$ is clearly an ideal.
- $I(\overline{S}) = I(S)$.

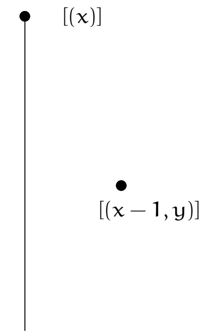


FIGURE 2. The set S of Exercise/example 2.A, pictured as a subset of $\mathbb{A}^2$

- $I(\cdot)$ is inclusion-reversing: if $S_1 \subset S_2$, then $I(S_2) \subset I(S_1)$.

2.A. EXERCISE/EXAMPLE. Let $A = k[x, y]$. If $S = \{[(x)], [(x-1, y)]\}$ (see Figure 2), then $I(S)$ consists of those polynomials vanishing on the y axis, and at the point $(1, 0)$. Give generators for this ideal.

2.B. TRICKY EXERCISE. Suppose $X \subset \mathbb{A}^3$ is the union of the three axes. (The x-axis is defined by $y = z = 0$, and the y-axis and z-axis are deined analogously.) Give generators for the ideal $I(X)$. Be sure to prove it! Hint: We will see later that this ideal is not generated by less than three elements.

2.C. EXERCISE. Show that $V(I(S)) = \overline{S}$. Hence $V(I(S)) = S$ for a closed set S. (Compare this to Exercise 2.D below.)

Note that $I(S)$ is always a radical ideal — if $f \in \sqrt{I(S)}$, then f^n vanishes on S for some $n > 0$, so then f vanishes on S, so $f \in I(S)$.

2.D. EXERCISE. Prove that if $I \subset A$ is an ideal, then $I(V(I)) = \sqrt{I}$.

This exercise and Exercise 2.C suggest that V and I are "almost" inverse. More precisely:

2.1. *Theorem.* — $V(\cdot)$ *and* $I(\cdot)$ *give a bijection between closed subsets of* $\operatorname{Spec} A$ *and radical ideals of* A *(where a closed subset gives a radical ideal by* $I(\cdot)$*, and a radical ideal gives a closed subset by* $V(\cdot)$*).*

2.E. IMPORTANT EXERCISE. Show that $V(\cdot)$ and $I(\cdot)$ give a bijection between *irreducible closed subsets* of $\operatorname{Spec} A$ and *prime* ideals of A. From this conclude that in $\operatorname{Spec} A$ there is a bijection between points of $\operatorname{Spec} A$ and irreducible closed subsets of $\operatorname{Spec} A$ (where a point determines an irreducible closed subset by taking the closure). Hence *each irreducible closed subset of* $\operatorname{Spec} A$ *has precisely one generic point* — any irreducible closed subset Z can be written uniquely as $\overline{\{z\}}$.

3. Distinguished open sets

If $f \in A$, define the **distinguished open set** $D(f) = \{[\mathfrak{p}] \in \operatorname{Spec} A : f \notin \mathfrak{p}\}$. It is the locus where f doesn't vanish. (I often privately write this as $D(f \neq 0)$ to remind myself of this. I also privately call this a "**D**oesn't-vanish set" in analogy with $V(f)$ being the **V**anishing set.) We have already seen this set when discussing $\operatorname{Spec} A_f$ as a subset of $\operatorname{Spec} A$. For example, we have observed that the Zariski-topology on the distinguished open set $D(f) \subset \operatorname{Spec} A$ coincides with the Zariski topology on $\operatorname{Spec} A_f$.

Here are some important but not difficult exercises to give you a feel for these important open sets.

3.A. Exercise. Show that the distinguished open sets form a base for the Zariski topology. (Hint: Given an ideal I, show that the complement of $V(I)$ is $\cup_{f \in I} D(f)$.)

3.B. Exercise. Suppose $f_i \in A$ as i runs over some index set J. Show that $\cup_{i \in J} D(f_i) = \operatorname{Spec} A$ if and only if $(f_i) = A$. (One of the directions will use the fact that any proper ideal of A is contained in some maximal ideal.)

3.C. Exercise. Show that if $\operatorname{Spec} A$ is an infinite union $\cup_{i \in J} D(f_i)$, then in fact it is a union of a finite number of these. (Hint: use the previous exercise 3.B.) Show that $\operatorname{Spec} A$ is quasicompact.

3.D. Exercise. Show that $D(f) \cap D(g) = D(fg)$.

3.E. Exercise. Show that if $D(f) \subset D(g)$, if and only if $f^n \in (g)$ for some n if and only if g is a unit in A_f. (Hint for the first equivalence: $f \in I(V((g)))$. We will use this shortly.

3.F. Exercise. Show that $D(f) = \varnothing$ if and only if $f \in \mathfrak{N}$.

4. The structure sheaf

The final ingredient in the definition of an affine scheme is the **structure sheaf** $\mathcal{O}_{\operatorname{Spec} A}$, which we think of as the "sheaf of algebraic functions". As motivation, in $\mathbb{A}^2$, we expect that on the open set $D(xy)$ (away from the two axes), $(3x^4 + y + 4)/x^7$ should be an algebraic function.

These functions will have values at points, but won't be determined by their values at points. But like all sheaves, they will indeed be determined by their germs. This is discussed in Section 4.4.

It suffices to describe the structure sheaf as a sheaf (of rings) on the base of distinguished open sets. Our strategy is as follows. We will define the sections on the base by

$$\mathcal{O}_{\operatorname{Spec} A}(D(f)) = A_f \tag{1}$$

We need to make sure that this is well-defined, i.e. that we have a natural isomorphism $A_f \to A_g$ if $D(f) = D(g)$. We will define the restriction maps $\operatorname{res}_{D(g),D(f)}$ as follows. If $D(f) \subset D(g)$, then we have shown that $D(fg) = D(f)$. There is a natural map $A_g \to A_{fg}$ given by $r/g^m \mapsto (rf^m)/(fg)^m$, and we will define

$$\operatorname{res}_{D(g),D(fg)=D(f)} : \mathcal{O}_{\operatorname{Spec} A}(D(g)) \to \mathcal{O}_{\operatorname{Spec} A}(D(fg))$$

to be this map. But it will be cleaner to state things a little differently.

If $D(f) \subset D(g)$, then by Exercise 3.E, g is a unit in A_f Thus by the universal property of localization, there is a natural map $A_g \to A_f$ which we temporarily denote $\operatorname{res}_{g,f}$, but which we secretly think of as $\operatorname{res}_{D(g),D(f)}$. If $D(f) \subset D(g) \subset D(h)$, then these restriction maps commute:

$$\begin{array}{ccc} A_h & \xrightarrow{\operatorname{res}_{h,g}} & A_g \\ & {\scriptstyle \operatorname{res}_{h,f}} \searrow \quad \swarrow {\scriptstyle \operatorname{res}_{g,f}} & \\ & A_f & \end{array} \tag{2}$$

commutes. (The map $A_h \to A_f$ is defined by universal property, and the composition $\operatorname{res}_{g,f} \circ \operatorname{res}_{h,g}$ satisfies this universal property.)

In particular, if $D(f) = D(g)$, then $\operatorname{res}_{g,f} \circ \operatorname{res}_{f,g}$ is the identity on A_f, (take $h = f$ in the above diagram (2)), and similarly $\operatorname{res}_{f,g} \circ \operatorname{res}_{g,f} = \operatorname{id}_{A_g}$. Thus we can define $\mathcal{O}_{\operatorname{Spec} A}(D(f)) = A_f$, and this is well-defined (independent of the choice of f).

By (2), we have defined a presheaf on the distinguished base.

We now come to a key theorem.

4.1. *Theorem. — The data just described gives a sheaf on the distinguished base, and hence determines a sheaf on the topological space* $\operatorname{Spec} A$.

This sheaf is called the **structure sheaf**, and will be denoted $\mathcal{O}_{\operatorname{Spec} A}$, or sometimes $\mathcal{O}$ if the scheme in question is clear from the context. Such a topological space, with sheaf, will be called an **affine scheme**. The notation $\operatorname{Spec} A$ will hereafter denote the data of a topological space with a structure sheaf.

Proof. We first check identity on the base. We deal with the case of a cover of the entire space A, and let you verify that essentially the same argument holds for a cover of some A_f. Suppose that $\operatorname{Spec} A = \cup_{i \in I} D(f_i)$ where i runs over some index set I. Then there is some finite subset of I, which we name $\{1, \dots, n\}$, such that $\operatorname{Spec} A = \cup_{i=1}^n D(f_i)$, i.e. $(f_1, \dots, f_n) = A$ (quasicompactness of $\operatorname{Spec} A$, Exercise 3.C). Suppose we are given $s \in A$ such that $\operatorname{res}_{\operatorname{Spec} A, D(f_i)} s = 0$ in A_{f_i} for all i. (We wish to show that $s = 0$.) Hence there is some m such that for each $i \in \{1, \dots, n\}$, $f_i^m s = 0$. Now $(f_1^m, \dots, f_n^m) = A$ ($\operatorname{Spec} A = \cup D(f_i) = \cup D(f_i^m)$), so there are $r_i \in A$ with $\sum_{i=1}^n r_i f_i^m = 1$ in A, from which

$$s = \left(\sum r_i f_i^m \right) s = \sum r_i (f_i^m s) = 0.$$

Thus we have checked the "base identity" axiom for $\operatorname{Spec} A$. (Serre has described this as a "partition of unity" argument, and if you look at it in the right way, his insight is very enlightening.)

4.A. EXERCISE. Make the tiny changes to the above argument to show base identity for any distinguished open $D(f)$. (Possible strategy: show that the argument is the same as the previous argument for $\operatorname{Spec} A_f$.)

We next show base gluability. As with base identity, we deal with the case where we wish to glue sections to produce a section over $\operatorname{Spec} A$. As before, we leave the general case where we wish to glue sections to produce a section over $D(f)$ to the reader (Exercise 4.B).

Suppose $\cup_{i \in I} D(f_i) = \operatorname{Spec} A$, where I is a index set (possibly horribly uncountably infinite). Suppose we are given elements in each A_{f_i} that agree on the overlaps $A_{f_i f_j}$. (Note that intersections of distinguished opens are also distinguished opens.)

Aside: experts might realize that we are trying to show exactness of

$$0 \to A \to \prod_i A_{f_i} \to \prod_{i \neq j} A_{f_i f_j}.$$

(What is the right-hand map?) Base identity corresponds to injectivity at A. The composition of the right two morphisms is trivially zero, and gluability is verifying exactness at $\prod_i A_{f_i}$.

Choose a finite subset $\{1, \dots, n\} \subset I$ with $(f_1, \dots, f_n) = A$ (i.e. use quasicompactness of $\operatorname{Spec} A$ to choose a finite subcover by $D(f_i)$). We have elements $a_i / f_i^{l_i} \in A_{f_i}$ agreeing on overlaps $A_{f_i f_j}$. Letting $g_i = f_i^{l_i}$, using $D(f_i) = D(g_i)$, we can simplify notation by considering our elements as of the form $a_i / g_i \in A_{g_i}$.

The fact that a_i/g_i and a_j/g_j "agree on the overlap" (i.e. in $A_{g_i g_j}$) means that for some m_{ij},

$$(g_i g_j)^{m_{ij}} (g_j a_i - g_i a_j) = 0$$

in A. By taking $m = \max m_{ij}$ (here we use the finiteness of I), we can simplify notation:

$$(g_i g_j)^m (g_j a_i - g_i a_j) = 0$$

for all i, j. Let $b_i = a_i g_i^m$ for all i, and $h_i = g_i^{m+1}$ (so $D(h_i) = D(g_i)$). Then we can simplify notation even more: on each $D(h_i)$, we have a function b_i/h_i, and the overlap condition is $h_j b_i - h_i b_j = 0$

Now $\cup_i D(h_i) = A$, implying that $1 = \sum_{i=1}^n r_i h_i$ for some $r_i \in A$. Define $r = \sum r_i b_i$. This will be the element of A that restricts to each b_j/h_j. Indeed,

$$r h_j - b_j = \sum_i r_i b_i h_j - \sum_i b_j r_i h_i = \sum_i r_i (b_i h_j - b_j h_i) = 0.$$

We are not quite done! We are supposed to have something that restricts to $a_i/f_i^{l_i}$ for *all* $i \in I$, not just $i = 1, \dots, n$. But a short trick takes care of this. We now show that for any $\alpha \in I - \{1, \dots, n\}$, r restricts to the desired element $a_\alpha\, A_{f_\alpha}$. Repeat the entire process above with $\{1, \dots, n, \alpha\}$ in place of $\{1, \dots, n\}$, to obtain $r' \in A$ which restricts to α_α for $i \in \{1, \dots, n, \alpha\}$. Then by base identity, $r' = r$. (Note that we use base identity to *prove* base gluability. This is an example of how base identity is "prior" to base gluability.) Hence r restricts to $a_\alpha/f_\alpha^{l_\alpha}$ as desired.

4.B. EXERCISE. Alter this argument appropriately to show base gluability for any distinguished open $D(f)$.

We have now completed the proof of Theorem 4.1.

□

The proof of Theorem 4.1 immediately generalizes, as the following exercise shows. This exercise will be essential for the definition of a quasicoherent sheaf later on [say where].

4.C. IMPORTANT EXERCISE/DEFINITION. Suppose M is an A-module. Show that the following construction describes a sheaf $\tilde{M}$ on the distinguished base. To $D(f)$ we associate $M_f = M \otimes_A A_f$; the restriction map is the "obvious" one. This is an $\mathcal{O}_{\operatorname{Spec} A}$-module! This sort of sheaf $\tilde{M}$ will be very important soon; it is an example of a *quasicoherent sheaf*.

Here is a useful fact for later: As a consequence, note that if $(f_1, \dots, f_r) = A$, we have identified M with a specific submodule of $M_{f_1} \times \cdots \times M_{f_r}$. Even though $M \to M_{f_i}$ may not be an inclusion for any f_i, $M \to M_{f_1} \times \cdots \times M_{f_r}$ *is* an inclusion. We don't care yet, but we'll care about this later, and I'll invoke this fact. (Reason: we'll want to show that if M has some nice property, then M_f does too, which will be easy. We'll also want to show that if $(f_1, \dots, f_n) = R$, then if M_{f_i} have this property, then M does too.)

4.2. Definition. We can now define *scheme* in general. First, define an **isomorphism of ringed spaces** $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ as (i) a homeomorphism $f : X \to Y$, and (ii) an isomorphism of sheaves $\mathcal{O}_X$ and $\mathcal{O}_Y$, considered to be on the same space via f. (Condition (ii), more precisely: an isomorphism $\mathcal{O}_X \to f^{-1}\mathcal{O}_Y$ of sheaves on X, or $f_*\mathcal{O}_X \to \mathcal{O}_Y$ of sheaves on Y.) In other words, we have a correspondence of sets, topologies, and structure sheaves. An **affine scheme** is a ringed space that is isomorphic to $(\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})$. A **scheme** $(X, \mathcal{O}_X)$ is a ringed space such that any point $x \in X$ has a neighborhood U such that $(U, \mathcal{O}_X|_U)$ is an affine scheme. The scheme can be denoted $(X, \mathcal{O}_X)$, although it is often denoted X, with the structure sheaf implicit.

An **isomorphism of two schemes** $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ is an isomorphism as ringed spaces.

4.3. *Remark.* From this definition of the structure sheaf on an affine scheme, several things are clear. First of all, if we are told that $(X, \mathcal{O}_X)$ is an affine scheme, we may recover its ring (i.e. find the ring A such that $\operatorname{Spec} A = X$) by taking the ring of global sections, as $X = D(1)$, so:

$$\begin{aligned} \Gamma(X, \mathcal{O}_X) &= \Gamma(D(1), \mathcal{O}_{\operatorname{Spec} A}) \quad \text{as } D(1) = \operatorname{Spec} A \\ &= A_1 \quad \text{(i.e. allow 1's in the denominator) by definition} \\ &= A. \end{aligned}$$

(You can verify that we get more, and can "recognize X as the scheme $\operatorname{Spec} A$": we get a natural isomorphism $f : (\operatorname{Spec} \Gamma(X, \mathcal{O}_X), \mathcal{O}_{\operatorname{Spec} \Gamma(X, \mathcal{O}_X)}) \to (X, \mathcal{O}_X)$. For example, if $\mathfrak{m}$ is a maximal ideal of $\Gamma(X, \mathcal{O}_X)$, $f([\mathfrak{m}]) = V(\mathfrak{m})$.) More generally, given $f \in A$, $\Gamma(D(f), \mathcal{O}_{\operatorname{Spec} A}) \cong A_f$. Thus under the natural inclusion of sets $\operatorname{Spec} A_f \hookrightarrow \operatorname{Spec} A$, the Zariski topology on $\operatorname{Spec} A$ restricts to give the Zariski topology on $\operatorname{Spec} A_f$ (as we've seen in an earlier Exercise), and the structure sheaf of $\operatorname{Spec} A$ restricts to the structure sheaf of $\operatorname{Spec} A_f$, as the next exercise shows.

4.D. IMPORTANT BUT EASY EXERCISE. Suppose $f \in A$. Show that under the identification of $D(f)$ in $\operatorname{Spec} A$ with $\operatorname{Spec} A_f$, there is a natural isomorphism of sheaves $(D(f), \mathcal{O}_{\operatorname{Spec} A}|_{D(f)}) \cong (\operatorname{Spec} A_f, \mathcal{O}_{\operatorname{Spec} A_f})$.

4.E. EXERCISE. Show that if X is a scheme, then the affine open sets form a base for the Zariski topology.

4.F. EXERCISE. If X is a scheme, and U is *any* open subset, prove that $(U, \mathcal{O}_X|_U)$ is also a scheme.

$(U, \mathcal{O}_X|_U)$ is called an *open subscheme of* U. If U is also an affine scheme, we often say U is an *affine open subset*, or an *affine open subscheme*, or sometimes informally just an *affine open*. For an example, $D(f)$ is an affine open subscheme of $\operatorname{Spec} A$.

4.4. Stalks of the structure sheaf: germs, and values at a point. Like every sheaf, the structure sheaf has stalks, and we shouldn't be surprised if they are interesting from an algebraic point of view. In fact, we have seen them before.

4.G. IMPORTANT EXERCISE. Show that the stalk of $\mathcal{O}_{\operatorname{Spec} A}$ at the point $[\mathfrak{p}]$ is the ring $A_\mathfrak{p}$.

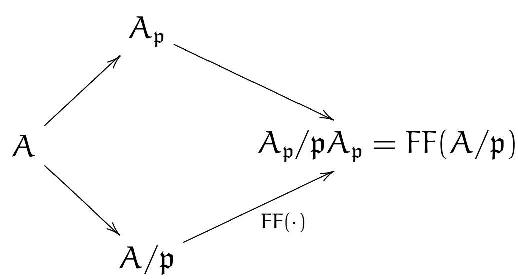
Essentially the same argument will show that the stalk of the sheaf $\tilde{M}$, defined in Exercise 4.C at $[\mathfrak{p}]$ is $M_\mathfrak{p}$. Here is an interesting consequence, or if you prefer, a geometric interpretation of an algebraic fact. A section is determined by it stalks (an earlier Exercise), meaning that $M \to \prod_{\mathfrak{p}} M_{\mathfrak{p}}$ is an inclusion. So for example an A-module is zero if and only if all its localizations at primes are zero.

The **residue field of a scheme at a point** is the local ring modulo its maximal ideal.

So now we can make some of our vague discussion earlier precise. Suppose $[\mathfrak{p}]$ is a point in some open set U of $\operatorname{Spec} A$. For example, say $A = k[x, y]$, $\mathfrak{p} = [(x)]$ [draw picture], and $U = \mathbb{A}^2 - (0, 0)$.

Then a function on U, i.e. a section of $\mathcal{O}_{\operatorname{Spec} A}$ over U, has a *germ near* $[\mathfrak{p}]$, which is an element of $A_{\mathfrak{p}}$. This stalk $A_{\mathfrak{p}}$ is a local ring, with maximal ideal $\mathfrak{p}A_{\mathfrak{p}}$. In our example, consider the function $(3x^4 + x^2 + xy + y^2)/(3x^2 + xy + y^2 + 1)$, which is defined on the open set $D(3x^2 + xy + y^2 + 1)$. Because there are no factors of x in the denominator, it is indeed in $A_{\mathfrak{p}}$.

A germ has a *value* at $[\mathfrak{p}]$, which is an element of $A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}$. This is isomorphic to $FF(A/\mathfrak{p})$, the fraction field of the quotient domain. It is useful to note that localization at $\mathfrak{p}$ and taking quotient by $\mathfrak{p}$ "commute", i.e. the following diagram commutes.



So the value of a function at a point always takes values in a field. In our example, to see the value of our germ at $x = 0$, we simply set $x = 0$. So we get the value $y^2/(y^2 + 1)$, which is certainly in $FF(k[y])$. (If you think you care only about complex schemes, and hence only about algebraically closed fields, let this be a first warning: $A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}$ won't be algebraically closed in general, even if A is a finitely generated $\mathbb{C}$-algebra!)

We say that the germ *vanishes* at $\mathfrak{p}$ if the value is zero. In our example, the germ doesn't vanish at $\mathfrak{p}$.

If anything makes you nervous, you should make up an example to assuage your nervousness. (Example: 27/4 is a regular function on $\operatorname{Spec} \mathbb{Z} - \{[(2)], [(7)]\}$. What is its value at $[(5)]$? Answer: $2/(-1) \equiv -2 \pmod 5$. What is its value at the generic point $[(0)]$? Answer: 27/4. Where does it vanish? At $[(3)]$.)

We now give three extended examples. Our short term goal is to see that we can really work with this sheaf, and can compute the ring of sections of interesting open sets that aren't just distinguished open sets of affine schemes. Our long-term goal is to see interesting examples that will come up repeatedly in the future. All three examples are non-affine schemes, so these examples are genuinely new to us.

4.5. Example: The plane minus the origin. I now want to work through an example with you, to show that this distinguished base is indeed something that you can work with. Let $A = k[x, y]$, so $\operatorname{Spec} A = \mathbb{A}_k^2$. If you want, you can let k be $\mathbb{C}$, but that won't be relevant. Let's work out the space of functions on the open set $U = \mathbb{A}^2 - (0, 0)$.

It is a non-obvious fact that you can't cut out this set with a single equation, so this isn't a distinguished open set. We'll see why fairly soon [where?]. But in any case, even if we're not sure that this is a distinguished open set, we can describe it as the union of two things which *are* distinguished open sets. If I throw out the x axis, i.e. the set $y = 0$, I get the distinguished open set $D(y)$. If I throw out the y axis, i.e. $x = 0$, I get the distinguished open set $D(x)$. So $U = D(x) \cup D(y)$. (Remark: $U = \mathbb{A}^2 - V(x, y)$ and $U = D(x) \cup D(y)$. Coincidence? I think not!) We will find the functions on U by gluing together functions on $D(x)$ and $D(y)$.

What are the functions on $D(x)$? They are, by definition, $A_x = k[x, y, 1/x]$. In other words, they are things like this: $3x^2 + xy + 3y/x + 14/x^4$. What are the functions on $D(y)$? They are, by definition, $A_y = k[x, y, 1/y]$. Note that $A \hookrightarrow A_x, A_y$. This is because x and y are not zero-divisors. (A is an integral domain — it has no zero-divisors, besides 0 — so localization is always an inclusion.) So we are looking for functions on $D(x)$ and $D(y)$ that agree on $D(x) \cap D(y) = D(xy)$, i.e. they are just the same Laurent polynomial. Which things of this first form are also of the second form? Just old-fashioned polynomials —

$$\Gamma(U, \mathcal{O}_{\mathbb{A}^2}) \equiv k[x, y]. \tag{3}$$

In other words, we get no extra functions by throwing out this point. Notice how easy that was to calculate!

4.6. (Aside: Notice that any function on $\mathbb{A}^2 - (0, 0)$ extends over all of $\mathbb{A}^2$. This is an analog of *Hartogs' Lemma* in complex geometry: you can extend a holomorphic function defined on the complement of a set of codimension at least two on a complex manifold over the missing set. This will work more generally in the algebraic setting: you can extend over points in codimension at least 2 not only if they are smooth, but also if they are mildly singular — what we will call *normal*. We will make this precise later. This fact will be very useful for us.)

We can now verify an interesting fact: $(U, \mathcal{O}_{\mathbb{A}^2}|_U)$ is a scheme, but it is not an affine scheme. Here's why: otherwise, if $(U, \mathcal{O}_{\mathbb{A}^2}|_U) = (\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})$, then we can recover A by taking global sections:

$$A = \Gamma(U, \mathcal{O}_{\mathbb{A}^2}|_U),$$

which we have already identified in (3) as $k[x, y]$. So if U is affine, then $U = \mathbb{A}_k^2$. But we get more: we can recover the points of $\operatorname{Spec} A$ by taking the primes of A. In particular, the prime ideal (x, y) of A should cut out a point of $\operatorname{Spec} A$. But on U, $V(x) \cap V(y) = \varnothing$. Conclusion: U is *not* an affine scheme. (If you are ever looking for a counterexample to something, and you are expecting one involving a non-affine scheme, keep this example in mind!)

You've seen two examples of non-affine schemes: an infinite disjoint union of non-empty schemes (Exercise 4.M), and now $\mathbb{A}^2 - (0,0)$. I want to give you two more important examples. They are important because they are the first examples of fundamental behavior, the first pathological, and the second central.

First, I need to tell you how to glue two schemes together. And before that, you should review how to glue topological spaces together along isomorphic open sets. Given two topological spaces X and Y, and open subsets $U \subset X$ and $V \subset Y$ along with a homeomorphism $U \cong V$, we can create a new topological space W, that we think of as gluing X and Y together along $U \cong V$. It is the quotient of the disjoint union $X \coprod Y$ by the equivalence relation $U \cong V$, where the quotient is given the quotient topology. Then X and Y are naturally (identified with) open subsets of W, and indeed cover W. Can you restate this with an arbitrary number of topological spaces glued together?

Now that we have discussed gluing topological spaces, let's glue schemes together. Suppose you have two schemes $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$, and open subsets $U \subset X$ and $V \subset Y$, along with a homeomorphism $f : U \xrightarrow{\sim} V$, and an isomorphism of structure sheaves $\mathcal{O}_X \cong f^*\mathcal{O}_Y$ (i.e. an isomorphism *of schemes* $(U, \mathcal{O}_X|_U) \cong (V, \mathcal{O}_Y|_V)$). Then we can glue these together to get a single scheme. Reason: let W be X and Y glued together using the isomorphism $U \cong V$. Then an earlier exercise on gluing sheaves shows that the structure sheaves can be glued together to get a sheaf of rings. Note that this is indeed a scheme: any point has a neighborhood that is an affine scheme. (Do you see why?)

4.H. EXERCISE. For later reference, show that you can glue together an arbitrary number of schemes together. Suppose we are given:

- schemes X_i (as i runs over some index set I, not necessarily finite),
- open subschemes $X_{ij} \subset X_i$,
- isomorphisms $f_{ij} : X_{ij} \to X_{ji}$
- such that the isomorphisms "agree along triple intersections", i.e. $f_{ik}|_{X_{ij} \cap X_{ik}} = f_{jk}|_{U_{ji} \cap U_{jk}} \circ f_{ij}|_{X_{ij} \cap X_{ik}}$.

Show that there is a unique scheme X (up to unique isomorphism) along with open subset isomorphic to X_i respecting this gluing data in the obvious sense.

I'll now give you two non-affine schemes. In both cases, I will glue together two copies of the affine line $\mathbb{A}^1_k$. Again, if it makes you feel better, let $k = \mathbb{C}$, but it really doesn't matter.

Let $X = \operatorname{Spec} k[t]$, and $Y = \operatorname{Spec} k[u]$. Let $U = D(t) = \operatorname{Spec} k[t, 1/t] \subset X$ and $V = D(u) = \operatorname{Spec} k[u, 1/u] \subset Y$. We will get both examples by gluing X and Y together along U and V. The difference will be in how we glue.

4.7. Extended example: the affine line with the doubled origin. Consider the isomorphism $U \cong V$ via the isomorphism $k[t, 1/t] \cong k[u, 1/u]$ given by $t \leftrightarrow u$. Let the resulting scheme be X. This is called the *affine line with doubled origin*. Figure 3 is a picture of it.

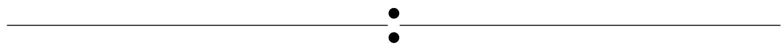


FIGURE 3. The affine line with doubled origin

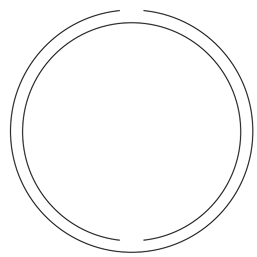


FIGURE 4. Gluing two affine lines together to get $\mathbb{P}^1$

As the picture suggests, intuitively this is an analogue of a failure of Hausdorffness. $\mathbb{A}^1$ itself is not Hausdorff, so we can't say that it is a failure of Hausdorffness. We see this as weird and bad, so we'll want to make up some definition that will prevent this from happening. This will be the notion of *separatedness*. This will answer other of our prayers as well. For example, on a separated scheme, the "affine base of the Zariski topology" is nice — the intersection of two affine open sets will be affine.

4.I. EXERCISE. Show that X is not affine. Hint: calculate the ring of global sections, and look back at the argument for $\mathbb{A}^2 - (0,0)$.

4.J. EXERCISE. Do the same construction with $\mathbb{A}^1$ replaced by $\mathbb{A}^2$. You'll have defined the *affine plane with doubled origin*. Use this example to show that the affine base of the Zariski topology isn't a nice base, by describing two affine open sets whose intersection is not affine.

4.8. Example 2: the projective line. Consider the isomorphism $U \cong V$ via the isomorphism $k[t, 1/t] \cong k[u, 1/u]$ given by $t \leftrightarrow 1/u$. Figure 4 is a suggestive picture of this gluing. Call the resulting scheme the **projective line over the field** k, $\mathbb{P}^1_k$.

Notice how the points glue. Let me assume that k is algebraically closed for convenience. (You can think about how this changes otherwise.) On the first affine line, we have the closed (= "old-fashioned") points $[(t-a)]$, which we think of as "a on the t-line", and we have the generic point. On the second affine line, we have closed points that are "b on the u-line", and the generic point. Then a on the t-line is glued to $1/a$ on the u-line (if $a \neq 0$ of course), and the generic point is glued to the generic point (the ideal (0) of $k[t]$ becomes the ideal (0) of $k[t, 1/t]$ upon localization, and the ideal (0) of

$k[u]$ becomes the ideal (0) of $k[u, 1/u]$. And (0) in $k[t, 1/t]$ is (0) in $k[u, 1/u]$ under the isomorphism $t \leftrightarrow 1/u$).

We can interpret the closed ("old-fashioned") points of $\mathbb{P}^1$ in the following way, which may make this sound closer to the way you have seen projective space defined earlier. The points are of the form $[a; b]$, where a and b are not both zero, and $[a; b]$ is identified with $[ac; bc]$ where $c \in k^*$. Then if $b \neq 0$, this is identified with a/b on the t-line, and if $a \neq 0$, this is identified with b/a on the u-line.

4.9. *Proposition. — $\mathbb{P}^1_k$ is not affine.*

Proof. We do this by calculating the ring of global sections.

The global sections correspond to sections over X and sections over Y that agree on the overlap. A section on X is a polynomial $f(t)$. A section on Y is a polynomial $g(u)$. If I restrict $f(t)$ to the overlap, I get something I can still call $f(t)$; and ditto for $g(u)$. Now we want them to be equal: $f(t) = g(1/t)$. How many polynomials in t are at the same time polynomials in $1/t$? Not very many! Answer: only the constants k. Thus $\Gamma(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}) = k$. If $\mathbb{P}^1$ were affine, then it would be $\operatorname{Spec} \Gamma(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}) = \operatorname{Spec} k$, i.e. one point. But it isn't — it has lots of points. $\square$

Note that we have proved an analog of a theorem: the only holomorphic functions on $\mathbb{CP}^1$ are the constants!

4.K. IMPORTANT EXERCISE. Figure out how to define projective n-space $\mathbb{P}^n_k$. Glue together $n+1$ opens each isomorphic to $\mathbb{A}^n_k$. Show that the only global sections of the structure sheaf are the constants, and hence that $\mathbb{P}^n_k$ is not affine if $n > 0$. (Hint: you might fear that you will need some delicate interplay among all of your affine opens, but you will only need two of your opens to see this. There is even some geometric intuition behind this: the complement of the union of two opens has codimension 2. But "Hartogs' Theorem" (to be stated rigorously later) says that any function defined on this union extends to be a function on all of projective space. Because we're expecting to see only constants as functions on all of projective space, we should already see this for this union of our two affine open sets.)

4.L. EXERCISE. Show that if k is algebraically closed, the closed points of $\mathbb{P}^n_k$ may be interpreted in the same way as we interpreted the points of $\mathbb{P}^1_k$. (The points are of the form $[a_0; \ldots; a_n]$, where the a_i are not all zero, and $[a_0; \ldots; a_n]$ is identified with $[ca_0; \ldots; ca_n]$ where $c \in k^*$.)

We will later give another definition of projective space . Your definition (from Exercise 4.K) will be handy for computing things. But there is something unnatural about it — projective space is highly symmetric, and that isn't clear from your point of view.

Note that your definition will give a definition of $\mathbb{P}^n_A$ for any *ring* A. This will be useful later.

4.10. Fun aside: The Chinese Remainder Theorem is a *geometric* fact. I want to show you that the Chinese Remainder theorem is embedded in what we've done, which shouldn't be obvious to you. I'll show this by example. The Chinese Remainder Theorem says that knowing an integer modulo 60 is the same as knowing an integer modulo 3, 4, and 5. Here's how to see this in the language of schemes. What is $\operatorname{Spec} \mathbb{Z}/(60)$? What are the primes of this ring? Answer: those prime ideals containing (60), i.e. those primes dividing 60, i.e. (2), (3), and (5). So here is my picture of the scheme [picture of 3 dots]. They are all closed points, as these are all maximal ideals, so the topology is the discrete topology. What are the stalks? You can check that they are $\mathbb{Z}/4$, $\mathbb{Z}/3$, and $\mathbb{Z}/5$. My picture is actually like this [draw a bit of one-dimensional fuzz on (2)]: the scheme has nilpotents here ($2^2 \equiv 0 \pmod 4$). I indicate nilpotents with "fuzz". So what are global sections on this scheme? They are sections on this open set (2), this other open set (3), and this third open set (5). In other words, we have a natural isomorphism of rings

$$\mathbb{Z}/60 \to \mathbb{Z}/4 \times \mathbb{Z}/3 \times \mathbb{Z}/5.$$

On a related note:

4.M. EXERCISE. (a) Show that the disjoint union of a *finite* number of affine schemes is also an affine scheme. (Hint: say what the ring is.)
(b) Show that an infinite disjoint union of (non-empty) affine schemes is not an affine scheme.

4.11. $\star$ *Example.* Here is an example of a function on an open subset of a scheme that is a bit surprising. On $X = \operatorname{Spec} k[w,x,y,z]/(wx-yz)$, consider the open subset $D(y) \cup D(w)$. Show that the function x/y on $D(y)$ agrees with z/w on $D(w)$ on their overlap $D(y) \cap D(w)$. Hence they glue together to give a section. You may have seen this before when thinking about analytic continuation in complex geometry — we have a "holomorphic" function which has the description x/y on an open set, and this description breaks down elsewhere, but you can still "analytically continue" it by giving the function a different definition on different parts of the space.

Follow-up for curious experts: This function has no "single description" as a well-defined expression in terms of w, x, y, z! There is lots of interesting geometry here. This example will be a constant source of interesting examples for us. We will later recognize it as the cone over the quadric surface. Here is a glimpse, in terms of words we have not yet defined. $\operatorname{Spec} k[w,x,y,z]$ is $\mathbb{A}^4$, and is, not surprisingly, 4-dimensional. We are looking at the set X, which is a hypersurface, and is 3-dimensional. It is a cone over a smooth quadric surface in $\mathbb{P}^3$. $D(y)$ is X minus some hypersurface, so we are throwing away a codimension 1 locus. $D(z)$ involves throwing another codimension 1 locus. You might think that their intersection is then codimension 2, and that maybe failure of extending this weird function to a global polynomial comes because of a failure of our Hartogs'-type theorem, which will be a failure of normality. But that's not true — $V(y) \cap V(z)$ is in fact codimension 1 — so no Hartogs-type theorem holds. Here is what is actually going on. $V(y)$ involves throwing away the (cone over the) union of two lines l and m_1, one in each "ruling" of the surface, and $V(z)$ also involves throwing away the (cone over the) union of two lines l and m_2. The intersection is the (cone over the) line l, which is a codimension 1 set. Neat fact: despite being "pure codimension 1", it is not cut out even set-theoretically by a single equation. (It is hard to get an example of this behavior. This example is the simplest example I know.) This means that any expression $f(w,x,y,z)/g(w,x,y,z)$ for our function cannot correctly describe our function on $D(y) \cup D(z)$ — at some point of $D(y) \cup D(z)$ it must be $0/0$. Here's why. Our function can't be defined on $V(y) \cap V(z)$, so g must vanish here. But g can't vanish just on the cone over l — it must vanish elsewhere too. (For the experts among the experts: here is why the cone over l is not cut out set-theoretically by a single equation. If $l = V(f)$, then $D(f)$ is affine. Let l' be another line in the same ruling as l, and let $C(l)$ (resp. l') be the cone over l (resp. l'). Then $C(l')$ can be given the structure of a closed subscheme of $\operatorname{Spec} k[w,x,y,z]$, and can be given the structure of $\mathbb{A}^2$. Then $C(l') \cap V(f)$ is a closed subscheme of $D(f)$. Any closed subscheme of an affine scheme is affine. But $l \cap l' = \varnothing$, so the cone over l intersects the cone over l' in a point, so $C(l') \cap V(f)$ is $\mathbb{A}^2$ minus a point, which we've seen is not affine, so we have a contradiction.)

We concluded with some initial discussion of properties of schemes, including irreducible, closed point, specialization, generization, generic point, connected component, and irreducible component.

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FOUNDATIONS OF ALGEBRAIC GEOMETRY CLASSES 9 AND 10

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CONTENTS

This week, we will define some useful properties of schemes.

1. TOPOLOGICAL PROPERTIES: IRREDUCIBILITY, CONNECTEDNESS, QUASICOMPACTNESS

We will start with some topological properties. The definitions of *irreducible*, *closed point*, *specialization*, *generalization*, *generic point*, *connected component*, and *irreducible component* were given earlier. You should have pictures in your mind of each of these notions.

An earlier exercise showed that $\mathbb{A}^n$ is irreducible (it was easy). This argument "behaves well under gluing", yielding:

1.A. EXERCISE. Show that $\mathbb{P}^n_k$ is irreducible.

1.B. EXERCISE. An earlier exercise showed that there is a bijection between irreducible closed subsets and points. Show that this is true of schemes as well.

1.C. EXERCISE. Prove that if X is a scheme that has a finite cover $X = \cup_{i=1}^n \operatorname{Spec} A_i$ where A_i is Noetherian, then X is a Noetherian topological space. (We will soon call such a scheme a *Noetherian scheme*, §3.5.)

Thus $\mathbb{P}^n_k$ and $\mathbb{P}^n_{\mathbb{Z}}$ are Noetherian topological spaces: we built them by gluing together a finite number of Spec's of Noetherian rings.

Date: Monday, October 22, 2007 and Wednesday, October 24, 2007. Updated Nov. 17, 2007.

1.1. Definition. A topological space X is **connected** if it cannot be written as the disjoint union of two non-empty open sets.

1.D. EXERCISE. Show that an irreducible topological space is connected.

1.E. EXERCISE. Give (with proof!) an example of a scheme that is connected but reducible. (Possible hint: a picture may help. The symbol "$\times$" has two "pieces" yet is connected.)

1.F. EXERCISE. If $A = \prod A_1 \times A_2 \times \cdots \times A_n$, describe an isomorphism $\operatorname{Spec} A = \operatorname{Spec} A_1 \coprod \operatorname{Spec} A_2 \coprod \cdots \coprod \operatorname{Spec} A_n$. Show that each $\operatorname{Spec} A_i$ is a distinguished open subset $D(f_i)$ of $\operatorname{Spec} A$. (Hint: let $f_i = (0, \cdots, 0, 1, 0, \cdots 0)$ where the 1 is in the ith component.) In other words, $\coprod_{i=1}^n \operatorname{Spec} A_i = \operatorname{Spec} \prod_{i=1}^n A_i$.

1.2. *Fun but irrelevant remark.* As affine schemes are quasicompact, $\coprod_{i=1}^\infty \operatorname{Spec} A_i$ cannot be isomorphic to $\operatorname{Spec} \prod_{i=1}^\infty A_i$. This lack of isomorphism has an entertaining consequence. Suppose the A_i are isomorphic to the field k. Then we certainly have an inclusion as sets

$$\coprod_{i=1}^\infty \operatorname{Spec} A_i \hookrightarrow \operatorname{Spec} \prod_{i=1}^\infty A_i$$

— there is a maximal ideal of $\operatorname{Spec} \prod A_i$ corresponding to each i (precisely those elements 0 in the ith component.) But there are other maximal ideals of $\prod A_i$. Hint: describe a proper ideal not contained in any of these maximal ideal. (One idea: consider elements $\prod a_i$ that are "eventually zero", i.e. $a_i = 0$ for $i \gg 0$.) This leads to the notion of *ultrafilters*, which are very useful, but irrelevant to our current discussion.

As long as we are on the topic of quasicompactness...

1.3. Definition. A scheme is **quasicompact** if its underlying topological space is quasicompact. This seems like a strong condition, but because Zariski-open sets are so large, almost any scheme naturally coming up in nature will be quasicompact.

1.G. EASY EXERCISE. Show that a scheme X is quasicompact if and only if it can be written as a finite union of affine schemes (Hence $\mathbb{P}_k^n$ is quasicompact.)

1.H. EXERCISE: QUASICOMPACT SCHEMES HAVE CLOSED POINTS. Show that if X is a nonempty quasicompact scheme, then it has a closed point. (Warning: there exist nonempty schemes with no closed points, so your argument had better use the quasicompactness hypothesis! We will see that in good situations, the closed points are dense, Exercise 3.H.)

1.4. Quasiseparatedness.

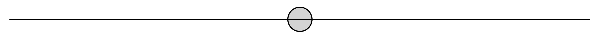

FIGURE 1. A picture of the scheme $\operatorname{Spec} k[x,y]/(xy,y^2)$

Quasiseparatedness is a weird notion that comes in handy for certain kinds of people. Most people, however, can ignore this notion. A scheme is **quasiseparated** if the intersection of any two quasicompact sets is quasicompact, or equivalently, if the intersection of any two affine open subsets is a finite union of affine open subsets.

1.I. SHORT EXERCISE. Prove this equivalence.

We will see later that this will be a useful hypothesis in theorems (in conjunction with quasicompactness), and that various interesting kinds of schemes (affine, locally Noetherian, separated, see Exercise 1.J, Exercise 3.B, and an exercise next quarter resp.) are quasiseparated, and this will allow us to state theorems more succinctly (e.g. "if X is quasicompact and quasiseparated" rather than "if X is quasicompact, and either this or that or the other thing hold").

1.J. EXERCISE. Show that affine schemes are quasiseparated.

"Quasicompact and quasiseparated" means something rather down to earth:

1.K. EXERCISE. Show that a scheme X is quasicompact and quasiseparated if and only if X can be covered by a finite number of affine open subsets, any two of which have intersection also covered by a finite number of affine open subsets.

2. REDUCEDNESS AND INTEGRALITY

Recall that one of the alarming things about schemes is that functions are not determined by their values at points, and that was because of the presence of *nilpotents*.

2.1. Definition. Recall that a ring is **reduced** if it has no nonzero nilpotents. A scheme X is **reduced** if $\mathcal{O}_X(U)$ has no nonzero nilpotents for any open set U of X.

An example of a nonreduced affine scheme is $\operatorname{Spec} k[x,y]/(y^2,xy)$. A useful representation of this scheme is given in Figure 1, although we will only explain in §5 why this is a good picture. The fuzz indicates that there is some nonreducedness going on at the origin. Here are two different functions: x and $x+y$. Their values agree at all points (all closed points $[(x-a,y)] = (a,0)$ and at the generic point $[(y)]$). They are actually the same function on the open set $D(x)$, which is not surprising, as $D(x)$ is reduced, as the next exercise shows. (This explains why the fuzz is only at the origin, where $y=0$.)

2.A. EXERCISE. Show that $\left(k[x,y]/(y^2,xy)\right)_x$ has no nilpotents. (Possible hint: show that it is isomorphic to another ring, by considering the geometric picture.)

2.B. EXERCISE (REDUCEDNESS IS STALK-LOCAL). Show that a scheme is reduced if and only if none of the stalks have nilpotents. Hence show that if f and g are two functions on a reduced scheme that agree at all points, then $f = g$. (Two hints: $\mathcal{O}_X(U) \hookrightarrow \prod_{x \in U} \mathcal{O}_{X,x}$ from an earlier Exercise, and the nilradical is intersection of all prime ideals.)

Warning: if a scheme X is reduced, then it is immediate from the definition that its ring of global sections is reduced. However, the converse is not true; we will meet an example later.

2.C. EXERCISE. Suppose X is quasicompact, and f is a function (a global section of $\mathcal{O}_X$) that vanishes at all points of x. Show that there is some n such that $f^n = 0$. Show that this may fail if X is not quasicompact. (This exercise is less important, but shows why we like quasicompactness, and gives a standard pathology when quasicompactness doesn't hold.) Hint: take an infinite disjoint union of $\operatorname{Spec} A_n$ with $A_n := k[\epsilon]/\epsilon^n$.

Definition. A scheme X is **integral** if $\mathcal{O}_X(U)$ is an integral domain for every open set U of X.

2.D. IMPORTANT EXERCISE. Show that a scheme X is integral if and only if it is irreducible and reduced.

2.E. EXERCISE. Show that an affine scheme $\operatorname{Spec} A$ is integral if and only if A is an integral domain.

2.F. EXERCISE. Suppose X is an integral scheme. Then X (being irreducible) has a generic point η. Suppose $\operatorname{Spec} A$ is any non-empty affine open subset of X. Show that the stalk at η, $\mathcal{O}_{X,\eta}$, is naturally $FF(A)$, the fraction field of A. This is called the **function field** $FF(X)$ of X. It can be computed on any non-empty open set of X, as any such open set contains the generic point. The symbol FFis deliberately ambiguous — it may stand for fraction field or function field.

2.G. EXERCISE. Suppose X is an integral scheme. Show that the restriction maps $\operatorname{res}_{U,V} : \mathcal{O}_X(U) \to \mathcal{O}_X(V)$ are inclusions so long as $V \neq \varnothing$. Suppose $\operatorname{Spec} A$ is any non-empty affine open subset of X (so A is an integral domain). Show that the natural map $\mathcal{O}_X(U) \to \mathcal{O}_{X,\eta} = FF(A)$ (where U is any non-empty open set) is an inclusion. Thus irreducible varieties (an important example of integral schemes defined later) have the convenient that sections over different open sets can be considered subsets of the same thing. This makes restriction maps and gluing easy to consider; this is one reason why varieties are usually introduced before schemes.

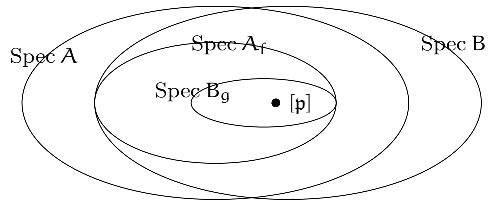

FIGURE 2. Trick to show that the intersection of two affine open sets may be covered by open sets that are simultaneously open in both affines

An almost-local criterion for integrality is given in 3.F.

3. PROPERTIES OF SCHEMES THAT CAN BE CHECKED "AFFINE-LOCALLY"

This section is intended to address something tricky and annoying in the definition of schemes. We've defined a scheme as a topological space with a sheaf of rings, that can be covered by affine schemes. Hence we have all of the affine opens in the cover, but we don't know how to communicate between any two of them. Somewhat more explicitly, if I have an affine cover, and you have an affine cover, and we want to compare them, and I calculate something on my cover, there should be some way of us getting together, and figuring out how to translate my calculation over onto your cover. The Affine Communication Lemma 3.3 will provide a convenient machine for doing this.

Thanks to this lemma, we can define a host of important properties of schemes. All of these are "affine-local" in that they can be checked on any affine cover, i.e. a covering by open affine sets. We like such properties because we can check them using any affine cover we like. If the scheme in question is quasicompact, then we need only check a finite number of affine open sets.

3.1. *Warning.* In our limited examples so far, any time we've had an affine open subset of an affine scheme $\operatorname{Spec} B \subset \operatorname{Spec} A$, in fact $\operatorname{Spec} B = D(f)$ for some f. But this is not always true, and we will eventually have an example, using elliptic curves.

3.2. *Proposition. — Suppose* $\operatorname{Spec} A$ *and* $\operatorname{Spec} B$ *are affine open subschemes of a scheme* X*. Then* $\operatorname{Spec} A \cap \operatorname{Spec} B$ *is the union of open sets that are simultaneously distinguished open subschemes of* $\operatorname{Spec} A$ *and* $\operatorname{Spec} B$.

Proof. (See Figure 2 for a sketch.) Given any point $[\mathfrak{p}] \in \operatorname{Spec} A \cap \operatorname{Spec} B$, we produce an open neighborhood of $[\mathfrak{p}]$ in $\operatorname{Spec} A \cap \operatorname{Spec} B$ that is simultaneously distinguished in both $\operatorname{Spec} A$ and $\operatorname{Spec} B$. Let $\operatorname{Spec} A_f$ be a distinguished open subset of $\operatorname{Spec} A$ contained in $\operatorname{Spec} A \cap \operatorname{Spec} B$. Let $\operatorname{Spec} B_g$ be a distinguished open subset of $\operatorname{Spec} B$ contained in $\operatorname{Spec} A_f$. Then $g \in \Gamma(\operatorname{Spec} B, \mathcal{O}_X)$ restricts to an element $g' \in \Gamma(\operatorname{Spec} A_f, \mathcal{O}_X) = A_f$. The points of $\operatorname{Spec} A_f$ where g vanishes are precisely the points of $\operatorname{Spec} A_f$ where g' vanishes, so

$$\begin{aligned}\operatorname{Spec} B_g &= \operatorname{Spec} A_f \setminus \{[\mathfrak{p}] : g' \in \mathfrak{p}\} \\ &= \operatorname{Spec}(A_f)_{g'}.\end{aligned}$$

If $g' = g''/f^n$ ($g'' \in A$) then $\operatorname{Spec}(A_f)_{g'} = \operatorname{Spec} A_{fg''}$, and we are done. □

The following easy result will be crucial for us.

3.3. *Affine Communication Lemma. — Let* P *be some property enjoyed by some affine open sets of a scheme* X*, such that*

(i) *if an affine open set* $\operatorname{Spec} A \hookrightarrow X$ *has* P *then for any* $f \in A$, $\operatorname{Spec} A_f \hookrightarrow X$ *does too.*
(ii) *if* $(f_1, \ldots, f_n) = A$*, and* $\operatorname{Spec} A_{f_i} \hookrightarrow X$ *has* P *for all* i*, then so does* $\operatorname{Spec} A \hookrightarrow X$.

Suppose that $X = \cup_{i \in I} \operatorname{Spec} A_i$ *where* $\operatorname{Spec} A_i$ *is an affine, and* A_i *has property* P*. Then every other open affine subscheme of* X *has property* P *too.*

We say such a property is **affine-local**. Note that any property that is stalk-local (a scheme has property P if and only if all its stalks have property Q) is necessarily affine-local (a scheme has property P if and only if all of its affines have property R, where an affine scheme has property R if and only if and only if all its stalks have property Q), but it is sometimes not so obvious what the right definition of Q is; see for example the discussion of normality in the next section.

Proof. Let $\operatorname{Spec} A$ be an affine subscheme of X. Cover $\operatorname{Spec} A$ with a finite number of distinguished opens $\operatorname{Spec} A_{g_j}$, each of which is distinguished in some $\operatorname{Spec} A_i$. This is possible by Proposition 3.2 and the quasicompactness of $\operatorname{Spec} A$. By (i), each $\operatorname{Spec} A_{g_j}$ has P. By (ii), $\operatorname{Spec} A$ has P. □

By choosing property P appropriately, we define some important properties of schemes.

3.4. *Proposition. — Suppose* A *is a ring, and* $(f_1, \ldots, f_n) = A$.

(a) *If* A *is a Noetherian ring, then so is* A_{f_i}*. If each* A_{f_i} *is Noetherian, then so is* A.
(b) *If* A *is reduced, then* A_{f_i} *is also reduced. If each* A_{f_i} *is reduced, then so is* A.
(c) *Suppose* B *is a ring, and* A *is an* B*-algebra. (Hence* A_g *is a* B*-algebra for all* B*.) If* A *is a finitely generated* B*-algebra, then so is* A_{f_i}*. If each* A_{f_i} *is a finitely-generated* B*-algebra, then so is* A.

We'll prove these shortly. But let's first motivate you to read the proof by giving some interesting definitions *assuming* Proposition 3.4 is true.

3.5. Important Definitions. Suppose X is a scheme. If X can be covered by affine opens $\operatorname{Spec} A$ where A is Noetherian, we say that X is a **locally Noetherian scheme**. If in addition X is quasicompact, or equivalently can be covered by finitely many such affine opens, we say that X is a **Noetherian scheme**. By Exercise 1.C, the underlying topological space of a Noetherian scheme is Noetherian. (We will see a number of definitions of the form "if X has this property, we say that it is locally Q; if further X is compact, we say that it is Q.")

3.A. EXERCISE. Show that all open subsets of a Noetherian topological space (hence a Noetherian scheme) are quasicompact.

3.B. EXERCISE. Show that locally Noetherian schemes are quasiseparated.

3.C. EXERCISE. Show that a Noetherian scheme has a finite number of irreducible components. Show that a Noetherian scheme has a finite number of connected components, each a finite union of irreducible components.

3.D. EXERCISE. If X is a Noetherian scheme, show that every point p has a closed point in its closure. (In particular, every non-empty Noetherian scheme has closed points; this is not true for every scheme, as remarked in Exercise 1.H.)

3.E. EXERCISE. If X is an affine scheme or Noetherian scheme, show that it suffices to check reducedness at *closed points*. (Hint: For the Noetherian case, recall Exercise 3.D.)

Integrality is not stalk-local, but it almost is, as is shown in the following believable exercise.

3.F. UNIMPORTANT EXERCISE. Show that a locally Noetherian scheme X is integral if and only if X is connected and all stalks $\mathcal{O}_{X,p}$ are integral domains (informally: "the scheme is locally integral"). Thus in "good situations" (when the scheme is Noetherian), integrality is the union of local (stalks are domains) and global (connected) conditions.

3.6. *Remark.* Joe Rabinoff gave a great example showing that "locally Noetherian" is not a stalk-local condition. Joe's counterexample: Let k be an algebraically closed field, let $b_1, b_2, b_3, \ldots \in k$ be a sequence of distinct elements, and let

$$A = k[s, a_1, a_2, \ldots]/((s - b_i)a_{i+1} - a_i, a_i^2)_{i=1,2,\ldots}$$

I claim that A is not noetherian, but that A_p is noetherian for every prime ideal. It suffices to check for maximal ideals, as Noetherianness is preserved by localization.. The nilradical $\mathfrak{N}$ of A is $(a_1, a_2, \ldots)$ (as the a_i clearly lie in the nilradical, and $A/(a_1, \ldots)$ is a domain so we've found it all), and $A/\mathfrak{N} = k[s]$, so the maximal ideals of A are the ideals of the form $\mathfrak{m} = (s - b, a_1, a_2, \ldots)$ for $b \in k$ Let $\mathfrak{m}$ be such an ideal.

- Suppose that $b = b_n$ for some n. For $i \neq n$, we have $a_{i+1} = a_i/(s - b_i)$ in $A_\mathfrak{m}$. Hence $A_\mathfrak{m}$ is the localization of a ring generated by the two variables s and a_n, so it's Noetherian.
- If b is distinct from all the b_i, then $A_\mathfrak{m}$ is the localization of a ring generated by s and a_1, as above.

Hence all stalks are Noetherian, but clearly the nilradical of A is not finitely generated.

3.G. EXERCISE. Show that X is reduced if and only if X can be covered by affine opens $\operatorname{Spec} A$ where A is reduced (nilpotent-free).

Our earlier definition required us to check that the ring of functions over *any* open set is nilpotent free. Our new definition lets us check a single affine cover. Hence for example $\mathbb{A}^n_k$ and $\mathbb{P}^n_k$ are reduced.

Suppose X is a scheme, and A is a ring (e.g. A is a field k), and $\Gamma(U, \mathcal{O}_X)$ has an A-algebra for all A, and the restriction maps respect the A-algebra structure. Then we say that X is an **A-scheme**, or a **scheme over** A. Suppose X is an A-scheme. If X can be covered by affine opens $\operatorname{Spec} B_i$ where each B_i is a *finitely generated* A-algebra, we say that X is **locally of finite type over** A, or that it is a **locally of finite type** A**-scheme**. (This is admittedly cumbersome terminology; it will make more sense later, once we know about morphisms.) If furthermore X is quasicompact, X is **finite type over** A, or a **finite type** A**-scheme**. Note that a scheme locally of finite type over k or $\mathbb{Z}$ (or indeed any Noetherian ring) is locally Noetherian, and similarly a scheme of finite type over any Noetherian ring is Noetherian. As our key "geometric" example: if $I \subset \mathbb{C}[x_1, \ldots, x_n]$ is an ideal, then $\operatorname{Spec} \mathbb{C}[x_1, \ldots, x_n]/I$ is a finite-type $\mathbb{C}$-scheme.

3.7. We now make a definition to make a connection to the language of varieties. An affine scheme that is reduced and finite type k-scheme is said to be an *affine variety (over* k*)*, or an *affine* k*-variety*. We are not yet ready to define varieties in general; we will need the notion of separatedness first, to exclude abominations of nature like the line with the doubled origin. We will define projective k-varieties before defining varieties in general (as separated finite type k-schemes). (Warning: in the literature, it is sometimes also required that the scheme be irreducible, or that k be algebraically closed.)

3.H. EXERCISE. Show that a point of a locally finite type k-scheme is a closed point if and only if the residue field of the stalk of the structure sheaf at that point is a finite extension of k. (Recall the following form of Hilbert's Nullstellensatz, richer than the version stated before: the maximal ideals of $k[x_1, \ldots, x_n]$ are precisely those with residue of the form a finite extension of k.) Show that the closed points are dense on such a scheme. (For another exercise on closed points, see 1.H.)

3.8. *Proof of Proposition 3.4.* (a) (i) If $I_1 \subsetneq I_2 \subsetneq I_3 \subsetneq \cdots$ is a strictly increasing chain of ideals of A_f, then we can verify that $J_1 \subsetneq J_2 \subsetneq J_3 \subsetneq \cdots$ is a strictly increasing chain of ideals of A, where

$$J_j = \{r \in A : r \in I_j\}$$

where $r \in I_j$ means "the image in A_f lies in I_j". (We think of this as $I_j \cap A$, except in general A needn't inject into A_{f_i}.) Clearly J_j is an ideal of A. If $x/f^n \in I_{j+1} \setminus I_j$ where $x \in A$, then $x \in J_{j+1}$, and $x \notin J_j$ (or else $x(1/f)^n \in J_j$ as well). (ii) Suppose $I_1 \subsetneq I_2 \subsetneq I_3 \subset \cdots$ is a strictly increasing chain of ideals of A. Then for each $1 \leq i \leq n$,

$$I_{i,1} \subset I_{i,2} \subset I_{i,3} \subset \cdots$$

is an increasing chain of ideals in A_{f_i}, where $I_{i,j} = I_j \otimes_A A_{f_i}$. It remains to show that for each j, $I_{i,j} \subsetneq I_{i,j+1}$ for some i; the result will then follow.

3.I. EXERCISE. Finish this argument.

3.J. EXERCISE. Prove (b).

(c) (i) is clear: if A is generated over B by $r_1, \ldots, r_n$, then A_f is generated over B by $r_1, \ldots, r_n, 1/f$.

(ii) Here is the idea. We have generators of A_i: r_{ij}/f_i^j, where $r_{ij} \in A$. I claim that $\{r_{ij}\}_{ij} \cup \{f_i\}_i$ generate A as a B-algebra. Here's why. Suppose you have any $r \in A$. Then in A_{f_i}, we can write r as some polynomial in the r_{ij}'s and f_i, divided by some huge power of f_i. So "in each A_{f_i}, we have described r in the desired way", except for this annoying denominator. Now use a partition of unity type argument to combine all of these into a single expression, killing the denominator. Show that the resulting expression you build still agrees with r in each of the A_{f_i}. Thus it is indeed r.

3.K. EXERCISE. Make this argument precise.

This concludes the proof of Proposition 3.4 □

4. NORMALITY AND FACTORIALITY

4.1. Normality.

We can now define a property of schemes that says that they are "not too far from smooth", called *normality*, which will come in very handy. We will see later that "locally Noetherian normal schemes satisfy Hartogs' theorem": functions defined away form a set of codimension 2 extend over that set, (2) Rational functions that have no poles are defined everywhere. We need definitions of dimension and/or poles to make this precise.

A scheme X is **normal** if all of its stalks $\mathcal{O}_{X,x}$ are normal (i.e. are domains, and integrally closed in their fraction fields). As reducedness is a stalk-local property (Exercise 2.B), normal schemes are reduced.

4.A. EXERCISE. Show that integrally closed domains behave well under localization: if A is an integrally closed domain, and S is a multiplicative subset, show that $S^{-1}A$ is an integrally closed domain. (The domain portion is easy. Hint for integral closure: assume that $x^n + a_{n-1}x^{n-1} + \cdots + a_0 = 0$ where $a_i \in S^{-1}A$ has a root in the fraction field. Turn this into another equation in A[x] that also has a root in the fraction field.)

It is no fun checking normality at every single point of a scheme. Thanks to this exercise, we know that if A is an integrally closed domain, then $\operatorname{Spec} A$ is normal. Also, for Noetherian schemes, normality can be checked at closed points, thanks to this exercise, and the fact that for such schemes, any point is a generization of a closed point (see Exercise 3.D)

It is not true that normal schemes are integral. For example, the disjoint union of two normal schemes is normal. Thus $\operatorname{Spec} k \coprod \operatorname{Spec} k \cong \operatorname{Spec}(k \times k) \cong \operatorname{Spec} k[x]/(x(x-1))$ is normal, but its ring of global sections is not a domain.

4.B. UNIMPORTANT EXERCISE. Show that a Noetherian scheme is normal if and only if it is the finite disjoint union of integral Noetherian normal schemes.

We are close to proving a useful result in commutative algebra, so we may as well go all the way.

4.2. *Proposition. — If A is an integral domain, then the following are equivalent.*

(1) A *integrally closed.*
(2) $A_{\mathfrak{p}}$ *is integrally closed for all prime ideals* $\mathfrak{p} \subset A$.
(3) $A_{\mathfrak{m}}$ *is integrally closed for all maximal ideals* $\mathfrak{m} \subset A$.

Proof. Clearly (2) implies (3). Exercise 4.A shows that integral closure is preserved by localization, so (1) implies (2).

It remains to show that (3) implies (1). This argument involves a very nice construction that we will use again. Suppose A is not integrally closed. We show that there is some $\mathfrak{m}$ such that $A_{\mathfrak{m}}$ is also not integrally closed. Suppose

$$x^n + a_{n-1}x^{n-1} + \cdots + a_0 = 0 \tag{1}$$

(with $a_i \in A$) has a solution s in $\mathrm{FF}(A)$. Let I be the **ideal of denominators of** s:

$$I := \{r \in A : rs \in A\}.$$

(Note that I is clearly an ideal of A.) Now $I \neq A$, as $1 \notin I$. Thus there is some maximal ideal $\mathfrak{m}$ containing I. Then $s \notin A_{\mathfrak{m}}$, so equation (1) in $A_{\mathfrak{m}}[x]$ shows that $A_{\mathfrak{m}}$ is not integrally closed as well, as desired. □

4.C. UNIMPORTANT EXERCISE. If A is an integral domain, show that $A = \cap A_{\mathfrak{m}}$, where the intersection runs over all maximal ideals of A. (We won't use this exercise, but it gives good practice with the ideal of denominators.)

4.D. UNIMPORTANT EXERCISE RELATING TO THE IDEAL OF DENOMINATORS. One might naively hope from experience with unique factorization domains that the ideal of denominators is principal. This is not true. As a counterexample, consider our new friend

$A = k[a, b, c, d]/(ad - bc)$ (which we will later recognize as the cone over the quadric surface), and $a/c = b/d \in FF(A)$. Show that $I = (c, d)$.

4.3. Factoriality.

We define a notion which implies normality.

4.4. Definition. If all the stalks of a scheme X are unique factorization domains, we say that X is **factorial**.

4.E. EXERCISE. Show that any localization of a Unique Factorization Domain is a Unique Factorization Domain.

Thus if A is a unique factorization domain, then $\operatorname{Spec} A$ is factorial. (The converse need not hold. Hence this property is *not* affine-local, as we will verify later. Here is a counterexample without proof: $\mathbb{Z}[\sqrt{17}]$.) Hence it suffices to check factoriality by finding an appropriate affine cover.

One of the reasons we like factoriality is that it implies normality.

4.F. IMPORTANT EXERCISE. Show that unique factorization domains are integrally closed. Hence factorial schemes are are normal, and if A is a unique factorization domain, then $\operatorname{Spec} A$ is normal. (However, rings can be integrally closed without being unique factorization domains, as we'll see in Exercise 4.I. An example without proof: $\mathbb{Z}[\sqrt{17}]$ again.)

4.G. EASY EXERCISE. Show that the following schemes are normal: $\mathbb{A}^n_k$, $\mathbb{P}^n_k$, $\operatorname{Spec} \mathbb{Z}$.

4.H. EXERCISE (WHICH WILL GIVE US A NUMBER OF ENLIGHTENING EXAMPLES LATER). Suppose A is a Unique Factorization Domain with 2 invertible, $f \in A$ has no repeated prime factors, and $z^2 - f$ is irreducible in $A[z]$. Show that $\operatorname{Spec} A[z]/(z^2 - f)$ is normal. Show that if f is *not* square-free, then $\operatorname{Spec} A[z]/(z^2 - f)$ is *not* normal. (Hint: $B := A[z]/(z^2 - f)$ is a domain, as $(z^2 - f)$ is prime in $A[z]$. Suppose we have monic $F(T) = 0$ with $F(T) \in B[T]$ which has a solution α in $FF(B)$. Then by replacing $F(T)$ by $\overline{F}(T)F(T)$, we can assume $F(T) \in A[T]$. Also, $\alpha = g + hz$ where $g, h \in FF(A)$. Now α is the solution of monic $Q(T) = T^2 - 2gT + (g^2 - h^2 f)T \in FF(A)[T]$, so we can factor $F(T) = P(T)Q(T)$ in $K[T]$. By Gauss' lemma, $2g$, $g^2 - h^2 f \in A$. Say $g = r/2$, $h = s/t$ (s and t have no common factors, $r, s, t \in A$). Then $g^2 - h^2 f = (r^2 t^2 - r s^2 f)/4t^2$. Then $t = 1$, and r is even.)

4.I. EXERCISE. Show that the following schemes are normal:

(a) $\operatorname{Spec} \mathbb{Z}[x]/(x^2 - n)$ where n is a square-free integer congruent to $3 \pmod 4$;
(b) $\operatorname{Spec} k[x_1, \dots, x_n]/x_1^2 + x_2^2 + \dots + x_m^2$ where $\operatorname{char} k \neq 2$, $m \geq 3$;

(c) $\operatorname{Spec} k[w, x, y, z]/(wz - xy)$ where $\operatorname{char} k \neq 2$ and k is algebraically closed. (This is our cone over a quadric surface example from Exercise 4.D.)

4.J. EXERCISE. Suppose A is a k-algebra where $\operatorname{char} k = 0$, and l/k is a finite field extension. Show that A is normal if and only if $A \otimes_k l$ is normal. Show that $\operatorname{Spec} k[w, x, y, z]/(wz - xy)$ is normal if k is characteristic 0. (In fact the hypothesis on the characteristic is unnecessary.) Possible hint: reduce to the case where l/k is Galois.

5. ASSOCIATED POINTS OF (LOCALLY NOETHERIAN) SCHEMES, AND DRAWING FUZZY PICTURES

Recall from just after Definition 2.1 (of *reduced*) our "fuzzy" pictures of the non-reduced scheme $\operatorname{Spec} k[x, y]/(y^2, xy)$ (see Figure 1). When this picture was introduced, we mentioned that the "fuzz" at the origin indicated that the non-reduced behavior was concentrated there; this was verified in Exercise 2.A, and indeed the origin is the only point where the stalk of the structure sheaf is non-reduced.

You might imagine that in a bigger scheme, we might have different closed subsets with different amount of "non-reducedness". This intuition will be made precise in this section. We will define *associated points* of a scheme, which will be the most important points of a scheme, encapsulating much of the interesting behavior of the structure sheaf. These will be defined for any locally Noetherian scheme. The primes corresponding to the associated points of an affine scheme $\operatorname{Spec} A$ will be called *associated primes of* A. (In fact this is backwards; we will define associated primes first, and then define associated points.)

The four properties about associated points that it will be most important to remember are as follows. Frankly, it is much more important to remember these four facts than it is to remember their proofs.

(1) *The generic points of the irreducible components are associated points.* The other associated points are called **embedded points**.

(2) *If X is reduced, then X has no embedded points.* (This jibes with the intuition of the picture of associated points described earlier.)

(3) Recall that one nice property of integral schemes X (such as irreducible affine varieties) not shared by all schemes is that for any open $U \subset X$, the natural map $\Gamma(U, \mathcal{O}_X) \to FF(X)$ is an inclusion (Exercise 2.G). Thus all sections over any open set (except $\varnothing$) and stalks can be thought of as lying in a single field $FF(X)$, which is the talk at the generic point.

More generally, if X is a locally Noetherian scheme, then for any $U \subset X$, the natural map

$$\Gamma(U, \mathcal{O}_X) \to \prod_{\text{associated } p \text{ in } U} \mathcal{O}_{X,p} \tag{2}$$

is an injection.

We define a **rational function** on a locally Noetherian scheme to be an element of the image of $\Gamma(U, \mathcal{O}_U)$ in (2) for some U containing all the associated points. The rational functions form a ring, called the **total fraction ring** of X, denoted $FF(X)$. If $X = \operatorname{Spec} A$ is affine, then this ring is called the **total fraction ring** of A, $FF(A)$. Note that if X is integral, this is the function field $FF(X)$, so this extends our earlier definition 2.F of $FF(\cdot)$. It can be more conveniently interpreted as follows, using the injectivity of (2). A rational function is a function defined on an open set containing all associated points, i.e. and ordered pair (U, f), where U is an open set containing all associated points, and $f \in \Gamma(U, \mathcal{O}_X)$. Two such data (U, f) and (U', f') define the same open rational function if and only if the restrictions of f and f' to $U \cap U'$ are the same. If X is reduced, this is the same as requiring that they are defined on an open set of each of the irreducible components. A rational function has a maximal domain of definition, because any two actual functions on an open set (i.e. sections of the structure sheaf over that open set) that agree as "rational functions" (i.e. on small enough open sets containing associated points) must be the same function, by the injectivity of (2). We say that a rational function f is **regular** at a point p if p is contained in this maximal domain of definition (or equivalently, if there is some open set containing p where f is defined).

The previous facts are intimately related to the following one.

(4) *A function on X is a zero divisor if and only if it vanishes at an associated point of X.*

Motivated by the above four properties, when sketching (locally Noetherian) schemes, we will draw the irreducible components (the closed subsets corresponding to maximal associated points), and then draw "additional fuzz" precisely at the closed subsets corresponding to embedded points. All of our earlier sketches were of this form.

Let's now get down to business of defining associated points, and showing that they the desired properties **(1)** through **(4)**.

We say an ideal $I \subset A$ in a ring is **primary** if $I \neq A$ and if $xy \in I$ implies either $x \in I$ or $y^n \in I$ for some $n > 0$.

It is useful to interpret maximal ideals as "the quotient is a field", and prime ideals as "the quotient is an integral domain". We can interpret primary ideals similarly as "the quotient is not 0, and every zero-divisor is nilpotent".

5.A. EXERCISE. Show that if $\mathfrak{q}$ is primary, then $\sqrt{\mathfrak{q}}$ is prime. If $\mathfrak{p} = \sqrt{\mathfrak{q}}$, we say that $\mathfrak{q}$ is $\mathfrak{p}$-*primary*. (Caution: $\sqrt{\mathfrak{q}}$ can be prime without $\mathfrak{q}$ being primary — consider our example (y^2, xy) in $k[x, y]$.)

5.B. EXERCISE. Show that if $\mathfrak{q}$ and $\mathfrak{q}'$ are $\mathfrak{p}$-primary, then so is $\mathfrak{q} \cap \mathfrak{q}'$.

5.C. EXERCISE (REALITY CHECK). Find all the primary ideals in $\mathbb{Z}$. (Answer: (0) and (p^n).)

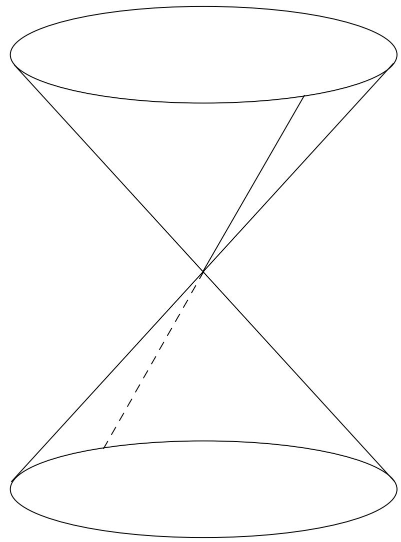

FIGURE 3. $V(x,z) \subset \operatorname{Spec} k[x,y,z]/(xy-z^2)$ is a ruling on a cone; $(x,z)^2$ is not (x,z)-primary.

5.1. ⋆ *Unimportant warning for experts (all others should skip this).* A prime power need not be primary. For example, let $A = k[x,y,z]/(xy-z^2)$, and $\mathfrak{p} = (x,z)$. Then $\mathfrak{p}$ is prime but $\mathfrak{p}^2$ is not primary. (Verify this — the algebra is easy! Why is $(x^2, xz, z^2, xy-z^2)$ not primary in $k[x,y,z]$?) We will soon be able to interpret $\operatorname{Spec} A$ as a "cone", and $V(x,z)$ as the "ruling" of the cone, see Figure 3, and the corresponding picture gives a geometric hint that there is something going on. We'll come back to this at a later date.

5.2. Primary decompositions.

A **primary decomposition** of an ideal $I \subset A$ is an expression of the ideal as a finite intersection of primary ideals.

$$I = \cap_{i=1}^n \mathfrak{q}_i$$

If there are "no redundant elements" (the $\sqrt{\mathfrak{q}_i}$ are all distinct, and for no i is $\mathfrak{q}_i \supset \cap_{j \neq i} \mathfrak{q}_j$), we say that the decomposition is **minimal**. Clearly any ideal with a primary decomposition has a minimal primary decomposition (using Exercise 5.B).

5.D. IMPORTANT EXERCISE (EXISTENCE OF PRIMARY DECOMPOSITION FOR NOETHERIAN RINGS). Suppose A is a Noetherian ring. Show that every proper ideal $I \subset A$ has a primary decomposition. (Hint: mimic the Noetherian induction argument we saw last week.)

5.E. IMPORTANT EXERCISE. (a) Find a minimal primary decomposition of (y^2, xy). (b) Find another one. (Possible hint: see Figure 1. You might be able to draw sketches of your different primary decompositions.)

In order to study these objects, we'll need a useful fact and a definition.

5.F. ESSENTIAL EXERCISE. (a) If $\mathfrak{p}$, $\mathfrak{p}_1$, ..., $\mathfrak{p}_n$ are prime ideals, and $\mathfrak{p} = \cap \mathfrak{p}_i$, show that $\mathfrak{p} = \mathfrak{p}_i$ for some i. (Hint: assume otherwise, choose $f_i \in \mathfrak{p}_i - \mathfrak{p}$, and consider $\prod f_i$.)
(b) If $\mathfrak{p} \supset \cap \mathfrak{p}_i$, then $\mathfrak{p} \supset \mathfrak{p}_i$ for some i.
(c) Suppose $I \subseteq \cup_{i=1}^n \mathfrak{p}_i$. (The right side is not an ideal!) Show that $I \subset \mathfrak{p}_i$ for some i. (Hint: by induction on n. Don't look in the literature — you might find a much longer argument!)

Parts (a) and (b) are "geometric facts"; try to draw pictures of what they mean.

If $I \subset A$ is an ideal, and $x \in A$, then define the **colon ideal** $(I : x) := \{a \in A : ax \in I\}$. (We will use this terminology only for this section.) For example, x is a *zero-divisor* if $(0 : x) \neq 0$.

5.3. *Theorem ("uniqueness" of primary decomposition). — Suppose $I \subset A$ has a minimal primary decomposition*

$$I = \cap_{i=1}^n \mathfrak{q}_i.$$

(For example, this is always true if A is Noetherian.) Then the $\sqrt{\mathfrak{q}_i}$ are precisely the prime ideals that are of the form

$$\sqrt{(I : x)}$$

for some $x \in A$. Hence this list of primes is independent of the decomposition.

These primes are called the **associated primes** of the ideal I. The **associated primes of** A are the associated primes of 0.

Proof. We make a very useful observation: for any $x \in A$,

$$(I : x) = (\cap \mathfrak{q}_i : x) = \cap (\mathfrak{q}_i : x),$$

from which

(3) $$\sqrt{(I : x)} = \cap \sqrt{(\mathfrak{q}_i : x)} = \cap_{x \notin \mathfrak{q}_j} \mathfrak{p}_j.$$

Now we prove the result.

Suppose first that $\sqrt{(I : x)}$ is prime, say $\mathfrak{p}$. Then $\mathfrak{p} = \cap_{x \notin \mathfrak{q}_j} \mathfrak{p}_j$ by (3), and by Exercise 5.F(a), $\mathfrak{p} = \mathfrak{p}_j$ for some j.

Conversely, given $\mathfrak{q}_i$, we find an x such that $\sqrt{(I : x)} = \sqrt{\mathfrak{q}_i}$ ($= \mathfrak{p}_i$). Take $x \in \cap_{j \neq i} \mathfrak{q}_j - \mathfrak{q}_i$ (which is possible by minimality of the primary decomposition). Then by (3), we're done. □

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FOUNDATIONS OF ALGEBRAIC GEOMETRY CLASSES 11 AND 12

RAVI VAKIL

CONTENTS

1. ASSOCIATED POINTS CONTINUED

Recall the four key facts to remember about associated points.

(1) *The generic points of the irreducible components are associated points.* The other associated points are called **embedded points**.

(2) *If X is reduced, then X has no embedded points.*

(3) If X is a locally Noetherian scheme, then for any $U \subset X$, the natural map

$$\Gamma(U, \mathcal{O}_X) \to \prod_{\text{associated } p \text{ in } U} \mathcal{O}_{X,p} \tag{1}$$

is an injection.

We define a **rational function** on a locally Noetherian scheme to be an element of the image of $\Gamma(U, \mathcal{O}_U)$ in (1) for some U containing all the associated points. The rational functions form a ring, called the **total fraction ring** of X, denoted $\mathrm{FF}(X)$. If $X = \operatorname{Spec} A$ is affine, then this ring is called the **total fraction ring** of A, $\mathrm{FF}(A)$.

(4) *A function on X is a zero divisor if and only if it vanishes at an associated point of X.*

Recall that an ideal $I \subset A$ in a ring is **primary** if $I \neq A$ and if $xy \in I$ implies either $x \in I$ or $y^n \in I$ for some $n > 0$. In other words, the quotient is not 0, and every zero-divisor is nilpotent. Hence the notion of "primary" should be seen as a condition on A/I, not on I.

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We know that if $\mathfrak{q}$ is primary, then $\sqrt{\mathfrak{q}}$ is prime, say $\mathfrak{p}$. We then say that $\mathfrak{q}$ is $\mathfrak{p}$-*primary*. We know that if $\mathfrak{q}$ and $\mathfrak{q}'$ are $\mathfrak{p}$-primary, then so is $\mathfrak{q} \cap \mathfrak{q}'$.

We also know that primary decompositions, and hence minimal primary decompositions, exist for any ideal of a Noetherian ring.

We proved:

1.1. *Theorem ("uniqueness" of primary decomposition). — Suppose* $I \subset A$ *has a minimal primary decomposition*

$$I = \cap_{i=1}^{n} \mathfrak{q}_i.$$

(For example, this is always true if A *is Noetherian.) Then the* $\sqrt{\mathfrak{q}_i}$ *are precisely the prime ideals that are of the form*

$$\sqrt{(I : x)}$$

for some $x \in A$*. Hence this list of primes is independent of the decomposition.*

These primes are called the **associated primes** of the ideal I. The **associated primes of** A are the associated primes of 0.

The proof involved the handy line

$$\sqrt{(I : x)} = \cap \sqrt{(\mathfrak{q}_i : x)} = \cap_{x \notin \mathfrak{q}_j} \mathfrak{p}_j. \tag{2}$$

So let's move forward!

1.A. EXERCISE (ASSOCIATED PRIMES BEHAVE WELL WITH RESPECT TO LOCALIZATION). Show that if A is a Noetherian ring, and S is a multiplicative subset (so there is an inclusion-preserving correspondence between the primes of $S^{-1}A$ and those primes of A not meeting S), then the associated primes of $S^{-1}A$ are just the associated primes of A not meeting S.

We then define the **associated points** of a locally Noetherian scheme X to be those points $p \in X$ such that, on any affine open set $\operatorname{Spec} A$ containing p, p corresponds to an associated prime of A. Note that this notion is well-defined: If p has two affine open neighborhoods $\operatorname{Spec} A$ and $\operatorname{Spec} B$ (say corresponding to primes $\mathfrak{p} \subset A$ and $\mathfrak{q} \subset B$ respectively), then p corresponds to an associated prime of A if and only if it corresponds to an associated prime of $A_{\mathfrak{p}} = \mathcal{O}_{X,p} = B_{\mathfrak{q}}$ if and only if it corresponds to an associated prime of B.

If furthermore X is quasicompact (i.e. X is a Noetherian scheme), then there are a finite number of associated points.

1.B. EXERCISE. (a) Show that the minimal primes of A are associated primes. We have now proved important fact **(1)**. (Hint: suppose $\mathfrak{p} \supset \cap_{i=1}^{n} \mathfrak{q}_i$. Then $\mathfrak{p} = \sqrt{\mathfrak{p}} \supset \sqrt{\cap_{i=1}^{n} \mathfrak{q}_i} = \cap_{i=1}^{n} \sqrt{\mathfrak{q}_i} = \cap_{i=1}^{n} \mathfrak{p}_i$, so by a previous exercise, $\mathfrak{p} \supset \mathfrak{p}_i$ for some i. If $\mathfrak{p}$ is minimal, then as $\mathfrak{p} \supset \mathfrak{p}_i \subset (0)$, we must have $\mathfrak{p} = \mathfrak{p}_i$.)

(b) Show that there can be other associated primes that are not minimal. (Hint: we've seen an example...) Your argument will show more generally that the minimal primes of I are associated primes of I.

1.C. EXERCISE. Show that if A is reduced, then the only associated primes are the minimal primes. (This establishes **(2)**.)

The $\mathfrak{q}_i$ corresponding to minimal primes are unique, but the $\mathfrak{q}_i$ corresponding to other associated primes are not unique. We will not need this fact, and hence won't prove it.

1.2. *Proposition. — The set of zero-divisors is the union of the associated primes.*

This establishes **(4)**: a function is a zero-divisor if and only if it vanishes at an associated point. Thus (for a Noetherian scheme) a function is a zero divisor if and only if its zero locus contains one of a finite set of points.

You may wish to try this out on the example of the affine line with fuzz at the origin.

Proof. If $\mathfrak{p}_i$ is an associated prime, then $\mathfrak{p}_i = \sqrt{(0 : x)}$ from the proof of Theorem 1.1, so $\cup \mathfrak{p}_i$ is certainly contained in the set Z of zero-divisors.

For the converse:

1.D. EXERCISE. Show that

$$Z = \cup_{x \neq 0}(0 : x) \subseteq \cup_{x \neq 0} \sqrt{(0 : x)} \subseteq Z.$$

Hence

$$Z = \cup_{x \neq 0} \sqrt{(0 : x)} = \cup_x \left(\cap_{x \notin \mathfrak{q}_j} \mathfrak{p}_j \right) \subseteq \cup \mathfrak{p}_j$$

using (2). □

1.E. UNIMPORTANT EXERCISE (RABINOFF'S THEOREM). Here is an interesting variation on **(4)**: show that $a \in A$ is nilpotent if and only if it vanishes at the associated points of $\operatorname{Spec} A$. Algebraically: we know that the nilpotents are the intersection of *all* prime ideals; now show that in the Noetherian case, the nilpotents are in fact the intersection of the (finite number of) associated prime ideals.

1.3. *Proposition. — The natural map* $A \to \prod_{\text{associated } \mathfrak{p}} A_{\mathfrak{p}}$ *is an inclusion.*

Proof. Suppose $r \neq 0$ maps to 0 under this map. Then there are $s_i \in A - \mathfrak{p}$ with $s_i r = 0$. Then $I := (s_1, \ldots, s_n)$ is an ideal consisting only of zero-divisors. Hence $I \subseteq \cup \mathfrak{p}_i$. Then $I \subset \mathfrak{p}_i$ for some i by an exercise from last week, contradicting $s_i \notin \mathfrak{p}_i$. □

1.F. EASIER AND LESS IMPORTANT EXERCISE. Prove fact **(3)**. (The previous Proposition establishes it for affine open sets.)

2. INTRODUCTION TO MORPHISMS OF SCHEMES

Whenever you learn about a new type of object in mathematics, you should naturally be curious about maps between them, which means understanding how they form a category. In order to satisfy this curiosity, we'll introduce the notion of morphism of schemes now, and at the same time we may as well define some easy-to-state properties of morphisms. However, we'll leave more subtle properties of morphisms for next quarter.

Recall that a scheme is (i) a set, (ii) with a topology, (iii) and a (structure) sheaf of rings, and that it is sometimes helpful to think of the definition as having three steps. In the same way, the notion of morphism of schemes $X \to Y$ may be defined (i) as a map of sets, (ii) that is continuous, and (iii) with some further information involving the sheaves of functions. In the case of affine schemes, we have already seen the map as sets, and later saw that this map is continuous.

Here are two motivations for how morphisms should behave. The first is algebraic, and the second is geometric.

(a) We'll want morphisms of affine schemes $\operatorname{Spec} B \to \operatorname{Spec} A$ to be precisely the ring maps $A \to B$. We have already seen that ring maps $A \to B$ induce maps of topological spaces in the opposite direction; the main new ingredient will be to see how to add the structure sheaf of functions into the mix. Then a morphism of schemes should be something that "on the level of affines, looks like this".

(b) We are also motivated by the theory of differentiable manifolds. Notice that if $\pi : X \to Y$ is a map of differentiable manifolds, then a differentiable function on Y pulls back to a differentiable function on X. More precisely, given an open subset $U \subset Y$, there is a natural map $\Gamma(U, \mathcal{O}_Y) \to \Gamma(\pi^{-1}(U), \mathcal{O}_X)$. This behaves well with respect to restriction (restricting a function to a smaller open set and pulling back yields the same result as pulling back and then restricting), so in fact we have a map of sheaves on Y: $\mathcal{O}_Y \to \pi_* \mathcal{O}_X$. Similarly a morphism of schemes $X \to Y$ should induce a map $\mathcal{O}_Y \to \pi_* \mathcal{O}_X$. But in fact in the category of differentiable manifolds a continuous map $X \to Y$ is a map of differentiable manifolds precisely when differentiable functions on Y pull back to differentiable functions on X (i.e. the pullback map from differentiable functions on Y to *functions* on X in fact lies in the subset of *differentiable functions*, i.e. the continuous map $X \to Y$ induces a pullback of differential functions $\mathcal{O}_Y \to \mathcal{O}_X$), so this map of sheaves *characterizes* morphisms in the differentiable category. So we could use this as the *definition* of morphism in the differentiable category.

But how do we apply this to the category of schemes? In the category of differentiable manifolds, a continuous map $X \to Y$ *induces* a pullback of (the sheaf of) functions, and we can ask when this induces a pullback of *differentiable* functions. However, functions are odder on schemes, and we can't recover the pullback map just from the map of topological spaces. A reasonable patch is to hardwire this into the definition of morphism, i.e. to have a continuous map $f : X \to Y$, along with a pullback map $f^\# : \mathcal{O}_Y \to f_*\mathcal{O}_X$. This leads to the definition of the category of ringed spaces.

One might hope to define morphisms of schemes as morphisms of ringed spaces. This isn't quite right, as then motivation (a) isn't satisfied: as desired, to each morphism $A \to B$ there is a morphism $\operatorname{Spec} B \to \operatorname{Spec} A$, but there can be additional morphisms of ringed spaces $\operatorname{Spec} B \to \operatorname{Spec} A$ not arising in this way (Exercise 3.C). A revised definition as morphisms of ringed spaces that locally looks of this form will work, but this is awkward to work with, and we take a different tack. However, we will check that our eventual definition actually is equivalent to this.

We'll begin by discussing morphisms of ringed spaces.

Before we do, we take this opportunity to use motivation (a) to motivate the definition of *equivalence of categories*. We wish to say that the category of rings and the category of affine schemes are opposite categories, i.e. that the "opposite category of affine schemes" (where all the arrows are reversed) is "essentially the same" as the category of rings. We indeed have a functor from rings to affine schemes (sending A to $\operatorname{Spec} A$), and a functor from affine schemes to rings (sending X to $\Gamma(X, \mathcal{O}_X)$). But if you think about it, you'll realize their composition isn't exactly the identity. (It all boils down to the meaning of "is" or "same", and this can get confusing.) Rather than trying to set things up so the composition *is* the identity, we just don't let this bother us, and make precise the notion that the composition is "essentially" the identity.

Suppose F and G are two functors from $\mathcal{A}$ to $\mathcal{B}$. A **natural transformation of functors** $F \to G$ is the data of a morphism $m_a : F(a) \to G(a)$ for each $a \in \mathcal{A}$ such that for each $f : a \to a'$ in $\mathcal{A}$, the diagram

$$\begin{array}{ccc} F(a) & \xrightarrow{F(f)} & F(a') \\ {\scriptstyle m_a}\downarrow & & \downarrow{\scriptstyle m_{a'}} \\ G(a) & \xrightarrow[G(f)]{} & G(a') \end{array}$$

A **natural isomorphism** of functors is a natural transformation such that each m_a is an isomorphism. The data of functors $F : \mathcal{A} \to \mathcal{B}$ and $F' : \mathcal{B} \to \mathcal{A}$ such that $F \circ F'$ is naturally isomorphic to the identity $I_{\mathcal{B}}$ on $\mathcal{B}$ and $F' \circ F$ is naturally isomorphic to $I_{\mathcal{A}}$ is said to be an **equivalence of categories**. This is the "right" notion of isomorphism of categories.

Two examples might make this strange concept more comprehensible. The double dual of a finite-dimensional vector space V is *not* V, but we learn early to say that it is canonically isomorphic to V. We make can that precise as follows. Let **f.d.** $\mathbf{Vec}_k$ be the category of finite-dimensional vector spaces over k. Note that this category contains oodles of vector spaces of each dimension.

2.A. EXERCISE. Let $\vee\vee$: **f.d.** $\mathbf{Vec}_k \to$ **f.d.** $\mathbf{Vec}_k$ be the double dual functor from the category of vector spaces over k to itself. Show that $\vee\vee$ is naturally isomorphic to the identity. (Without the finite-dimensional hypothesis, we only get a natural transformation of functors from id to $\vee\vee$.)

Let $\mathcal{V}$ be the category whose objects are k^n for each n (there is one vector space for each n), and whose morphisms are linear transformations. This latter space can be thought of as vector spaces with bases, and the morphisms are honest matrices. There is an obvious functor $\mathcal{V} \to \mathbf{f.d.Vec}_k$, as each k^n *is* a finite-dimensional vector space.

2.B. EXERCISE. Show that $\mathcal{V} \to \mathbf{f.d.Vec}_k$ gives an equivalence of categories, by describing an "inverse" functor. (You'll need the axiom of choice, as you'll simultaneously choose bases for each vector space in $\mathbf{f.d.Vec}_k$!)

Once you have come to terms with the notion of equivalence of categories, you will quickly see that rings and affine schemes are basically the same thing, with the arrows reversed:

2.C. EXERCISE. Assuming that morphisms of schemes are defined so that Motivation (a) holds, show that the category of rings and the opposite category of affine schemes are equivalent.

3. MORPHISMS OF RINGED SPACES

3.1. Definition. A **morphism** $\pi : X \to Y$ **of ringed spaces** is a continuous map of topological spaces (which we unfortunately also call π) along with a "pullback map" $\mathcal{O}_Y \to \pi_* \mathcal{O}_X$. By adjointness, this is the same as a map $\pi^{-1}\mathcal{O}_Y \to \mathcal{O}_X$. There is an obvious notion of composition of morphisms; hence there is a category of ringed spaces. Hence we have notion of automorphisms and isomorphisms. You can easily verify that an isomorphism $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a homeomorphism $f : X \to Y$ along with an isomorphism $\mathcal{O}_Y \to f_*\mathcal{O}_X$ (or equivalently $f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$).

If $U \subset Y$ is an open subset, then there is a natural morphism of ringed spaces $(U, \mathcal{O}_Y|_U) \to (Y, \mathcal{O}_Y)$. (Check this! The f^{-1} interpretation is cleaner to use here.) This is our model for an open immersion. More precisely, if $U \to Y$ is an isomorphism of U with an open subset V of Y, and we are given an isomorphism $(U, \mathcal{O}_U) \cong (V, \mathcal{O}_V)$ (via the isomorphism $U \cong V$), then the resulting map of ringed spaces is called an **open immersion** of ringed spaces.

3.A. EXERCISE (MORPHISMS OF RINGED SPACES GLUE). Suppose $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ are ringed spaces, $X = \cup_i U_i$ is an open cover of X, and we have morphisms of ringed spaces $f_i : U_i \to Y$ that "agree on the overlaps", i.e. $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$. Show that there is a unique morphism of ringed spaces $f : X \to Y$ such that $f|_{U_i} = f_i$. (An earlier exercise essentially showed this for topological spaces.)

3.B. EASY IMPORTANT EXERCISE. Given a morphism of ringed spaces $f : X \to Y$ with $f(p) = q$, show that there is a map of stalks $(\mathcal{O}_Y)_q \to (\mathcal{O}_X)_p$.

3.2. Key Exercise. Suppose $f^\# : B \to A$ is a morphism of rings. Define a morphism of ringed spaces $f : \operatorname{Spec} A \to \operatorname{Spec} B$ as follows. The map of topological spaces was given earlier. To describe a morphism of sheaves $\mathcal{O}_B \to f_* \mathcal{O}_A$ on $\operatorname{Spec} B$, it suffices to describe a morphism of sheaves on the distinguished base of $\operatorname{Spec} B$. On $D(g) \subset \operatorname{Spec} B$, we define

$$\mathcal{O}_B(D(g)) \to \mathcal{O}_A(f^{-1} D(g)) = \mathcal{O}_A(D(f^\# g))$$

by $B_g \to A_{f^\# g}$. Verify that this makes sense (e.g. is independent of g), and that this describes a morphism of sheaves on the distinguished base. (This is the third in a series of exercises. We showed that a morphism of rings induces a map of sets first, a map of topological spaces later, and now a map of ringed spaces here.)

This will soon be an example of morphism of schemes! In fact we could make that definition right now.

3.3. Definition we won't start with. A morphism of schemes $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a morphism of ringed spaces that "locally looks like" the maps of affine schemes described in Key Exercise 3.2. Precisely, for each choice of affine opens $\operatorname{Spec} A \subset X$, $\operatorname{Spec} B \subset Y$, such that $f(\operatorname{Spec} A) \subset \operatorname{Spec} B$, the induced map of ringed spaces should be of the form shown in Key Exercise 3.2.

We would like this definition to be checkable on an affine cover, and we might hope to use the affine communication lemma to develop the theory in this way. This works, but it will be more convenient to use a clever trick: in the next section, we will use the notion of locally ringed spaces, and then once we have used it, we will discard it like yesterday's garbage.

The map of ringed spaces of Key Exercise 3.2 is really not complicated. Here is an example. Consider the ring map $\mathbb{C}[x] \to \mathbb{C}[y]$ given by $x \mapsto y^2$. We are mapping the affine line with co-ordinate y to the affine line with co-ordinate x. The map is (on closed points) $a \mapsto a^2$. For example, where does $[(y-3)]$ go to? Answer: $[(x-9)]$, i.e. $3 \mapsto 9$. What is the preimage of $[(x-4)]$? Answer: those prime ideals in $\mathbb{C}[y]$ containing $[(y^2-4)]$, i.e. $[(y-2)]$ and $[(y+2)]$, so the preimage of 4 is indeed ± 2. This is just about the map of sets, which is old news, so let's now think about functions pulling back. What is the pullback of the function $3/(x-4)$ on $D([(x-4)]) = \mathbb{A}^1 - \{4\}$? Of course it is $3/(y^2-4)$ on $\mathbb{A}^1 - \{-2, 2\}$.

We conclude with an example showing that not every morphism of ringed spaces between affine schemes is of the form of Key Exercise 3.2.

3.C. UNIMPORTANT EXERCISE. Recall that $\operatorname{Spec} k[x]_{(x)}$ has two points, corresponding to (0) and (x), where the second point is closed, and the first is not. Consider the map *of ringed spaces* $\operatorname{Spec} k(x) \to \operatorname{Spec} k[x]_{(x)}$ sending the point of $\operatorname{Spec} k(x)$ to $[(x)]$, and the pullback map $f^\# \mathcal{O}_{\operatorname{Spec} k(x)} \to \mathcal{O}_{\operatorname{Spec} k[x]_{(x)}}$ is induced by $k \hookrightarrow k(x)$. Show that this map of ringed spaces is not of the form described in Key Exercise 3.2.

In order to prove that morphisms behave in a way we hope, we will introduce the notion of a *locally ringed space*. It will not be used later, although it is useful elsewhere in geometry. The notion of locally ringed spaces is inspired by what we know about manifolds. If $\pi : X \to Y$ is a morphism of manifolds, with $\pi(p) = q$, and f is a function on Y vanishing at q, then the pulled back function $\pi^\# f$ on X should vanish on p. Put differently: germs of functions (at $q \in Y$) vanishing at q should pull back to germs of functions (at $p \in X$) vanishing at p.

A **locally ringed space** is a ringed space $(X, \mathcal{O}_X)$ such that the stalks $\mathcal{O}_{X,x}$ are all local rings. A **morphism of locally ringed spaces** $f : X \to Y$ is a morphism of ringed spaces such that the induced map of stalks $\mathcal{O}_{Y,q} \to \mathcal{O}_{X,p}$ (Exercise 3.B) sends the maximal ideal of the former into the maximal ideal of the latter (a "local morphism of local rings"). This means something rather concrete and intuitive: "if $p \mapsto q$, and g is a function vanishing at q, then it will pull back to a function vanishing at p." Note that locally ringed spaces form a category.

4.A. EXERCISE. Show that morphisms of locally ringed spaces glue (cf. Exercise 3.A). (Hint: Basically, the proof of Exercise 3.A works.)

4.B. EASY IMPORTANT EXERCISE. (a) Show that $\operatorname{Spec} A$ is a locally ringed space. (b) The morphism of ringed spaces $f : \operatorname{Spec} A \to \operatorname{Spec} B$ defined by a ring morphism $f^\# : B \to A$ is a morphism of locally ringed spaces.

4.1. *Key Proposition. — If $f : \operatorname{Spec} A \to \operatorname{Spec} B$ is a morphism of locally ringed spaces then it is the morphism of locally ringed spaces induced by the map $f^\# : B = \Gamma(\operatorname{Spec} B, \mathcal{O}_{\operatorname{Spec} B}) \to \Gamma(\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A}) = A$ as in Exercise 4.B(b).*

Proof. Suppose $f : \operatorname{Spec} A \to \operatorname{Spec} B$ is a morphism of locally ringed spaces. Then we wish to show that $f^\# : \mathcal{O}_{\operatorname{Spec} B} \to f_* \mathcal{O}_{\operatorname{Spec} A}$ is the morphism of sheaves given by Exercise 3.2 (cf. Exercise 4.B(b)). It suffices to checked this on the distinguished base.

Note that if $b \in B$, $f^{-1}(D(b)) = D(f^\# b)$; this is where we use the hypothesis that f is a morphism of locally ringed spaces.

The commutative diagram

$$\begin{array}{ccc} \Gamma(\operatorname{Spec} B, \mathcal{O}_{\operatorname{Spec} B}) & \xrightarrow{f^\#_{\operatorname{Spec} B}} & \Gamma(\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A}) \\ \downarrow & & \downarrow \otimes_B B_b \\ \Gamma(D(b), \mathcal{O}_{\operatorname{Spec} B}) & \xrightarrow{f^\#_{D(b)}} & \Gamma(D(f^\# b), \mathcal{O}_{\operatorname{Spec} A}) \end{array}$$

may be written as

$$\begin{array}{ccc} B & \xrightarrow{f^{\#}_{\operatorname{Spec} B}} & A \\ \downarrow & & \downarrow \otimes_B B_{\mathfrak{b}} \\ B_{\mathfrak{b}} & \xrightarrow{f^{\#}_{D(\mathfrak{b})}} & A_{f^{\#}\mathfrak{b}}. \end{array}$$

We want that $f^{\#}_{D(\mathfrak{b})} = (f^{\#}_{\operatorname{Spec} B})_{\mathfrak{b}}$. This is clear from the commutativity of that last diagram. □

We are ready for our definition.

4.2. Definition. If X and Y are schemes, then a morphism of locally ringed spaces is called a **morphism of schemes**. We have thus defined a *category* of schemes. (We then have notions of **isomorphism** — just the same as before — and **automorphism**.)

The definition in terms of locally ringed spaces easily implies tentative definition 3.3:

4.C. IMPORTANT EXERCISE. Show that a morphism of schemes $f : X \to Y$ is a morphism of ringed spaces that looks locally like morphisms of affines. Precisely, if $\operatorname{Spec} A$ is an affine open subset of X and $\operatorname{Spec} B$ is an affine open subset of Y, and $f(\operatorname{Spec} A) \subset \operatorname{Spec} B$, then the induced morphism of ringed spaces is a morphism of affine schemes. Show that it suffices to check on a set $(\operatorname{Spec} A_i, \operatorname{Spec} B_i)$ where the $\operatorname{Spec} A_i$ form an open cover X.

In practice, we will use the fact the affine cover interpretation, and forget completely about locally ringed spaces.

It is also clear (from the corresponding facts about locally ringed spaces) that morphisms glue (Exercise 4.A), and the composition of two morphisms is a morphism. Isomorphisms in this category are precise what we defined them to be earlier (homeomorphism of topological spaces with isomorphisms of structure sheaves).

4.3. The category of schemes (or k-schemes, or A-schemes, or Z-schemes). It is often convenient to consider subcategories. For example, the category of k-schemes (where k is a field) is defined as follows. The objects are morphisms of the form

$$\begin{array}{c} X \\ \Big\downarrow {\scriptstyle \text{structure morphism}} \\ \operatorname{Spec} k \end{array}$$

(This is the same definition as earlier, but in a more satisfactory form.) The morphisms in the category of k-schemes are commutative diagrams

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow & & \downarrow \\ \operatorname{Spec} k & \xrightarrow{=} & \operatorname{Spec} k \end{array}$$

which is more conveniently written as a commutative diagram

$$\begin{array}{ccc} X & \longrightarrow & Y \\ & \searrow \quad \swarrow & \\ & \operatorname{Spec} k. & \end{array}$$

For example, complex geometers may consider the category of $\mathbb{C}$-schemes.

When there is no confusion, simply the top row of the diagram is given. More generally, if A is a ring, the category of A-schemes is defined in the same way, with A replacing k. And if Z is a scheme, the category of Z-schemes is defined in the same way, with Z replacing $\operatorname{Spec} k$.

4.4. Examples.

4.D. IMPORTANT EXERCISE. (This exercise will give you some practice with understanding morphisms of schemes by cutting up into affine open sets.) Make sense of the following sentence: "$\mathbb{A}^{n+1} \setminus \{\vec{0}\} \to \mathbb{P}^n$ given by

$$(x_0, x_1, \ldots, x_{n+1}) \mapsto [x_0; x_1; \ldots; x_n]$$

is a morphism of schemes." Caution: you can't just say where points go; you have to say where functions go. So you'll have to divide these up into affines, and describe the maps, and check that they glue.

4.E. IMPORTANT EXERCISE. Show that morphisms $X \to \operatorname{Spec} A$ are in natural bijection with ring morphisms $A \to \Gamma(X, \mathcal{O}_X)$. (Hint: Show that this is true when X is affine. Use the fact that morphisms glue.)

In particular, there is a canonical morphism from a scheme to Spec of its space of global sections. (Warning: Even if X is a finite-type k-scheme, the ring of global sections might be nasty! In particular, it might not be finitely generated.)

4.5. *Side fact for experts: Γ and* Spec *are adjoints.* We have a functor Spec from rings to locally ringed spaces, and a functor Γ from locally ringed spaces to rings. Exercise 4.E implies $(\Gamma, \operatorname{Spec})$ is an adjoint pair! Thus we could have defined Spec by requiring it to be adjoint to Γ.

4.F. EXERCISE. Show that $\operatorname{Spec} \mathbb{Z}$ is the final object in the category of schemes. In other words, if X is any scheme, there exists a unique morphism to $\operatorname{Spec} \mathbb{Z}$. (Hence the category of schemes is isomorphic to the category of $\mathbb{Z}$-schemes.)

4.G. EXERCISE. Show that morphisms $X \to \operatorname{Spec} \mathbb{Z}[t]$ correspond to global sections of the structure sheaf.

4.6. $\star$ **Representable functors.** This is one of our first explicit examples of an important idea, that of representable functors. This is a very useful idea, whose utility isn't immediately apparent. We have a contravariant functor from schemes to sets, taking a scheme to its set of global sections. We have another contravariant functor from schemes to sets, taking X to $\mathrm{Hom}(X, \mathrm{Spec}\, \mathbb{Z}[t])$. This is describing a (natural) isomorphism of the functors. More precisely, we are describing an isomorphism $\Gamma(X, \mathcal{O}_X) \cong \mathrm{Hom}(X, \mathrm{Spec}\, \mathbb{Z}[t])$ that behaves well with respect to morphisms of schemes: given any morphism $f : X \to Y$, the diagram

$$\begin{array}{ccc} \Gamma(Y, \mathcal{O}_Y) & \xrightarrow{\sim} & \mathrm{Hom}(Y, \mathrm{Spec}\, \mathbb{Z}[t]) \\ \downarrow f^* & & \downarrow f\circ \\ \Gamma(X, \mathcal{O}_X) & \xrightarrow{\sim} & \mathrm{Hom}(X, \mathrm{Spec}\, \mathbb{Z}[t]) \end{array}$$

commutes. *Given a contravariant functor from schemes to sets, by the usual universal property argument, there is only one possible scheme* Y*, up to isomorphism, such that there is a natural isomorphism between this functor and* $\mathrm{Hom}(\cdot, Y)$. If there is such a Y, we say that the functor is **representable**.

Here are a couple of examples of something you've seen to put it in context. (i) The contravariant functor $h^Y = \mathrm{Hom}(\cdot, Y)$ (i.e. $X \mapsto \mathrm{Hom}(X, Y)$) is representable by Y. (ii) Suppose we have morphisms $X, Y \to Z$. The contravariant functor $\mathrm{Hom}(\cdot, X) \times_{\mathrm{Hom}(\cdot, Z)} \mathrm{Hom}(\cdot, Y)$ is representable if and only if the fibered product $X \times_Z Y$ exists (and indeed then the contravariant functor is represented by $\mathrm{Hom}(\cdot, X \times_Z Y)$). This is purely a translation of the definition of the fibered product — you should verify this yourself.

Remark for experts: The global sections form something better than a set — they form a scheme. You can define the notion of ring scheme, and show that if a contravariant functor from schemes to rings is representable (as a contravariant functor from schemes to sets) by a scheme Y, then Y is guaranteed to be a ring scheme. The same is true for group schemes.

4.H. RELATED EXERCISE. Show that global sections of $\mathcal{O}_X^*$ correspond naturally to maps to $\mathrm{Spec}\, \mathbb{Z}[t, t^{-1}]$. ($\mathrm{Spec}\, \mathbb{Z}[t, t^{-1}]$ is a *group scheme*.)

5. SOME TYPES OF MORPHISMS

(This section has been moved forward to class 13.)

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FOUNDATIONS OF ALGEBRAIC GEOMETRY CLASS 13

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CONTENTS

We now define a bunch of types of morphisms. (These notes include some topics discussed the previous class.)

1. SOME TYPES OF MORPHISMS: QUASICOMPACT AND QUASISEPARATED; OPEN IMMERSION; AFFINE, FINITE, CLOSED IMMERSION; LOCALLY CLOSED IMMERSION

In this section, we'll give some analogues of open subsets, closed subsets, and locally closed subsets. This will also give us an excuse to define affine and finite morphisms (closed immersions are a special case). It will also give us an excuse to define some important special closed immersions, in the next section. In section after that, we'll define some more types of morphisms.

1.1. Quasicompact and quasiseparated morphisms.

A morphism $f : X \to Y$ is **quasicompact** if for every open affine subset U of Y, $f^{-1}(U)$ is quasicompact. Equivalently, the preimage of any quasicompact open subset is quasicompact. We will like this notion because (i) we know how to take the maximum of a finite set of numbers, and (ii) most reasonable schemes will be quasicompact.

1.A. EASY EXERCISE. Show that the composition of two quasicompact morphisms is quasicompact.

1.B. EXERCISE. Show that any morphism from a Noetherian scheme is quasicompact.

Date: Monday, November 5, 2007. Updated Nov. 13, 2007. Minor update Nov. 15.

1.C. EXERCISE (QUASICOMPACTNESS IS AFFINE-LOCAL ON THE TARGET). Show that a morphism $f : X \to Y$ is quasicompact if there is cover of Y by open affine sets U_i such that $f^{-1}(U_i)$ is quasicompact. (Hint: easy application of the affine communication lemma!)

Along with quasicompactness comes the weird notion of quasiseparatedness. A morphism $f : X \to Y$ is **quasiseparated** if for every open affine subset U of Y, $f^{-1}(U)$ is a quasiseparated scheme. This will be a useful hypothesis in theorems (in conjunction with quasicompactness), and that various interesting kinds of morphisms (locally Noetherian source, affine, separated, see Exercise 1.D, Exercise 1.J, and an exercise next quarter resp.) are quasiseparated, and this will allow us to state theorems more succinctly.

1.D. EXERCISE. Show that any morphism from a locally Noetherian scheme is quasiseparated. (Hint: locally Noetherian schemes are quasiseparated.) Thus those readers working only with Noetherian schemes may take this as a standing hypothesis.

1.E. EASY EXERCISE. Show that the composition of two quasiseparated morphisms is quasiseparated.

1.F. EXERCISE (QUASISEPARATEDNESS IS AFFINE-LOCAL ON THE TARGET). Show that a morphism $f : X \to Y$ is quasiseparated if there is cover of Y by open affine sets U_i such that $f^{-1}(U_i)$ is quasiseparated. (Hint: easy application of the affine communication lemma!)

Following Grothendieck's philosophy of thinking that the important notions are properties of morphisms, not of objects, we can restate the definition of quasicompact (resp. quasiseparated) scheme as a scheme that is quasicompact (resp. quasiseparated) over the final object $\operatorname{Spec} \mathbb{Z}$ in the category of schemes.

1.2. Open immersions.

An **open immersion of schemes** is defined to be an open immersion as ringed spaces. In other words, a morphism $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ of schemes is an open immersion if f factors as

$$(X, \mathcal{O}_X) \xrightarrow[\sim]{g} (U, \mathcal{O}_Y|_U) \xrightarrow{h} (Y, \mathcal{O}_Y)$$

where g is an isomorphism, and $U \hookrightarrow Y$ is an inclusion of an open set. It is immediate that isomorphisms are open immersions. We say that $(U, \mathcal{O}_Y|_U)$ is an **open subscheme** of $(Y, \mathcal{O}_Y)$, and often sloppily say that $(X, \mathcal{O}_X)$ is an open subscheme of $(Y, \mathcal{O}_Y)$.

1.G. EXERCISE. Suppose $i : U \to Z$ is an open immersion, and $f : Y \to Z$ is any morphism. Show that $U \times_Z Y$ exists. (Hint: I'll even tell you what it is: $(f^{-1}(U), \mathcal{O}_Y|_{f^{-1}(U)})$.)

1.H. EASY EXERCISE. Show that open immersions are monomorphisms.

1.I. EASY EXERCISE. Suppose $f : X \to Y$ is an open immersion. Show that if Y is locally Noetherian, then X is too. Show that if Y is Noetherian, then X is too. However, show that if Y is quasicompact, X need not be. (Hint: let Y be affine but not Noetherian.)

"Open immersions" are scheme-theoretic analogues of open subsets. "Closed immersions" are scheme-theoretic analogues of closed subsets, but they are of a quite different flavor, as we'll see soon.

1.3. Affine morphisms.

A morphism $f : X \to Y$ is **affine** if for every affine open set U of Y, $f^{-1}(U)$ is an affine scheme. We have immediately that affine morphisms are quasicompact.

1.J. FAST EXERCISE. Show that affine morphisms are quasiseparated. (Hint: affine schemes are quasiseparated, an earlier exercise.)

1.4. *Proposition (the property of "affineness" is affine-local on the target). — A morphism $f : X \to Y$ is affine if there is a cover of Y by affine open sets U such that $f^{-1}(U)$ is affine.*

For part of the proof, it will be handy to have a lemma.

1.5. *Lemma. — If X is a quasicompact quasiseparated scheme and $s \in \Gamma(X, \mathcal{O}_X)$, then the natural map $\Gamma(X, \mathcal{O}_X)_s \to \Gamma(X_s, \mathcal{O}_X)$ is an isomorphism.*

A brief reassuring comment on the "quasicompact quasiseparated" hypothesis: This just means that X can be covered by a finite number of affine open subsets, any two of which have intersection also covered by a finite number of affine open subsets. The hypothesis applies in lots of interesting situations, such as if X is affine or Noetherian.

Proof. Cover X with finitely many affine open sets $U_i = \operatorname{Spec} A_i$. Let $U_{ij} = U_i \cap U_j$. Then

$$\Gamma(X, \mathcal{O}_X) \to \prod_i A_i \rightrightarrows \prod_{i,j} \Gamma(U_{ij}, \mathcal{O}_X)$$

is exact. Localizing at s gives

$$\Gamma(X, \mathcal{O}_X)_s \to \left(\prod_i A_i\right)_s \rightrightarrows \left(\prod_{i,j} \Gamma(U_{ij}, \mathcal{O}_X)\right)_s$$

As localization commutes with finite products,

$$\Gamma(X, \mathcal{O}_X)_s \to \prod_i (A_i)_{s_i} \rightrightarrows \prod_{i,j} \Gamma(U_{ij}, \mathcal{O}_X)_s$$

is exact, where the global function s induces functions $s_i \in A_i$. If $\Gamma(U_{ij}, \mathcal{O}_X)_s \cong \Gamma((U_{ij})_s, \mathcal{O}_X)$, then it is clear that $\Gamma(X, \mathcal{O}_x)_s$ are the sections over X_s. Note that U_{ij} are quasicompact, by the quasiseparatedness hypothesis, and also quasiseparated, as open subsets of quasiseparated schemes are quasiseparated. Therefore we can reduce to the case where $X \subseteq \operatorname{Spec} A$

is a (quasicompact quasiseparated) open subset of an affine scheme. Then $U_{ij} = \operatorname{Spec} A_{f_i f_j}$ is affine and $\Gamma(U_{ij}, \mathcal{O}_x)_s = \Gamma((U_{ij})_s, \mathcal{O}_X)$ so the same exact sequence implies the result. $\square$

Proof of Proposition 1.4. As usual, we use the Affine Communication Lemma. We check our two criteria. First, suppose $f : X \to Y$ is affine over $\operatorname{Spec} B$, i.e. $f^{-1}(\operatorname{Spec} B) = \operatorname{Spec} A$. Then $f^{-1}(\operatorname{Spec} B_s) = \operatorname{Spec} A_{f\# s}$.

Second, suppose we are given $f : X \to \operatorname{Spec} B$ and $(f_1, \dots, f_n) = B$ with X_{f_i} affine ($\operatorname{Spec} A_i$, say). We wish to show that X is affine too. X is quasi-compact (as it is covered by n affine open sets). Let $t_i \in \Gamma(X, \mathcal{O}_X)$ be the pullback of the sections $s_i \in B$. The morphism f factors as $h \circ g$ where $g : X \to \operatorname{Spec} \Gamma(X, \mathcal{O}_X)$ and $h : \operatorname{Spec} \Gamma(X, \mathcal{O}_X) \to \operatorname{Spec} B$ are the natural maps. Then Lemma 1.5 implies that $g|_{f^{-1}(\operatorname{Spec} B_{s_i})} : X_{t_i} \to \operatorname{Spec} \Gamma(X, \mathcal{O}_X)_{t_i}$ are isomorphisms. Therefore, g is an isomorphism and X is affine. $\square$

1.6. Finite morphisms.

An affine morphism $f : X \to Y$ is **finite** if for every affine open set $\operatorname{Spec} B$ of Y, $f^{-1}(\operatorname{Spec} B)$ is the spectrum of an B-algebra that is a finitely-generated B-*module*. Warning about terminology (finite vs. finitely-generated): Recall that if we have a ring homomorphism $A \to B$ such that B is a finitely-generated A-*module* then we say that B is a **finite** A-algebra. This is stronger than being a finitely-generated A-*algebra*.

By definition, finite morphisms are affine.

1.K. EXERCISE (THE PROPERTY OF FINITENESS IS AFFINE-LOCAL ON THE TARGET). Show that a morphism $f : X \to Y$ is finite if there is a cover of Y by affine open sets $\operatorname{Spec} A$ such that $f^{-1}(\operatorname{Spec} A)$ is the spectrum of a finite A-algebra.

1.L. EASY EXERCISE. Show that the composition of two finite morphisms is also finite.

We now give four examples of finite morphisms, to give you some feeling for how finite morphisms behave. In each example, you'll notice two things. In each case, the maps are always finite-to-one. We'll verify this in Exercise 3.E. You'll also notice that the morphisms are closed, i.e. the image of closed sets are closed. This argument uses the going-up theorem, and we'll verify this when we discuss that. Intuitively, you should think of finite as being closed plus finite fibers, although this isn't quite true. We'll make this precise later.

Example 1: Branched covers. If $p(t) \in k[t]$ is a non-zero polynomial, then $\operatorname{Spec} k[t] \to \operatorname{Spec}[u]$ given by $u \mapsto p(t)$ is a finite morphism, see Figure 1.

Example 2: Closed immersions (to be defined soon, in §1.8). The morphism $\operatorname{Spec} k \to \operatorname{Spec} k[t]$ given by $t \mapsto 0$ is a finite morphism, see Figure 2.

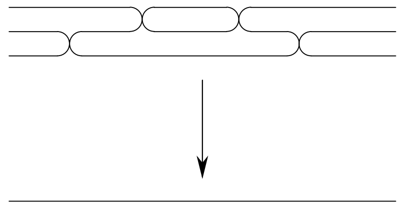

FIGURE 1. The "branched cover" of $\mathbb{A}^1$ given by $u \mapsto p(t)$ is finite

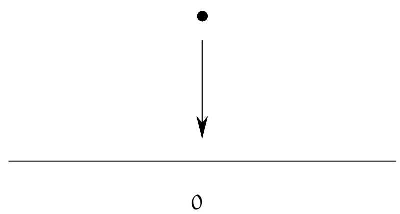

FIGURE 2. The "closed immersion" $\operatorname{Spec} k \to \operatorname{Spec} k[t]$ is finite

Example 3: Normalization (to be defined later). The morphism $\operatorname{Spec} k[t] \to \operatorname{Spec} k[x,y]/(y^2 - x^2 - x^3)$ given by $(x,y) \mapsto (t^2-1, t^3-t)$ (check that this is a well-defined ring map!) is a finite morphism, see Figure 3.

1.M. IMPORTANT EXERCISE (EXAMPLE 4, FINITE MORPHISMS TO $\operatorname{Spec} k$). Show that if $X \to \operatorname{Spec} k$ is a finite morphism, then X is a discrete finite union of points, each with residue field a finite extension of k, see Figure 4. (An example is $\operatorname{Spec} \mathbb{F}_8 \times \mathbb{F}_4[x,y]/(x^2,y^4) \times \mathbb{F}_4[t]/t^9 \times \mathbb{F}_2 \to \operatorname{Spec} F_2$.)

1.7. *Example.* The natural map $\mathbb{A}^2 - \{(0,0)\} \to \mathbb{A}^2$ is an open immersion, and has finite fibers, but is not affine (as $\mathbb{A}^2 - \{(0,0)\}$ isn't affine) and hence not finite.

1.8. Closed immersions and closed subschemes.

Just as open immersions (the scheme-theoretic version of open set) are locally modeled on open sets $U \subset Y$, the analogue of closed subsets also has a local model. This was foreshadowed by our understanding of closed subsets of $\operatorname{Spec} B$ as roughly corresponding to ideals. If $I \subset B$ is an ideal, then $\operatorname{Spec} B/I \hookrightarrow \operatorname{Spec} B$ is a morphism of schemes, and this is our prototypical example of a closed immersion.

A morphism $f : X \to Y$ is a **closed immersion** if it is an affine morphism, and for each open subset $\operatorname{Spec} B \subset Y$, with $f^{-1}(\operatorname{Spec} B) \cong \operatorname{Spec} A$, $B \to A$ is a surjective map (i.e. of the form $B \to B/I$, our desired local model). We often say that X is a **closed subscheme** of Y.

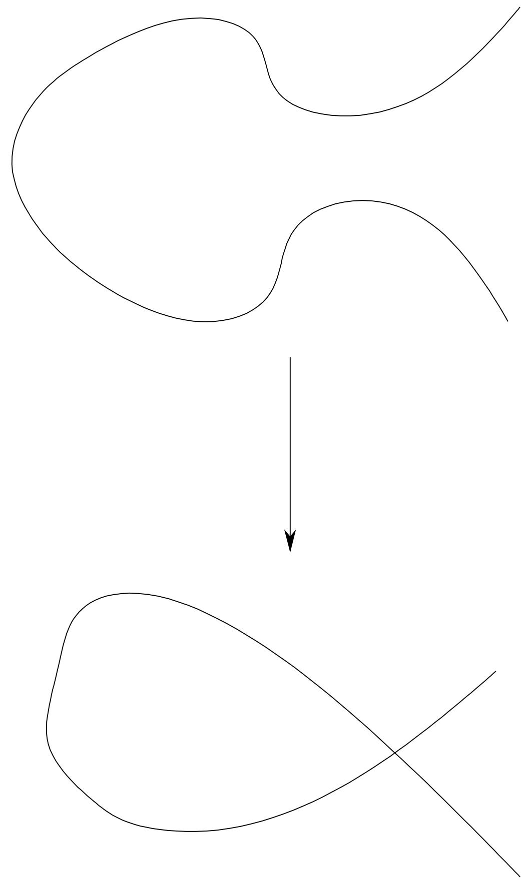

FIGURE 3. The "normalization" $\operatorname{Spec} k[t] \to \operatorname{Spec} k[x, y]/(y^2 - x^2 - x^3)$ given by $(x, y) \mapsto (t^2 - 1, t^3 - t)$ is finite

FIGURE 4. A picture of a finite morphism to $\operatorname{Spec} k$. Notice that bigger fields are written as bigger dots. [I'd like to add some fuzz to some of these points at some point.]

1.N. EASY EXERCISE. Show that closed immersions are finite.

1.O. EXERCISE. Show that the property of being a closed immersion is affine-local on the target.

A closed immersion $f : X \hookrightarrow Y$ determines an *ideal sheaf* on Y, as the kernel $\mathcal{I}_{X/Y}$ of the map of $\mathcal{O}_Y$-modules

$$\mathcal{O}_Y \to f_* \mathcal{O}_X$$

(An **ideal sheaf** on Y is what it sounds like: it is a sheaf of ideals. It is a sub-$\mathcal{O}_Y$-module $\mathcal{I} \hookrightarrow \mathcal{O}_Y$. On each open subset, it gives an ideal $\mathcal{I}(U) \hookrightarrow \mathcal{O}_Y(U)$.) We thus have an exact sequence $0 \to \mathcal{I}_{X/Y} \to \mathcal{O}_Y \to f_* \mathcal{O}_X \to 0$.

1.P. IMPORTANT EXERCISE: A USEFUL CRITERION FOR WHEN IDEALS IN AFFINE OPEN SETS DEFINE A CLOSED SUBSCHEME. It will be convenient (for example in §2) to define certain closed subschemes of Y by defining on any affine open subset $\operatorname{Spec} B$ of Y an ideal $I_B \subset B$. Show that these $\operatorname{Spec} B/I_B \hookrightarrow \operatorname{Spec} B$ glue together to form a closed subscheme precisely if for each affine open subset $\operatorname{Spec} B \hookrightarrow Y$ and each $f \in B$, $I_{(B_f)} = (I_B)_f$.

Warning: you might hope that closed subschemes correspond to ideal sheaves of $\mathcal{O}_Y$. Sadly not every ideal sheaf arises in this way. Here is an example.

1.Q. UNIMPORTANT EXERCISE. Let $X = \operatorname{Spec} k[x]_{(x)}$, the germ of the affine line at the origin, which has two points, the closed point and the generic point η. Define $\mathcal{I}(X) = \{0\} \subset \mathcal{O}_X(X) = k[x]_{(x)}$, and $\mathcal{I}(\eta) = k(x) = \mathcal{O}_X(\eta)$. Show that this sheaf of ideals does not correspond to a closed subscheme (see Exercise 1.P).

We will see later that closed subschemes correspond to *quasicoherent* sheaves of ideals; the mathematical content of this statement will turn out to be precisely Exercise 1.P.

1.R. IMPORTANT EXERCISE. (a) In analogy with closed subsets, define the notion of a finite union of closed subschemes of X, and an arbitrary intersection of closed subschemes.
(b) Describe the scheme-theoretic intersection of $(y - x^2)$ and y in $\mathbb{A}^2$. See Figure 5 for a picture. (For example, explain informally how this corresponds to two curves meeting at a single point with multiplicity 2 — notice how the 2 is visible in your answer. Alternatively, what is the non-reducedness telling you — both its "size" and its "direction"?) Describe the scheme-theoretic union.
(c) Describe the scheme-theoretic intersection of $(y^2 - x^2)$ and y in $\mathbb{A}^2$. Draw a picture. (Are you surprised? Did you expect the intersection to be multiplicity one or multiplicity two?) Hence show that if X, Y, and Z are closed subschemes of W, then $(X \cap Z) \cup (Y \cap Z) \neq (X \cup Y) \cap Z$ in general.
(d) Show that the underlying set of a finite union of closed subschemes is the finite union of the underlying sets, and similarly for arbitrary intersections.

1.S. IMPORTANT EXAMPLE THAT SHOULD HAVE BEEN DONE EARLIER. We now make a preliminary definition of projective n-space $\mathbb{P}^n_k$, by gluing together $n+1$ open sets each isomorphic to $\mathbb{A}^n_k$. Judicious choice of notation for these open sets will make our life easier. Our motivation is as follows. In the construction of $\mathbb{P}^1$ above, we thought of points of projective space as $[x_0; x_1]$, where (x_0, x_1) are only determined up to scalars, i.e. (x_0, x_1)

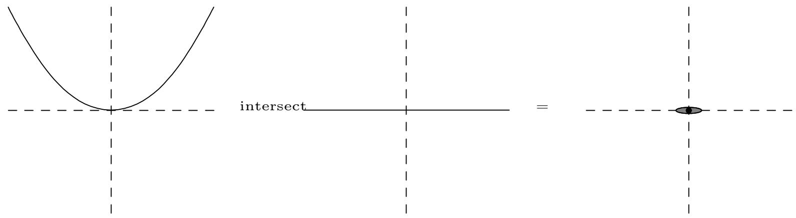

FIGURE 5. The scheme-theoretic intersection of the parabola $y = x^2$ and the x-axis is a non-reduced scheme (with fuzz in the x-direction)

is considered the same as $(\lambda x_0, \lambda x_1)$. Then the first patch can be interpreted by taking the locus where $x_0 \neq 0$, and then we consider the points $[1; t]$, and we think of t as x_1/x_0; even though x_0 and x_1 are not well-defined, x_1/x_0 *is*. The second corresponds to where $x_1 \neq 0$, and we consider the points $[u; 1]$, and we think of u as x_0/x_1. It will be useful to instead use the notation $x_{1/0}$ for t and $x_{0/1}$ for u.

For $\mathbb{P}^n$, we glue together $n+1$ open sets, one for each of $i = 0, \dots, n+1$. The ith open set U_i will have co-ordinates $x_{0/i}$, …, $x_{(i-1)/i}$, $x_{(i+1)/i}$, …, $x_{n/i}$. It will be convenient to write this as

$$\operatorname{Spec} k[x_{0/i}, x_{1/i}, \dots, x_{n/i}]/(x_{i/i} - 1)$$

(so we have introduced a "dummy variable" $x_{i/i}$ which we set to 1). We glue the distinguished open set $D(x_{j/i})$ of U_i to the distinguished open set $D(x_{i/j})$ of U_j, by identifying these two schemes by describing the identification of rings

$$\operatorname{Spec} k[x_{0/i}, x_{1/i}, \dots, x_{n/i}, 1/x_{j/i}]/(x_{i/i} - 1) \cong$$

$$\operatorname{Spec} k[x_{0/j}, x_{1/j}, \dots, x_{n/j}, 1/x_{i/j}]/(x_{j/j} - 1)$$

via $x_{k/i} = x_{k/j}/x_{i/j}$ and $x_{k/j} = x_{k/i}/x_{j/i}$ (which implies $x_{i/j}x_{j/i} = 1$). We need to check that this gluing information agrees over triple overlaps.

1.T. EXERCISE. Check this, as painlessly as possible. (Possible hint: the triple intersection is affine; describe the corresponding ring.)

Note that our definition doesn't use the fact that k is a field. Hence we may as well define $\mathbb{P}^n_A$ for any *ring* A. This will be useful later.

1.9. *Example: Closed immersions of projective space $\mathbb{P}^n_A$.* Consider the definition of projective space $\mathbb{P}^n_A$ given above. Any *homogeneous* polynomial f in $x_0, \dots, x_n$ defines a closed subscheme. (Thus even though x_0, …, x_n don't make sense as functions, their vanishing locus still makes sense.) On open set U_i, the closed subscheme is $f(x_{0/i}, \dots, x_{n/i})$, which we think of as $f(x_0, \dots, x_n)/x_i^{\deg f}$. On the overlap

$$U_i \cap U_j = \operatorname{Spec} A[x_{0/i}, \dots, x_{n/i}, x_{j/i}^{-1}]/(x_{i/i} - 1),$$

these functions on U_i and U_j don't exactly agree, but they agree up to a non-vanishing scalar, and hence cut out the same subscheme of $U_i \cap U_j$:

$$f(x_{0/i}, \ldots, f_{n/i}) = x_{j/i}^{\deg f} f(x_{0/j}, \ldots, x_{n/j}).$$

Thus by intersecting such closed subschemes, we see that any collection of homogeneous polynomials in $A[x_0, \ldots, x_n]$ cut out a closed subscheme of $\mathbb{P}^n_A$. We could take this as a provisional definition of a *projective* A*-scheme* (or a projective scheme over A). (We'll give a better definition in the next Chapter.)

Notice: piggybacking on the annoying calculation that $\mathbb{P}^n$ consists of $n+1$ pieces glued together nicely is the fact that any closed subscheme of $\mathbb{P}^n$ cut out by a bunch of homogeneous polynomials consists of $n+1$ pieces glued together nicely.

Notice also that this subscheme is not in general cut out by a single global function on $\mathbb{P}^n_A$. For example, if $A = k$, there *are* no nonconstant global functions. We take this opportunity to introduce some related terminology. A closed subscheme is **locally principal** if on each open set in a small enough open cover it is cut out by a single equation. Thus each homogeneous polynomial in $n+1$ variables defines a locally principal closed subscheme. (Warning: one can check this on a fine enough affine open cover, but this is not an affine-local condition! We will see an example in the next day's notes — one $\mathbb{P}^2$ minus a conic, consider a line.) A case that will be important later is when the ideal sheaf is not just locally generated by a function, but is generated by a function that is not a zero-divisor. In this case (once we have defined our terms) we will call this an *invertible ideal sheaf*, and the closed subscheme will be an *effective Cartier divisor*.

A closed subscheme cut out by a single (homogeneous) equation is called a **hypersurface** in $\mathbb{P}^n_k$. The **degree of a hypersurface** is the degree of the polynomial. (Implicit in this is that this notion can be determined from the subscheme itself; we haven't yet checked this.) A hypersurface of degree 1 (resp. degree 2, 3, ...) is called a **hyperplane** (resp. **quadric, cubic, quartic, quintic, sextic, septic, octic, ... hypersurface**). If $n = 2$, a degree 1 hypersurface is called a **line**, and a degree 2 hypersurface is called a **conic curve**, or a **conic** for short. If $n = 3$, a hypersurface is called a **surface**.) (In a couple of weeks, we will justify the terms *curve* and *surface*.)

1.U. EXERCISE. (a) Show that $wz = xy, x^2 = wy, y^2 = xz$ describes an irreducible curve in $\mathbb{P}^3_k$. This curve is called the *twisted cubic*. The twisted cubic is a good non-trivial example of many things, so it you should make friends with it as soon as possible. (b) Show that the twisted cubic is isomorphic to $\mathbb{P}^1_k$.

1.V. UNIMPORTANT EXERCISE. The usual definition of a closed immersion is a morphism $f : X \to Y$ such that f induces a homeomorphism of the underlying topological space of Y onto a closed subset of the topological space of X, and the induced map $f^\# : \mathcal{O}_X \to f_*\mathcal{O}_Y$ of sheaves on X is surjective. Show that this definition agrees with the one given above. (To show that our definition involving surjectivity on the level of affine open sets implies this definition, you can use the fact that surjectivity of a morphism of sheaves can be checked on a base, which you can verify yourself.)

1.10. $\star$ **A fun example.** The affine-locality of affine morphisms (Proposition 1.4) has some non-obvious consequences, as shown in the next exercise.

1.W. EXERCISE. Suppose X is an affine scheme, and Y is a closed subscheme locally cut out by one equation (e.g. if Y is an effective Cartier divisor). Show that $X - Y$ is affine. (This is clear if Y is globally cut out by one equation f; then if $X = \operatorname{Spec} A$ then $Y = \operatorname{Spec} A_f$. However, Y is not always of this form.)

1.11. *Example.* Here is an explicit consequence. We showed earlier that on the cone over the smooth quadric surface $\operatorname{Spec} k[w, x, y, z]/(wz - xy)$, the cone over a ruling $w = x = 0$ is not cut out scheme-theoretically by a single equation, by considering Zariski-tangent spaces. We now show that it isn't even cut out set-theoretically by a single equation. For if it were, its complement would be affine. But then the closed subscheme of the complement cut out by $y = z = 0$ would be affine. But this is the scheme $y = z = 0$ (also known as the wx-plane) minus the point $w = x = 0$, which we've seen is non-affine. (For comparison, on the cone $\operatorname{Spec} k[x, y, z]/(xy - z^2)$, see Figure 6, the ruling $x = z = 0$ is not cut out scheme-theoretically by a single equation, but it *is* cut out set-theoretically by $x = 0$.) Verify all this! (Hint: Use Exercise 1.4.)

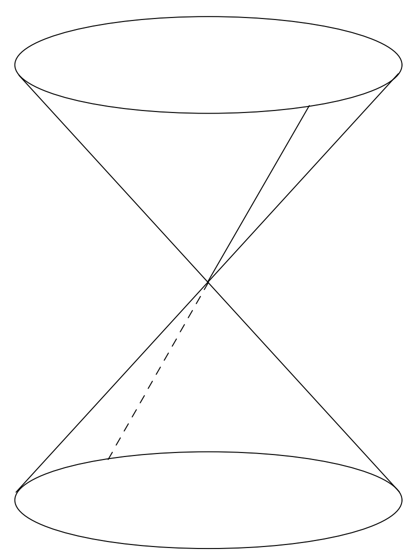

FIGURE 6. $V(x, z) \subset \operatorname{Spec} k[x, y, z]/(xy - z^2)$ is a ruling on a cone

We have now defined the analog of open subsets and closed subsets in the land of schemes. Their definition is slightly less "symmetric" than in the usual topological setting: the "complement" of a closed subscheme is a unique open subscheme, but there are many "complementary" closed subschemes to a given open subscheme in general. (We'll soon define one that is "best", that has a reduced structure, §2.6.)

1.12. Locally closed immersions and locally closed subschemes.

Now that we have defined analogs of open and closed subsets, it is natural to define the analog of locally closed subsets. Recall that locally closed subsets are intersections of open subsets and closed subsets. Hence they are closed subsets of open subsets, or equivalently open subsets of closed subsets. That equivalence will be a little subtle in the land of schemes.

We say a morphism $X \to Y$ is a **locally closed immersion** if it can factored into $X \xrightarrow{f} Z \xrightarrow{g} Y$ where f is a closed immersion and g is an open immersion. (Warning: The term *immersion* is often used instead of *locally closed immersion*, but this is unwise terminology, as the differential geometric notion of immersion is closer to what algebraic geometers call *unramified*, which we'll define next quarter. The algebro-geometric notion of locally closed immersion is closest to the differential geometric notion of *embedding*.) It is often said that X is a **locally closed subscheme** of Y.

For example, $\operatorname{Spec} k[t, t^{-1}] \to \operatorname{Spec} k[x, y]$ where $(x, y) \mapsto (t, 0)$ is a locally closed immersion (see Figure 7).

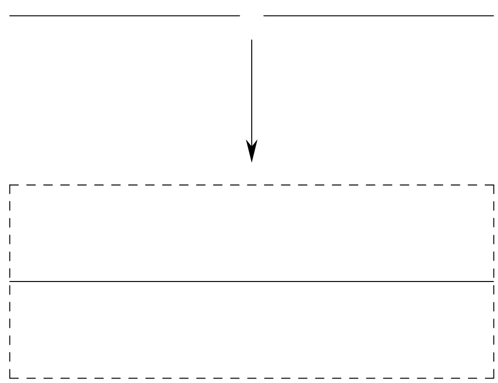

FIGURE 7. The locally closed immersion $\operatorname{Spec} k[t, t^{-1}] \to k[x, y]$ ($t \mapsto (t, 0) = (x, y)$, i.e. $(x, y) \to (t, 0)$)

We can make sense of finite intersections of locally closed immersions.

Clearly a open subscheme U of a closed subscheme V of X can be interpreted as a closed subscheme of an open subscheme: as the topology on V is induced from the topology on X, the underlying set of U is the intersection of some open subset U' on X with V. We can take $V' = V \cap U$, and then $V' \to U'$ is a closed immersion, and $U' \to X$ is an open immersion.

It is less clear that a closed subscheme V' of an open subscheme U' can be expressed as an open subscheme U of a closed subscheme V. In the category of topological spaces, we would take V as the closure of V', so we are now motivated to define the analogous construction, which will give us an excuse to introduce several related ideas, in the next section. We will then resolve this issue in good cases (e.g. if X is Noetherian) in Exercise 2.D.

2. Constructions related to "smallest closed subschemes": scheme-theoretic image, scheme-theoretic closure, induced reduced subscheme, and the reduction of a scheme

We now define a series of notions that are all of the form "the smallest closed subscheme such that something or other is true". One example will be the notion of scheme-theoretic closure of a locally closed immersion, which will allow us to interpret locally closed immersions in three equivalent ways (open subscheme intersect closed subscheme; open subscheme of closed subscheme; and closed subscheme of open subscheme).

2.1. Scheme-theoretic image.

We start with the notion of scheme-theoretic image. If $f : X \to Y$ is a morphism of schemes, the notion of the image of f as *sets* is clear: we just take the points in Y that are the image of points in X. But if we would like the image as a scheme, then the notion becomes more problematic. (For example, what is the image of $\mathbb{A}^2 \to \mathbb{A}^2$ given by $(x, y) \mapsto (x, xy)$?) We will come back to the notion of image later, but for now we will define the "scheme-theoretic image". This will incorporate the notion that the image of something with non-reduced structure ("fuzz") can also have non-reduced structure.

Definition. Suppose $i : Z \hookrightarrow Y$ is a closed subscheme, giving an exact sequence $0 \to \mathcal{I}_{Z/Y} \to \mathcal{O}_Y \to i_*\mathcal{O}_Z \to 0$. We say that the image of $f : X \to Y$ lies in Z if the composition $\mathcal{I}_{Z/Y} \to \mathcal{O})_Y \to f_*\mathcal{O}_X$ is zero. Informally, locally functions vanishing on Z pull back to the zero function on X. If the image of f lies in two subschemes Z_1 and Z_2, it clearly lies in their intersection $Z_1 \cap Z_2$. We then define the **scheme-theoretic image of** f of f, a closed subscheme on Y, as the "smallest closed subscheme containing the image", i.e. the intersection of all closed subschemes containing the image.

Example 1. Consider $\operatorname{Spec} k[\epsilon]/\epsilon^2 \to \operatorname{Spec} k[x] = \mathbb{A}^1_k$ given by $x \mapsto \epsilon$. Then the scheme-theoretic image is given by $k[x]/x^2$ (the polynomials pulling back to 0 are precisely multiples of x^2). Thus the image of the fuzzy point still has some fuzz.

Example 2. Consider $f : \operatorname{Spec} k[\epsilon]/\epsilon^2 \to \operatorname{Spec} k[x] = \mathbb{A}^1_k$ given by $x \mapsto 0$. Then the scheme-theoretic image is given by $k[x]/x$: the image is reduced. In this picture, the fuzz is "collapsed" by f.

Example 3. Consider $f : \operatorname{Spec} k[t, t^{-1}] = \mathbb{A}^1 - \{0\} \to \mathbb{A}^1 = \operatorname{Spec} k[u]$ given by $u \mapsto t$. Any function $g(u)$ which pulls back to 0 as a function of t must be the zero-function. Thus the scheme-theoretic image is everything. The set-theoretic image, on the other hand, is the distinguished open set $\mathbb{A}^1 - \{0\}$. Thus in not-too-pathological cases, the underlying set of the scheme-theoretic image is not the set-theoretic image. But the situation isn't terrible: the underlying set of the scheme-theoretic image must be closed, and indeed it is the closure of the set-theoretic image. We might imagine that in reasonable cases this will be true, and in even nicer cases, the underlying set of the scheme-theoretic image will be set-theoretic image. We will later see that this is indeed the case.

But we feel obliged to show that pathologies can happen.

Example 4. Let $X = \coprod k[\epsilon_n]/(\epsilon_n^n)$ and $Y = \operatorname{Spec} k[x]$, and define $X \to Y$ by $x \to \epsilon_n$ on the nth component of X. Then if a function $g(x)$ on Y pulls back to 0 on X, then its Taylor expansion is 0 to order n (by examining the pullback to the nth component of X, so $g(x)$ must be 0. Thus the scheme-theoretic image is Y, while the set-theoretic image is easily seen to be just the origin.

This example clearly is weird though, and we can show that in "reasonable circumstances" such pathology doesn't occur. It would be great to compute the scheme-theoretic image affine-locally. On affine open set $\operatorname{Spec} B \subset Y$, define the ideal $I_B \subset B$ of functions which pullback to 0 on X. (Formally, $I_B := \ker(B \to \Gamma(f_*(\mathcal{O}_X), \operatorname{Spec} B)$.) Then if for each such B, and each $g \in B$, $I_B \otimes_B B_g \to I_{B_g}$ is an isomorphism, then we will have defined the pushforward subscheme (see Exercise 1.P). Clearly each function on $\operatorname{Spec} B$ that vanishes when pulled back to $f^{-1}(\operatorname{Spec} B)$ also vanishes when restricted to $D(g)$ and then pulled back to $f^{-1}(D(g))$. So the question is: given a function r/g^n on $D(g)$ that pulls back to $f^{-1}D(g)$, is it true that for some m, $rg^m = 0$ when pulled back to $f^{-1}(\operatorname{Spec} B)$? (i) The answer is clearly yes if $f^{-1}(\operatorname{Spec} B)$ is reduced: we simply take rg. (ii) The answer is also yes if $f^{-1}(\operatorname{Spec} B)$ is affine, say $\operatorname{Spec} A$: if $r' = f^{\#}r$ and $g' = f^{\#}g$ in A, then if $r' = 0$ on $D(g')$, then there is an m such that $r'(g')^m = 0$: $r' = 0$ in $D(g')$, which means precisely this fact. (iii) Furthermore, the answer is yes if $f^{-1}(\operatorname{Spec} B)$ is quasicompact: cover $f^{-1}(\operatorname{Spec} B)$ with finitely many affine open sets. For each one there will be some m_i so that $rg^{m_i} = 0$ when pulled back to this open set. Then let $m = \max(m_i)$. (We now see why quasicompactness is our friend!)

In conclusion, we have proved the following theorem.

2.2. *Theorem. — Suppose $f : X \to Y$ is a morphism of schemes. If X is reduced or f is quasicompact (e.g. if X is Noetherian, Exercise 1.B), then the scheme-theoretic image of f may be computed affine-locally.*

2.3. *Corollary. — Under the hypotheses of the previous theorem, the closure of the set-theoretic image of f is the underlying set of the scheme-theoretic image.*

Example 4 above shows that we cannot excise these hypotheses.

Proof. The set-theoretic image is clearly in the underlying set of the scheme-theoretic image. The underlying set of the scheme-theoretic image is closed, so the closure of the set-theoretic image is contained in underlying set of the scheme-theoretic image. On the other hand, if U is the complement of the closure of the set-theoretic image, $f^{-1}(U) = \varnothing$. As under these hypotheses, the scheme theoretic image can be computed locally, the scheme-theoretic image is the empty set on U. □

We conclude with a few stray remarks.

2.A. Easy Exercise. If X is reduced, show that the scheme-theoretic image of $f : X \to Y$ is also reduced.

More generally, you might expect there to be no unnecessary non-reduced structure on the image not forced by non-reduced structure on the source. We make this precise in the locally Noetherian case, when we can talk about associated points.

2.B. $\star$ UNIMPORTANT EXERCISE. If $f : X \to Y$ is a morphism of locally Noetherian schemes, show that the associated points of the image subscheme are a subset of the image of the associated points of X.

2.4. Aside: set-theoretic images can be nice too. I want to say a little more on what the set-theoretic image of a morphism can look like, although we'll hold off before proving these statements. We know that the set-theoretic image can be open (open immersion), and closed (closed immersions), and locally closed (locally closed immersions). But it can be weirder still: consider the example $\mathbb{A}^2 \to \mathbb{A}^2$ given by $(x, y) \mapsto (x, xy)$ mentioned earlier. The image is the plane, minus the y-axis, plus the origin. The image can be stranger still, and indeed if S is *any* subset of a scheme Y, it can be the image of a morphism: let X be the disjoint union of spectra of the residue fields of all the points of S, and let $f : X \to Y$ be the natural map. This is quite pathological, and in fact that if we are in any reasonable situation, the image is essentially no worse than arose in the previous example.

We define a **constructible subset of a Noetherian scheme** to be a subset which belongs to the smallest family of subsets such that (i) every open set is in the family, (ii) a finite intersection of family members is in the family, and (iii) the complement of a family member is also in the family. So for example the image of $(x, y) \mapsto (x, xy)$ is constructible.

Note that if $X \to Y$ is a morphism of schemes, then the preimage of a constructible set is a constructible set.

2.C. EXERCISE. Suppose X is a Noetherian scheme. Show that a subset of X is constructible if and only if it is the finite disjoint union of locally closed subsets.

Then if $f : X \to Y$ is a finite type morphism of Noetherian schemes, the image of any constructible set is constructible. This is *Chevalley's Theorem*, and we will prove it later. We will also have reasonable criteria for when the image is closed.

(For hardened experts only: [EGA 0_{III}.9.1] gives a definition of constructible in more generality. A *constructible subset of a topological space* X is a member of the Boolean algebra generated by open subsets U of X such that the inclusion $U \hookrightarrow X$ is quasicompact. If X is an affine scheme, or more generally quasicompact and quasiseparated, this is equivalent to U being quasicompact. A subset $Z \subset X$ is *locally constructible* if X admits an open covering $\{V_i\}$ such that $Z \cap V_i \subset V_i$ is constructible for each i. If X is quasicompact and quasiseparated, this is the same as $Z \subset X$ being constructible, so if X is a scheme, then it is equivalent to say that $Z \cap V$ is constructible for every affine open set V. The general form of Chevalley's constructibility theorem [EGA IV_1.1.8.4] is that the image of a locally constructible set under a finitely presented map, is also locally constructibility. Thanks to Brian Conrad for this!)

2.5. Scheme-theoretic closure of a locally closed subscheme.

We define the **scheme-theoretic closure** of a locally closed immersion $f : X \to Y$ as the scheme-theoretic image of X.

2.D. EXERCISE. If $X \to Y$ is quasicompact (e.g. if X is Noetherian, Exercise 1.B) or if X is reduced, show that the following three notions are the same. (Hint: Theorem niceschemetheoreticimage.)

(a) V is an open subscheme of X intersect a closed subscheme of X
(b) V is an open subscheme of a closed subscheme of X
(c) V is a closed subscheme of an open subscheme of X.

(Hint: it will be helpful to note that the scheme-theoretic image may be computed on each open subset of the base.)

2.E. UNIMPORTANT EXERCISE USEFUL FOR INTUITION. If $f : X \to Y$ is a locally closed immersion into a locally Noetherian scheme (so X is also locally Noetherian), then the associated points of the scheme-theoretic image are (naturally in bijection with) the associated points of X. (Hint: Exercise 2.B.) Informally, we get no non-reduced structure on the scheme-theoretic closure not "forced by" that on X.

2.6. The induced reduced subscheme structure on a closed subset.

Suppose X^{set} is a closed sub*set* of a scheme Y. Then we can define a canonical scheme structure X on X^{set}, that is reduced. We could describe it as being cut out by those functions whose values are zero at all the points of X^{set}. On affine open subset $\operatorname{Spec} B$ of Y, if the set X^{set} corresponds to the radical ideal $I = I(X^{set})$, the scheme X corresponds to $\operatorname{Spec} B/I$. We could also consider this construction as an example of a scheme-theoretic image in the following crazy way: let W be the scheme that is a disjoint union of all the points of X^{set}, where the point corresponding to p in X^{set} is Spec of the residue field of $\mathcal{O}_{Y,p}$. Let $f : W \to Y$ be the "canonical" map sending "p to p", and giving an isomorphism on residue fields. Then the scheme structure on X is the scheme-theoretic image of f. A third definition: it is the smallest closed subscheme whose underlying set contains X^{set}.

This construction is called the **induced reduced subscheme structure** on the closed subset X^{set}. (Vague exercise: Make a definition of the induced reduced subscheme structure precise and rigorous to your satisfaction.)

2.F. EXERCISE. Show that the underlying set of the induced reduced subscheme $X \to Y$ is indeed the closed subset X^{set}. Show that X is reduced.

2.7. Reduced version of a scheme.

In the special case where X^{set} all of Y, we obtain a *reduced closed subscheme* $Y^{red} \to Y$, called the **reduction** of Y. On affine open subset $\operatorname{Spec} B \hookrightarrow Y$, $Y^{red} \hookrightarrow Y$ corresponds to the nilradical $\mathfrak{N}(B)$ of B. The *reduction* of a scheme is the "reduced version" of the scheme, and informally corresponds to "shearing off the fuzz".

An alternative equivalent definition: on the affine open subset $\operatorname{Spec} B \hookrightarrow Y$, the reduction of Y corresponds to the ideal $\mathcal{N}(B) \subset Y$. As for any $f \in B$, $\mathcal{N}(B)_f = \mathcal{N}(B_f)$, by Exercise 1.P this defines a closed subscheme.

2.G. UNIMPORTANT EXERCISE (BUT USEFUL FOR VISUALIZATION). Show that if Y is a locally Noetherian scheme, the "reduced locus" of Y (where $Y^{red} \to Y$ is an isomorphism) is an open subset of Y. (In fact the non-reduced locus is a closure of certain associated points.)

3. MORE FINITENESS CONDITIONS ON MORPHISMS: (LOCALLY) OF FINITE TYPE, QUASIFINITE, (LOCALLY) OF FINITE PRESENTATION

3.1. Morphisms (locally of) finite type.

A morphism $f : X \to Y$ is **locally of finite type** if for every affine open set $\operatorname{Spec} B$ of Y, $f^{-1}(\operatorname{Spec} B)$ can be covered with open sets $\operatorname{Spec} A_i$ so that the induced morphism $B \to A_i$ expresses A_i as a finitely generated B-algebra. By the affine-locality of finite-typeness of B-schemes, this is equivalent to: for *every* affine open set $\operatorname{Spec} A_i$ in X, A_i is a finitely generated B-algebra.

A morphism is **of finite type** if it is locally of finite type and quasicompact. Translation: for every affine open set $\operatorname{Spec} B$ of Y, $f^{-1}(\operatorname{Spec} B)$ can be covered with *a finite number of* open sets $\operatorname{Spec} A_i$ so that the induced morphism $B \to A_i$ expresses A_i as a finitely generated B-algebra.

3.A. EXERCISE (THE NOTIONS "LOCALLY OF FINITE TYPE" AND "FINITE TYPE" ARE AFFINE-LOCAL ON THE TARGET). Show that a morphism $f : X \to Y$ is locally of finite type if there is a cover of Y by affine open sets $\operatorname{Spec} B_i$ such that $f^{-1}(\operatorname{Spec} B_i)$ is locally of finite type over B_i.

3.B. EXERCISE. Show that a morphism $f : X \to Y$ is locally of finite type if for *every* affine open subsets $\operatorname{Spec} A \subset X$, $\operatorname{Spec} B \subset Y$, with $f(\operatorname{Spec} A) \subset \operatorname{Spec} B$, A is a finitely generated B-algebra. (Hint: use the affine communication lemma on $f^{-1}(\operatorname{Spec} B)$.)

Example: the "structure morphism" $\mathbb{P}^n_A \to \operatorname{Spec} A$ is of finite type, as $\mathbb{P}^n_A$ is covered by $n+1$ open sets of the form $\operatorname{Spec} A[x_1, \dots, x_n]$. More generally, $\operatorname{Proj} S_* \to \operatorname{Spec} A$ (where $S_0 = A$) is of finite type.

More generally still: our earlier definition of schemes of "finite type over k" (or "finite type k-schemes") is now a special case of this more general notion: a scheme X is of finite type over k means that we are given a morphism $X \to \operatorname{Spec} k$ (the "structure morphism") that is of finite type.

Here are some properties enjoyed by morphisms of finite type.

3.C. EASY EXERCISE. Show that finite morphisms are of finite type. Hence closed immersions are of finite type.

3.D. EXERCISES (NOT HARD, BUT IMPORTANT).

(a) Show that an open immersion is locally of finite type. Show that an open immersion into a locally Noetherian scheme is of finite type. More generally, show that a quasicompact open immersion is of finite type.
(b) Show that the composition of two morphisms of locally finite type is locally of finite type. (Hence as quasicompact morphisms also compose, the composition of two morphisms of finite type is also of finite type.)
(c) Suppose we have morphisms $X \xrightarrow{f} Y \xrightarrow{g} Z$, with f quasicompact, and $g \circ f$ of finite type. Show that f is finite type.
(d) Suppose $f : X \to Y$ is finite type, and Y is Noetherian. Show that X is also Noetherian.

A morphism f is **quasifinite** if it is of finite type, and for all $y \in Y$, $f^{-1}(y)$ is a finite set. The main point of this definition is the "finite fiber" part; the "finite type" part is there so this notion is "preserved by fibered product" (an exercise in the class on fiber products next week).

3.E. EXERCISE. Show that finite morphisms are quasifinite. (This is a useful exercise, because you will have to figure out how to figure out how to get at points in a fiber of a morphism: given $f : X \to Y$, and $y \in Y$, what are the points of $f^{-1}(y)$? Here is a hint: if $X = \operatorname{Spec} A$ and $Y = \operatorname{Spec} B$ are both affine, and $y = [\mathfrak{p}]$, then we can throw out everything in A outside $\overline{y}$ by modding out by $\mathfrak{p}$; you can show that the preimage is $A/\mathfrak{p}$. Then we have reduced to the case where Y is the Spec of an integral domain, and $[\mathfrak{p}] = [0]$ is the generic point. We can throw out the rest of the points by localizing at 0. You can show that the preimage is $(A_{\mathfrak{p}})/\mathfrak{p}A_{\mathfrak{p}}$. Then, once you have shown that finiteness behaves well with respect to the operations you made done, you have reduced the problem to Exercise 1.M.)

There are quasifinite morphisms which are not finite, for example $\mathbb{A}^2 - \{(0,0\} \to \mathbb{A}^2$ (Example 1.7). The key example of a morphism with finite fibers that is not quasifinite is $\operatorname{Spec} \overline{\mathbb{Q}} \to \operatorname{Spec} \mathbb{Q}$.

How to picture quasifinite morphisms, thanks go Brian Conrad. If $X \to Y$ is a finite morphism, then quasi-compact open subset $U \subset X$ is quasi-finite over Y. In fact *every* reasonable quasifinite morphism arises in this way. Thus the right way to visualize quasifiniteness is as a finite map with some (closed locus of) points removed.

3.2. $\star$ *Morphisms (locally) of finite presentation.* There is a variant often of use to non-Noetherian people. A morphism $f : X \to Y$ is **locally of finite presentation** (or **locally finitely presented**) if for each affine open subset $\operatorname{Spec} B$ of Y, $f^{-1}(\operatorname{Spec} B) == \cup_i \operatorname{Spec} A_i$ with $B \to A_i$ finitely presented (finitely generated with a finite number of relations). A morphism is of **finite presentation** (or **finitely presented**) if it is locally of finite presentation and quasicompact.

If X is locally Noetherian, then locally of finite presentation is the same as locally of finite type, and finite presentation is the same as finite type. So if you are a Noetherian person, you needn't worry about this notion.

3.F. EXERCISE. Show that the notion of "locally finite presentation" is affine-local.

3.G. $\star\star$ EXERCISE: LOCALLY OF FINITE PRESENTATION IS A PURELY CATEGORICAL NOTION. Show that "locally of finite presentation" is equivalent to the following. If $F : (Sch/Y) \to (Sets)$, $S \mapsto \operatorname{Hom}_Y(S, X)$, we require F to commute with direct limits, i.e. if $\{A_i\}$ is a direct system, then $F(\varinjlim A_i) = \varinjlim F(A_i)$.

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FOUNDATIONS OF ALGEBRAIC GEOMETRY CLASS 14

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CONTENTS

Today: projective schemes.

1. INTRODUCTION

At this point, we know that we can construct schemes by gluing affine schemes together. If a large number of affine schemes are involved, this can obviously be a laborious and tedious process. Our example of closed subschemes of projective space showed that we could piggyback on the construction of projective space to produce complicated and interesting schemes. In this chapter, we formalize this notion of *projective schemes*. Projective schemes over the complex numbers give good examples (in the classical topology) of compact complex varieties. In fact they are such good examples that it is quite hard to come up with an example of a compact complex variety that is provably not projective. (We will see examples later, although we won't concern ourselves with the relationship to the classical topology.) Similarly, it is quite hard to come up with an example of a complex variety that is provably not an open subset of a projective variety. In particular, most examples of complex varieties that come up in nature are of this form. More generally, projective schemes will be the key example of the algebro-geometric analogue of compactness (*properness*). Thus one advantage of the notion of projective scheme is that it encapsulates much of the algebraic geometry arising in nature.

In fact our example from last day already gives the notion of projective A-schemes in full generality. Recall that any collection of homogeneous elements of $A[x_0, \dots, x_n]$ describes a closed subscheme of $\mathbb{P}^n_A$. Any closed subscheme of $\mathbb{P}^n_A$ cut out by a set of homogeneous polynomials will be called a *projective A-scheme*. (You may be initially most interested in the "classical" case where A is an algebraically closed field.) If I is the ideal

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in $A[x_0, \dots, x_n]$ generated by these homogeneous polynomials, the scheme we have constructed will be called $\operatorname{Proj} A[x_0, \dots, x_n]/I$. Then $x_0, \dots, x_n$ are informally said to be *projective coordinates* on the scheme. Warning: they are not functions on the scheme. (We will later interpret them as sections of a line bundle.) This lecture will reinterpret this example in a more useful language. For example, just as there is a rough dictionary between rings and affine schemes, we will have an analogous dictionary between graded rings and projective schemes. Just as one can work with affine schemes by instead working with rings, one can work with projective schemes by instead working with graded rings.

1.1. A motivating picture from classical geometry.

We motivate a useful way of picturing projective schemes by recalling how one thinks of projective space "classically" (in the classical topology, over the real numbers). $\mathbb{P}^n$ can be interpreted as the lines through the origin in $\mathbb{R}^{n+1}$. Thus subsets of $\mathbb{P}^n$ correspond to unions of lines through the origin of $\mathbb{R}^{n+1}$, and closed subsets correspond to such unions which are closed. (The same is not true with "closed" replaced by "open"!)

One often pictures $\mathbb{P}^n$ as being the "points at infinite distance" in $\mathbb{R}^{n+1}$, where the points infinitely far in one direction are associated with the points infinitely far in the opposite direction. We can make this more precise using the decomposition

$$\mathbb{P}^{n+1} = \mathbb{R}^{n+1} \coprod \mathbb{P}^n \tag{1}$$

by which we mean that there is an open subset in $\mathbb{P}^{n+1}$ identified with $\mathbb{R}^{n+1}$ (the points with last projective co-ordinate non-zero), and the complementary closed subset identified with $\mathbb{P}^n$ (the points with last projective co-ordinate zero).

Then for example any equation cutting out some set V of points in $\mathbb{P}^n$ will also cut out some set of points in $\mathbb{R}^n$ that will be a closed union of lines. We call this the *affine cone* of V. These equations will cut out some union of $\mathbb{P}^1$'s in $\mathbb{P}^{n+1}$, and we call this the *projective cone* of V. The projective cone is the disjoint union of the affine cone and V. For example, the affine cone over $x^2 + y^2 = z^2$ in $\mathbb{P}^2$ is just the "classical" picture of a cone in $\mathbb{R}^2$, see Figure 1.

We will make this analogy precise in our algebraic setting in §2.3.

2. THE Proj CONSTRUCTION

Let's abstract these notions, just as we abstracted the notion of the Spec of a ring with given generators and relations over k to the Spec of a ring in general.

In the examples we've seen, we have a graded ring $A[x_0, \dots, x_n]/I$ where I is a **homogeneous ideal** (i.e. I is generated by homogeneous elements of $A[x_0, \dots, x_n]$). Here we are taking the usual grading on $A[x_0, \dots, x_n]$, where each x_i has weight 1. Then $A[x_0, \dots, x_n]/I$ is also a graded ring $S_\bullet$, and we'll call its graded pieces S_0, S_1, etc. (The subscript $\bullet$ in $S_\bullet$ is intended to remind us of the indexing. In a graded ring, multiplication

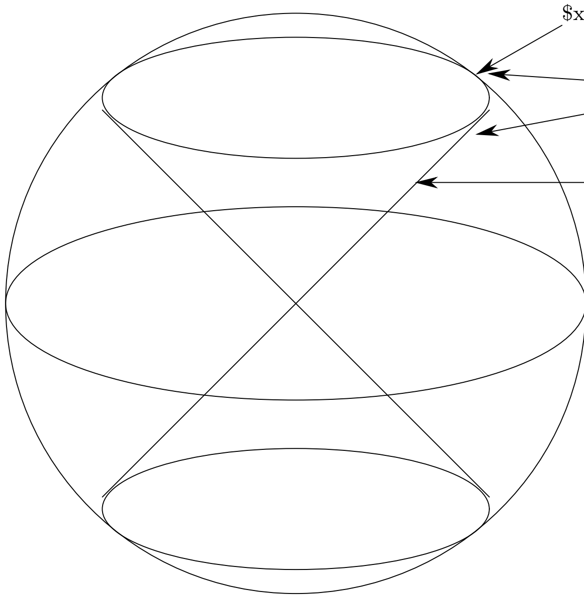

FIGURE 1. The affine and projective cone of $x^2 + y^2 = z^2$ in classical geometry

sends $S_m \times S_n$ to S_{m+n}. Note that S_0 is a subring, and S is a S_0-algebra.) In our examples that $S_0 = A$, and $S_\bullet$ is generated over S_0 by S_1.

2.1. Standing assumptions about graded rings. We make some standing assumptions on graded rings. Fix a ring A (the **base ring**). Our motivating example is $S_\bullet = A[x_0, x_1, x_2]$, with the usual grading. **Assume that $S_\bullet$ is graded by $\mathbb{Z}^{\geq 0}$, with $S_0 = A$.** Hence each S_n is an A-module. The subset $S_+ := \oplus_{i>0} S_i \subset S_\bullet$ is an ideal, called the **irrelevant ideal**. The reason for the name "irrelevant" will be clearer soon. **Assume that the irrelevant ideal S_+ is a finitely-generated ideal.**

2.A. EXERCISE. Show that $S_\bullet$ is a finitely-generated graded ring if and only if $S_\bullet$ is a finitely-generated graded A-algebra, i.e. generated over $A = S_0$ by a finite number of homogeneous elements of positive degree. (Hint for the forward implication: show that the generators of S_+ as an ideal are also generators of $S_\bullet$ as an algebra.)

If these assumptions hold, we say that $S_\bullet$ is a **finitely generated graded ring**.

We now define a scheme $\operatorname{Proj} S_\bullet$. You won't be surprised that we will define it as a *set*, with a *topology*, and a *structure sheaf*.

The set. The points of $\operatorname{Proj} S_\bullet$ are defined to be those homogeneous prime ideals *not containing the irrelevant ideal* S_+. The homogeneous primes containing the irrelevant ideal are irrelevant.

For example, if $S_\bullet = k[x,y,z]$ with the usual grading, then $(z^2 - x^2 - y^2)$ is a homogeneous prime ideal. We picture this as a subset of $\operatorname{Spec} S_\bullet$; it is a cone (see Figure 1). We picture $\mathbb{P}^2_k$ as the "plane at infinity". Thus we picture this equation as cutting out a conic "at infinity". We will make this intuition somewhat more precise in §2.3.

The topology. As with affine schemes, we define the Zariski topology by describing the closed subsets. They are of the form $V(I)$, where I is a homogeneous ideal. (Here $V(I)$ has essentially the same definition as before: those *homogeneous* prime ideals containing I.) Particularly important open sets will the **distinguished open sets** $D(f) = \operatorname{Proj} S_\bullet \setminus V(f)$, where $f \in S_+$ is homogeneous.

2.B. EASY EXERCISE. Verify that the distinguished open sets form a base of the topology. (The argument is essentially identical to the affine case.)

As with the affine case, if $D(f) \subset D(g)$, then $f^n \in (g)$ for some n, and vice versa. Clearly $D(f) \cap D(g) = D(fg)$, by the same immediate argument as in the affine case.

The structure sheaf. We define $\boxed{\mathcal{O}_{\operatorname{Proj} S_\bullet}(D(f)) = ((S_\bullet)_f)_0}$ where $((S_\bullet)_f)_0$ means the 0-graded piece of the graded ring $(S_\bullet)_f$. (The notation $((S_\bullet)_f)_0$ is admittedly unfortunate — the first and third subscripts refer to the grading, and the second refers to localization.) As in the affine case, we define restriction maps, and verify that this is well-defined (i.e. if $D(f) = D(f')$, then we are defining the same ring, and that the restriction maps are well-defined).

For example, if $S_\bullet = k[x_0, x_1, x_2]$ and $f = x_0$, we get $(k[x_0, x_1, x_2]_{x_0})_0 := k[x_{1/0}, x_{2/0}]$ (using our earlier language for projective patches).

We now check that this is a sheaf. We could show that this is a sheaf on the base, and the argument would be as in the affine case (which was not easy). Here instead is a sneakier argument. We first note that the topological space $D(f)$ and $\operatorname{Spec}((S_\bullet)_f)_0$ are canonically homeomorphic: they have matching distinguished bases. (To the distinguished open $D(g) \cap D(f)$ of $D(f)$, we associate $D(g^{\deg f}/f^{\deg g})$ in $\operatorname{Spec}(S_f)_0$. To $D(h)$ in $\operatorname{Spec}(S_f)_0$, we associate $D(f^n h) \subset D(f)$, where n is chosen large enough so that $f^n h \in S_\bullet$.) Second, we note that the sheaf of rings on the distinguished base of $D(f)$ can be associated (via this homeomorphism just described) with the sheaf of rings on the distinguished base of $\operatorname{Spec}((S_\bullet)_f)_0$: the sections match (the ring of sections $((S_\bullet)_{fg})_0$ over $D(g) \cap D(f) \subset D(f)$, those homogeneous degree 0 quotients of $S_\bullet$ with f's and g's in the denominator, is naturally identified with the ring of sections over the corresponding open set of $\operatorname{Spec}((S_\bullet)_f)_0$) and the restriction maps clearly match (think this through yourself!). Thus we have described an isomorphism of schemes

$$(D(f), \mathcal{O}_{\operatorname{Proj} S_\bullet}) \cong \operatorname{Spec}(S_f)_0.$$

2.C. EASY EXERCISE. Describe a natural "structure morphism" $\operatorname{Proj} S_\bullet \to \operatorname{Spec} A$.

2.2. Projective and quasiprojective schemes.

We call a scheme of the form $\operatorname{Proj} S_\bullet$ (where $S_0 = A$) a **projective scheme over** A, or a **projective** A**-scheme**. A **quasiprojective** A**-scheme** is an open subscheme of a projective A-scheme. The "A" is omitted if it is clear from the context; often A is some field.

We now make a connection to classical terminology. A **projective variety (over** k**)**, or an **projective** k**-variety** is a *reduced* projective k-scheme. (Warning: in the literature, it is sometimes also required that the scheme be irreducible, or that k be algebraically closed.) A **quasiprojective** k**-variety** is an open subscheme of a projective k-variety. We defined affine varieties earlier, and you can check that affine open subsets of projective k-varieties are affine k-varieties. We will define varieties in general later.

The notion of quasiprojective k-scheme is a good one, covering most interesting cases which come to mind. We will see before long that the affine line with the doubled origin is not quasiprojective for somewhat silly reasons ("non-Hausdorffness"), but we'll call that kind of bad behavior "non-separated". Here is a surprisingly subtle question: *Are there quasicompact* k*-schemes that are not quasiprojective?* Translation: if we're gluing together a finite number of schemes each sitting in some $\mathbb{A}^n_k$, can we ever get something not quasiprojective? We will finally answer this question in the negative in the next quarter.

2.D. EASY EXERCISE. Show that all projective A-schemes are quasicompact. (Translation: show that any projective A-scheme is covered by a finite number of affine open sets.) Show that $\operatorname{Proj} S_\bullet$ is finite type over $A = S_0$. If S_0 is a Noetherian ring, show that $\operatorname{Proj} S_\bullet$ is a Noetherian scheme, and hence that $\operatorname{Proj} S_\bullet$ has a finite number of irreducible components. Show that any quasiprojective scheme is locally of finite type over A. If A is Noetherian, show that any quasiprojective A-scheme is quasicompact, and hence of finite type over A. Show this need not be true if A is not Noetherian. Better: give an example of a quasiprojective A-scheme that is not quasicompact (necessarily for some non-Noetherian A). (Hint: Flip ahead to silly example 3.2.)

2.3. Affine and projective cones.

If $S_\bullet$ is a finitely-generated graded ring, then the **affine cone** of $\operatorname{Proj} S_\bullet$ is $\operatorname{Spec} S_\bullet$. Note that this construction depends on $S_\bullet$, not just of $\operatorname{Proj} S_\bullet$. As motivation, consider the graded ring $S_\bullet = \mathbb{C}[x, y, z]/(z^2 - x^2 - y^2)$. Figure 2 is a sketch of $\operatorname{Spec} S_\bullet$. (Here we draw the "real picture" of $z^2 = x^2 + y^2$ in $\mathbb{R}^3$.) It is a cone in the most traditional sense; the origin $(0, 0, 0)$ is the "cone point".

This gives a useful way of picturing Proj (even over arbitrary rings than $\mathbb{C}$). Intuitively, you could imagine that if you discarded the origin, you would get something that would project onto $\operatorname{Proj} S_\bullet$. The following exercise makes that precise.

2.E. EXERCISE. If $S_\bullet$ is a projective scheme over a field k, Describe a natural morphism $\operatorname{Spec} S_\bullet \setminus \{0\} \to \operatorname{Proj} S_\bullet$.

This has the following generalization to A-schemes, which you might find geometrically reasonable. This again motivates the terminology "irrelevant".

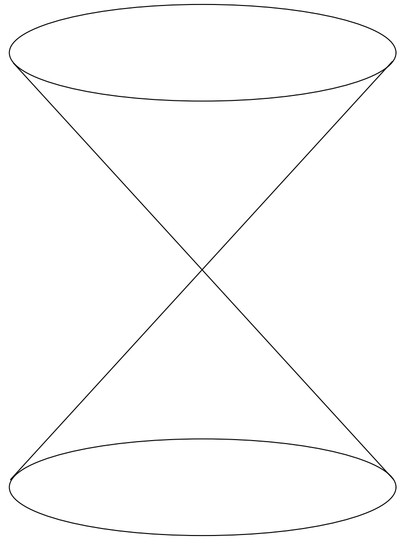

FIGURE 2. A sketch of the cone $\operatorname{Spec} k[x, y, z]/(z^2 - x^2 - y^2)$.

2.F. EXERCISE. If $S_\bullet$ is a projective A-scheme, describe a natural morphism $\operatorname{Spec} S_\bullet \setminus V(S_+) \to \operatorname{Proj} S_\bullet$.

In fact, it can be made precise that $\operatorname{Proj} S_\bullet$ is the affine cone, minus the origin, modded out by multiplication by scalars.

The **projective cone** of $\operatorname{Proj} S_\bullet$ is $\operatorname{Proj} S_\bullet[T]$, where T is a new variable of degree 1. For example, the cone corresponding to the conic $\operatorname{Proj} k[x, y, z]/(z^2 - x^2 - y^2)$ is $\operatorname{Proj} k[x, y, z, T]/(z^2 - x^2 - y^2)$.

2.G. EXERCISE (CF. (1)). Show that the projective cone of $\operatorname{Proj} S_\bullet[T]$ has a closed subscheme isomorphic to $\operatorname{Proj} S_\bullet$ (corresponding to $T = 0$), whose complement (the distinguished open set $D(T)$) is isomorphic to the affine cone $\operatorname{Spec} S_\bullet$.

You can also check that $\operatorname{Proj} S_\bullet$ is a locally principal closed subscheme, and is also locally not a zero-divisor (an *effective Cartier divisor*).

This construction can be usefully pictured as the affine cone union some points "at infinity", and the points at infinity form the Proj. The reader may which to ponder Figure 2, and try to visualize the conic curve "at infinity".

We have thus completely discribed the algebraic analog of the classical picture of 1.1.

3. EXAMPLES

3.1. *Example.* We (re)define projective space by $\mathbb{P}^n_A := \operatorname{Proj} A[x_0, \ldots, x_n]$. This definition involves no messy gluing, or choice of special patches.

3.A. EXERCISE. Check that this agrees with our earlier version of projective space.

3.2. *Silly example.* Note that $\mathbb{P}^0_A = \operatorname{Proj} A[T] \cong \operatorname{Spec} A$. Thus "$\operatorname{Spec} A$ is a projective A-scheme".

Here is a useful generalization of this example that I forgot to say in class:

3.B. EXERCISE: FINITE MORPHISMS TO $\operatorname{Spec} A$ ARE PROJECTIVE. If B is a finitely generated A-algebra, define $S_\bullet$ by $S_0 = A$, and $S_n = B$ for $n > 0$ (with the obvious graded ring structure). Describe an isomorphism

$$\begin{array}{ccc} \operatorname{Proj} S_\bullet & \overset{\sim}{\longleftrightarrow} & \operatorname{Spec} B \\ & \searrow \quad \swarrow & \\ & \operatorname{Spec} A & \end{array}$$

3.C. EXERCISE. Show that $X = \mathbb{P}^2_k \setminus \{x^2 + y^2 = z^2\}$ is an affine scheme. Show that $x = 0$ cuts out a locally principal closed subscheme that is not principal.

3.3. Example: $\mathbb{P}V$. We can make this definition of projective space even more choice-free as follows. Let V be an $(n+1)$-dimensional vector space over k. (Here k can be replaced by any ring A as usual.) Let $\operatorname{Sym}^\bullet V^\vee = k \oplus V^\vee \oplus \operatorname{Sym}^2 V^\vee \oplus \cdots$. (The reason for the dual is explained by the next exercise.) If for example V is the dual of the vector space with basis associated to $x_0, \ldots, x_n$, we would have $\operatorname{Sym}^\bullet V^\vee = k[x_0, \ldots, x_n]$. Then we can define $\mathbb{P}V := \operatorname{Proj} \operatorname{Sym}^\bullet V^\vee$. In this language, we have an interpretation for x_0, ..., x_n: they are the linear functionals on the underlying vector space V.

3.D. UNIMPORTANT EXERCISE. Suppose k is algebraically closed. Describe a natural bijection between one-dimensional subspaces of V and the points of $\mathbb{P}V$. Thus this construction canonically (in a basis-free manner) describes the one-dimensional subspaces of the vector space $\operatorname{Spec} V$.

On a related note: you can also describe a natural bijection between points of V and the points of $\operatorname{Spec} \operatorname{Sym}^\bullet V^\vee$. This construction respects the affine/projective cone picture of §2.3.

As maps of rings correspond to maps of affine schemes in the opposite direction, maps of graded rings sometimes give maps of projective schemes in the opposite direction. Before we make this precise, let's see an example to see what can go wrong. There isn't quite a map $\mathbb{P}^2_k \to \mathbb{P}^1_k$ given by $[x;y;z] \to [x;y]$, because this alleged map isn't defined only at the point $[0;0;1]$. What has gone wrong? The map $\mathbb{A}^3 = \operatorname{Spec} k[x,y,z] \to \mathbb{A}^2 = \operatorname{Spec} k[x,y]$ makes perfect sense. However, the z-axis in $\mathbb{A}^3$ maps to the origin in $\mathbb{A}^2$, so the point of $\mathbb{P}^2$ corresponding to the z-axis maps to the "cone point" of the affine cone, and hence not to the projective scheme. The image of this point of $\mathbb{A}^2$ contains the irrelevant ideal. If this problem doesn't occur, a map of rings gives a map of projective schemes in the opposite direction.

4.A. IMPORTANT EXERCISE. (a) Suppose that $f : S_\bullet \longrightarrow R_\bullet$ is a morphism of finitely-generated graded rings (i.e. a map of rings preserving the grading) over A. Suppose further that

$$f(S_+) \supset R_+. \tag{2}$$

Show that this induces a morphism of schemes $\operatorname{Proj} R_\bullet \to \operatorname{Proj} S_\bullet$. (Warning: not every morphism arises in this way.)
(b) Suppose further that $S_\bullet \longrightarrow\!\!\!\!\!\rightarrow R_\bullet$ is a surjection of finitely-generated graded rings (so (2) is automatic). Show that the induced morphism $\operatorname{Proj} R_\bullet \to \operatorname{Proj} S_\bullet$ is a closed immersion. (Warning: not every closed immersion arises in this way!)

Hypothesis (2) can be replaced with the weaker hypothesis $\sqrt{f(S_+)} \supset R_+$, but in practice this hypothesis (2) suffices.

4.B. EXERCISE. Show that an injective linear map of k-vector spaces $V \hookrightarrow W$ induces a closed immersion $\mathbb{P}V \hookrightarrow \mathbb{P}W$.

This closed subscheme is called a **linear space**. Once we know about dimension, we will call this a linear space of dimension $\dim V - 1 = \dim \mathbb{P}V$. A linear space of dimension 1 (resp. 2, n, $\dim \mathbb{P}W - 1$) is called a **line** (resp. **plane**, **n-plane**, **hyperplane**). (If the linear map in the previous exercise is not injective, then the hypothesis (2) of Exercise 4.A fails.)

4.1. A particularly nice case: when $S_\bullet$ is generated in degree 1. If $S_\bullet$ is generated by S_1 as an A-algebra, we say that $S_\bullet$ is **generated in degree** 1.

4.C. EXERCISE. Suppose $S_\bullet$ is a finitely generated graded ring generated in degree 1. Show that S_1 is a finitely-generated module, and the irrelevant ideal S_+ is generated in degree 1.

4.D. EXERCISE. Show that if $S_\bullet$ is generated by S_1 (as an A-algebra) by $n+1$ elements $x_0, \dots, x_n$, then $\operatorname{Proj} S_\bullet$ may be described as a closed subscheme of $\mathbb{P}^n_A$ as follows. Consider A^{n+1} as a free module with generators $t_0, \dots, t_n$ associated to $x_0, \dots, x_n$. The surjection of

$$\operatorname{Sym}^\bullet A^{n+1} = A[t_0, t_1, \dots, t_n] \twoheadrightarrow S_\bullet$$

implies $S_\bullet = A[t_0, t_1, \dots t_n]/I$, where I is a homogeneous ideal.

This is completely analogous to the fact that if R is a finitely-generated A-algebra, then choosing n generators of R as an algebra is the same as describing $\operatorname{Spec} R$ as a closed subscheme of $\mathbb{A}^n_A$. In the affine case this is "choosing coordinates"; in the projective case this is "choosing projective coordinates".

For example, $\operatorname{Proj} k[x, y, z]/(z^2 - x^2 - y^2)$ is a closed subscheme of $\mathbb{P}^2_k$. (A picture is shown in Figure 2.)

Recall that we can interpret the closed points of $\mathbb{P}^n$ as the lines through the origin in $\mathbb{A}^{n+1}$. The following exercise states this more generally.

4.E. EXERCISE. Suppose $S_\bullet$ is a finitely-generated graded ring over an algebraically closed field k, generated in degree 1 by $x_0, \dots, x_n$, inducing closed immersions $\operatorname{Proj} S_\bullet \hookrightarrow \mathbb{P}^n$ and $\operatorname{Spec} S_\bullet \hookrightarrow \mathbb{A}^n$. Describe a natural bijection between the closed points of $\operatorname{Proj} S_\bullet$ and the "lines through the origin" in $\operatorname{Spec} S_\bullet \subset \mathbb{A}^n$.

5. IMPORTANT EXERCISES

There are many fundamental properties that are best learned by working through problems.

5.1. Analogues of results on affine schemes.

5.A. EXERCISE.

(a) Suppose I is any homogeneous ideal, and f is a homogeneous element. Show that f vanishes on $V(I)$ if and only if $f^n \in I$ for some n. (Hint: Mimic the affine case; see an earlier exercise.)
(b) If $Z \subset \operatorname{Proj} S_\bullet$, define $I(\cdot)$. Show that it is a homogeneous ideal. For any two subsets, show that $I(Z_1 \cup Z_2) = I(Z_1) \cap I(Z_2)$.
(c) For any subset $Z \subset \operatorname{Proj} S_\bullet$, show that $V(I(Z)) = \overline{Z}$.

5.B. EXERCISE. Show that the following are equivalent. (This is more motivation for the S_+ being "irrelevant": any ideal whose radical contains it is "geometrically irrelevant".)

(a) $V(I) = \varnothing$

(b) for any f_i (i in some index set) generating I, $\cup D(f_i) = \operatorname{Proj} S_\bullet$
(c) $\sqrt{I} \supset S_+$.

5.2. Scaling the grading, and the Veronese embedding.

Here is a useful construction. Define $S_{n\bullet} = \oplus_{j=0}^{\infty} S_{nj}$. (We could rescale our degree, so "old degree" n is "new degree" 1.)

5.C. EXERCISE. Show that $\operatorname{Proj} S_{n\bullet}$ is isomorphic to $\operatorname{Proj} S_\bullet$.

5.D. EXERCISE. Suppose $S_\bullet$ is generated over S_0 by $f_1, \dots, f_n$. Find a d such that $S_{d\bullet}$ is generated in "new" degree 1 (= "old" degree d). This is handy, as it means that, using the previous Exercise 5.C, we can assume that any finitely-generated graded ring is generated in degree 1. In particular, we can place every Proj in some projective space via the construction of Exercise 4.D.

Example: Suppose $S_\bullet = k[x,y]$, so $\operatorname{Proj} S_\bullet = \mathbb{P}^1_k$. Then $S_{2\bullet} = k[x^2, xy, y^2] \subset k[x,y]$. We identify this subring as follows.

5.E. EXERCISE. Let $u = x^2$, $v = xy$, $w = y^2$. Show that $S_{2\bullet} = k[u,v,w]/(uw - v^2)$.

We have a graded ring generated by three elements in degree 1. Thus we think of it as sitting "in" $\mathbb{P}^2$, via the construction of §4.D. This can be interpreted as "$\mathbb{P}^1$ as a conic in $\mathbb{P}^2$".

Thus if k is algebraically closed of characteristic not 2, using the fact that we can diagonalize quadrics, the conics in $\mathbb{P}^2$, up to change of co-ordinates, come in only a few flavors: sums of 3 squares (e.g. our conic of the previous exercise), sums of 2 squares (e.g. $y^2 - x^2 = 0$, the union of 2 lines), a single square (e.g. $x^2 = 0$, which looks set-theoretically like a line, and is non-reduced), and 0 (not really a conic at all). Thus we have proved: any plane conic (over an algebraically closed field of characteristic not 2) that can be written as the sum of three squares is isomorphic to $\mathbb{P}^1$.

We now soup up this example.

5.F. EXERCISE. Show that $\operatorname{Proj} S_{3\bullet}$ is the *twisted cubic* "in" $\mathbb{P}^3$.

5.G. EXERCISE. Show that $\operatorname{Proj} S_{d\bullet}$ is given by the equations that

$$\begin{pmatrix} y_0 & y_1 & \cdots & y_{d-1} \\ y_1 & y_2 & \cdots & y_d \end{pmatrix}$$

is rank 1 (i.e. that all the 2×2 minors vanish). This is called the **degree d rational normal curve** "in" $\mathbb{P}^d$.

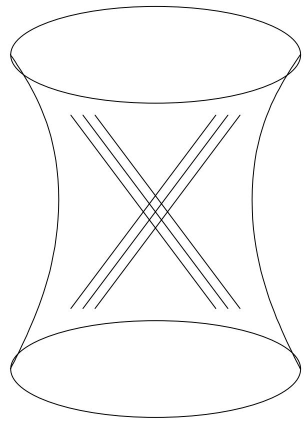

FIGURE 3. The two rulings on the quadric surface $V(wz - xy) \subset \mathbb{P}^3$. One ruling contains the line $V(w, x)$ and the other contains the line $V(w, y)$.

More generally, if $S_\bullet = k[x_0, \dots, x_n]$, then $\operatorname{Proj} S_{d\bullet} \subset \mathbb{P}^{N-1}$ (where N is the number of degree d polynomials in $x_0, \dots, x_n$) is called the **d-uple embedding** or **d-uple Veronese embedding**. It is enlightening to interpret this closed immersion as a map of graded rings.

5.H. COMBINATORIAL EXERCISE. Show that $N = \binom{n+d}{d}$.

5.I. UNIMPORTANT EXERCISE. Find five linearly independent quadric equations vanishing on the **Veronese surface** $\operatorname{Proj} S_{2\bullet}$ where $S_\bullet = k[x_0, x_1, x_2]$, which sits naturally in $\mathbb{P}^5$. (You needn't show that these equations generate all the equations cutting out the Veronese surface, although this is in fact true.)

5.3. Entertaining geometric exercises.

5.J. USEFUL GEOMETRIC EXERCISE. Describe all the lines on the quadric surface $wz-xy = 0$ in $\mathbb{P}^3_k$. (Hint: they come in two "families", called the **rulings** of the quadric surface.) This construction arises all over the place in nature.

Hence (by diagonalization of quadrics), if we are working over an algebraically closed field of characteristic not 2, we have shown that all rank 4 quadric surfaces have two rulings of lines.

5.K. EXERCISE. Show that $\mathbb{P}^n_k$ is normal. More generally, show that $\mathbb{P}^n_A$ is normal if A is a Unique Factorization Domain.

5.4. *Example.* If we put a non-standard weighting on the variables of $k[x_1, \dots, x_n]$ — say we give x_i degree d_i — then $\operatorname{Proj} k[x_1, \dots, x_n]$ is called **weighted projective space** $\mathbb{P}(d_1, d_2, \dots, d_n)$.

5.L. EXERCISE. Show that $\mathbb{P}(m, n)$ is isomorphic to $\mathbb{P}^1$. Show that

$$\mathbb{P}(1, 1, 2) \cong \operatorname{Proj} k[u, v, w, z]/(uw - v^2).$$

Hint: do this by looking at the even-graded parts of $k[x_0, x_1, x_2]$, cf. Exercise 5.C.

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FOUNDATIONS OF ALGEBRAIC GEOMETRY CLASSES 15 AND 16

RAVI VAKIL

CONTENTS

This week we discussed fibered products and separatedness.

1. FIBERED PRODUCTS OF SCHEMES EXIST

We will now construct the fibered product in the category of schemes. In other words, given $X, Y \to Z$, we will show that $X \times_Z Y$ exists. (Recall that the *absolute product* in a category is the fibered product over the final object, so $X \times Y = X \times_{\mathbb{Z}} Y$ in the category of schemes, and $X \times Y = X \times_S Y$ if we are implicitly working in the category of S-schemes, for example if S is the spectrum of a field.) Notational warning: lazy people wanting to save chalk and ink will write $\times_k$ for $\times_{\operatorname{Spec} k}$, and similarly for $\times_{\mathbb{Z}}$. It already happened in the paragraph above!

Before we get started, we'll make a few random remarks.

Remark 1. We've made a big deal about schemes being *sets*, endowed with a *topology*, upon which we have a *structure sheaf*. So you might think that we'll construct the product in this order. However, here is a sign that something interesting happens at the level of sets that will mess up this strategy. you should believe that if we take the product of two affine lines (over your favorite algebraically closed field k, say), you should get the affine plane: $\mathbb{A}^1_k \times_k \mathbb{A}^1_k$ should be $\mathbb{A}^2_k$. And we'll see that this is indeed true. But the underlying set of the latter is *not* the underlying set of the former —- we get additional points! Thus products of schemes do something a little subtle on the level of sets.

Date: Monday, November 12, 2007. Updated Dec. 10.

1.A. EXERCISE. If k is algebraically closed, describe a natural map of *sets* $\mathbb{A}^1_k \times \mathbb{A}^1_k \to \mathbb{A}^2_k$. Show that this map is not surjective. On the other hand, show that it is a bijection on closed points.

Remark 2. Recall that the diagram of a fibered square

$$\begin{array}{ccc} W & \longrightarrow & Y \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & Z \end{array}$$

goes by a number of names, including *fibered diagram*, *Cartesian diagram*, *fibered square*, and *Cartesian square*. Because of its geometric interpretation, in algebraic geometry it is often called a **base change diagram** or a **pullback diagram**, and $W \to X$ is called the **pullback** of $Y \to Z$ by f, and W is called the **pullback** of Y by f.

The reason for the phrase "base change" or "pullback" is the following. If X is a point of Z (i.e. f is the natural map of Spec of the residue field of a point of Z into Z), then W is interpreted as the fiber of the family.

1.B. EXERCISE. Show that in the category of topological spaces, this is true, i.e., if $Y \to Z$ is a continuous map, and X is a point p of Z, then the fiber of Y over p is naturally identified with $X \times_Z Y$.

More generally, for general $X \to Z$, the fiber of $W \to X$ over a point p of X is naturally identified with the fiber of $Y \to Z$ over $f(p)$.

Let's now show that fibered products always exist in the category of schemes.

1.1. *Big Theorem (fibered products always exist).* — *Suppose $f : X \to Z$ and $g : Y \to Z$ are morphisms of schemes. Then the fibered product*

$$\begin{array}{ccc} X \times_Z Y & \xrightarrow{f'} & Y \\ \downarrow{\scriptstyle g'} & & \downarrow{\scriptstyle g} \\ X & \xrightarrow{f} & Z \end{array}$$

exists in the category of schemes.

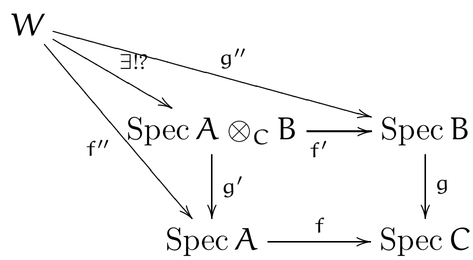
As always when showing that certain objects defined by universal properties exist, we have two ways of looking at the objects in practice: by using the universal property, or by using the details of the construction.

The key idea, roughly, is this: we cut everything up into affine open sets, do fibered products in that category (where it turns out we have seen the concept before in a different guise), and show that everything glues nicely. The conceptually difficult part of the proof comes from the gluing, and realizing that we have to check almost nothing.

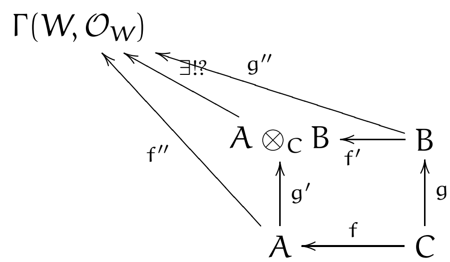
The proof will be a little long, but you will notice that we repeat a kind of argument several times. A much shorter proof is possible by interpreting this in the language of representable functors, and we give this proof afterward for experts.

Proof. We have an extended proof by universal property. We divide the proof up into a number of bite-sized pieces. Between bites, we will often take a break for some side comments.

Step 1: everything affine. First, if X, Y, Z are affine schemes, say $X = \operatorname{Spec} A$, $Y = \operatorname{Spec} B$, $Z = \operatorname{Spec} C$, the fibered product exists, and is $\operatorname{Spec} A \otimes_C B$. Here's why. Suppose W is any scheme, along with morphisms $f'' : W \to X$ and $g'' : W \to Y$ such that $f \circ f'' = g \circ g''$ as morphisms $W \to Z$. We hope that there exists a unique $h : W \to \operatorname{Spec} A \otimes_C B$ such that $f'' = g' \circ h$ and $g'' = f' \circ h$.



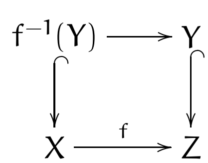
But maps to affine schemes correspond precisely to maps of global sections in the other direction (earlier exercise):



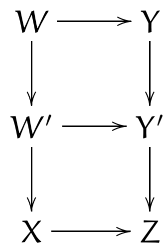
But this is precisely the universal property for tensor product! (The tensor product is the cofibered product in the category of rings.)

1.2. *Side remark (cf. Exercise 1.A).* Thus indeed $\mathbb{A}^1 \times \mathbb{A}^1 \cong \mathbb{A}^2$, and more generally $(\mathbb{A}^1)^n \cong \mathbb{A}^n$.

Step 2: fibered products with open immersions. Second, we note that the fibered product with open immersions always exists: if $Y \hookrightarrow Z$ an open immersion, then for any $f : X \to Z$, $X \times_Z Y$ is the open subset $f^{-1}(Y)$. (More precisely, this open subset satisfies the universal property.) This was an earlier exercise (which wasn't hard).



Step 3: fibered products of affine with almost-affine over affine. We can combine steps 1 and 2 as follows. Suppose X and Z are affine, and $Y \to Z$ factors as $Y \stackrel{i}{\hookrightarrow} Y' \stackrel{g}{\longrightarrow} Z$ where i is an open immersion and Y' is affine. Then $X \times_Z Y$ exists. This is because if the two smaller squares of

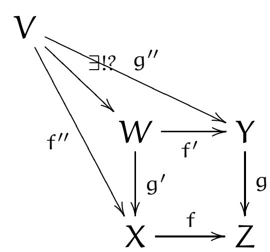


are fibered diagrams, then the "outside rectangle" is also a fibered diagram. (This was an earlier exercise, although you should be able to see this on the spot.)

Key Step 4: fibered product of affine with arbitrary over affine exists. We now come to the key part of the argument: if X and Z are affine, and Y is arbitrary. This is confusing when you first see it, so we'll first deal with a special case, when Y is the union of two affine open sets $Y_1 \cup Y_2$. Let $Y_{12} = Y_1 \cap Y_2$.

Now for $i = 1, 2$, $X \times_Z Y_i$ exists by Step 1; call this W_i. Also, $X \times_Z Y_{12}$ exists by Step 3 (call it W_{12}), and comes with natural open immersions into W_1 and W_2. Thus we can glue W_1 to W_2 along W_{12}; call this resulting scheme W.

We'll check that this is the fibered product by verifying that it satisfies the universal property. Suppose we have maps $f'' : V \to X$, $g'' : V \to Y$ that compose (with f and g respectively) to the same map $V \to Z$. We need to construct a unique map $h : V \to W$, so that $f' \circ h = g''$ and $g' \circ h = f''$.



For $i = 1, 2$, define $V_i := (g'')^{-1}(Y_i)$. Define $V_{12} := (g'')^{-1}(Y_{12}) = V_1 \cap V_2$. Then there is a unique map $V_i \to W_i$ such that the composed maps $V_i \to X$ and $V_i \to Y_i$ are desired (by the universal product of the fibered product $X \times_Z Y_i = W_i$), hence a unique map $h_i : V_i \to W$. Similarly, there is a unique map $h_{12} : V_{12} \to W$ such that the composed maps $V_{12} \to X$ and $V_{12} \to Y$ are as desired. But the restriction of h_i to V_{12} is one such map, so it must be h_{12}. Thus the maps h_1 and h_2 agree on V_{12}, and glue together to a unique map $h : V \to W$. We have shown existence and uniqueness of the desired h. (We are using the fact that "morphisms glue", which corresponds to the fact that maps to a scheme form a sheaf. This leads to a shorter explanation of the proof, which we give at the end of this long proof.)

We have thus shown that if Y is the union of two affine open sets, and X and Z are affine, then $X \times_Z Y$ exists.

We now tackle the general case. (The reader may prefer to first think through the case where "two" is replaced by "three".) We now cover Y with open sets Y_i, as i runs over some index set (not necessarily finite!). As before, we define W_i and W_{ij}. We can glue these together to produce a scheme W along with open sets we identify with W_i (Exercise 4.H in the current revised version of the class 7/8 notes).

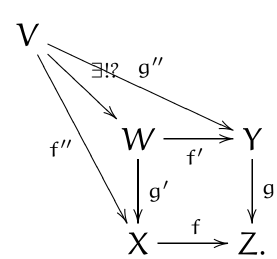
As in the two-affine case, we show that W is the fibered product by showing that it satisfies the universal property. Suppose we have maps $f'' : V \to X$, $g'' : V \to Y$ that compose to the same map $V \to Z$. We construct a unique map $h : V \to W$, so that $f' \circ h = g''$ and $g' \circ h = f''$. Define $V_i = (g'')^{-1}(Y_i)$ and $V_{ij} := (g'')^{-1}(Y_{ij}) = V_i \cap V_j$. Then there is a unique map $V_i \to W_i$ such that the composed maps $V_i \to X$ and $V_i \to Y_i$ are desired, hence a unique map $h_i : V_i \to W$. Similarly, there is a unique map $h_{ij} : V_{ij} \to W$ such that the composed maps $V_{ij} \to X$ and $V_{ij} \to Y$ are as desired. But the restriction of h_i to V_{ij} is one such map, so it must be h_{ij}. Thus the maps h_i and h_j agree on V_{ij}. Thus the h_i glue together to a unique map $h : V \to W$. We have shown existence and uniqueness of the desired h, completing this step.

Side remark. One special case of it is called **extending the base field**: if X is a k-scheme, and k' is a field extension (often k' is the algebraic closure of k), then $X \times_{\operatorname{Spec} k} \operatorname{Spec} k'$ (sometimes informally written $X \times_k k'$ or $X_{k'}$) is a k'-scheme. Often properties of X can be checked by verifying them instead on $X_{k'}$. This is the subject of **descent** — certain properties "descend" from $X_{k'}$ to X. We have already seen that the property of being *normal* descends in this way (in an earlier exercise).

Step 5: Z affine, X and Y arbitrary. We next show that if Z is affine, and X and Y are arbitrary schemes, then $X \times_Z Y$ exists. We just follow Step 4, with the roles of X and Y reversed, using the fact that by the previous step, we can assume that the fibered product with an affine scheme with an arbitrary scheme over an affine scheme exists.

Step 6: Z admits an open immersion into an affine scheme Z', X and Y arbitrary. This is akin to Step 3: $X \times_Z Y$ satisfies the universal property of $X \times_{Z'} Y$.

Step 7: the general case. We again employ the trick from Step 4. Say $f : X \to Z$, $g : Y \to Z$ are two morphisms of schemes. Cover Z with affine open subsets Z_i. Let $X_i = f^{-1}X_i$ and $Y_i = g^{-1}Y_i$. Define $Z_{ij} = Z_i \cap Z_j$, and X_{ij} and Y_{ij} analogously. Then $W_i := X_i \times_{Z_i} Y_i$ exists for all i, and has as open sets $W_{ij} := X_{ij} \times_{Z_{ij}} Y_{ij}$ along with gluing information satisfying the cocycle condition (arising from the gluing information for Z from the Z_i and Z_{ij}). Once again, we show that this satisfies the universal property. Suppose V is any scheme, along with maps to X and Y that agree when they are composed to Z. We need to show that there is a unique morphism $V \to W$ completing the diagram



Now break V up into open sets $V_i = g'' \circ f^{-1}(Z_i)$. Then by the universal property for W_i, there is a unique map $V_i \to W_i$ (which we can interpret as $V_i \to W$). Thus we have already shown uniqueness of $V \to W$. These must agree on $V_i \cap V_j$, because there is only one map $V_i \cap V_j$ to W making the diagram commute. Thus all of these morphisms $V_i \to W$ glue together, so we are done. □

1.3. For experts only!: Describing the existence of fibered products using high-falutin' language.

(Thanks to Jarod for suggesting that I include this, and helping me think through how best to present it. If you have suggestions to make this clearer — to experts of course — please let me know!)

The previous proof can be described more cleanly in the language of representable functors. You'll find this enlightening only after you have absorbed the argument above and meditated on it for a long time. For experts, we include the more abstract picture here. You might find that this is most useful to shed light on representable functors, rather than on the existence of the fibered product.

Recall that to each scheme X we have a contravariant functor h^X from the category of schemes **Sch** to the category of **Sets**, taking a scheme Y to $\mathrm{Mor}(Y, X)$. It may be more convenient to think of it as a covariant functor $h^X : \mathbf{Sch}^{\mathrm{opp}} \to \mathbf{Sets}$.

But this functor h^X is better than a functor. We know that if $\{U_i\}$ is an open cover of Y, a morphism $Y \to X$ is determined by its restrictions $U_i \to X$, and given morphisms $U_i \to X$ that agree on the overlap $U_i \cap U_j \to X$, we can glue them together to get a morphism $Y \to X$. (This is roughly our statement that "morphisms glue".) In the language of equalizer exact sequences,

$$\cdot \longrightarrow \mathrm{Hom}(Y, X) \longrightarrow \prod \mathrm{Hom}(U_i, X) \rightrightarrows \prod \mathrm{Hom}(U_i \cap U_j, X) \ .$$

Thus morphisms to X (i.e. the functor h^X) form a sheaf on every scheme X. If this holds, we say that *the functor is a sheaf*. (If you want to impress your friends and frighten your enemies, you can tell them that this is a *sheaf on the big Zariski site*.)

We can repeat this discussion for the category $\mathbf{Sch}_S$ of schemes over a given base scheme S.

Notice that the definition of fibered product also gives a contravariant functor

$$h_{X \times_Z Y} : \mathbf{Sch} \to \mathbf{Sets} :$$

to the scheme W we associate the set of commutative diagrams

$$\begin{array}{ccc} W & & Y \\ \downarrow & & \downarrow \\ X & \longrightarrow & Z \end{array}$$

(What is the image of $W \to W'$ under this functor?) The existence of fibered product is precise the statement that there is a natural isomorphism of functors $h_{X \times_Z Y} \cong h_W$ for some scheme W. In that case, we say that $h_{X \times_Z Y}$ is a **representable functor**, and that it is **representable by** W. The usual universal property argument shows that this determines W up to unique isomorphism.

We can now interpret Key Step 4 of the proof of Theorem 1.1 as follows. Suppose X and Z are affine, and Y_i is an affine open cover of Y. Suppose the covariant functor $F_Y : (\mathbf{Sch}_Y)^{opp} \to \mathbf{Sets}$ is a sheaf on the category of Y-schemes $\mathbf{Sch}_Y$, and F_{Y_i} is the "restriction of the sheaf to Y_i" (where we include only those Y-schemes that are in fact Y_i-schemes, i.e. those $T \to Y$ whose structure morphisms factor through Y_i, $T \to Y_i \to Y$).

1.C. EXERCISE. Show that if F_{Y_i} is representable, then so is F_Y. (Hint: this is basically just the proofs of Steps 3 and 4.)

We then apply this in the special case where F_Y is given by

$$\left(T \xrightarrow{f} Y \right) \mapsto \left(\begin{array}{ccc} T & \xrightarrow{f} & Y \\ \downarrow & & \downarrow \\ X & \longrightarrow & Z \end{array} \right).$$

[I don't see how to make that diagram on the right look good...]

1.D. EXERCISE. Check that this F_Y is a sheaf. (This is not hard once you realize what this is asking.)

Then Steps 5 through 7 are one-liners; you should think these through. (For Step 5, you'll replace Y by X. For Steps 6/7, you'll replace Y by Z.)

We can make this argument slicker still (and not have to repeat three similar arguments) as follows. (This is frighteningly abstract.) One of Grothendieck's insights is that we should hope to treat contravariant functors $\mathbf{Sch} \to \mathbf{Sets}$ as "geometric spaces", even if we don't know if they are representable. For this reason, I'll call such a functor (for this section only!) a functor-space, to emphasize that we are thinking of it as some sort of spaces. Many notions carry over to this more general setting without change, and some notions are easier. For example, a morphism of functor-spaces $h \to h'$ is just a natural transformation of functors. The following exercise shows that this extends the notion of morphisms of schemes.

1.E. EXERCISE. Show that if X and Y are schemes, then there is a natural bijection between morphisms of schemes $X \to Y$ and morphisms of functor spaces $h^X \to h^Y$. (Hint: this has nothing to do with schemes; your argument will work in any category.)

Also, fibered products of functor-spaces always exist: $h \times_{h''} h'$ may be defined by

$$h \times_{h''} h'(W) = h(W) \times_{h''(W)} h'(W)$$

(where the fibered product on the right is a fibered product of sets, and those always exist). Notice that this didn't use any properties of schemes; this works with **Sch** replaced by any category.

We can make some other definitions that extend notions from schemes to functor-spaces. We say that $h \to h'$ express h as an **open subfunctor** of h' if for all representative morphisms h^X and maps $h^X \to h'$, the fibered product $h^X \times_{h'} h$ is representable, by u say, and $h^U \to h^X$ is an open immersion. the following fibered square may help.

$$\begin{array}{ccc} h^Y & \longrightarrow & h \\ {\scriptstyle \text{open}}\downarrow & & \downarrow \\ h^X & \longrightarrow & h' \end{array}$$

Notice that a morphism of representable functor spaces $h^W \to h^Z$ is an open immersion if and only if $W \to Z$ is an open immersion, so this indeed extends the notion of open immersion to these functors.

A collection h_i of open subfunctors of h' is said to **cover** h' if for each map $h^X \to h'$ from a representable subfunctor, the corresponding open subsets $U_i \hookrightarrow X$ cover X.

1.F. KEY EXERCISE. If a functor-space h is a sheaf that has an open cover by representable functor-spaces ("is covered by schemes"), then h is representable.

Given this formalism, we can now give a quick description of the proof of the existence of fibered products. Exercise 1.D showed that $h_{X \times_Z Y}$ is a sheaf.

1.G. EXERCISE. Suppose $(Z_i)_i$ is an affine cover of Z, $(X_{ij})_j$ is an affine cover of the preimage of Z_i in X, and $(Y_{ik})_k$ is an affine cover of the preimage of Z_i in Y. Show that $(h_{X_{ij} \times_{Z_i} Y_{ik}})_{ijk}$ is an open cover of the functor $h_{X \times_Z Y}$. (Hint: use the definition of open covers!)

But $(h_{X_{ij} \times_{Z_i} Y_{ik}})_{ijk}$ is representable (fibered products of affines over and affine exist, Step 1 of the proof of Theorem 1.1), so we are done.

2. COMPUTING FIBERED PRODUCTS IN PRACTICE

Before giving a bunch of examples, we should first see how to actually compute fibered products in practice.

There are four types of morphisms that it is particularly easy to take fibered products with, and all morphisms can be built from these four atomic components.

(1) Base change by open immersions.

We've already done this, and we used it in the proof that fibered products of schemes exist.

$$\begin{array}{ccc} f^{-1}(Y) & \longrightarrow & Y \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & Z \end{array}$$

I'll describe the remaining three on the level of affine open sets, because we obtain general fibered products by gluing.

(2) Adding an extra variable.

2.A. EASY ALGEBRA EXERCISE.. Show that $B \otimes_A A[t] \cong B[t]$.

Hence the following is a fibered diagram.

$$\begin{array}{ccc} \operatorname{Spec} B[t] & \longrightarrow & \operatorname{Spec} A[t] \\ \downarrow & & \downarrow \\ \operatorname{Spec} B & \longrightarrow & \operatorname{Spec} A \end{array}$$

(3) Base change by closed immersions

2.B. EXERCISE. Suppose $\phi : A \to B$ is a ring homomorphism, and $I \subset A$ is an ideal. Let $I^e := \langle \phi(i) \rangle_{i \in I} \subset B$ be the *extension of* I *to* B. Describe a natural isomorphism $B/I^e \cong B \otimes_A (A/I)$. (Hint: consider $I \to A \to A/I \to 0$, and use the right-exactness of $\otimes_A B$.)

As an immediate consequence: the fibered product with a subscheme is the subscheme of the fibered product in the obvious way. We say that "closed immersions are preserved by base change".

As an application, we can compute tensor products of finitely generated k algebras over k. For example, we have a canonical isomorphism

$$k[x_1, x_2]/(x_1^2 - x_2) \otimes_k k[y_1, y_2]/(y_1^3 + y_2^3) \cong k[x_1, x_2, y_1, y_2]/(x_1^2 - x_2, y_1^3 + y_2^3).$$

2.1. *Example.* We can also use now compute $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$:

$$\begin{aligned} \mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} &\cong \mathbb{C} \otimes_{\mathbb{R}} (\mathbb{R}[x]/(x^2+1)) \\ &\cong (\mathbb{C} \otimes_{\mathbb{R}} \mathbb{R}[x])/(x^2+1) \qquad \text{by } \mathbf{(3)} \\ &\cong \mathbb{C}[x]/(x^2+1) \qquad \text{by } \mathbf{(2)} \\ &\cong \mathbb{C}[x]/(x-i)(x+i) \\ &\cong \mathbb{C} \times \mathbb{C} \end{aligned}$$

Thus $\operatorname{Spec}\mathbb{C} \times_{\mathbb{R}} \operatorname{Spec}\mathbb{C} \cong \operatorname{Spec}\mathbb{C} \coprod \operatorname{Spec}\mathbb{C}$. This example is the first example of many different behaviors. Notice for example that two points somehow correspond to the Galois group of $\mathbb{C}$ over $\mathbb{R}$; for one of them, x (the "i" in one of the copies of $\mathbb{C}$) equals i (the "i" in the other copy of $\mathbb{C}$), and in the other, $x = -i$.

(4) Base change of affine schemes by localization.

2.C. EXERCISE. Suppose $\phi : A \to B$ is a ring homomorphism, and $S \subset A$ is a multiplicative subset of A, which implies that $\phi(S)$ is a multiplicative subset of B. Describe a natural isomorphism $\phi(S)^{-1}B \cong B \otimes_A (S^{-1}A)$.

Translation: the fibered product with a localization is the localization of the fibered product in the obvious way. We say that "localizations are preserved by base change". This is handy if the localization is of the form $A \hookrightarrow A_f$ (corresponding to taking distinguished open sets) or $A \hookrightarrow FF(A)$ (from A to the fraction field of A, corresponding to taking generic points), and various things in between.

These four facts let you calculate lots of things in practice, as we will see throughout the rest of this chapter.

2.D. EXERCISE: THE THREE IMPORTANT TYPES OF MONOMORPHISMS OF SCHEMES. Show that the following are monomorphisms: open immersions, closed immersions, and localization of affine schemes. As monomorphisms are closed under composition, compositions of the above are also monomorphisms (e.g. locally closed immersions, or maps from Spec of stalks at points of X to X).

3. PULLING BACK FAMILIES AND FIBERS OF MORPHISMS

3.1. Pulling back families.

We can informally interpret fibered product in the following geometric way. Suppose $Y \to Z$ is a morphism. We interpret this as a "family of schemes parametrized by a **base scheme** (or just plain **base**) Z." Then if we have another morphism $X \to Z$, we interpret the induced map $X \times_Z Y \to X$ as the "pulled back family".

$$\begin{array}{ccc} X \times_Z Y & \longrightarrow & Y \\ {\scriptstyle \text{pulled back family}} \downarrow & & \downarrow {\scriptstyle \text{family}} \\ X & \longrightarrow & Z \end{array}$$

We sometimes say that $X \times_Z Y$ is the **scheme-theoretic pullback of** Y, **scheme-theoretic inverse image**, or **inverse image scheme of** Y. For this reason, fibered product is often called **base change** or **change of base** or **pullback**.

3.2. Fibers of morphisms.

Suppose $p \to Z$ is the inclusion of a point (not necessarily closed). (If K is the residue field of a point, we mean the canonical map $\operatorname{Spec} K \to Z$.) Then if $g : Y \to Z$ is any morphism, the base change with $p \to Z$ is called the **fiber of** g **above** p or the **preimage of** p, and is denoted $g^{-1}(p)$. If Z is irreducible, the fiber above the generic point is called the **generic fiber**. In an affine open subscheme $\operatorname{Spec} A$ containing p, p corresponds to some prime ideal $\mathfrak{p}$, and the morphism corresponds to the ring map $A \to A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}$. this is the composition if localization and closed immersion, and thus can be computed by the tricks above.

(Quick remark: $p \to Z$ is a monomorphism, by Exercise 2.D.)

3.3. *Example.* The following example has many enlightening aspects. Consider the projection of the parabola $y^2 = x$ to the x axis over $\mathbb{Q}$, corresponding to the map of rings $\mathbb{Q}[x] \to \mathbb{Q}[y]$, with $x \mapsto y^2$. (If $\mathbb{Q}$ alarms you, replace it with your favorite field and see what happens.)

Then the preimage of 1 is two points:

$$\begin{aligned}\operatorname{Spec} \mathbb{Q}[x,y]/(y^2-x) \otimes_{\mathbb{Q}} \operatorname{Spec} \mathbb{Q}[x]/(x-1) &\cong \operatorname{Spec} \mathbb{Q}[x,y]/(y^2-x, x-1) \\ &\cong \operatorname{Spec} \mathbb{Q}[y]/(y^2-1) \\ &\cong \operatorname{Spec} \mathbb{Q}[y]/(y-1) \coprod \operatorname{Spec} \mathbb{Q}[y]/(y+1).\end{aligned}$$

The preimage of 0 is one nonreduced point:

$$\operatorname{Spec} \mathbb{Q}[x,y]/(y^2-x, x) \cong \operatorname{Spec} \mathbb{Q}[y]/(y^2).$$

The preimage of -1 is one reduced point, but of "size 2 over the base field".

$$\operatorname{Spec} \mathbb{Q}[x,y]/(y^2-x, x+1) \cong \operatorname{Spec} \mathbb{Q}[y]/(y^2+1) \cong \operatorname{Spec} \mathbb{Q}[i].$$

The preimage of the generic point is again one reduced point, but of "size 2 over the residue field", as we verify now.

$$\operatorname{Spec} \mathbb{Q}[x,y]/(y^2-x) \otimes \mathbb{Q}(x) \cong \operatorname{Spec} \mathbb{Q}[y] \otimes \mathbb{Q}(y^2)$$

i.e. you take elements polynomials in y, and you are allowed to invert polynomials in y^2. A little thought shows you that you are then allowed to invert polynomials in y, as if $f(y)$ is any polynomial in y, then

$$\frac{1}{f(y)} = \frac{f(-y)}{f(y)f(-y)},$$

and the latter denominator is a polynomial in y^2. Thus

$$\operatorname{Spec} \mathbb{Q}[x,y]/(y^2-x) \otimes \mathbb{Q}(x) \cong \mathbb{Q}(y)$$

which is a degree 2 field extension of $\mathbb{Q}(x)$.

Notice the following interesting fact: in each case, the number of preimages can be interpreted as 2, where you count to two in several ways: you can count points (as in the case of the preimage of 1); you can get non-reduced behavior (as in the case of the preimage of 0); or you can have a field extension of degree 2 (as in the case of the preimage of -1 or the generic point). In each case, the fiber is an affine scheme whose dimension as a vector space over the residue field of the point is 2. Number theoretic readers may have seen this behavior before. This is going to be symptomatic of a very special and important kind of morphism (a finite flat morphism).

Try to draw a picture of this morphism if you can, so you can develop a pictoral shorthand for what is going on.

Here are some other examples.

3.A. EXERCISE. Prove that $\mathbb{A}^n_R \cong \mathbb{A}^n_{\mathbb{Z}} \times_{\operatorname{Spec} \mathbb{Z}} \operatorname{Spec} R$. Prove that $\mathbb{P}^n_R \cong \mathbb{P}^n_{\mathbb{Z}} \times_{\operatorname{Spec} \mathbb{Z}} \operatorname{Spec} R$.

3.B. EXERCISE. Show that the underlying topological space of the (scheme-theoretic) fiber $X \to Y$ above a point p is naturally identified with the topological fiber of $X \to Y$ above p.

3.C. EXERCISE. Show that for finite-type schemes over $\mathbb{C}$, the closed points (=complex-valued points by the Nullstellensatz) of the fibered product correspond to the fibered product of the complex-valued points. (You will just use the fact that $\mathbb{C}$ is algebraically closed.)

3.4. Here is a definition in common use. The terminology is a bit unfortunate, because it is a second (different) meaning of "points of a scheme". (Sadly, we'll even see a third different meaning soon, §4.2.) If T is a scheme, the **T-valued points of a scheme** X are defined to be the morphism $T \to X$. They are sometimes denoted $X(T)$. If A is a ring (most commonly in this context a field), the **A-valued points of a scheme** X are defined to be the morphism $\operatorname{Spec} A \to X$. They are sometimes denoted $X(A)$. For example, if k is an algebraically closed field, then the k-valued points of a finite type scheme are just the closed points; but in general, things can be weirder. (When we say "points of a scheme", and not A-valued points, we will always mean the usual meaning, not this meaning.)

3.D. EXERCISE. Describe a natural bijection $(X \times_Z Y)(T) \cong X(T) \times_{Z(T)} Y(T)$. (The right side is a fibered product of sets.) In other words, fibered products behaves well with respect to T-valued points. This is one of the motivations for this notion. (This generalizes Exercise 3.C.)

3.E. EXERCISE. Consider the morphism of schemes $X = \operatorname{Spec} k[t] \to Y = \operatorname{Spec} k[u]$ corresponding to $k[u] \to k[t]$, $t = u^2$, where $\operatorname{char} k \neq 2$. Show that $X \times_Y X$ has 2 irreducible components. (What happens if $\operatorname{char} k = 2$?)

3.F. EXERCISE GENERALIZING $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$. Suppose L/K is a finite Galois field extension. What is $L \otimes_K L$?

3.G. Hard but fascinating exercise for those familiar with the Galois group of $\overline{\mathbb{Q}}$ over $\mathbb{Q}$. Show that the points of $\operatorname{Spec} \overline{\mathbb{Q}} \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}$ are in natural bijection with $\operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, and the Zariski topology on the former agrees with the profinite topology on the latter.

3.H. Weird Exercise. Show that $\operatorname{Spec} \mathbb{Q}(t) \otimes_{\mathbb{Q}} \mathbb{C}$ has closed points in natural correspondence with the transcendental complex numbers. (If the description $\operatorname{Spec} \mathbb{C}[t] \otimes_{\mathbb{Q}[t]} \mathbb{Q}(t)$ is more striking, you can use that instead.) This scheme doesn't come up in nature, but it is certainly neat!

4. Properties preserved by base change

We now discuss a number of properties that behave well under base change.

We've already shown that the notion of "open immersion" is preserved by base change. We did this by explicitly describing what the fibered product of an open immersion is: if $Y \hookrightarrow Z$ is an open immersion, and $f : X \to Z$ is any morphism, then we checked that the open subscheme $f^{-1}(Y)$ of X satisfies the universal property of fibered products.

We have also shown that the notion of "closed immersion" is preserved by base change (§2 **(3)**). In other words, given a fiber diagram

$$\begin{array}{ccc} W & \longrightarrow & X \\ \downarrow & & \downarrow \\ Y & \overset{\text{cl. imm.}}{\hookrightarrow} & Z \end{array}$$

where $Y \hookrightarrow Z$ is a closed immersion, $W \to X$ is as well.

4.A. Easy Exercise. Show that locally principal closed subschemes pull back to locally principal closed subschemes.

Similarly, other important properties are preserved by base change.

4.B. Exercise. Show that the following properties of morphisms are preserved by base change.

(a) quasicompact
(b) quasiseparated
(c) affine morphism
(d) finite
(e) locally of finite type
(f) finite type
(g) locally of finite presentation
(h) finite presentation

4.C. EXERCISE. Show that the notion of "quasifinite morphism" (finite type + finite fibers) is preserved by base change. (Warning: the notion of "finite fibers" is not preserved by base change. $\operatorname{Spec}\overline{\mathbb{Q}} \to \operatorname{Spec}\mathbb{Q}$ has finite fibers, but $\operatorname{Spec}\overline{\mathbb{Q}} \otimes_{\mathbb{Q}} \overline{\mathbb{Q}} \to \operatorname{Spec}\overline{\mathbb{Q}}$ has one point for each element of $\operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, see Exercise 3.G.)

4.D. EXERCISE. Show that surjectivity is preserved by base change. (**Surjectivity** has its usual meaning: surjective as a map of sets.) (You may end up using the fact that for any fields k_1 and k_2 containing k_3, $k_1 \otimes_{k_3} k_2$ is non-zero, and also the axiom of choice.)

4.E. EXERCISE. If P is a property of morphisms preserved by base change, and $X \to Y$ and $X' \times Y'$ are two morphisms of S-schemes with property P, show that $X \times_S X' \to Y \times_S Y'$ has property P as well.

4.1. $\star$ **Properties not preserved by base change, and how to fix them.**

There are some notions that you should reasonably expect to be preserved by pullback based on your geometric intuition. Given a family in the topological category, fibers pull back in reasonable ways. So for example, any pullback of a family in which all the fibers are irreducible will also have this property; ditto for connected. Unfortunately, both of these fail in algebraic geometry, as the Example 2.1 shows:

$$\begin{array}{ccc} \operatorname{Spec}\mathbb{C} \coprod \operatorname{Spec}\mathbb{C} & \longrightarrow & \operatorname{Spec}\mathbb{C} \\ \downarrow & & \downarrow \\ \operatorname{Spec}\mathbb{C} & \longrightarrow & \operatorname{Spec}\mathbb{R} \end{array}$$

The family on the right (the vertical map) has irreducible and connected fibers, and the one on the left doesn't. The same example shows that the notion of "integral fibers" also doesn't behave well under pullback.

4.F. EXERCISE. Suppose k is a field of characteristic p, so $k(u^p)/k(u)$ is an inseparable extension. By considering $k(u^p) \otimes_{k(u)} k(u^p)$, show that the notion of "reduced fibers" does not necessarily behave well under pullback. (The fact that I'm giving you this example should show that this happens only in characteristic p, in the presence of something as strange as inseparability.)

We rectify this problem as follows.

4.2. A **geometric point** of a scheme X is defined to be a morphism $\operatorname{Spec} k \to X$ where k is an algebraically closed field. Awkwardly, this is now the third kind of "point" of a scheme! There are just plain points, which are elements of the underlying set; there are T-valued points, which are maps $T \to X$, §3.4; and there are geometric points. Geometric points are clearly a flavor of a T-valued point, but they are also an enriched version of a (plain) point: they are the data of a point with an inclusion of the residue field of the point in an algebraically closed field.

A **geometric fiber** of a morphism $X \to Y$ is defined to be the fiber over a geometric point of Y. A morphism has **connected** (resp. **irreducible**, **integral**, **reduced**) **geometric fibers** if all its geometric fibers are connected (resp. irreducible, integral, reduced).

4.G. EXERCISE. Show that the notion of "connected (resp. irreducible, integral, reduced)" geometric fibers behaves well under base change.

4.H. EXERCISE FOR THE ARITHMETICALLY-MINDED. Show that for the morphism $\operatorname{Spec} \mathbb{C} \to \operatorname{Spec} \mathbb{R}$, all geometric fibers consist of two reduced points. (Cf. Example 2.1.)

4.I. EXERCISE. Recall Example 3.3, the projection of the parabola $y^2 = x$ to the x axis, corresponding to the map of rings $\mathbb{Q}[x] \to \mathbb{Q}[y]$, with $x \mapsto y^2$. Show that the geometric fibers of this map are always two points, except for those geometric fibers over $0 = [(x)]$.

Checking whether a k-scheme is geometrically connected etc. seems annoying: you need to check every single algebraically closed field containing k. However, in each of these four cases, the failure of nice behavior of geometric fibers can already be detected after a finite field extension. For example, $\operatorname{Spec} \mathbb{Q}(i) \to \operatorname{Spec} \mathbb{Q}$ is not geometrically connected, and in fact you only need to base change by $\operatorname{Spec} \mathbb{Q}(i)$ to see this. We make this precise as follows.

4.J. EXERCISE. Suppose X is a k-scheme.

(a) Show that X is geometrically irreducible if and only if $X \times_k k^s$ is irreducible if and only if $X \times_k K$ is irreducible for all field extensions K/k. (Here k^s is the separable closure of k.)
(b) Show that X is geometrically connected if and only if $X \times_k k^s$ is connected if and only if $X \times_k K$ is connected for all field extensions K/k.
(c) Show that X is geometrically reduced if and only if $X \times_k k^p$ is reduced if and only if $X \times_k K$ is reduced for all field extensions K/k. (Here k^p is the perfect closure of k.) Thus if $\operatorname{char} k = 0$, then X is geometrically reduced if and only if it is reduced.
(d) Combining (a) and (c), show that X is geometrically integral if and only if $X \times_k K$ is geometrically integral for all field extensions K/k.

5. PRODUCTS OF PROJECTIVE SCHEMES: THE SEGRE EMBEDDING

I will next describe products of projective A-schemes over A. The case of greatest initial interest is if $A = k$. In order to do this, I need only describe $\mathbb{P}^m_A \times_A \mathbb{P}^n_A$, because any projective scheme has a closed immersion in some $\mathbb{P}^m_A$, and closed immersions behave well under base change, so if $X \hookrightarrow \mathbb{P}^m_A$ and $Y \hookrightarrow \mathbb{P}^n_A$ are closed immersions, then $X \times_A Y \hookrightarrow \mathbb{P}^m_A \times_A \mathbb{P}^n_A$ is also a closed immersion, cut out by the equations of X and Y.

We'll describe $\mathbb{P}^m_A \times_A \mathbb{P}^n_A$, and see that it too is a projective A-scheme.

Before we do this, we'll get some motivation from classical projective spaces (non-zero vectors modulo non-zero scalars) in a special case. Our map will send $[x_0; x_1; x_2] \times [y_0; y_1]$ to a point in $\mathbb{P}^5$, whose co-ordinates we think of as being entries in the "multiplication table"

$$\begin{bmatrix} x_0y_0; & x_1y_0; & x_2y_0; \\ x_0y_1; & x_1y_1; & x_2y_1 \end{bmatrix}$$

This is indeed a well-defined map of sets. Notice that the resulting matrix is rank one, and from the matrix, we can read off $[x_0; x_1; x_2]$ and $[y_0; y_1]$ up to scalars. For example, to read off the point $[x_0; x_1; x_2] \in \mathbb{P}^2$, we just take the first row, unless it is all zero, in which case we take the second row. (They can't both be all zero.) In conclusion: in classical projective geometry, given a point of $\mathbb{P}^m$ and $\mathbb{P}^n$, we have produced a point in $\mathbb{P}^{mn+m+n}$, and from this point in $\mathbb{P}^{mn+m+n}$, we can recover the points of $\mathbb{P}^m$ and $\mathbb{P}^n$.

Suitably motivated, we return to algebraic geometry. We define a map

$$\mathbb{P}^m_A \times_A \mathbb{P}^n_A \to \mathbb{P}^{mn+m+n}_A$$

by

$$([x_0; \dots; x_m], [y_0; \dots; y_n]) \mapsto [z_{00}; z_{01}; \cdots; z_{ij}; \cdots; z_{mn}]$$

$$= [x_0y_0; x_0y_1; \cdots; x_iy_j; \cdots x_my_n].$$

More explicitly, we consider the map from the affine open set $U_i \times V_j$ (where $U_i = D(x_i)$ and $V_j = D(y_j)$ to the affine open set $W_{ij} = D(z_{ij})$ by

$$(x_{0/i}, \dots, x_{m/i}, y_{0/j}, \dots, y_{n/j}) \mapsto (x_{0/i}y_{0/j}; \dots; x_{i/i}y_{j/j}; \dots; x_{m/i}y_{n/j})$$

or, in terms of algebras, $z_{ab/ij} \mapsto x_{a/i}y_{b/j}$.

5.A. EXERCISE. Check that these maps glue to give a well-defined morphism $\mathbb{P}^m_A \times_A \mathbb{P}^n_A \to \mathbb{P}^{mn+m+n}_A$.

I claim this morphism is a closed immersion. We can check this on an open cover of the target (the notion of being a closed immersion is affine-local, an earlier exercise). Let's check this on the open set where $z_{ij} \neq 0$. The preimage of this open set in $\mathbb{P}^m_A \times \mathbb{P}^n_A$ is the locus where $x_i \neq 0$ and $y_j \neq 0$, i.e. $U_i \times V_j$. As described above, the map of rings is given by $z_{ab/ij} \mapsto x_{a/i}y_{b/j}$; this is clearly a surjection, as $z_{aj/ij} \mapsto x_{a/i}$ and $z_{ib/ij} \mapsto y_{b/j}$.

This map is called the **Segre morphism** or **Segre embedding**. If A is a field, the image is called the **Segre variety**.

Here are some useful comments.

5.B. EXERCISE. Show that the Segre scheme (the image of the Segre morphism) is cut out by the equations corresponding to

$$\operatorname{rank} \begin{pmatrix} a_{00} & \cdots & a_{0n} \\ \vdots & \ddots & \vdots \\ a_{m0} & \cdots & a_{mn} \end{pmatrix} = 1,$$

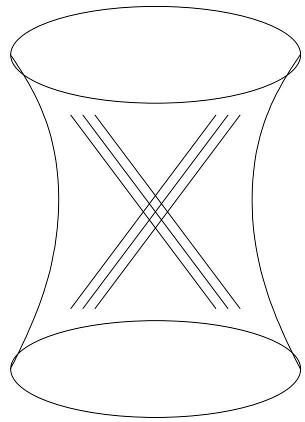


FIGURE 1. The two rulings on the quadric surface $V(wz - xy) \subset \mathbb{P}^3$. One ruling contains the line $V(w, x)$ and the other contains the line $V(w, y)$.

i.e. that all 2×2 minors vanish. (Hint: suppose you have a polynomial in the a_{ij} that becomes zero upon the substitution $a_{ij} = x_i y_j$. Give a recipe for subtracting polynomials of the form monomial times 2×2 minor so that the end result is 0.)

5.1. *Important Example.* Let's consider the first non-trivial example, when $m = n = 1$. We get $\mathbb{P}^1 \times \mathbb{P}^1 \hookrightarrow \mathbb{P}^3$. We get a single equation

$$\operatorname{rank} \begin{pmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{pmatrix} = 1,$$

i.e. $a_{00}a_{11} - a_{01}a_{10} = 0$. We get our old friend, the quadric surface! Hence: the nonsingular quadric surface $wz - xy = 0$ is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ (Figure 1). One family of lines corresponds to the image of $\{x\} \times \mathbb{P}^1$ as x varies, and the other corresponds to the image $\mathbb{P}^1 \times \{y\}$ as y varies.

Since (by diagonalizability of quadratics) all nonsingular quadratics over an algebraically closed field are isomorphic, we have that all nonsingular quadric surfaces over an algebraically closed field are isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$.

Note that this is not true over a field that is not algebraically closed. For example, over $\mathbb{R}$, $w^2 + x^2 + y^2 + z^2 = 0$ is not isomorphic to $\mathbb{P}^1_{\mathbb{R}} \times_{\mathbb{R}} \mathbb{P}^1_{\mathbb{R}}$. Reason: the former has no real points, while the latter has lots of real points.

5.C. EXERCISE: A CO-ORDINATE-FREE DESCRIPTION OF THE SEGRE EMBEDDING. Show that the Segre embedding can be interpreted as $\mathbb{P}V \times \mathbb{P}W \to \mathbb{P}(V \otimes W)$ via the surjective map of graded rings

$$\operatorname{Sym}^\bullet(V^\vee \otimes W^\vee) \twoheadrightarrow \sum_{i=0}^\infty \left(\operatorname{Sym}^i V^\vee\right) \otimes \left(\operatorname{Sym}^i W^\vee\right)$$

"in the opposite direction".

Can you define the Segre embedding for the product of three projective spaces?

6. Separated morphisms

The notion of a separated morphism is fundamentally important. It looks weird the first time you see it, but it is highly motivated.

6.1. Motivation. Separation is the analogue of the Hausdorff condition for manifolds (see Exercise 6.A), so let's review why we like Hausdorffness. Recall that a topological space is *Hausdorff* if for every two points x and y, there are disjoint open neighborhoods of x and y. The real line is Hausdorff, but the "real line with doubled origin" is not. Many proofs and results about manifolds use Hausdorffness in an essential way. For example, the classification of compact one-dimensional real manifolds is very simple, but if the Hausdorff condition were removed, we would have a very wild set.

So armed with this definition, we can cheerfully exclude the line with doubled origin from civilized discussion, and we can (finally) define the notion of a *variety*, in a way that corresponds to the classical definition.

With our motivation from manifolds, we shouldn't be surprised that all of our affine and projective schemes are separated: certainly, in the land of real manifolds, the Hausdorff condition comes for free for "subsets" of manifolds. (More precisely, if Y is a manifold, and X is a subset that satisfies all the hypotheses of a manifold except possibly Hausdorffness, then Hausdorffness comes for free.)

As an unexpected added bonus, a separated morphism to an affine scheme has the property that the intersection of a two affine open sets in the source is affine (Proposition 6.8). This will make Cech cohomology work very easily on (quasicompact) schemes. You should see this as the analogue of the fact that in $\mathbb{R}^n$, the intersection of two convex sets is also convex. In fact affine schemes will be trivial from the point of view of quasicoherent cohomology, just as convex sets in $\mathbb{R}^n$ are, so this metaphor is quite apt.

A lesson arising from the construction is the importance of the diagonal morphism. More precisely given a morphism $X \to Y$, nice consequences can be leveraged from good behavior of the diagonal morphism $\delta : X \to X \times_Y X$, usually through fun diagram chases. This is a lesson that applies across many fields of mathematics. (Another nice gift the diagonal morphism: it will soon give us a good algebraic definition of differentials.)

Grothendieck taught us that one should try to define properties of morphisms, not of objects; then we can say that an object has that property if the morphism to the final object has that property. We saw this earlier with the notion of quasicompact. In this spirit, separation will be a property of morphisms, not schemes.

Before we define separation, we make an observation about all diagonal maps.

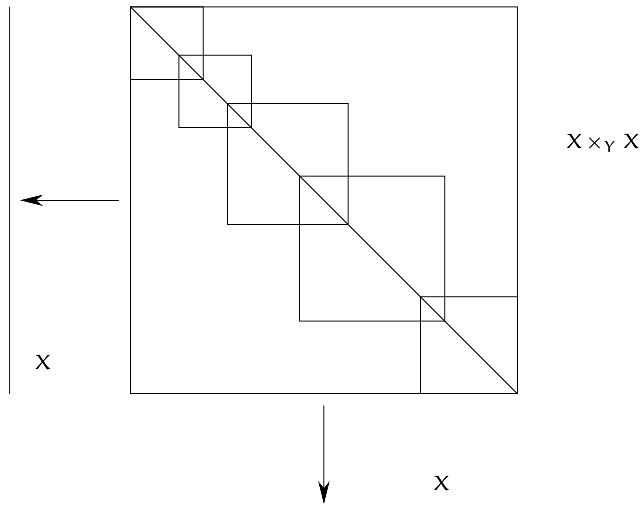


FIGURE 2. A neighborhood of the diagonal is covered by $U_{ij} \times_{V_j} U_{ij}$

6.2. *Proposition. — Let* $X \to Y$ *be a morphism of schemes. Then the diagonal morphism* $\delta : X \to X \times_Y X$ *is a locally closed immersion.*

This locally closed subscheme of $X \times_Y X$ (the diagonal) will be denoted Δ.

Proof. We will describe a union of open subsets of $X \times_Y X$ covering the image of X, such that the image of X is a closed immersion in this union.

6.3. Say Y is covered with affine open sets V_i and X is covered with affine open sets U_{ij}, with $\pi : U_{ij} \to V_i$. Then the diagonal is covered by $U_{ij} \times_{V_i} U_{ij}$. (Any point $p \in X$ lies in some U_{ij}; then $\delta(p) \in U_{ij} \times_{V_i} U_{ij}$. Figure 2 may be helpful.) As a reality check: $U_{ij} \times_{V_i} U_{ij}$ is indeed an affine open subscheme of $X \times_Y X$, by considering the factorization

$$U_{ij} \times_{V_i} U_{ij} \to U_{ij} \times_Y U_{ij} \to U_{ij} \times_Y X \to X \times_Y X$$

where the first arrow is an isomorphism as $V_i \hookrightarrow Y$ is a monomorphism (as it is an open immersion, Exercise 2.D). The second and third arrows are open immersions as open immersions are preserved by base change.

Finally, we'll check that $U_{ij} \to U_{ij} \times_{V_i} U_{ij}$ is a closed immersion. Say $V_i = \operatorname{Spec} S$ and $U_{ij} = \operatorname{Spec} R$. Then this corresponds to the natural ring map $R \times_S R \to R$, which is obviously surjective. □

The open subsets we described may not cover $X \times_Y X$, so we have not shown that δ is a closed immersion.

6.4. *Definition.* A morphism $X \to Y$ is **separated** if the diagonal morphism $\delta : X \to X \times_Y X$ is a closed immersion. An A-scheme X is said to be **separated over** A if the structure morphism $X \to \operatorname{Spec} A$ is separated. When people say that a scheme (rather than a morphism) X is separated, they mean implicitly that some morphism is separated. For example, if they are talking about A-schemes, they mean that X is separated over A.

Thanks to Proposition 6.2, a morphism is separated if and only if the diagonal is closed. This is reminiscent of a definition of Hausdorff, as the next exercise shows.

6.A. EXERCISE (FOR THOSE SEEKING TOPOLOGICAL MOTIVATION). Show that a topological space X is Hausdorff if the diagonal is a closed subset of $X \times X$. (The reason separatedness of schemes doesn't give Hausdorffness — i.e. that for any two open points x and y there aren't necessarily disjoint open neighborhoods — is that in the category of schemes, the topological space $X \times X$ is not in general the product of the topological space X with itself. For example, Exercise 1.A showed that $\mathbb{A}^2_k$ does not have the product topology on $\mathbb{A}^1_k \times_k \mathbb{A}^1_k$.)

6.B. IMPORTANT EASY EXERCISE. Show that open immersions and closed immersions are separated. (Hint: Just do this by hand. Alternatively, show that monomorphisms are separated. Open and closed immersions are monomorphisms, by Exercise 2.D.)

6.C. IMPORTANT EASY EXERCISE. Show that every morphism of affine schemes is separated. (Hint: this was essentially done in Proposition 6.2.)

I'll now give you an example of something separated that is not affine. The following single calculation will imply that all quasiprojective A-schemes are separated (once we know that the composition of separated morphisms are separated, after Thanksgiving).

6.5. *Proposition. — $\mathbb{P}^n_A \to \operatorname{Spec} A$ is separated.*

We give two proofs. The first is by direct calculation. The second requires no calculation, and just requires that you remember some classical constructions described earlier.

Proof 1: direct calculation. We cover $\mathbb{P}^n_A \times_A \mathbb{P}^n_A$ with open sets of the form $U_i \times U_j$, where $U_0, \dots, U_n$ form the "usual" affine open cover. The case $i = j$ was taken care of before, in the proof of Proposition 6.2. If $i \neq j$ then

$$U_i \times_A U_j \cong \operatorname{Spec} A[x_{0/i}, \dots, x_{n/i}, y_{0/j}, \dots, y_{n/j}]/(x_{i/i} - 1, y_{j/j} - 1).$$

Now the restriction of the diagonal Δ is contained in U_i (as the diagonal map composed with projection to the first factor is the identity), and similarly is contained in U_j. Thus the diagonal map over $U_i \times_A U_j$ is $U_i \cap U_j \to U_i \times_A U_j$. This is a closed immersion, as the corresponding map of rings

$$\operatorname{Spec} A[x_{0/i}, \dots, x_{n/i}, y_{0/j}, \dots, y_{n/j}] \to \operatorname{Spec} A[x_{0/i}, \dots, x_{n/i}, x_{j/i}^{-1}]/(x_{i/i} - 1)$$

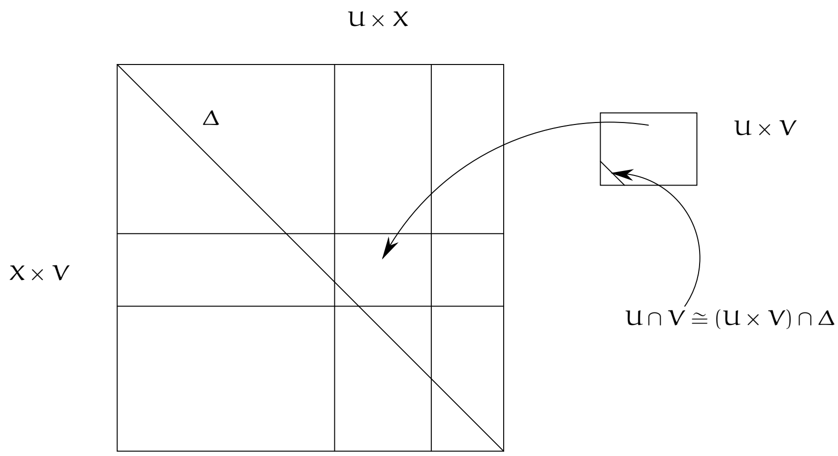
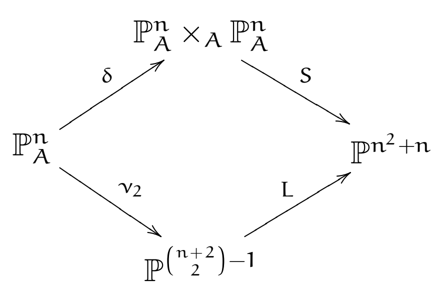


FIGURE 3. Small Proposition 6.6

(given by $x_{k/i} \mapsto x_{k/i}$, $y_{k/j} \mapsto x_{k/i}/x_{j/i}$) is clearly a surjection (as each generator of the ring on the right is clearly in the image — note that $x_{j/i}^{-1}$ is the image of $y_{i/j}$). $\square$

Proof 2: classical geometry, pointed out by Jarod. Note that the diagonal map $\delta : \mathbb{P}^n_A \to \mathbb{P}^n_A \times_A \mathbb{P}^n_A$ followed by the Segre embedding $S : \mathbb{P}^n_A \times_A \mathbb{P}^n_A \to \mathbb{P}^{n^2+n}$ (a closed immersion) can also be factored as the second Veronese map $\nu_2 : \mathbb{P}^n_A \to \mathbb{P}^{\binom{n+2}{2}-1}$ followed by a linear map $L : \mathbb{P}^{\binom{n+2}{2}-1} \to \mathbb{P}^{n^2+n}$ (an earlier exercise, from when we discussed morphisms of projective schemes via morphisms of graded rings), both of which are closed immersions. You should verify this. This forces δ to send closed sets to closed sets (or else $S \circ \delta$ won't, but $L \circ \nu_2$ to).



We note for future reference a minor result proved in the course of Proof 1. Figure 3 may help show why this is natural.

6.6. *Small Proposition. — If* U *and* V *are open subsets of an* A*-scheme* X*, then* $\Delta \cap (U \times_A V) \cong U \cap V$.

6.D. EXERCISE. Show that the line with doubled origin X is not separated, by verifying that the image of the diagonal morphism is not closed.

We finally define then notion of variety!

6.7. *Definition.* A **variety** over a field k, or k**-variety**, is a reduced, separated scheme of finite type over k. For example, a reduced finite type affine k-scheme is a variety. In other words, to check if $\operatorname{Spec} k[x_1, \ldots, x_n]/(f_1, \ldots, f_r)$ is a variety, you need only check reducedness.

Notational caution: In some sources, the additional condition of irreducibility is imposed. We will not do this. Also, it is often assumed that k is algebraically closed. We will not do this either.

Here is a very handy consequence of separatedness.

6.8. *Proposition. — Suppose $X \to \operatorname{Spec} A$ is a separated morphism to an affine scheme, and U and V are affine open sets of X. Then $U \cap V$ is an affine open subset of X.*

Before proving this, we state a consequence that is otherwise nonobvious. If $X = \operatorname{Spec} A$, then the intersection of any two affine open sets is open (just take $A = \mathbb{Z}$ in the above proposition). This is certainly not an obvious fact! We know that the intersection of any two distinguished affine open sets is affine (from $D(f) \cap D(g) = D(fg)$), but we have very little handle on affine open sets in general.

Warning: this property does not characterize separatedness. For example, if $A = \operatorname{Spec} k$ and X is the line with doubled origin over k, then X also has this property.

Proof. By Proposition 6.6, $(U \times_A V) \cap \Delta = U \cap V$, where Δ is the diagonal. But $U \times_A V$ is affine (the fibered product is two affine schemes over an affine scheme is affine, Step 1 of our construction of fibered products, Theorem 1.1), and Δ is a closed subscheme of an affine scheme, and hence affine. □

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FOUNDATIONS OF ALGEBRAIC GEOMETRY CLASS 17

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CONTENTS

Hi everyone — Welcome back! We had last introduced the algebraic analogue of Hausdorffness, called *separation* or *separatedness*. This is a bit weird, but frankly, it is because the notion of Hausdorff involves some mild contortions, and it is easy to forget that.

1. Review of earlier discussion on separation

Let me remind you how it works. Our motivating example of what we are ejecting from civilized discourse is the line with the doubled origin.

We said that a morphism $X \to Y$ is **separated** if the diagonal morphism $\delta : X \to X \times_Y X$ is a closed immersion. An A-scheme X is said to be **separated over** A if the structure morphism $X \to \operatorname{Spec} A$ is separated.

A **variety** over a field k, or k**-variety**, is a reduced, separated scheme of finite type over k. For example, a reduced finite type affine k-scheme is a variety. In other words, to check if $\operatorname{Spec} k[x_1, \dots, x_n]/(f_1, \dots, f_r)$ is a variety, you need only check reducedness.

As diagonals are always locally closed immersions, a morphism is separated if and only if the diagonal is closed. This is reminiscent of a definition of Hausdorff., as the next exercise shows.

We saw that the following types of morphisms are separated:

- open and closed immersions (more generally, monomorphisms)
- morphisms of affine schemes

Date: Monday, November 26, 2007.

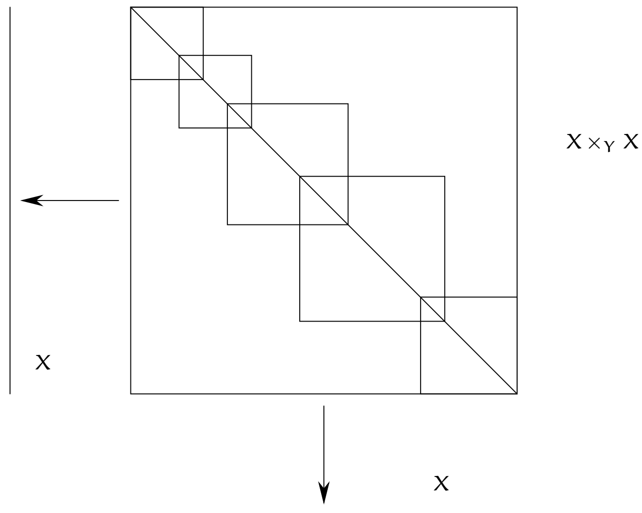


FIGURE 1. A neighborhood of the diagonal is covered by $U_{ij} \times_{V_j} U_{ij}$

- projective A-schemes (over A)

In the course of proving the projective fact, we showed:

1.1. *Small Proposition. — If U and V are open subsets of an A-scheme X, then $\Delta \cap (U \times_A V) \cong U \cap V$.*

We used this to show a handy consequence of separatedness.

1.2. *Proposition. — Suppose $X \to \operatorname{Spec} A$ is a separated morphism to an affine scheme, and U and V are affine open sets of X. Then $U \cap V$ is an affine open subset of X.*

2. QUASISEPARATED MORPHISMS (AND QUASISEPARATED SCHEMES)

We now define a handy relative of separation, that is also given in terms of a property of the diagonal morphism, and has similar properties. The reason it is less famous is because it automatically holds for the sorts of schemes that people usually deal with. We say a morphism $f : X \to Y$ is **quasiseparated** if the diagonal morphism $\delta : X \to X \times_Y X$ is quasicompact. I'll give a more insightful translation shortly, in Exercise 2.A.

Most algebraic geometers will only see quasiseparated morphisms, so this may be considered a very weak assumption. Here are two large classes of morphisms that are quasiseparated. (a) As closed immersions are quasicompact (easy, and an earlier exercise), separated implies quasiseparated. (b) If X is a Noetherian scheme, then any morphism to another scheme is quasicompact (easy, an earlier exercise), so any $X \to Y$ is quasiseparated. Hence those working in the category of Noetherian schemes need never worry about this issue.

The following characterization makes quasiseparation a useful hypothesis in proving theorems.

2.A. EXERCISE. Show that $f : X \to Y$ is quasiseparated if and only if for any affine open $\operatorname{Spec} A$ of Y, and two affine open subsets U and V of X mapping to $\operatorname{Spec} A$, $U \cap V$ is a *finite* union of affine open sets. (Hint: compare this to Proposition 1.2.)

In particular, a morphism $f : X \to Y$ is quasicompact and quasiseparated if and only if the preimage of any affine open subset of Y is a *finite* union of affine open sets in X, whose pairwise intersections are all *also* finite unions of affine open sets. The condition of quasiseparation is often paired with quasicompactness in hypotheses of theorems.

2.B. EXERCISE (A NONQUASISEPARATED SCHEME). Let $X = \operatorname{Spec} k[x_1, x_2, \dots]$, and let U be $X - [\mathfrak{m}]$ where $\mathfrak{m}$ is the maximal ideal $(x_1, x_2, \dots)$. Take two copies of X, glued along U. Show that the result is not quasiseparated. (This open immersion $U \hookrightarrow X$ came up earlier, as an example of a nonquasicompact open subset of an affine scheme.)

3. BACK TO SEPARATION

3.1. *Theorem. — Both separatedness and quasiseparatedness are preserved by base change.*

Proof. Suppose

$$\begin{array}{ccc} W & \longrightarrow & X \\ \downarrow & & \downarrow \\ Y & \longrightarrow & Z \end{array}$$

is a fiber square. We will show that if $Y \to Z$ is separated or quasiseparated, then so is $W \to X$. The reader should verify that

$$\begin{array}{ccc} W & \xrightarrow{\delta_W} & W \times_X W \\ \downarrow & & \downarrow \\ Y & \xrightarrow{\delta_Y} & Y \times_Z Y \end{array}$$

is a fiber diagram. (This is a categorical fact, and holds true in any category with fibered products.) As the property of being a closed immersion is preserved by base change, if δ_Y is a closed immersion, so is δ_X.

Quasiseparatedness follows in the identical manner, as quasicompactness is also preserved by base change. □

3.2. *Proposition. — The condition of being separated is local on the target. Precisely, a morphism* $f : X \to Y$ *is separated if and only if for any cover of* Y *by open subsets* U_i*,* $f^{-1}(U_i) \to U_i$ *is separated for each* i*.*

3.3. Hence affine morphisms are separated, as maps from affine schemes to affine schemes are separated by an exercise from last day. In particular, finite morphisms are separated.

Proof. If $X \to Y$ is separated, then for any $U_i \hookrightarrow Y$, $f^{-1}(U_i) \to U_i$ is separated, as separatedness is preserved by base change (Theorem 3.1). Conversely, to check if $\Delta \hookrightarrow X \times_Y X$ is a closed subset, it suffices to check this on an open cover. If $g : X \times_Y X \to Y$ is the natural morphism, our open cover U_i of Y induces an open cover $f^{-1}(U_i) \times_{U_i} f^{-1}(U_i)$ of $X \times_Y X$. Then $f^{-1}(U_i) \to U_i$ separated implies $f^{-1}(U_i) \to f^{-1}(U_i) \times_{U_i} f(U_i)$ is a closed immersion by definition of separatedness. □

3.A. EXERCISE. Prove that the condition of being quasiseparated is local on the target. (Hint: the condition of being quasicompact is local on the target; use a similar argument.)

3.4. *Proposition. — (a) The condition of being separated is closed under composition. In other words, if* $f : X \to Y$ *is separated and* $g : Y \to Z$ *is separated, then* $g \circ f : X \to Z$ *is separated.*
(b) The condition of being quasiseparated is closed under composition.

Proof. (a) We are given that $\delta_f : X \hookrightarrow X \times_Y X$ and $\delta_g : Y \to Y \times_Z Y$ are closed immersions, and we wish to show that $\delta_h : X \to X \times_Z X$ is a closed immersion. Consider the diagram

$$\begin{array}{ccccc} X & \xrightarrow{\delta_f} & X \times_Y X & \xrightarrow{c} & X \times_Z X \\ & & \downarrow & & \downarrow \\ & & Y & \xrightarrow{\delta_g} & Y \times_Z Y. \end{array}$$

The square is the magic fibered diagram I've discussed before. As δ_g is a closed immersion, c is too (closed immersions are preserved by base change). Thus $c \circ \delta_f$ is a closed immersion (the composition of two closed immersions is also a closed immersion, an earlier exercise).

(b) The identical argument (with "closed immersion" replaced by "quasicompact") shows that the condition of being quasiseparated is closed under composition. □

3.5. *Proposition. — Any quasiprojective* A*-scheme is separated over* A*.*

As a corollary, any reduced quasiprojective k-scheme is a k-variety.

Proof. Suppose $X \to \operatorname{Spec} A$ is a quasiprojective A-scheme. The structure morphism can be factored into an open immersion composed with a closed immersion followed by $\mathbb{P}^n_A \to A$. Open immersions and closed immersions are separated (an earlier exercise, from last day I think), and $\mathbb{P}^n_A \to A$ is separated (a Proposition from last day). Separated morphisms are separated (Proposition 3.4), so we are done. □

3.6. *Proposition. — Suppose* $f : X \to Y$ *and* $f' : X' \to Y'$ *are separated (resp. quasiseparated) morphisms of S-schemes (where S is a scheme). Then the product morphism* $f \times f' : X \times_S X' \to Y \times_S Y'$ *is separated (resp. quasiseparated).*

Proof. An earlier exercise showed that the product of two morphisms having a property has the same property, so long as that property is preserved by base change, and composition. □

3.7. Applications.

As a first application, we define the *graph morphism*.

3.8. *Definition.* Suppose $f : X \to Y$ is a morphism of Z-schemes. The morphism $\Gamma_f : X \to X \times_Z Y$ given by $\Gamma_f = (\mathrm{id}, f)$ is called the **graph morphism**. Then f factors as $\mathrm{pr}_2 \circ \Gamma_f$, where pr_2 is the second projection (see Figure 2).

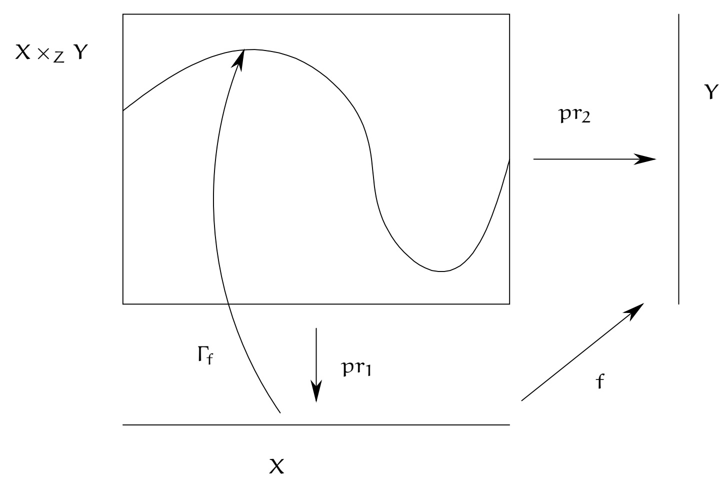


FIGURE 2. The graph morphism

3.9. *Proposition. — The graph morphism* Γ *is always a locally closed immersion. If* Y *is a separated Z-scheme (i.e. the structure morphism* $Y \to Z$ *is separated), then* Γ *is a closed immersion.*

This will be generalized in Exercise 3.B.

Proof by Cartesian diagram.

$$\begin{array}{ccc} X & \longrightarrow & X \times_Z Y \\ \big\downarrow & & \big\downarrow \\ Y & \xrightarrow{\delta} & Y \times_Z Y \end{array}$$

The notions of locally closed immersion and closed immersion are preserved by base change, so if the bottom arrow δ has one of these properties, so does the top. □

We now come to a very useful, but bizarre-looking, result.

3.10. *Cancellation Theorem for a Property P of Morphisms. — Let P be a class of morphisms that is preserved by base change and composition. Suppose*

$$\begin{array}{ccccc} X & & \xrightarrow{\quad f \quad} & & Y \\ & {\scriptstyle h}\searrow & & \swarrow{\scriptstyle g} & \\ & & Z & & \end{array}$$

is a commuting diagram of schemes.

(a) *Suppose that the diagonal morphism $\delta_g : Y \to Y \times_Z Y$ is in P and $h : X \to Z$ is in P. The $f : X \to Y$ is in P.*
(b) *In particular, suppose that closed immersions are in P. Then if h is in P and g is separated, then f is in P.*

When you plug in different P, you get very different-looking (and non-obvious) consequences.

For example, locally closed immersions are separated, so by part (a), if you factor a locally closed immersion $X \to Z$ into $X \to Y \to Z$, then $X \to Y$ *must* be a locally closed immersion.

Possibilities for P in case (b) include: finite morphisms, morphisms of finite type, closed immersions, affine morphisms.

Proof of (a). By the fibered square

$$\begin{array}{ccc} X & \xrightarrow{\Gamma_f} & X \times_Z Y \\ {\scriptstyle f}\big\downarrow & & \big\downarrow \\ Y & \xrightarrow{\delta_g} & Y \times_Z Y \end{array}$$

we see that the graph morphism $\Gamma : X \to X \times_Z Y$ is in P (Definition 3.8), as P is closed under base change. By the fibered square

$$\begin{array}{ccc} X \times_Z Y & \xrightarrow{h'} & Y \\ \downarrow & & \downarrow g \\ X & \xrightarrow{h} & Z \end{array}$$

the projection $h' : X \times_Z Y \to Y$ is in P as well. Thus $f = h' \circ \Gamma$ is in P □

Here now are some fun and useful exercises.

3.B. EXERCISE. Suppose $\pi : Y \to X$ is a morphism, and $s : X \to Y$ is a *section* of a morphism, i.e. $\pi \circ s$ is the identity on X. Show that s is a locally closed immersion. Show that if π is separated, then s is a closed immersion. (This generalizes Proposition 3.9.) Give an example to show that s needn't be a closed immersion if π isn't separated.

3.C. EXERCISE. Show that a A-scheme is separated (over A) if and only if it is separated over $\mathbb{Z}$. (In particular, a complex scheme is separated over $\mathbb{C}$ if and only if it is separated over $\mathbb{Z}$, so complex geometers and arithmetic geometers can communicate about separated schemes without confusion.)

3.D. USEFUL EXERCISE: THE LOCUS WHERE TWO MORPHISMS AGREE. Suppose f and g are two morphisms $X \to Y$, over some scheme Z. We can now give meaning to the phrase 'the locus where f and g agree', and that in particular there is a smallest locally closed subscheme where they agree. Suppose $h : W \to X$ is some morphism (perhaps a locally closed immersion). We say that f and g agree on h if $f \circ h = g \circ h$. Show that there is a locally closed subscheme $i : V \hookrightarrow X$ such that any morphism $h : W \to X$ on which f and g agree factors uniquely through i, i.e. there is a unique $j : W \to V$ such that $h = i \circ j$. (You may recognize this as a universal property statement.) Show further that if $V \to Z$ is separated, then $i : V \hookrightarrow X$ is a closed immersion. Hint: define V to be the following fibered product:

$$\begin{array}{ccc} V & \longrightarrow & Y \\ \downarrow & & \downarrow \delta \\ X & \xrightarrow{(f,g)} & Y \times_Z Y. \end{array}$$

As δ is a locally closed immersion, $V \to X$ is too. Then if $h : W \to X$ is any scheme such that $g \circ h = f \circ h$, then h factors through V.

Minor Remarks. 1) In the previous exercise, we are describing $V \hookrightarrow X$ by way of a universal property. Taking this as the definition, it is not a priori clear that V is a locally closed subscheme of X, or even that it exists.)

2) In the case of reduced finite type k-schemes, the locus where f and g agree can be interpreted as follows. f and g agree at x if $f(x) = g(x)$, and the two maps of residue fields are the same.

3) Notice that Z arises as part of the hypothesis, but is not present in the conclusion!

3.E. EXERCISE. Show that the line with doubled origin X is not separated, by finding two morphisms $f_1, f_2 : W \to X$ whose domain of agreement is not a closed subscheme. (Another argument was given in an exercise, I believe last day.)

3.F. LESS IMPORTANT EXERCISE. Suppose P is a class of morphisms such that closed immersions are in P, and P is closed under fibered product and composition. Show that if $f : X \to Y$ is in P then $f^{red} : X^{red} \to Y^{red}$ is in P. (Two examples are the classes of separated morphisms and quasiseparated morphisms.) Hint:

$$\begin{array}{ccccc} X^{red} & \longrightarrow & X \times_Y Y^{red} & \longrightarrow & Y^{red} \\ & \searrow & \downarrow & & \downarrow \\ & & X & \longrightarrow & Y \end{array}$$

4. RATIONAL MAPS

This is a historically ancient topic. It has appeared late for us because we have just learned about separatedness. Informally: a rational map is a "morphism $X \to Y$ defined almost everywhere". We will see that in good situations that where a rational map is defined, it is uniquely defined.

When discussing rational maps, unless otherwise stated, *we will assume* X *and* Y *to be integral and separated*, although the notions we will introduce can be useful in more general circumstances. The reader interested in more general notions should consider first the case where the schemes in question are reduced and separated, but not necessarily irreducible. Many notions can make sense in more generality (without reducedness hypotheses for example), but I'm not sure if there is a widely accepted definition.

A key example will be irreducible varieties, and the language of rational maps is most often used in this case.

A **rational map** from X to Y, denoted $X \dashrightarrow Y$, is a morphism on a dense open set, with the equivalence relation: $(f : U \to Y) \sim (g : V \to Y)$ if there is a dense open set $Z \subset U \cap V$ such that $f|_Z = g|_Z$. (In a moment, we will improve this to: if $f|_{U \cap V} = g|_{U \cap V}$.) People often use the word "map" for "morphism", which is quite reasonable. But then a rational map need not be a map. So to avoid confusion, when one means "rational map", one should never just say "map".

An obvious example of a rational map is a morphism. Another example is the following.

4.A. EASY EXERCISE. Interpret rational functions on a separated integral scheme as rational maps to $\mathbb{A}^1_{\mathbb{Z}}$. (This is analogous to functions corresponding to morphisms to $\mathbb{A}^1_{\mathbb{Z}}$, an earlier exercise.)

4.1. *Important Theorem. — Two* S*-morphisms* $f_1, f_2 : U \to Z$ *from a reduced scheme to a separated* S*-scheme agreeing on a dense open subset of* U *are the same.*

4.B. EXERCISE. Give examples to show how this breaks down when we give up reducedness of the base or separatedness of the target. Here are some possibilities. For the first, consider the two maps $\operatorname{Spec} k[x,y]/(y^2, xy) \to \operatorname{Spec} k[t]$, where we take f_1 given by $t \mapsto x$ and f_2 given by $t \mapsto x + y$; f_1 and f_2 agree on the distinguished open set $D(x)$. (See Figure 3.) For the second, consider the two maps from $\operatorname{Spec} k[t]$ to the line with the doubled origin, one of which maps to the "upper half", and one of which maps to the "lower half". these to morphisms agree on the dense open set $D(f)$. (See Figure 4.)

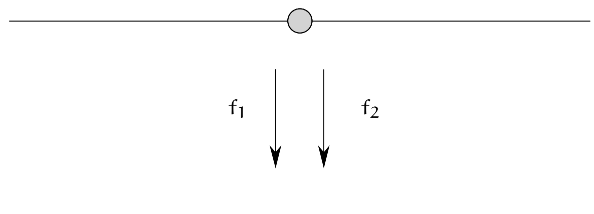


FIGURE 3. Two different maps from a nonreduced scheme agreeing on an open set

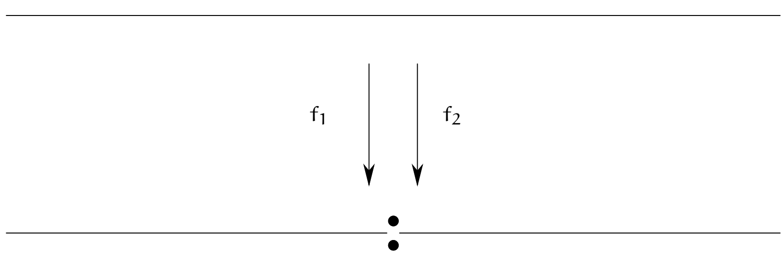


FIGURE 4. Two different maps to a nonseparated scheme agreeing on an open set

Proof. Let V be the locus where f_1 and f_2 agree. It is a closed subscheme of U by Exercise 3.D, which contains the generic point. But the only closed subscheme of a reduced scheme U containing the generic point is all of U. □

Consequence 1. Hence (as X is reduced and Y is separated) if we have two morphisms from open subsets of X to Y, say $f : U \to Y$ and $g : V \to Y$, and they agree on a dense open subset $Z \subset U \cap V$, then they necessarily agree on $U \cap V$.

Consequence 2. Also: a rational map has a largest **domain of definition** on which $f : U \dashrightarrow Y$ is a morphism, which is the union of all the domains of definition.

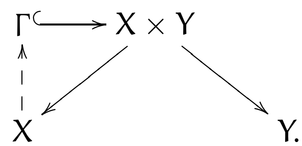
In particular, a rational function from a reduced scheme has a largest domain of definition.

4.2. *The graph of a rational map.*

Define the **graph** of a rational map $f : X \dashrightarrow Y$ as follows. Let (U, f') be any representative of this rational map (so $f' : U \to Y$ is a morphism). Let Γ_f be the scheme-theoretic closure of $\Gamma_{f'} \hookrightarrow U \times Y \hookrightarrow X \times Y$, where the first map is a closed immersion, and the second is an open immersion.

4.C. EXERCISE. Show that the graph of a rational map is independent of the choice of representative of the rational map.

In analogy with graphs of morphisms (e.g. Figure 2), the following diagram of a graph of a rational map can be handy.



5. DOMINANT AND BIRATIONAL MAPS

A rational map $f : X \dashrightarrow Y$ is **dominant** if for some (and hence every) representative $U \to Y$, the image is dense in Y. Equivalently, f is dominant if it sends the generic point of X to the generic point of Y.

5.A. EXERCISE. Show that you can compose two rational maps $f : X \dashrightarrow Y$, $g : Y \dashrightarrow Z$ if f is dominant.

In particular, integral separated schemes and dominant rational maps between them form a category which is geometrically interesting.

5.B. EASY EXERCISE. Show that dominant rational maps give morphisms of function fields in the opposite direction.

It is not true that morphisms of function fields give dominant rational maps, or even rational maps. For example, $\operatorname{Spec} k[x]$ and $\operatorname{Spec} k(x)$ have the same function field $(k(x))$, but there is no rational map $\operatorname{Spec} k[x] \dashrightarrow \operatorname{Spec} k(x)$. Reason: that would correspond to a morphism from an open subset U of $\operatorname{Spec} k[x]$, say $k[x, 1/f(x)]$, to $k(x)$. But there is no map of rings $k(x) \to k[x, 1/f(x)]$ for any one $f(x)$.

However, maps of function fields indeed give dominant rational maps in the case of varieties, see Proposition 5.1 below.

A rational map $f : X \to Y$ is said to be **birational** if it is dominant, and there is another rational map (a "rational inverse") that is also dominant, such that $f \circ g$ is (in the same equivalence class as) the identity on Y, and $g \circ f$ is (in the same equivalence class as) the identity on X. This is the notion of isomorphism in the category of integral separated schemes and dominant rational maps.

A *morphism* is **birational** if it is birational as a rational map. We say X and Y are **birational** (to each other) if there exists a birational map $X \dashrightarrow Y$. Birational maps induce isomorphisms of function fields. Proposition 5.1 will imply that a map between k-varieties that induces an isomorphism of function fields is birational.

We now prove a Proposition promised earlier.

5.1. *Proposition. — Suppose X, Y are irreducible varieties, and we are given $f^{\#} : FF(Y) \hookrightarrow FF(Y)$. Then there exists a dominant rational map $f : X \dashrightarrow Y$ inducing $f^{\#}$.*

Proof. By replacing Y with an affine open set, we may assume Y is affine, say $Y = \operatorname{Spec} k[x_1, \ldots, x_n]/(f_1, \ldots, f_r)$. Then we have $x_1, \ldots, x_n \in K(X)$. Let U be an open subset of the domains of definition of these rational functions. Then we get a morphism $U \to \mathbb{A}^n_k$. But this morphism factors through $Y \subset \mathbb{A}^n$, as $x_1, \ldots, x_n$ satisfy the relations $f_1, \ldots, f_r$. □

5.C. EXERCISE. Let K be a finitely generated field extension of k. Show there exists an irreducible k-variety with function field K. (Hint: let $x_1, \ldots, x_n$ be generators for K over k. Consider the map $k[t_1, \ldots, t_n] \to K$ given by $t_i \mapsto x_i$, and show that the kernel is a prime ideal $\mathfrak{p}$, and that $k[t_1, \ldots, t_n]/\mathfrak{p}$ has fraction field K. This can be interpreted geometrically: consider the map $\operatorname{Spec} K \to \operatorname{Spec} k[t_1, \ldots, t_n]$ given by the ring map $t_i \mapsto x_i$, and take the closure of the image.)

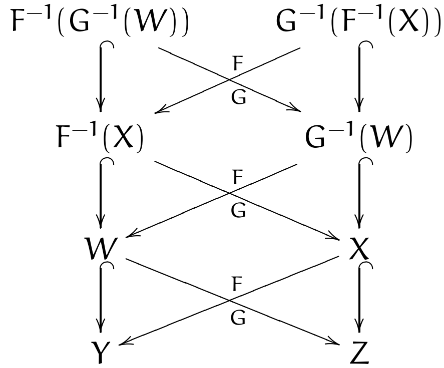
5.2. *Proposition. — Suppose Y and Z are integral k-varieties. Then Y and Z are birational if and only if there is a dense (=non-empty) open subscheme U of Y and a dense open subscheme V of Z such that $U \cong V$.*

This gives you a good idea of how to think of birational maps.

Proof. I find this proof kind of surprising and unexpected.

Clearly if Y and Z have isomorphic open sets U and V respectively, then they are birational (with birational maps given by the isomorphisms $U \to V$ and $V \to U$ respectively).

For the other direction, assume that $f : Y \dashrightarrow Z$ is a birational map, with inverse birational map $g : Z \dashrightarrow Y$. Choose representatives for these rational maps $F : W \to Y$ (where W is an open subscheme of Y) and $G : X \to Z$ (where Z is an open subscheme of Z). We will see that $F^{-1}(G^{-1}(W) \subset Y$ and $G^{-1}(F^{-1}(X)) \subset Z$ are isomorphic open subschemes.



The two morphisms $G \circ F$ and the identity from $F^{-1}(G^{-1}(W)) \to W$ represent to the same rational map, so by Theorem 4.1 they are the same morphism. Thus $G \circ F$ gives the identity map from $F^{-1}(G^{-1}(W))$ to itself. Similarly $F \circ G$ gives the identity map on $G^{-1}(F^{-1}(X))$. All that remains is to show that F maps $F^{-1}(G^{-1}(W))$ into $G^{-1}(F^{-1}(X))$, and that G maps $G^{-1}(F^{-1}(X))$ into $F^{-1}(G^{-1}(W))$, and by symmetry it suffices to show the former. Suppose $q \in F^{-1}(G^{-1}(W))$. Then $F(G(F(q)) = F(q) \in X$, from which $F(q) \in G^{-1}(F^{-1}(X))$. □

6. Examples of rational maps

Here are some examples of rational maps. A recurring theme is that domains of definition of rational maps to projective schemes extend over nonsingular codimension one points. We'll make this precise when we discuss curves next quarter.

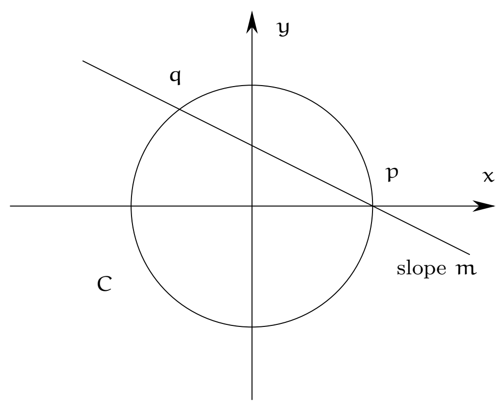


FIGURE 5. Finding primitive Pythagorean triples using geometry

The first example is how you find a formula for Pythagorean triples. Suppose you are looking for rational points on the circle C given by $x^2 + y^2 = 1$ (Figure 5). One rational point is $p = (1, 0)$. If q is another rational point, then pq is a line of rational (non-infinite) slope. This gives a rational map from the conic C to $\mathbb{A}^1$. Conversely, given a line of slope m through p, where m is rational, we can recover q as follows: $y = m(x-1)$, $x^2 + y^2 = 1$.

We substitute the first equation into the second, to get a quadratic equation in x. We know that we will have a solution $x = 1$ (because the line meets the circle at $(x, y) = (1, 0)$), so we expect to be able to factor this out, and find the other factor. This indeed works:

$$\begin{aligned} x^2 + (m(x-1))^2 &= 1 \\ \Rightarrow \quad (m^2+1)x^2 + (-2)x + (m^2-1) &= 0 \\ \Rightarrow \quad (x-1)((m^2+1)x - (m^2-1)) &= 0 \end{aligned}$$

The other solution is $x = (m^2-1)/(m^2+1)$, which gives $y = 2m/(m^2+1)$. Thus we get a birational map between the conic C and $\mathbb{A}^1$ with coordinate m, given by $f : (x, y) \mapsto y/(x-1)$ (which is defined for $x \neq 1$), and with inverse rational map given by $m \mapsto ((m^2-1)/(m^2+1), 2m/(m^2+1))$ (which is defined away from $m^2+1 = 0$).

We can extend this to a rational map $C \dashrightarrow \mathbb{P}^1$ via the inclusion $\mathbb{A}^1 \to \mathbb{P}^1$. Then f is given by $(x, y) \mapsto [y; x-1]$. We then have an interesting question: what is the domain of definition of f? It appears to be defined everywhere except for where $y = x - 1 = 0$, i.e. everywhere but p. But in fact it can be extended over p! Note that $(x, y) \mapsto [x+1; -y]$ (where $(x, y) \neq (-1, y)$) agrees with f on their common domains of definition, as $[x+1; -y] = [y; x-1]$. Hence this rational map can be extended farther than we at first thought. This will be a special case of a result we'll see later .

(For the curious: we are working with schemes over $\mathbb{Q}$. But this works for any scheme over a field of characteristic not 2. What goes wrong in characteristic 2?)

6.A. EXERCISE. Use the above to find a "formula" yielding all Pythagorean triples.

6.B. EXERCISE. Show that the conic $x^2 + y^2 = z^2$ in $\mathbb{P}^2_k$ is isomorphic to $\mathbb{P}^1_k$ for any field k of characteristic not 2. (We've done this earlier in the case where k is algebraically closed, by diagonalizing quadrics.)

In fact, any conic in $\mathbb{P}^2_k$ with a k-valued point (i.e. a point with residue field k) is isomorphic to $\mathbb{P}^1_k$. (This hypothesis is certainly necessary, as $\mathbb{P}^1_k$ certainly has k-valued points. $x^2 + y^2 + z^2 = 0$ over $k = \mathbb{R}$ gives an example of a conic that is not isomorphic to $\mathbb{P}^1_k$.)

6.C. EXERCISE. Find all rational solutions to $y^2 = x^3 + x^2$, by finding a birational map to $\mathbb{A}^1$, mimicking what worked with the conic.

You will obtain a rational map to $\mathbb{P}^1$ that is not defined over the node $x = y = 0$, and *can't* be extended over this codimension 1 set. This is an example of the limits of our future result showing how to extend rational maps to projective space over codimension 1 sets: the codimension 1 sets have to be nonsingular. More on this soon!

6.D. EXERCISE. Use something similar to find a birational map from the quadric $Q = \{x^2 + y^2 = w^2 + z^2\}$ to $\mathbb{P}^2$. Use this to find all rational points on Q. (This illustrates a good way of solving Diophantine equations. You will find a dense open subset of Q that is isomorphic to a dense open subset of $\mathbb{P}^2$, where you can easily find all the rational points. There will be a closed subset of Q where the rational map is not defined, or not an isomorphism, but you can deal with this subset in an ad hoc fashion.)

6.E. IMPORTANT CONCRETE EXERCISE (A FIRST VIEW OF A BLOW-UP). Let k be an algebraically closed field. (We make this hypothesis in order to not need any fancy facts on nonsingularity.) Consider the rational map $\mathbb{A}^2_k \dashrightarrow \mathbb{P}^1_k$ given by $(x, y) \mapsto [x; y]$. I think you have shown earlier that this rational map cannot be extended over the origin. Consider the graph of the birational map, which we denote $\mathrm{Bl}_{(0,0)} \mathbb{A}^2_k$. It is a subscheme of $\mathbb{A}^2_k \times \mathbb{P}^1_k$. Show that if the coordinates on $\mathbb{A}^2$ are x, y, and the coordinates on $\mathbb{P}^1$ are u, v, this subscheme is cut out in $\mathbb{A}^2 \times \mathbb{P}^1$ by the single equation $xv = yu$. Describe the fiber of the morphism $\mathrm{Bl}_{(0,0)} \mathbb{A}^2_k \to \mathbb{P}^1_k$ over each closed point of $\mathbb{P}^1_k$. Describe the fiber of the morphism $\mathrm{Bl}_{(0,0)} \mathbb{A}^2_k \to \mathbb{A}^2_k$. Show that the fiber over $(0, 0)$ is an effective Cartier divisor (a closed subscheme that is locally principal and not a zero-divisor). It is called the *exceptional divisor*.

6.F. EXERCISE (THE CREMONA TRANSFORMATION, A USEFUL CLASSICAL CONSTRUCTION). Consider the rational map $\mathbb{P}^2 \dashrightarrow \mathbb{P}^2$, given by $[x; y; z] \to [1/x; 1/y; 1/z]$. What is the the domain of definition? (It is bigger than the locus where $xyz \neq 0$!) You will observe that you can extend it over codimension 1 sets. This will again foreshadow a result we will soon prove.

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FOUNDATIONS OF ALGEBRAIC GEOMETRY CLASS 18

RAVI VAKIL

CONTENTS

1. PROPER MORPHISMS

I'll now tell you about a new property of morphisms, the notion of *properness*. You can think about this in several ways.

Recall that a map of topological spaces (also known as a continuous map!) is said to be proper if the preimage of compact sets is compact. Properness of morphisms is an analogous property. For example, proper varieties over $\mathbb{C}$ will be the same as compact in the "usual" topology. Alternatively, we will see that projective A-schemes are proper over A — this is the hardest thing we will prove — so you can see this as nice property satisfied by projective schemes, and quite convenient to work with.

A (continuous) map of topological spaces $f : X \to Y$ is **closed** if for each closed subset $S \subset X$, $f(S)$ is also closed. A morphism of schemes is closed if the underlying continuous map is closed. We say that a morphism of schemes $f : X \to Y$ is **universally closed** if for every morphism $g : Z \to Y$, the induced morphism $Z \times_Y X \to Z$ is closed. In other words, a morphism is universally closed if it remains closed under any base change. (A note on terminology: if P is some property of schemes, then a morphism of schemes is said to be "universally P" if it remains P under any base change.)

A morphism $f : X \to Y$ *is* **proper** *if it is separated, finite type, and universally closed.* A scheme X is often said to be proper if some implicit structure morphism is proper. For example, a k-scheme X is often described as proper if $X \to \operatorname{Spec} k$ is proper. (A k-scheme is often said to be *complete* if it is proper. We will not use this terminology.)

Let's try this idea out in practice. We expect that $\mathbb{A}^1_{\mathbb{C}} \to \operatorname{Spec} \mathbb{C}$ is not proper, because the complex manifold corresponding to $\mathbb{A}^1_{\mathbb{C}}$ is not compact. However, note that this map is separated (it is a map of affine schemes), finite type, and closed. So the "universally" is what matters here.

Date: Wednesday, November 28, 2007.

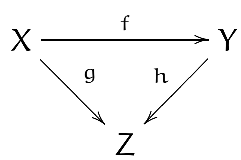
1.A. EXERCISE. Show that $\mathbb{A}^1_{\mathbb{C}} \to \operatorname{Spec} \mathbb{C}$ is not proper, by finding a base change that turns this into a non-closed map. (Hint: Consider $\mathbb{A}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}} \to \mathbb{P}^1_{\mathbb{C}}$.)

1.1. As a first example: closed immersions are proper. They are clearly separated, as affine morphisms are separated. They are finite type. After base change, they remain closed immersions, and closed immersions are always closed.

1.2. *Proposition.* —

(a) *The notion of "proper morphism" is stable under base change.*
(b) *The notion of "proper morphism" is local on the target (i.e. $f : X \to Y$ is proper if and only if for any affine open cover $U_i \to Y$, $f^{-1}(U_i) \to U_i$ is proper). Note that the "only if" direction follows from (a) — consider base change by $U_i \hookrightarrow Y$.*
(c) *The notion of "proper morphism" is closed under composition.*
(d) *The product of two proper morphisms is proper (i.e. if $f : X \to Y$ and $g : X' \to Y'$ are proper, where all morphisms are morphisms of Z-schemes) then $f \times g : X \times_Z X' \to Y \times_Z Y'$ is proper.*
(e) *Suppose*

(1)



is a commutative diagram, and g is proper, and h is separated. Then f is proper.

A sample application of (e): A morphism (over $\operatorname{Spec} k$) from a proper k-scheme to a separated k-scheme is always proper.

Proof. (a) We have already shown that the notions of separatedness and finite type are local on the target. The notion of closedness is local on the target, and hence so is the notion of universal closedness.

(b) The notions of separatedness, finite type, and universal closedness are all preserved by fibered product. (Notice that this is why universal closedness is better than closedness — it is automatically preserved by base change!)

(c) The notions of separatedness, finite type, and universal closedness are all preserved by composition.

(d) By (a) and (c), this follows from an earlier exercise showing that a property of morphisms preserved by composition and base change is also preserved by products.

(e) Closed immersions are proper, so we invoke the Cancellation Theorem for properties of morphisms. □

We now come to the most important example of proper morphisms.

1.3. *Theorem.* — *Projective A-schemes are proper over A.*

It is not easy to come up with an example of an A-scheme that is proper but not projective! We will see a simple example of a proper but not projective surface, . Once we discuss blow-ups, I'll give Hironaka's example of a proper but not projective nonsingular ("smooth") threefold over $\mathbb{C}$.

Proof. The structure morphism of a projective A-scheme $X \to \operatorname{Spec} A$ factors as a closed immersion followed by $\mathbb{P}^n_A$. Closed immersions are proper, and compositions of proper morphisms are proper, so it suffices to show that $\mathbb{P}^n_A \to \operatorname{Spec} A$ is proper. We have already seen that this morphism is finite type (an earlier easy exercise) and separated (shown last week by hand), so it suffices to show that $\mathbb{P}^n_A \to \operatorname{Spec} A$ is universally closed. As $\mathbb{P}^n_A = \mathbb{P}^n_{\mathbb{Z}} \times_{\mathbb{Z}} \operatorname{Spec} A$, it suffices to show that $\mathbb{P}^n_X := \mathbb{P}^n_{\mathbb{Z}} \times_{\mathbb{Z}} X \to X$ is closed for any scheme X. But the property of being closed is local on the target on X, so by covering X with affine open subsets, it suffices to show that $\mathbb{P}^n_A \to \operatorname{Spec} A$ is closed. This is important enough to merit being stated as a Theorem.

1.4. *Theorem.* — $\pi : \mathbb{P}^n_A \to \operatorname{Spec} A$ *is a closed morphism.*

This is sometimes called the fundamental theorem of elimination theory. Here are some examples to show you that this is a bit subtle.

First, let $A = k[a, b, c, \ldots, i]$, and consider the closed subscheme of $\mathbb{P}^2_A$ (taken with coordinates x, y, z) corresponding to $ax + by + cz = 0$, $dx + ey + fz = 0$, $gx + hy + iz = 0$. Then we are looking for the locus in $\operatorname{Spec} A$ where these equations have a non-trivial solution. This indeed corresponds to a Zariski-closed set — where

$$\det \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = 0.$$

As a second example, let $A = k[a_0, a_1, \ldots, a_m, b_0, b_1, \ldots, b_n]$. Now consider the closed subscheme of $\mathbb{P}^1_A$ (taken with coordinates x and y) corresponding to $a_0 x^m + a_1 x^{m-1} y + \cdots + a_m y^m = 0$ and $b_0 x^n + b_1 x^{m-1} y + \cdots + b_n y^n = 0$. Then we are looking at the locus in $\operatorname{Spec} A$ where these two polynomials have a common root — this is known as the *resultant*.

More generally, this question boils down to the following question. Given a number of homogeneous equations in $n+1$ variables with indeterminate coefficients, Proposition 1.4 implies that one can write down equations in the coefficients that will precisely determine when the equations have a nontrivial solution.

Proof of Theorem 1.4. Suppose $Z \hookrightarrow \mathbb{P}^n_A$ is a closed *subset*. We wish to show that $\pi(Z)$ is closed.

Suppose $y \notin \pi(Z)$ is a *closed* point of $\operatorname{Spec} A$. We'll check that there is a distinguished open neighborhood $D(f)$ of y in $\operatorname{Spec} A$ such that $D(f)$ doesn't meet $\pi(Z)$. (If we could show this for *all* points of $\pi(Z)$, we would be done. But I prefer to concentrate on closed points for now.) Suppose y corresponds to the maximal ideal $\mathfrak{m}$ of A. We seek $f \in A - \mathfrak{m}$ such that $\pi^* f$ vanishes on Z.

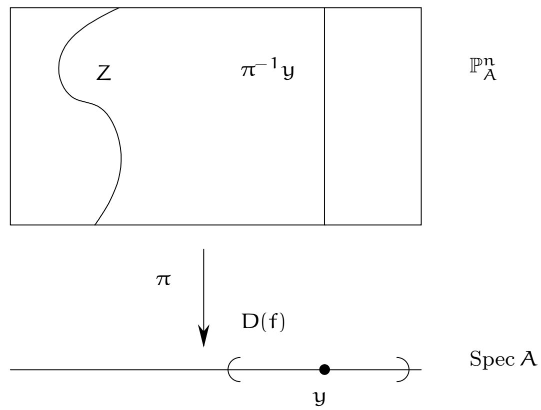


FIGURE 1

Let $U_0, \dots, U_n$ be the usual affine open cover of $\mathbb{P}^n_A$. The closed subsets $\pi^{-1}y$ and Z do not intersect (see Figure 1). On the affine open set U_i, we have two closed subsets $Z \cap U_i$ and $\pi^{-1}y \cap U_i$ that do not intersect, which means that the ideals corresponding to the two closed sets generate the unit ideal, so in the ring of functions $A[x_{0/i}, x_{1/i}, \dots, x_{n/i}]$. on U_i, we can write

$$1 = a_i + \sum m_{ij} g_{ij}$$

where $m_{ij} \in \mathfrak{m}$, and a_i vanishes on Z. Note that $a_i, g_{ij} \in A[x_{0/i}, \dots, x_{n/i}]$, so by multiplying by a sufficiently high power x_i^n of x_i, we have an equality

$$x_i^N = a_i' + \sum m_{ij} g_{ij}'$$

on U_i, where both sides are expressions in $S_\bullet = A[x_0, \dots, x_n]$. We may take N large enough so that it works for all i. Thus for N' sufficiently large, we can write any monomial in $x_1, \dots, x_n$ of degree N' as something vanishing on Z plus a linear combination of elements of $\mathfrak{m}$ times other polynomials. Hence

$$S_{N'} = I(Z)_{N'} + \mathfrak{m} S_{N'}$$

where $I(Z)_*$ is the graded ideal of functions vanishing on Z. We now need Nakayama's lemma. If you haven't seen this result before, we will prove it next week. We will use the following form of it: if M is a finitely generated module over A such that $M = \mathfrak{m}M$ for some maximal ideal $\mathfrak{m}$, then there is some $f \notin \mathfrak{m}$ such that $fM = 0$. Applying this in the case where $M = S_{N'}/I(Z)_{N'}$, we see that there exists $f \in A - \mathfrak{m}$ such that

$$fS_{N'} \subset I(Z)_{N'}.$$

Thus we have found our desired f.

We now tackle Theorem 1.4 in general. Suppose $y = [\mathfrak{p}]$ not in the image of Z. Applying the above argument in $\operatorname{Spec} A_\mathfrak{p}$, we find $S_{N'} \otimes A_\mathfrak{p} = I(Z)_{N'} \otimes A_\mathfrak{p} + \mathfrak{m} S_{N'} \otimes A_\mathfrak{p}$, from which $g(S_{N'}/I(Z)_{N'}) \otimes A_\mathfrak{p} = 0$ for some $g \in A_\mathfrak{p} - \mathfrak{p}A_\mathfrak{p}$, from which $(S_{N'}/I(Z)_{N'}) \otimes A_\mathfrak{p} = 0$. As $S_{N'}$ is a finitely generated A-module, there is some $f \in A - \mathfrak{p}$ with $fS_N \subset I(Z)$ (if the module-generators of $S_{N'}$, and $f_1, \dots, f_a$ are annihilate the generators respectively, then take $f = \prod f_i$), so once again we have found $D(f)$ containing $\mathfrak{p}$, with (the pullback of) f vanishing on Z. □

Notice that projectivity was essential to the proof: we used graded rings in an essential way.

This also concludes the proof of Theorem 1.3.

1.5. *Corollary. — Finite morphisms are proper.*

Proof. Suppose $f : X \to Y$ is a finite morphism. As properness is local on the base, to check properness of f, we may assume Y is affine. But finite morphisms to $\operatorname{Spec} A$ are projective , and projective morphisms are proper. □

In particular, as promised in our initial discussion of finiteness:

1.6. *Corollary. — Finite morphisms are closed.*

1.7. Unproved facts that may help you correctly think about finiteness.

We conclude with some interesting facts that we will prove later. They may shed some light on the notion of finiteness.

A morphism is finite if and only if it is proper and affine, if and only if it is proper and quasifinite. We have verify the "only if" parts of this statement; the "if" parts are harder.

As an application: quasifinite morphisms from proper schemes to separated schemes are finite. Here is why: suppose $f : X \to Y$ is a quasifinite morphism over Z, where X is proper over Z. Then by the Cancellation Theorem for properties of morphisms, $X \to Y$ is proper. Hence as f is quasifinite and proper, f is finite.

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FOUNDATIONS OF ALGEBRAIC GEOMETRY CLASS 19

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CONTENTS

1. DIMENSION AND CODIMENSION

The notion of *dimension* is the first of two algebraically "hard" properties of schemes, the other being smoothness = nonsingularity (coming at the start of next quarter).

1.1. Dimension. One rather basic notion we expect to have of geometric objects is *dimension*, and our goal in this chapter is to define the dimension of schemes. This should agree with, and generalize, our geometric intuition. Keep in mind that although we think of this as a basic notion in geometry, it is a slippery concept, and has been so for historically. (For example, how do we know that there isn't an isomorphism between some 1-dimensional manifold and some 2-dimensional manifold?)

A caution for those thinking over the complex numbers: our dimensions will be algebraic, and hence half that of the "real" picture. For example, $\mathbb{A}^1_{\mathbb{C}}$, which you may picture as the complex numbers (plus one generic point), has dimension 1.

Surprisingly, the right definition is purely topological — it just depends on the topological space, and not on the structure sheaf. We define the **dimension** of a topological space X as the supremum of lengths of chains of closed irreducible sets, starting the indexing with 0. (This dimension may be infinite.) Scholars of the empty set can take the dimension of the empty set to be $-\infty$. Define the **dimension** of a ring as the Krull dimension of its spectrum — the $\sup$ of the lengths of the chains of nested prime ideals (where indexing starts at zero). These two definitions of dimension are sometimes called **Krull dimension**. (You might think a Noetherian ring has finite dimension because all chains of prime ideals are finite, but this isn't necessarily true — see Exercise 1.6.)

As we have a natural homeomorphism between $\operatorname{Spec} A$ and $\operatorname{Spec} A/\mathfrak{n}(A)$ (the Zariski topology doesn't care about nilpotents), we have $\dim A = \dim A/\mathfrak{n}(A)$.

Date: Monday, December 4, 2007.

Examples. We have identified all the prime ideals of $k[t]$ (they are 0, and $(f(t))$ for irreducible polynomials $f(t)$), $\mathbb{Z}$ (0 and (p)), k (only 0), and $k[x]/(x^2)$ (only 0), so we can quickly check that $\dim \mathbb{A}^1_k = \dim \operatorname{Spec} \mathbb{Z} = 1$, $\dim \operatorname{Spec} k = 0$, $\dim \operatorname{Spec} k[x]/(x^2) = 0$.

We must be careful with the notion of dimension for reducible spaces. If Z is the union of two closed subsets X and Y, then $\dim_Z = \max(\dim X, \dim Y)$. In particular, if Z is the disjoint union of something of dimension 2 and something of dimension 1, then it has dimension 2. Thus dimension is not a "local" characteristic of a space. This sometimes bothers us, so we will often talk about dimensions of irreducible topological spaces. If a topological space can be expressed as a finite union of irreducible subsets, then say that it is **equidimensional** or **pure dimensional** (resp. equidimensional of dimension n or pure dimension n) if each of its components has the same dimension (resp. they are all of dimension n).

An equidimensional dimension 1 (resp. 2, n) topological space is said to be a **curve** (resp. **surface**, **n-fold**).

1.2. Codimension. Because dimension behaves oddly for disjoint unions, we need some care when defining codimension, and in using the phrase. For example, if Y is a closed subset of X, we might define the codimension to be $\dim X - \dim Y$, but this behaves badly. For example, if X is the disjoint union of a point Y and a curve Z, then $\dim X - \dim Y = 1$, but the reason for this has nothing to do with the local behavior of X near Y.

A better definition is as follows. In order to avoid excessive pathology, we define the codimension of Y in X *only when Y is irreducible*. We define the **codimension of an irreducible closed subset** $Y \subset X$ of a topological space as the supremum of lengths of *increasing* chains of irreducible closed subsets starting at Y (where indexing starts at 0). The **codimension of a point** is defined to be the codimension of its closure.

We say that a prime ideal $\mathfrak{p}$ in a ring has **codimension** equal to the supremum of lengths of the chains of decreasing prime ideals starting at $\mathfrak{p}$, with indexing starting at 0. Thus in an integral domain, the ideal (0) has codimension 0; and in $\mathbb{Z}$, the ideal (23) has codimension 1. Note that the codimension of the prime ideal $\mathfrak{p}$ in A is $\dim A_{\mathfrak{p}}$. (This notion is often called *height*.) Thus the codimension of $\mathfrak{p}$ in A is the codimension of $[\mathfrak{p}]$ in $\operatorname{Spec} A$.

1.A. EXERCISE. Show that if Y is an irreducible subset of a scheme X with generic point y, show that the codimension of Y is the dimension of the local ring $\mathcal{O}_{X,y}$.

Note that Y is codimension 0 in X if it is an irreducible component of X. Similarly, Y is codimension 1 if it is strictly contained in an irreducible component Y', and there is no irreducible subset strictly between Y and Y'. (See Figure 1 for examples.) An closed subset all of whose irreducible components are codimension 1 in some ambient space X is said to be a **hypersurface** in X.

1.B. EASY EXERCISE. Show that

$$\operatorname{codim}_X Y + \dim Y \leq \dim X. \tag{1}$$

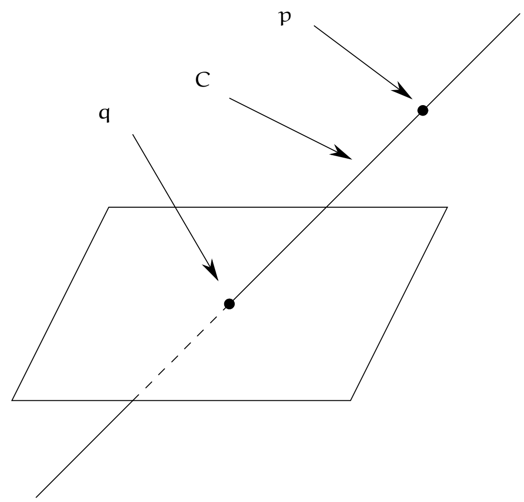

FIGURE 1. Behavior of codimension

We will see next day that equality always holds if X and Y are varieties, but equality doesn't always hold.

Warnings. (1) We have *only* defined codimension for *irreducible* Y in X. Exercise extreme caution in using this word in any other setting. We may use it in the case where the irreducible components of Y *each* have the same codimension.

(2) The notion of codimension still can behave slightly oddly. For example, consider Figure 1. (You should think of this as an intuitive sketch, but once we define dimension correctly, this will be precise.) Here the total space X has dimension 2, but point p is dimension 0, and codimension 1. We also have an example of a codimension 2 subset q contained in a codimension 0 subset C with no codimension 1 subset "in between".

Worse things can happen; we will soon see an example of a closed point in an irreducible surface that is nonetheless codimension 1, not 2. However, for irreducible varieties (finitely generated domains over a field), this can't happen, and the inequality (1) must be an inequality. We'll show this next day.

1.3. What will happen in this chapter.

In this chapter, we'll explore the notions of dimension and codimension, and show that they satisfy properties that we find desirable, and (later) useful. In particular, we'll learn some techniques for computing dimension.

We would certainly want affine n-space to have dimension n. We will indeed show (next day) that $\dim \mathbb{A}^n_k = n$, and show more generally that the dimension of an irreducible variety over k is its transcendence degree. En route, we will see some useful facts, including the Going-Up Theorem, and Noether Normalization. (While proving the Going-Up Theorem, we will see a trick that will let us prove many forms of Nakayama's Lemma, which will be useful to us in the future.) Related to the Going-Up Theorem is the fact that certain nice ("integral") morphisms $X \to Y$ will have the property that $\dim X = \dim Y$ (Exercise 2.H).

Noether Normalization will let us prove Chevalley's Theorem, stating that the image of a finite type morphism of Noetherian schemes is always constructable. We will also give a short proof of the Nullstellensatz.

We then briefly discuss two useful facts about codimension one. A linear function on a vector space is either vanishes in codimension 0 (if it is the 0-function) or else in codimension 1. The same is true much more generally for functions on Noetherian schemes. Informally: a function on a Noetherian scheme also vanishes in pure codimension 0 or 1. More precisely, the irreducible components of its vanishing locus are all codimension at most 1. This is Krull's Principal Ideal Theorem. A second fact, that we'll call "Algebraic Hartogs' Lemma", informally states that on a normal scheme, any rational function with no poles is in fact a regular function. These two codimension one facts will come in very handy in the future.

We end this introductory section with a first property about codimensions (and hypersurfaces) that we'll find useful, and a pathology.

1.4. *Warm-up proposition. — In a unique factorization domain A, all codimension 1 prime ideals are principal.*

We will see next day that the converse (in the case where A is Noetherian domain) holds as well.

Proof. Suppose $\mathfrak{p}$ is a codimension 1 prime. Choose any $f \neq 0$ in $\mathfrak{p}$, and let g be any irreducible/prime factor of f that is in $\mathfrak{p}$ (there is at least one). Then (g) is a prime ideal contained in $\mathfrak{p}$, so $(0) \subset (g) \subset \mathfrak{p}$. As $\mathfrak{p}$ is codimension 1, we must have $\mathfrak{p} = (g)$, and thus $\mathfrak{p}$ is principal. $\square$

1.5. *A fun but unimportant counterexample.* As a Noetherian ring has no infinite chain of prime ideals, you may think that Noetherian rings must have finite dimension. Here is an example of a Noetherian ring with infinite dimension, due to Nagata, the master of counterexamples.

1.6. Exercise $\star$. Choose an increasing sequence of positive integers $m_1, m_2, \ldots$ whose *differences* are also increasing ($m_{i+1} - m_i > m_i - m_{i-1}$). Let $P_i = (x_{m_i+1}, \ldots, x_{m_{i+1}})$ and $S = A - \cup_i P_i$. Show that S is a multiplicative set. Show that $S^{-1}A$ is Noetherian. Show that each $S^{-1}P$ is the smallest prime ideal in a chain of prime ideals of length $m_{i+1} - m_i$. Hence conclude that $\dim S^{-1}A = \infty$.

A ring homomorphism $\phi : B \to A$ is **integral** if every element of A is integral over $\phi(B)$. In other words, if a is any element of A, then a satisfies some monic polynomial

$$a^n + ?a^{n-1} + \cdots + ? = 0$$

where all the coefficients lie in $\phi(B)$. We call ϕ an **integral extension** if ϕ is an inclusion of rings.

2.A. EXERCISE. Show that if $f : B \to A$ is a ring homomorphism, and $(b_1, \ldots, b_n) = 1$ in B, and $B_{b_i} \to A_{f(b_i)}$ is integral, then f is integral. Thus we can define the notion of **integral morphism** of schemes.

2.B. EXERCISE. Show that the notion of integral homomorphism is well behaved with respect to localization and quotient of B, and quotient of A, but not localization of A. Show that the notion of integral extension is well behaved with respect to localization and quotient of B, but not quotient of A. If possible, draw pictures of your examples.

2.C. EXERCISE. Show that if B is an integral extension of A, and C is an integral extension of B, then C is an integral extension of A.

2.1. *Proposition (finite implies integral). — If A is a finite B-algebra, then ϕ is an integral homomorphism.*

The converse is false: integral does not imply finite, as $\mathbb{Q} \hookrightarrow \overline{\mathbb{Q}}$ is an integral homomorphism, but $\overline{\mathbb{Q}}$ is not a finite $\mathbb{Q}$-module.

2.D. UNIMPORTANT EXERCISE: FINITE = INTEGRAL + FINITE TYPE. Show that a morphism is finite if and only if it is integral and finite type.

Proof. The proof involves a useful trick.

Choose a finite generating set $m_1, \ldots, m_n$ of A as a B-module. Then $a m_i = \sum b_{ij} m_j$, for some $b_{ij} \in B$. Thus

$$(a I_{n \times n} - [b_{ij}]_{ij}) \begin{pmatrix} m_1 \\ \vdots \\ m_n \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}. \tag{2}$$

We can't quite invert this matrix $(a I_{n\times n} - [b_{ij}]_{ij})$, but we almost can. Recall that any $n \times n$ matrix M has an *adjoint matrix* $\operatorname{adj}(M)$ such that $\operatorname{adj}(M) M = \det(M) \mathrm{Id}_n$. (The ijth entry of $\operatorname{adj}(M)$ is the determinant of the matrix obtained from M by deleting the ith column and jth row, times $(-1)^{i+j}$.) The coefficients of $\operatorname{adj}(M)$ are polynomials in the coefficients of M. (You've likely seen this in the form of a formula for M^{-1} when there *is* an inverse.)

Multiplying both sides of (3) on the left by $\operatorname{adj}(\mathrm{Id}_n - A)$, we obtain

$$\det(\mathrm{Id}_n - A) \begin{pmatrix} m_1 \\ \vdots \\ m_n \end{pmatrix} = 0.$$

Multiplying (2) by the adjoint of $(aI_{n\times n} - [b_{ij}]_{ij})$, we get

$$\det(aI_{n\times n} - [b_{ij}]_{ij}) \begin{pmatrix} m_1 \\ \vdots \\ m_n \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}.$$

So $\det(aI - M)$ annihilates A, i.e. $\det(aI - M) = 0$. But expanding the determinant yields an integral equation for a with coefficients in B. □

We now state the Going-up theorem.

2.2. *The Cohen-Seidenberg Going up theorem. — Suppose* $\phi : B \to A$ *is an integral extension. Then for any prime ideal* $\mathfrak{q} \subset B$*, there is a prime ideal* $\mathfrak{p} \subset A$ *such that* $\mathfrak{p} \cap B = \mathfrak{q}$.

Although this is a theorem in algebra, the name can be interpreted geometrically: the theorem asserts that the corresponding morphism of schemes is surjective, and that "above" every prime $\mathfrak{q}$ "downstairs", there is a prime $\mathfrak{q}$ "upstairs", see Figure 2. (For this reason, it is often said that $\mathfrak{q}$ is "above" $\mathfrak{p}$ if $\mathfrak{p} \cap B = \mathfrak{q}$.)

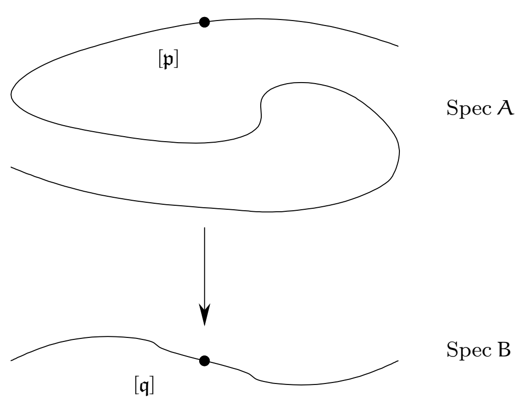

FIGURE 2. A picture of the Going-up theorem

2.E. EXERCISE (REALITY CHECK). The morphism $k[t] \to k[t]_{(t)}$ is not integral, as $1/t$ satisfies no monic polynomial with coefficients in $k[t]$. Show that the conclusion of the Going-up theorem 2.2 fails.

Proof of the Cohen-Seidenberg Going-Up theorem 2.2 $\star$. This proof is eminently readable, but could be skipped on first reading. We start with an exercise.

2.F. EXERCISE. Show that the special case where A is a field translates to: if $B \subset A$ is a subring with A integral over B, then B is a field. Prove this. (Hint: all you need to do is show that all nonzero elements in B have inverses in B. Here is the start: If $b \in B$, then $1/b \in A$, and this satisfies some integral equation over B.)

Proof of the Going-Up Theorem 2.2. We first make a reduction: by localizing at $\mathfrak{q}$, so we can assume that $(B, \mathfrak{q})$ is a local ring.

Then let $\mathfrak{p}$ be any *maximal* ideal of A. We will see that $\mathfrak{p} \cap B = \mathfrak{q}$. Consider the following diagram.

$$\begin{array}{ccc} A & \twoheadrightarrow & A/\mathfrak{p} \\ \uparrow & & \uparrow \\ B & \twoheadrightarrow & B/(B \cap \mathfrak{p}) \end{array} \qquad \text{field}$$

By the Exercise above, the lower right is a field too, so $B \cap \mathfrak{p}$ is a maximal ideal, hence $\mathfrak{q}$. □

2.G. IMPORTANT BUT STRAIGHTFORWARD EXERCISE (SOMETIMES ALSO CALLED THE GOING-UP THEOREM). Show that if $\mathfrak{q}_1 \subset \mathfrak{q}_2 \subset \cdots \subset \mathfrak{q}_n$ is a chain of prime ideals of B, and $\mathfrak{p}_1 \subset \cdots \subset \mathfrak{p}_m$ is a chain of prime ideals of A such that $\mathfrak{p}_i$ "lies over" $\mathfrak{q}_i$ (and $m < n$), then the second chain can be extended to $\mathfrak{p}_1 \subset \cdots \subset \mathfrak{p}_n$ so that this remains true.

This version of the Going-up Theorem has an important consequence.

2.H. IMPORTANT EXERCISE. Show that if $f : \operatorname{Spec} A \to \operatorname{Spec} B$ corresponds to an integral *extension of rings*, then $\dim \operatorname{Spec} A = \dim \operatorname{Spec} B$. (Hint: show that a chain of prime ideals downstairs gives a chain upstairs, by the previous exercise, of the same length. Conversely, a chain upstairs gives a chain downstairs. We need to check that no two elements of the chain upstairs goes to the same element $[\mathfrak{q}] \in \operatorname{Spec} B$ of the chain downstairs. As integral extensions are well-behaved by localization and quotients of $\operatorname{Spec} B$ (Exercise 2.B), we can replace B by $B_\mathfrak{q}/\mathfrak{q}B_\mathfrak{q}$ (and A by $A \otimes_B (B_\mathfrak{q}/\mathfrak{q}B_\mathfrak{q})$). Thus we can assume B is a field. Hence we must show that if $\phi : k \to A$ is an integral extension, then $\dim A = 0$. Outline of proof: Suppose $\mathfrak{p} \subset \mathfrak{m}$ are two prime ideals of $\mathfrak{p}$. Mod out by $\mathfrak{p}$, so we can assume that A is a domain. I claim that any non-zero element is invertible: Say $x \in A$, and $x \neq 0$. Then the minimal monic polynomial for x has non-zero constant term. But then x is invertible — recall the coefficients are in a field.)

The trick in the proof of Proposition 2.1 is very handy, and can be used to quickly prove Nakayama's lemma. This name is used for several different but related results. Nakayama isn't especially closely related to dimension, but we may as well prove it while the trick is fresh in our minds.

3.1. *Nakayama's Lemma version 1. — Suppose* A *is a ring,* I *an ideal of* A*, and* M *is a finitely-generated* A*-module. Suppose* $M = IM$*. Then there exists an* $a \in A$ *with* $a \equiv 1 \pmod{I}$ *with* $aM = 0$*.*

Proof. Say M is generated by $m_1, \ldots, m_n$. Then as $M = IM$, we have $m_i = \sum_j a_{ij} m_j$ for some $a_{ij} \in I$. Thus

$$(\mathrm{Id}_n - A) \begin{pmatrix} m_1 \\ \vdots \\ m_n \end{pmatrix} = 0 \tag{3}$$

where Id_n is the $n \times n$ identity matrix in A, and $A = (a_{ij})$. Multiplying both sides of (3) on the left by $\operatorname{adj}(\mathrm{Id}_n - A)$, we obtain

$$\det(\mathrm{Id}_n - A) \begin{pmatrix} m_1 \\ \vdots \\ m_n \end{pmatrix} = 0.$$

But when you expand out $\det(\mathrm{Id}_n - A)$, you get something that is $1 \pmod{I}$. □

Here is why you care: Suppose I is contained in all maximal ideals of A. (The intersection of all the maximal ideals is called the *Jacobson radical*, but we won't use this phrase. For comparison, recall that the nilradical was the intersection of the *prime ideals* of A.) Then I claim that any $a \equiv 1 \pmod{I}$ is invertible. For otherwise $(a) \neq A$, so the ideal (a) is contained in some maximal ideal $\mathfrak{m}$ — but $a \equiv 1 \pmod{\mathfrak{m}}$, contradiction. Then as a is invertible, we have the following.

3.2. *Nakayama's Lemma version 2. — Suppose* A *is a ring,* I *an ideal of* A *contained in all maximal ideals, and* M *is a finitely-generated* A*-module. (The most interesting case is when* A *is a local ring, and* I *is the maximal ideal.) Suppose* $M = IM$*. Then* $M = 0$*.*

3.A. EXERCISE (NAKAYAMA'S LEMMA VERSION 3). Suppose A is a ring, and I is an ideal of A contained in all maximal ideals. Suppose M is a *finitely generated* A-module, and $N \subset M$ is a submodule. If $N/IN \to M/IM$ an isomorphism, then $M = N$. (This can be useful, although it won't come up again for us.)

3.B. IMPORTANT EXERCISE (NAKAYAMA'S LEMMA VERSION 4). Suppose $(A, \mathfrak{m})$ is a local ring. Suppose M is a finitely-generated A-module, and $f_1, \ldots, f_n \in M$, with (the images of) $f_1, \ldots, f_n$ generating $M/\mathfrak{m}M$. Then $f_1, \ldots, f_n$ generate M. (In particular, taking $M = \mathfrak{m}$, if we have generators of $\mathfrak{m}/\mathfrak{m}^2$, they also generate $\mathfrak{m}$.)

3.C. UNIMPORTANT EXERCISE (NAKAYAMA'S LEMMA VERSION 5). Prove Nakayama version 1 (Lemma 3.1) without the hypothesis that M is finitely generated, but with the hypothesis that $I^n = 0$ for some n. (This argument does *not* use the trick.) This result is quite useful, although we won't use it.

3.D. IMPORTANT EXERCISE THAT WE WILL USE SOON. Suppose S is a subring of a ring A, and $r \in A$. Suppose there is a faithful $S[r]$-module M that is finitely generated as an S-module. Show that r is integral over S. (Hint: look carefully at the proof of Nakayama's Lemma version 1, and change a few words.)

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FOUNDATIONS OF ALGEBRAIC GEOMETRY CLASS 20

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CONTENTS

This is the last class of the quarter! We will finish with dimension theory today.

1. DIMENSION AND TRANSCENDENCE DEGREE

We now prove an alternative interpretation for dimension for irreducible varieties.

1.1. *Theorem (dimension = transcendence degree). — Suppose* A *is a finitely-generated domain over a field* k*. Then* $\dim \operatorname{Spec} A$ *is the transcendence degree of the fraction field* $FF(A)$ *over* k*.*

By "finitely generated domain over k", we mean "a finitely generated k-algebra that is an integral domain".

In case you haven't seen the notion of transcendence degree, here is a quick summary of the relevant facts. Suppose K/k is a finitely generated field extension. Then any two maximal sets of algebraically independent elements of K over k (i.e. any set with no algebraic relation) have the same size (a non-negative integer or ∞). If this size is finite, say n, and $x_1, \dots, x_n$ is such a set, then $K/k(x_1, \dots, x_n)$ is necessarily a finitely generated algebraic extension, i.e. a finite extension. (Such a set $x_1, \dots, x_n$ is called a transcendence basis, and n is called the *transcendence degree*.)

In particular, we see that $\dim \mathbb{A}^n_k = n$. However, our proof of Theorem 1.1 will go *through* this fact, so it isn't really a Corollary.

Date: Wednesday, December 6, 2007.

1.2. Sample consequences. We will prove Theorem 1.1 shortly. But we first show that it is useful by giving some immediate consequences. We begin with a proof of the Nullstellensatz, promised earlier.

1.A. EXERCISE: NULLSTELLENSATZ FROM DIMENSION THEORY.
(a) Suppose $A = k[x_1, \ldots, x_n]/I$, where k is an algebraically closed field and I is some ideal. Then the maximal ideals are precisely those of the form $(x_1 - a_1, \ldots, x_n - a_n)$, where $a_i \in k$. This version (the "weak Nullstellensatz") was stated earlier.
(b) Suppose $A = k[x_1, \ldots, x_n]/I$ where k is not necessarily algebraically closed. Show that every maximal ideal of A has a residue field that is a finite extension of k. This version was stated in earlier. (Hint for both parts: the maximal ideals correspond to dimension 0 points, which correspond to transcendence degree 0 extensions of k, i.e. finite extensions of k. If $k = \overline{k}$, the maximal ideals correspond to surjections $f : k[x_1, \ldots, x_n] \to k$. Fix one such surjection. Let $a_i = f(x_i)$, and show that the corresponding maximal ideal is $(x_1 - a_1, \ldots, x_n - a_n)$.)

1.3. *Points of* $\mathbb{A}^2_k$. We can now confirm that we have named all the primes of $k[x, y]$ where k is algebraically closed (as promised earlier when $k = \mathbb{C}$). Recall that we have discovered the primes (0), $f(x, y)$ where f is irreducible, and $(x - a, y - b)$ where $a, b \in k$. As $\mathbb{A}^2_k$ is irreducible, there is only one irreducible subset of codimension 0. By the Proposition from last day about UFDs, all codimension 1 primes are principal. By the inequality $\dim X + \operatorname{codim}_Y X = \dim Y$, there are no primes of codimension greater than 2, and any prime of codimension 2 must be maximal. We have identified all the maximal ideals of $k[x, y]$ by the Nullstellensatz.

1.B. IMPORTANT EXERCISE. Suppose X is an irreducible variety. Show that $\dim X$ is the transcendence degree of the function field (the stalk at the generic point) $\mathcal{O}_{X,\eta}$ over k. Thus (as the generic point lies in all non-empty open sets) the dimension can be computed in any open set of X. (This is not true in general, see §3.4.)

Here is an application that you might reasonably have wondered about before thinking about algebraic geometry.

1.C. EXERCISE. Suppose $f(x, y)$ and $g(x, y)$ are two complex polynomials ($f, g \in \mathbb{C}[x, y]$). Suppose f and g have no common factors. Show that the system of equations $f(x, y) = g(x, y) = 0$ has a finite number of solutions. (This isn't essential for what follows. But it is a basic fact, and very believable.)

1.D. EXERCISE. Suppose $X \subset Y$ is an inclusion of irreducible k-varieties, and η is the generic point of X. Show that $\dim X + \dim \mathcal{O}_{Y,\eta} = \dim Y$. Hence show that $\dim X + \operatorname{codim}_Y X = \dim Y$. Thus for varieties, the inequality $\dim X + \operatorname{codim}_Y X \leq \dim Y$ is always an equality.

1.E. EXERCISE. Show that $\operatorname{Spec} k[w,x,y,z]/(wz-xy, wy-x^2, xz-y^2)$ is an integral *surface*. You might expect it to be a curve, because it is cut out by three equations in 4-space. (You may recognize this as the affine cone over the twisted cubic.) It turns out that you actually need three equations to cut out this surface. The first equation cuts out a threefold in four-space (by Krull's theorem 3.2, see later). The second equation cuts out a surface: our surface, along with another surface. The third equation cuts out our surface, and removes the "extraneous component". One last aside: notice once again that the cone over the quadric surface $k[w,x,y,z]/(wz-xy)$ makes an appearance.)

1.4. Noether Normalization.

Hopefully you are now motivated to understand the proof of Theorem 1.1 on dimension and transcendence degree. To set up the argument, we introduce another important and ancient result, Noether's Normalization Lemma.

1.5. *Noether Normalization Lemma. — Suppose* A *is an integral domain, finitely generated over a field* k. *If* $\operatorname{tr.deg.}_k A = n$, *then there are elements* $x_1, \dots, x_n \in A$, *algebraically independent over* k, *such that* A *is a finite (hence integral) extension of* $k[x_1, \dots, x_n]$.

The geometric content behind this result is that given any integral affine k-scheme X, we can find a surjective finite morphism $X \to \mathbb{A}^n_k$, where n is the transcendence degree of the function field of X (over k). Surjectivity follows from the Going-Up Theorem.

Nagata's proof of Noether normalization $\star$. Suppose we can write $A = k[y_1, \dots, y_m]/\mathfrak{p}$, i.e. that A can be chosen to have m generators. Note that $m \geq n$. We show the result by induction on m. The base case $m = n$ is immediate.

Assume now that $m > n$, and that we have proved the result for smaller m. We will find $m-1$ elements $z_1, \dots, z_{m-1}$ of A such that A is finite over $A' := k[z_1, \dots, z_{m-1}]$ (i.e. the subring of A generated by $z_1, \dots, z_{m-1}$). Then by the inductive hypothesis, A' is finite over some $k[x_1, \dots, x_n]$, and A is finite over A', so A is finite over $k[x_1, \dots, x_n]$.

$$\begin{array}{c} A \\ \Big| \text{ finite} \\ A' = k[z_1, \dots, z_{m-1}]/\mathfrak{p} \\ \Big| \text{ finite} \\ k[x_1, \dots, x_n] \end{array}$$

As $y_1, \dots, y_m$ are algebraically dependent, there is some non-zero algebraic relation $f(y_1, \dots, y_m) = 0$ among them (where f is a polynomial in m variables).

Let $z_1 = y_1 - y_m^{r_1}$, $z_2 = y_2 - y_m^{r_2}$, $\dots$, $z_{m-1} = y_{m-1} - y_m^{r_{m-1}}$, where $r_1, \dots, r_{m-1}$ are positive integers to be chosen shortly. Then

$$f(z_1 + y_m^{r_1}, z_2 + y_m^{r_2}, \dots, z_{m-1} + y_m^{r_{m-1}}, y_m) = 0.$$

Then upon expanding this out, each monomial in f (as a polynomial in m variables) will yield a single term in that is a constant times a power of y_m (with no z_i factors). By choosing the r_i so that $0 \ll r_1 \ll r_2 \ll \cdots \ll r_{m-1}$, we can ensure that the powers of y_m appearing are all distinct, and so that in particular there is a leading term y_m^N, and all other terms (including those with z_i-factors) are of smaller degree in y_m. Thus we have described an integral dependence of y_m on $z_1, \dots, z_{m-1}$ as desired. □

1.6. *Aside: the geometric idea behind Nagata's proof.* There is some geometric intuition behind this. Suppose we have an m-dimensional variety in $\mathbb{A}_k^n$ with $m < n$, for example $xy = 1$ in $\mathbb{A}^2$. One approach is to project it to a hyperplane via a finite morphism. In the case of $xy = 1$, if we projected to the x-axis, it wouldn't be finite, roughly speaking because the asymptote $x = 0$ prevents the map from being closed. But if we projected to a line, we might hope that we would get rid of this problem, and indeed we usually can: this problem arises for only a finite number of directions. But we might have a problem if the field were finite: perhaps the finite number of directions in which to project each have a problem. (The reader may show that if k is an infinite field, then the substitution in the above proof $z_i = y_i - y_m^{r_i}$ can be replaced by the linear substitution $z_i = y_i - a_i y_m$ where $a_i \in k$, and that for a non-empty Zariski-open choice of a_i, we indeed obtain a finite morphism.) Nagata's trick in general is to "jiggle" the variables in a non-linear way, and that this is enough to prevent non-finiteness of the map.

Proof of Theorem 1.1 on dimension and transcendence degree. Suppose X is an integral affine k-scheme. We show that $\dim X$ equals the transcendence degree n of its function field, by induction on n. Fix X, and assume the result is known for all transcendence degrees less than n.

By Noether normalization, there exists a surjective finite morphism map $X \to \mathbb{A}_k^n$. By the Going-Up theorem, $\dim X = \dim \mathbb{A}_k^n$. If $n = 0$, we are done, as $\dim \mathbb{A}_k^0 = 0$.

We now show that $\dim \mathbb{A}_k^n = n$ for $n > 0$, by induction. Clearly $\dim \mathbb{A}_k^n \geq n$, as we can describe a chain of irreducible subsets of length $n + 1$: if $x_1, \dots, x_n$ are coordinates on $\mathbb{A}^n$, consider the chain of ideals

$$(0) \subset (x_1) \subset \cdots \subset (x_1, \dots, x_n)$$

in $k[x_1, \dots, x_n]$. Suppose we have a chain of prime ideals of length at least n:

$$(0) = \mathfrak{p}_0 \subset \cdots \subset \mathfrak{p}_m.$$

where $\mathfrak{p}_1$ is a codimension 1 prime ideal. Then $\mathfrak{p}_1$ is principal (as $k[x_1, \dots, x_n]$ is a unique factorization domain, a Proposition proved on Monday) say $\mathfrak{p}_1 = (f(x_1, \dots, x_n))$, where f is an irreducible polynomial. Then $k[x_1, \dots, x_n]/(f(x_1, \dots, x_n))$ has transcendence degree $n - 1$, so by induction,

$$\dim k[x_1, \dots, x_n]/(f) = n - 1.$$

□

We can now prove Chevalley's Theorem 2.1, discussed earlier.

2.1. *Chevalley's Theorem. — Suppose* $f : X \to Y$ *is a morphism of finite type of Noetherian schemes. Then the image of any constructable set is constructable.*

The proof will use Noether normalization. This is remarkable: Noether normalization is about finitely generated algebras over a field, but there is no field in the statement of Chevalley's theorem. Hence if you prefer to work over arbitrary rings (or schemes), this shows that you still care about facts about finite type schemes over a field. Conversely, even if you are interested in finite type schemes over a given field (like $\mathbb{C}$), the field that comes up in the proof of Chevalley's theorem is *not* that field, so even if you prefer to work over $\mathbb{C}$, this argument shows that you still care about working over arbitrary fields, not necessarily algebraically closed.

2.A. HARD EXERCISE. Reduce the proof of Chevalley's theorem 2.1 to the following statement: suppose $f : X = \operatorname{Spec} A \to Y = \operatorname{Spec} B$ is a dominant morphism, where A and B are domains, and f corresponds to $\phi : B \to B[x_1, \dots, x_n]/I \cong A$. Then the image of f contains a dense open subset of $\operatorname{Spec} B$. (Hint: Make a series of reductions. The notion of constructable is local, so reduce to the case where Y is affine. Then X can be expressed as a finite union of affines; reduce to the case where X is affine. X can be expressed as the finite union of irreducible components; reduce to the case where X is irreducible. Reduce to the case where X is reduced. By considering the closure of the image of the generic point of X, reduce to the case where Y also is integral (irreducible and reduced), and $X \to Y$ is dominant. Use Noetherian induction in some way on Y.)

Proof. We prove the statement given in the previous exercise. Let $K := FF(B)$. Now $A \otimes_B K$ is a localization of A with respect to B^* (interpreted as a subset of A), so it is a domain, and it is finitely generated over K (by $x_1, \dots, x_n$), so it has finite transcendence degree r over K. Thus by Noether normalization, we can find a subring $K[y_1, \dots, y_r] \subset A \otimes_B K$, so that $A \otimes_B K$ is integrally dependent on $K[y_1, \dots, y_r]$. We can choose the y_i to be in A: each is in $(B^*)^{-1}A$ to begin with, so we can replace each y_i by a suitable K-multiple.

Sadly A is not necessarily integrally dependent on $A[y_1, \dots, y_r]$ (as this would imply that $\operatorname{Spec} A \to \operatorname{Spec} B$ is surjective by the Going-Up Theorem). However, each x_i satisfies some integral equation

$$x_i^n + f_1(y_1, \dots, y_r) x_i^{n-1} + \cdots + f_n(y_1, \dots, y_r) = 0$$

where f_j are polynomials with coefficients in $K = FF(B)$. Let g be the product of the denominators of all the coefficients of all these polynomials (a finite set). Then A_g is integral over $B_g[y_1, \dots, y_r]$, and hence $\operatorname{Spec} A_g \to \operatorname{Spec} B_g$ is surjective; $\operatorname{Spec} B_g$ is our open subset. □

3. FUN IN CODIMENSION ONE: KRULL'S PRINCIPAL IDEAL THEOREM, ALGEBRAIC HARTOGS' LEMMA, AND MORE

In this section, we'll explore a number of results related to codimension one.

Codimension one primes of $\mathbb{Z}$ and $k[x, y]$ correspond to prime numbers and irreducible polynomials respectively. We will make this link precise for unique factorization domains. Then we introduce two results that apply in more general situations, and link functions and the codimension one points where they vanish, Krull's Principal Ideal Theorem 3.2, and Algebraic Hartogs' Lemma 3.6. We will find these two theorems very useful. For example, Krull's Principal Ideal Theorem will help us compute codimensions, and will show us that codimension can behave oddly, and Algebraic Hartogs' Lemma will give us a useful characterization of Unique Factorization Domains (Proposition 3.8).

The results in this section will require (locally) Noetherian hypotheses.

3.1. Krull's Principal Ideal Theorem. As described earlier in the chapter, in analogy with linear algebra, we have the following.

3.2. *Krull's Principal Ideal Theorem (geometric version). — Suppose X is a Noetherian scheme, and f is a function. Then the irreducible components of $V(f)$ are codimension 0 or 1.*

This is clearly equivalent to the following algebraic statement.

3.3. *Krull's Principal Ideal Theorem (algebraic version). — Suppose A is a Noetherian ring, and $f \in A$. Then every minimal prime $\mathfrak{p}$ containing f has codimension at most 1. If furthermore f is not a zero-divisor, then every minimal prime $\mathfrak{p}$ containing f has codimension precisely 1.*

The full proof is technical, so I'll postpone it to §4, and you shouldn't read it unless you really want to.

But this immediately useful. For example, consider the scheme $\operatorname{Spec} k[w, x, y, z]/(wx - yz)$. What is its dimension? It is cut out by one non-zero equation $wx - yz$ in $\mathbb{A}^4$, so it is a threefold.

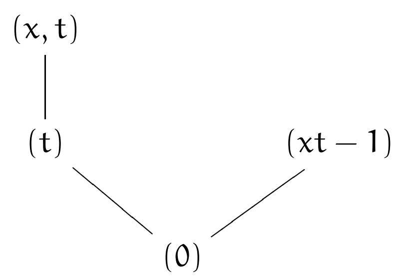
3.A. EXERCISE. What is the dimension of $\operatorname{Spec} k[w, x, y, z]/(wz - xy, y^{17} + z^{17})$? (Be careful to check they hypotheses before invoking Krull!)

3.B. EXERCISE. Show that an irreducible homogeneous polynomial in $n + 1$ variables over a field k describes an integral scheme of dimension $n - 1$.

3.C. EXERCISE (IMPORTANT FOR LATER). (a) *(Hypersurfaces meet everything of dimension at least 1 in projective space — unlike in affine space.)* Suppose X is a closed subset of $\mathbb{P}^n_k$ of dimension at least 1, and H a nonempty hypersurface in $\mathbb{P}^n_k$. Show that H meets X. (Hint:

consider the affine cone, and note that the cone over H contains the origin. Use Krull's Principal Ideal Theorem 3.3.)
(b) (Definition: Subsets in $\mathbb{P}^n$ cut out by linear equations are called **linear subspaces**. Dimension 1, 2 linear subspaces are called **lines** and **planes** respectively.) Suppose $X \hookrightarrow \mathbb{P}^n_k$ is a closed subset of dimension r. Show that any codimension r linear space meets X. Hint: Refine your argument in (a). (In fact any two things in projective space that you might expect to meet for dimensional reasons do in fact meet. We won't prove that here.)
(c) Show further that there is an intersection of $r+1$ hypersurfaces missing X. (The key step: show that there is a hypersurface of sufficiently high degree that doesn't contain every generic point of X. Show this by induction on the number of generic points. To get from n to $n+1$: take a hypersurface not vanishing on $p_1, \dots, p_n$. If it doesn't vanish on p_{n+1}, we're done. Otherwise, call this hypersurface f_{n+1}. Do something similar with $n+1$ replaced by i ($1 \le i \le n$). Then consider $\sum_i f_1 \cdots \widehat{f_i} \cdots f_{n+1}$.)

3.4. Pathologies of the notion of "codimension". We can use Krull's Principal Ideal Theorem to produce the long-promised example of pathology in the notion of codimension. Let $A = k[x]_{(x)}[t]$. In other words, elements of A are polynomials in t, whose coefficients are quotients of polynomials in x, where no factors of x appear in the denominator. (Warning: A is not isomorphic to $k[x,t]_{(x)}$.) Clearly, A is a domain, and $(xt-1)$ is not a zero divisor. You can verify that $A/(xt-1) \cong k[x]_{(x)}[1/x] \cong k(x)$ — "in $k[x]_{(x)}$, we may divide by everything but x, and now we are allowed to divide by x as well" — so $A/(xt-1)$ is a field. Thus $(xt-1)$ is not just prime but also maximal. By Krull's theorem, $(xt-1)$ is codimension 1. Thus $(0) \subset (xt-1)$ is a maximal chain. However, A has dimension at least 2: $(0) \subset (t) \subset (x,t)$ is a chain of primes of length 3. (In fact, A has dimension precisely 2, although we don't need this fact in order to observe the pathology.) Thus we have a codimension 1 prime in a dimension 2 ring that is dimension 0. Here is a picture of this lattice of ideals.



This example comes from geometry; it is enlightening to draw a picture see Figure 1. $\operatorname{Spec} k[x]_{(x)}$ corresponds to a germ of $\mathbb{A}^1_k$ near the origin, and $\operatorname{Spec} k[x]_{(x)}[t]$ corresponds to "this $\times$ the affine line". You may be able to see from the picture some motivation for this pathology — note that $V(xt-1)$ doesn't meet $V(x)$, so it can't have any specialization on $V(x)$, and there nowhere else for $V(xt-1)$ to specialize.

It is disturbing that this misbehavior turns up even in a relative benign-looking ring.

3.D. UNIMPORTANT EXERCISE. Show that it is false that if X is an integral scheme, and U is a non-empty open set, then $\dim U = \dim X$.

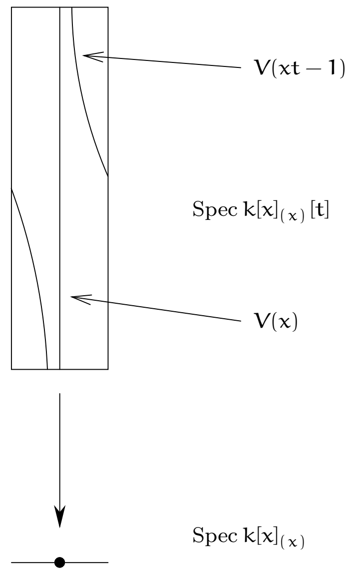


FIGURE 1. Dimension and codimension behave oddly on the surface $\operatorname{Spec} k[x]_{(x)}[t]$

3.5. Algebraic Hartogs' Lemma for Noetherian normal schemes.

Hartogs' Lemma in several complex variables states (informally) that a holomorphic function defined away from a codimension two set can be extended over that. We now describe an algebraic analog, for Noetherian normal schemes.

3.6. *Algebraic Hartogs' Lemma. — Suppose* A *is a Noetherian normal domain.*

$$A = \cap_{\mathfrak{p} \text{ codimension } 1} A_{\mathfrak{p}}.$$

The equality takes place inside $FF(A)$; recall that any localization of a domain A is naturally a subset of $FF(A)$. Warning: No one else calls this Algebraic Hartogs' Lemma. I've called it this because I find the that it parallels the statement in complex geometry. The proof is technical, so we postpone it to §3.9. (One can state Algebraic Hartogs' Lemma more generally in the case that $\operatorname{Spec} A$ is a Noetherian normal scheme, meaning that A is a product of Noetherian normal domains; the reader may wish to do so. A more general statement is that if A is a Noetherian domain, then $\cap_{\operatorname{codim} P=1} A_P$ is the integral closure of A (Atiyah-Macdonald, Cor. 5.22). We won't need this. And this "domain" condition can also be relaxed.)

One might say that if $f \in FF(A)$ does not lie in $A_{\mathfrak{p}}$ where $\mathfrak{p}$ has codimension 1, then f has a pole at $[\mathfrak{p}]$, and if $f \in FF(A)$ lies in $\mathfrak{p}A_{\mathfrak{p}}$ where $\mathfrak{p}$ has codimension 1, then f has a zero at $[\mathfrak{p}]$. It is worth interpreting Algebraic Hartogs' Lemma as saying that a rational function on a normal scheme with no poles is in fact regular (an element of A). More generally, if X is a Noetherian normal scheme, we can define zeros and poles of rational functions on X. (We will soon define the *order* of a zero or a pole.)

3.E. EXERCISE. Suppose f is an element of a normal domain A, and f is contained in no codimension 1 primes. Show that f is a unit.

3.F. EXERCISE. Suppose f and g are two global sections of a Noetherian normal scheme, not vanishing at any associated point, with the same poles and zeros. Show that each is a unit times the other.

3.7. A useful characterization of unique factorization domains.

We can use Algebraic Hartogs' Lemma 3.6 to prove one of the four things you need to know about unique factorization domains.

3.8. *Proposition. — Suppose that* A *is a Noetherian domain. Then* A *is a Unique Factorization Domain if and only if all codimension* 1 *primes are principal.*

This contains the Proposition last day showing that in a UFD, all height 1 primes are principal, and (in some sense) its converse.

Proof. We have already shown in last day (in the Proposition mentioned in the previous sentence) that if A is a Unique Factorization Domain, then all codimension 1 primes are principal. Assume conversely that all codimension 1 primes of A are principal. I claim that the generators of these ideals are irreducible, and that we can uniquely factor any element of A into these irreducibles, and a unit. First, suppose (f) is a codimension 1 prime ideal $\mathfrak{p}$. Then if $f = gh$, then either $g \in \mathfrak{p}$ or $h \in \mathfrak{p}$. As $\operatorname{codim} \mathfrak{p} > 0$, $\mathfrak{p} \neq 0$, so by Nakayama's Lemma (as $\mathfrak{p}$ is finitely generated), $\mathfrak{p} \neq \mathfrak{p}^2$. Thus both g and h cannot be in $\mathfrak{p}$. Say $g \notin \mathfrak{p}$. Then g is contained in no codimension 1 primes (as f was contained in only one, namely $\mathfrak{p}$), and hence is a unit by Exercise 3.E.

Finally, we show that any non-zero element f of A can be factored into irreducibles. Now $V(f)$ is contained in a finite number of codimension 1 primes, as (f) as a finite number of associated primes, and hence a finite number of minimal primes. We show that any nonzero f can be factored into irreducibles by induction on the number of codimension 1 primes containing f. In the base case where there are none, then f is a unit by Exercise 3.E. For the general case where there is at least one, say $f \in \mathfrak{p} = (g)$. Then $f = g^n h$ for some $h \notin (g)$. (Reason: otherwise, we have an ascending chain of ideals $(f) \subset (f/g) \subset (f/g^2) \subset \cdots$, contradicting Noetherianness.) Thus $f/g^n \in A$, and is contained in one fewer codimension 1 primes. □

3.9. *Proof of Algebraic Hartogs' Lemma 3.6 ⋆.* This proof does not shed light on any of the other discussion in this section, and thus should not be read. However, you should sleep soundly at night knowing that the proof is this short. Obviously the right side is contained in the left. Assume we have some x in all A_P but not in A. Let I be the "ideal of denominators":

$$I := \{r \in A : rx \in A\}.$$

(The ideal of denominators arose in an earlier discussion about normality, when we proved the stalk-locality of normality.) We know that $I \neq A$, so choose $\mathfrak{q}$ a minimal prime containing I.

Observe that this construction behaves well with respect to localization (i.e. if $\mathfrak{p}$ is any prime, then the ideal of denominators x in $A_{\mathfrak{p}}$ is the $I_{\mathfrak{p}}$, and it again measures the failure of 'Algebraic Hartogs' Lemma for x,' this time in $A_{\mathfrak{p}}$). But Hartogs' Theorem is vacuously true for dimension 1 rings, so hence no codimension 1 prime contains I. Thus $\mathfrak{q}$ has codimension at least 2. By localizing at $\mathfrak{q}$, we can assume that A is a local ring with maximal ideal $\mathfrak{q}$, and that $\mathfrak{q}$ is the only prime containing I. Thus $\sqrt{I} = \mathfrak{q}$, so there is some n with $I \subset \mathfrak{q}^n$. Take a minimal such n, so $I \not\subset \mathfrak{q}^{n-1}$, and choose any $y \in \mathfrak{q}^{n-1} - \mathfrak{q}^n$. Let $z = yx$. Then $z \notin A$ (so $\mathfrak{q}z \notin \mathfrak{q}$), but $\mathfrak{q}z \subset A$: $\mathfrak{q}z$ is an ideal of A.

I claim $\mathfrak{q}z$ is not contained in $\mathfrak{q}$. Otherwise, we would have a finitely-generated A-module (namely $\mathfrak{q}$) with a faithful $A[z]$-action, forcing z to be integral over A (and hence in A) by an Exercise in the Nakayama section last day.

Thus $\mathfrak{q}z$ is an ideal of A not contained in $\mathfrak{q}$, so it must be A! Thus $\mathfrak{q}z = A$ from which $\mathfrak{q} = A(1/z)$, from which $\mathfrak{q}$ is principal. But then $\operatorname{codim} Q = \dim A \leq \dim_{A/Q} Q/Q^2 \leq 1$ by Nakayama's lemma, contradicting the fact that $\mathfrak{q}$ has codimension at least 2. □

4. Proof of Krull's Principal Ideal Theorem 3.3 ⋆⋆

The details of this proof won't matter much to us, so you should probably not read it. It is included so you can glance at it and believe that the proof is fairly short, and you could read it if you really wanted to.

4.1. *Lemma. — If A is a Noetherian ring with one prime ideal. Then A is Artinian, i.e., it satisfies the descending chain condition for ideals.*

The notion of Artinian rings is very important, but we will get away without discussing it much.

Proof. If A is a ring, we define more generally an *Artinian A-module*, which is an A-module satisfying the descending chain condition for submodules. Thus A is an Artinian ring if it is Artinian over itself as a module.

If $\mathfrak{m}$ is a maximal ideal of R, then any finite-dimensional $(R/\mathfrak{m})$-vector space (interpreted as an R-module) is clearly Artinian, as any descending chain

$$M_1 \supset M_2 \supset \cdots$$

must eventually stabilize (as $\dim_{R/\mathfrak{m}} M_i$ is a non-increasing sequence of non-negative integers).

4.A. EXERCISE. Show that for any n, $\mathfrak{m}^n/\mathfrak{m}^{n+1}$ is a finitely-dimensional $A/\mathfrak{m}$-vector space. (Hint: show it for $n = 0$ and $n = 1$. Use the dimension for $n = 1$ to bound the dimension for general n.) Hence $\mathfrak{m}^n/\mathfrak{m}^{n+1}$ is an Artinian A-module.

As $\sqrt{0}$ is prime, it must be $\mathfrak{m}$.

4.B. EXERCISE. Prove that $\mathfrak{m}^n = 0 = 0$ for some n. (Hint: suppose $\mathfrak{m}$ can be generated by m elements, each of which has kth power 0, and show that $\mathfrak{m}^{m(k-1)+1} = 0$.)

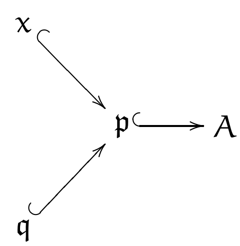
4.C. EXERCISE. Show that if $0 \to M' \to M \to M'' \to 0$ is an exact sequence of modules. then M is Artinian if and only if M' and M'' are Artinian. (Hint: think about the corresponding question about Noetherian modules, which we've seen before.)

Thus as we have a finite filtration

$$A \supset \mathfrak{m} \supset \cdots \supset \mathfrak{m}^n = 0$$

all of whose quotients are Artinian, so A is Artinian as well. This completes the proof of the Lemma. □

Proof of Krull's principal ideal theorem 3.3. Suppose we are given $x \in A$, with $\mathfrak{p}$ a minimal prime containing x. By localizing at $\mathfrak{p}$, we may assume that A is a local ring, with maximal ideal $\mathfrak{p}$. Suppose $\mathfrak{q}$ is another prime strictly contained in $\mathfrak{p}$.



For the first part of the theorem, we must show that $A_\mathfrak{q}$ has dimension 0. The second part follows from our earlier work: if any minimal primes are height 0 (minimal primes of A), f is a zero-divisor, as minimal primes of A are all associated primes of A, and elements of associated primes of A are zero-divisors.

Now $\mathfrak{p}$ is the only prime ideal containing (x), so $A/(x)$ has one prime ideal. By Lemma 4.1, $A/(x)$ is Artinian.

We invoke a useful construction, the n*th symbolic power of a prime ideal*: if A is a ring, and $\mathfrak{q}$ is a prime ideal, then define

$$\mathfrak{q}^{(n)} := \{r \in A : rs \in \mathfrak{q}^n \text{ for some } s \in A - \mathfrak{q}\}.$$

We have a descending chain of ideals in A

$$\mathfrak{q}^{(1)} \supset \mathfrak{q}^{(2)} \supset \cdots,$$

so we have a descending chain of ideals in $A/(x)$

$$\mathfrak{q}^{(1)} + (x) \supset \mathfrak{q}^{(2)} + (x) \supset \cdots$$

which stabilizes, as $A/(x)$ is Artinian. Say $\mathfrak{q}^{(n)} + (x) = \mathfrak{q}^{(n+1)} + (x)$, so

$$\mathfrak{q}^{(n)} \subset \mathfrak{q}^{(n+1)} + (x).$$

Hence for any $f \in \mathfrak{q}^{(n)}$, we can write $f = ax + g$ with $g \in \mathfrak{q}^{(n+1)}$. Hence $ax \in \mathfrak{q}^{(n)}$. As $\mathfrak{p}$ is minimal over x, $x \notin \mathfrak{q}$, so $a \in \mathfrak{q}^{(n)}$. Thus

$$\mathfrak{q}^{(n)} = (x)\mathfrak{q}^{(n)} + \mathfrak{q}^{(n+1)}.$$

As x is in the maximal ideal $\mathfrak{p}$, the second version of Nakayama's lemma gives $\mathfrak{q}^{(n)} = \mathfrak{q}^{(n+1)}$.

We now shift attention to the local ring $A_{\mathfrak{q}}$, which we are hoping is dimension 0. We have $\mathfrak{q}^{(n)}A_{\mathfrak{q}} = \mathfrak{q}^{(n+1)}A_{\mathfrak{q}}$ (the symbolic power construction clearly construction commutes with respect to localization). For any $r \in \mathfrak{q}^n A_{\mathfrak{q}} \subset \mathfrak{q}^{(n)}A_{\mathfrak{q}}$, there is some $s \in A_{\mathfrak{q}} - \mathfrak{q}A_{\mathfrak{q}}$ such that $rs \in \mathfrak{q}^{n+1}A_{\mathfrak{q}}$. As s is invertible, $r \in \mathfrak{q}^{n+1}A_{\mathfrak{q}}$ as well. Thus $\mathfrak{q}^n A_{\mathfrak{q}} \subset \mathfrak{q}^{n+1}A_{\mathfrak{q}}$, but as $\mathfrak{q}^{n+1}A_{\mathfrak{q}} \subset \mathfrak{q}^n A_{\mathfrak{q}}$, we have $\mathfrak{q}^n A_{\mathfrak{q}} = \mathfrak{q}^{n+1}A_{\mathfrak{q}}$. By Nakayama's Lemma version 4,

$$\mathfrak{q}^n A_{\mathfrak{q}} = 0.$$

Finally, any local ring $(R, \mathfrak{m})$ such that $\mathfrak{m}^n = 0$ has dimension 0, as $\operatorname{Spec} A$ consists of only one point: $[\mathfrak{m}] = V(\mathfrak{m}) = V(\mathfrak{m}^n) = V(0) = \operatorname{Spec} A$. ☐

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