FOUNDATIONS OF ALGEBRAIC GEOMETRY CLASS 42

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CONTENTS

Last day we saw three proofs of:

0.1. *The "curve to projective" extension Theorem. — Suppose* C *is a pure dimension* 1 *Noetherian scheme over a base* S*, and* $p \in C$ *is a nonsingular closed point of it. Suppose* Y *is a projective* S*-scheme. Then any morphism* $C - p \to Y$ *extends to* $C \to Y$*.*

We now use the "Curve-to-projective" Extension Theorem 0.1 to show the following.

0.2. *Theorem. — If* C *is an irreducible nonsingular curve over a field* k*, then there is an open immersion* $C \hookrightarrow C'$ *into some projective nonsingular curve* C' *(over* k*).*

We'll use make particular use of the fact that one-dimensional Noetherian schemes have a boring topology.

Proof. We begin by finding a nonconstant map to $\mathbb{P}^1$. Given a nonsingular irreducible k-curve C, take a non-empty (=dense) affine open set, and take any non-constant function f on that affine open set to get a rational map $C \dashrightarrow \mathbb{P}^1$ given by $[1; f]$. As a dense open set of a dimension 1 scheme consists of everything but a finite number of points, by the "Curve-to-projective" Extension Theorem 0.1, this extends to a morphism $C \to \mathbb{P}^1$.

We now take the normalization of $\mathbb{P}^1$ in the function field $\mathrm{FF}(C)$ of C (a finite extension of $\mathrm{FF}(\mathbb{P}^1)$), to obtain $C' \to \mathbb{P}^1$. (Normalization in a field extension was discussed in Exercise last day.)

Now C' is normal, hence nonsingular (as nonsingular = normal in dimension 1). By the finiteness of integral closure, $C' \to \mathbb{P}^1$ is a finite morphism. Moreover, finite morphisms are projective, so by considering the composition of projective morphisms $C' \to \mathbb{P}^1 \to \operatorname{Spec} k$, we see that C' is projective over k. Thus we have an isomorphism $\mathrm{FF}(C') \to \mathrm{FF}(C)$, hence a rational map $C \dashrightarrow C'$, which extends to a morphism $C \to C'$ by the "Curve-to-Projective" Extension Theorem 0.1.

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Finally, I claim that $C \to C'$ is an open immersion. If we can prove this, then we are done. I note first that this is an injection of sets:

- the generic point goes to the generic point
- the closed points of C correspond to distinct valuations on $FF(C)$ (as C is separated, by the easy direction of the valuative criterion of separatedness)

Thus as sets, C is C' minus a finite number of points. As the topology on C and C' is the "cofinite topology" (i.e. the open sets include the empty set, plus everything minus a finite number of closed points), the map $C \to C'$ of topological spaces expresses C as a homeomorphism of C onto its image $\mathfrak{im}(C)$. Let $f : C \to \mathfrak{im}(C)$ be this morphism of schemes. Then the morphism $\mathcal{O}_{\mathfrak{im}(C)} \to f_*\mathcal{O}_C$ can be interpreted as $\mathcal{O}_{\mathfrak{im}(C)} \to \mathcal{O}_C$ (where we are identifying C and $\mathfrak{im}(C)$ via the homeomorphism f). This morphism of sheaves is an isomorphism of stalks at all points $p \in \mathfrak{im}(C)$ (the stalks are both isomorphic to the discrete valuation ring corresponding to $p \in C'$), and is hence an isomorphism. Thus $C \to \mathfrak{im}(C)$ is an isomorphism of schemes, and thus $C \to C'$ is an open immersion. □

1. Various categories of "curves" are all essentially the same

1.1. *Theorem. — The following categories are equivalent.*

(i) *irreducible nonsingular projective curves* /k*, and surjective* k*-morphisms.*
(ii) *irreducible nonsingular projective curves* /k*, and dominant* k*-morphisms.*
(iii) *irreducible nonsingular projective curves* /k*, and dominant rational maps* /k*.*
(iv) *irreducible reduced* /k *curves, and dominant rational maps* /k*.*
(v) *the opposite category of fields of transcendence degree* 1 *over* k*, and* k*-homomorphisms.*

For simplicity of notation, all morphisms and maps in the following discussion are assumed to be defined over k.

This Theorem has a lot of implications. For example, each quasiprojective reduced curve is birational to precisely one projective nonsingular curve. Also, we now see that transcendence degree 1 field extensions have a genus, through their equivalence to curves. Thus we know for the first time that there exist transcendence degree 1 extensions of k that are not generated by a single element.

1.A. Exercise. Show that all nonsingular proper curves are projective. (Hint: suppose C is such a curve. It admits an open immersion $i : C \hookrightarrow C'$. Argue that i is proper, and hence has closed image.)

The interested reader can tweak the proof below to show the following variation of the theorem: in (i)–(iv), consider only geometrically irreducible curves, and in (v), consider only fields K such that $\overline{k} \cap K = k$ in $\overline{K}$. This variation allows us to exclude "weird" curves

we may not want to consider. For example, if $k = \mathbb{R}$, then we are allowing curves such as $\mathbb{P}^1_{\mathbb{C}}$ which are not geometrically irreducible (as $\mathbb{P}^1_{\mathbb{C}} \times_{\mathbb{R}} \mathbb{C} \cong \mathbb{P}^1_{\mathbb{C}} \coprod \mathbb{P}^1_{\mathbb{C}}$).

Proof. Any surjective morphism is a dominant morphism, and any dominant morphism is a dominant rational map, and each nonsingular projective curve is a quasiprojective curve, so we've shown (i) $\to$ (ii) $\to$ (iii) $\to$ (iv). To get from (iv) to (i), we first note that the nonsingular points on a quasiprojective reduced curve are dense. (One way to see this: normalization is an isomorphism away from a closed subset, an Exercise last day.) Given a dominant rational map between quasiprojective reduced curves $C \to C'$, we get a dominant rational map between their normalizations, which in turn gives a dominant rational map between their projective models $D \dashrightarrow D'$. The dominant rational map is necessarily a morphism by the "Curve-to-Projective" Extension Theorem 0.1, and then this morphism is necessarily projective and hence closed, and hence surjective (as the image contains the generic point of D', and hence its closure). Thus we have established (iv) $\to$ (i).

It remains to connect (v). Each dominant rational map of quasiprojective reduced curves indeed yields a map of function fields of dimension 1 (their fraction fields). Each function field of dimension 1 yields a reduced affine (hence quasiprojective) curve over k, and each map of two such yields a dominant rational map of the curves. □

1.2. Degree of a morphism between projective nonsingular curves.

You might already have a reasonable sense that a map of compact Riemann surfaces has a well-behaved degree, that the number of preimages of a point of C' is constant, so long as the preimages are counted with appropriate multiplicity. For example, if f locally looks like $z \mapsto z^m = y$, then near $y = 0$ and $z = 0$ (but not *at* $z = 0$), each point has precisely m preimages, but as y goes to 0, the m preimages coalesce.

We now show the algebraic version of this fact. Suppose $f : C \to C'$ is a surjective (or equivalently, dominant) map of nonsingular projective curves. We will show that f has a well-behaved degree, in a sense that we will now make precise.

Then f is finite, as f is a projective morphism with finite fibers. Alternatively, we can see the finiteness of f as follows. Let C'' be the normalization of C' in the function field of C. Then we have an isomorphism $\mathrm{FF}(C) \cong \mathrm{FF}(C'')$ which leads to birational maps $C \dashleftarrow\!\!\dashrightarrow C''$ which extend to morphisms as both C and C'' are nonsingular and projective. Thus this yields an isomorphism of C and C''. But $C'' \to C$ is a finite morphism by the finiteness of integral closure.

1.3. *Proposition. — Suppose that $\pi : C \to C'$ is a surjective finite morphism, where C is an integral curve, and C' is an integral nonsingular curve. Then $\pi_* \mathcal{O}_C$ is locally free of finite rank.*

As π is finite, $\pi_* \mathcal{O}_C$ is a finite type sheaf on $\mathcal{O}'_C$.

Before proving the proposition. I want to remind you what this means. Suppose d is the rank of this allegedly locally free sheaf. Then the fiber over any point of C with residue field K is the Spec of an algebra of dimension d over K. This means that the number of points in the fiber, counted with appropriate multiplicity, is always d.

As a motivating example, consider the map $\mathbb{Q}[y] \to \mathbb{Q}[x]$ given by $x \mapsto y^2$. (We've seen this example before.) I picture this as the projection of the parabola $x = y^2$ to the x-axis.

(i) The fiber over $x = 1$ is $\mathbb{Q}[y]/(y^2 - 1)$, so we get 2 points.
(ii) The fiber over $x = 0$ is $\mathbb{Q}[y]/(y^2)$ — we get one point, with multiplicity 2, arising because of the nonreducedness.
(iii) The fiber over $x = -1$ is $\mathbb{Q}[y]/(y^2+1) \cong \mathbb{Q}[i]$ — we get one point, with multiplicity 2, arising because of the field extension.
(iv) Finally, the fiber over the generic point $\mathrm{Spec}\,\mathbb{Q}(x)$ is $\mathrm{Spec}\,\mathbb{Q}(y)$, which is one point, with multiplicity 2, arising again because of the field extension (as $\mathbb{Q}(y)/\mathbb{Q}(x)$ is a degree 2 extension).

We thus see three sorts of behaviors (as (iii) and (iv) are the same behavior). Note that even if you only work with algebraically closed fields, you will still be forced to this third type of behavior, because residue fields at generic points tend not to be algebraically closed (witness case (iv) above).

Note that we need C' to be nonsingular for this to be true. Otherwise, the normalization of a nodal curve (Figure 1) shows an example where most points have one preimage, and one point (the node) has two.

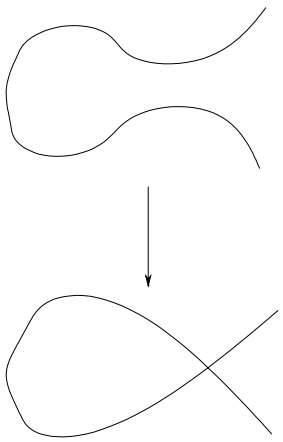

FIGURE 1. Normalization of a node shows that degree need not be well-behaved if the target is not smooth

Proof of Proposition 1.3. (For experts: we will later see that what matters here is that the morphism is finite and *flat*. But we don't yet know about flatness.)

The question is local on the target, so we may assume that C' is affine. Note that $\pi_*\mathcal{O}_C$ is torsion-free (as $\Gamma(C, \mathcal{O}_C)$ is an integral domain). Our plan is as follows: by an important exercise from ages ago, if the rank of the coherent sheaf $\pi_*\mathcal{O}_C$ is constant, then (as C' is reduced) $\pi_*\mathcal{O}_C$ is locally free. We'll show this by showing the rank at any closed point p of C' is the same as the rank at the generic point.

The notion of "rank at a point" behaves well under base change, so we base change to the discrete valuation ring $\mathcal{O}_{C',p}$. Then $\pi_*\mathcal{O}_C$ is a finitely generated module over a discrete valuation ring which is torsion-free. By the classification of finitely generated modules over a principal ideal domain, any finitely generate module over a principal ideal domain A is a direct sum of modules of the form $A/(d)$ for various $d \in A$. But if A is a discrete valuation ring, and $A/(d)$ is torsion-free, then $A/(d)$ is necessarily A (as for example all ideals of A are of the form 0 or a power of the maximal ideal). Thus we are done. □

Remark. Degrees maps of complex algebraic curves in this algebro-geometric sense agrees with the usual topological degree, which can after all be computed in the same way, by "counting preimages" appropriately.

1.B. EXERCISE. Suppose $f : C \to C'$ is a degree d morphism of integral projective nonsingular curves, and $\mathcal{L}$ is an invertible sheaf on C'. Show that $\deg_C f^*\mathcal{L} = d \deg_{C'} \mathcal{L}$. (Hint: compute $\deg_{\mathcal{L}}$ using any non-zero rational section s of $\mathcal{L}$, and compute $\deg f^*\mathcal{L}$ using the rational section f^*s of $f^*\mathcal{L}$. Note that zeros pull back to zeros, and poles pull back to poles.)

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