FOUNDATIONS OF ALGEBRAIC GEOMETRY CLASS 39

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CONTENTS

We'll conclude this quarter by discussing derived functor cohomology, which was introduced by Grothendieck in his celebrated Tôhoku article. For quasicoherent sheaves on quasicompact separated schemes, derived functor will agree with Cech cohomology. Furthermore, Cech cohomology will suffice for most of our purposes, and is quite down to earth and computable. But derived functor cohomology is worth seeing for a number of reasons. First of all, it generalizes readily to a wide number of situations. Second, it will easily provide us with some useful notions, such as Ext-groups and the Leray spectral sequence.

But to be honest, we won't use it much for the rest of the course, so feel free to just skim these notes, and come back to them later.

1. THE TOR FUNCTORS

We begin with a warm-up: the case of Tor. This is a hands-on example. But if you understand it well, you will understand derived functors in general. Tor will be useful to prove facts about flatness, which we'll discuss later. Tor is short for "torsion". The reason for this name is that the 0th and/or 1st Tor-group measures common torsion in abelian groups (aka $\mathbb{Z}$-modules).

If you have never seen this notion before, you may want to just remember its properties, which are natural. But I'd like to prove everything anyway — it is surprisingly easy.

The idea behind Tor is as follows. Whenever we see a right-exact functor, we always hope that it is the end of a long-exact sequence. Informally, given a short exact sequence,

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we are hoping to see a long exact sequence

(1)
$$\begin{tikzcd}
\cdots \arrow[r] & \mathrm{Tor}_i^A(M,N') \arrow[r] & \mathrm{Tor}_i^A(M,N) \arrow[r] & \mathrm{Tor}_i^A(M,N'') \arrow[r] & \cdots \\
\arrow[r] & \mathrm{Tor}_1^A(M,N') \arrow[r] & \mathrm{Tor}_1^A(M,N) \arrow[r] & \mathrm{Tor}_1^A(M,N'') & \\
\arrow[r] & M \otimes_A N' \arrow[r] & M \otimes_A N \arrow[r] & M \otimes_A N'' \arrow[r] & 0.
\end{tikzcd}$$

More precisely, we are hoping for *covariant functors* $\mathrm{Tor}_i^A(\cdot, N)$ from A-modules to A-modules (giving 2/3 of the morphisms in that long exact sequence), with $\mathrm{Tor}_0^A(M,N) \equiv M \otimes_A N$, and natural δ morphisms $\mathrm{Tor}_{i+1}^A(M,N'') \to \mathrm{Tor}_i^A(M,N')$ for every short exact sequence giving the long exact sequence. (In case you care, "natural" means: given a morphism of short exact sequences, the natural square you would write down involving the δ-morphism must commute. I'm not going to state this explicitly.)

It turns out to be not too hard to make this work, and this will also motivate derived functors. Let's now define $\mathrm{Tor}_i^A(M,N)$.

Take any resolution $\mathcal{R}$ of N by free modules:

$$\cdots \longrightarrow A^{\oplus n_2} \longrightarrow A^{\oplus n_1} \longrightarrow A^{\oplus n_0} \longrightarrow N \longrightarrow 0.$$

More precisely, build this resolution from right to left. Start by choosing generators of N as an A-module, giving us $A^{\oplus n_0} \to N \to 0$. Then choose generators of the kernel, and so on. Note that we are not requiring the n_i to be finite, although if N is a finitely-generated module and A is Noetherian (or more generally if N is coherent and A is coherent over itself), we can choose the n_i to be finite. Truncate the resolution, by stripping off the last term. Then tensor with M (which may lose exactness!). Let $\mathrm{Tor}^i_A(M,N)_{\mathcal{R}}$ be the homology of this complex at the ith stage ($i \geq 0$). The subscript $\mathcal{R}$ reminds us that our construction depends on the resolution, although we will soon see that it is independent of the resolution.

We make some quick observations.

• $\mathrm{Tor}_0^A(M,N)_{\mathcal{R}} \cong M \otimes_A N$, and this isomorphism is canonical. Reason: as tensoring is right exact, and $A^{\oplus n_1} \to A^{\oplus n_0} \to N \to 0$ is exact, we have that $M^{\oplus n_1} \to M^{\oplus n_0} \to M \otimes_A N \to 0$ is exact, and hence that the homology of the truncated complex $M^{\oplus n_1} \to M^{\oplus n_0} \to 0$ is $M \otimes_A N$.

• If $M \otimes \cdot$ is exact (i.e. M is *flat*), then $\mathrm{Tor}_i^A(M,N)_{\mathcal{R}} = 0$ for all i.

Now given two modules N and N' and resolutions $\mathcal{R}$ and $\mathcal{R}'$ of N and N', we can "lift" any morphism $N \to N'$ to a morphism of the two resolutions:

$$\begin{tikzcd}
\cdots \arrow[r] \arrow[d] & A^{\oplus n_i} \arrow[r] \arrow[d] & \cdots \arrow[r] \arrow[d] & A^{\oplus n_1} \arrow[r] \arrow[d] & A^{\oplus n_0} \arrow[r] \arrow[d] & N \arrow[r] \arrow[d] & 0 \\
\cdots \arrow[r] & A^{\oplus n_i'} \arrow[r] & \cdots \arrow[r] & A^{\oplus n_1'} \arrow[r] & A^{\oplus n_0'} \arrow[r] & N' \arrow[r] & 0
\end{tikzcd}$$

Here we are using the freeness of $A^{\otimes n_i}$: if $a_1, \ldots, a_{n_i}$ are generators of $A^{\otimes n_i}$, to lift the map $b : A^{\otimes n_i} \to A^{\otimes n'_{i-1}}$ to $c : A^{\otimes n_i} \to A^{\otimes n'_i}$, we arbitrarily lift $b(a_i)$ from $A^{\otimes n'_{i-1}}$ to $A^{\otimes n'_i}$, and declare this to be $c(a_i)$.

Denote the choice of lifts by $\mathcal{R} \to \mathcal{R}'$. Now truncate both complexes (remove column $N \to N'$) and tensor with M. Maps of complexes induce maps of homology, so we have described maps (a priori depending on $\mathcal{R} \to \mathcal{R}'$)

$$\mathrm{Tor}_i^A(M, N)_{\mathcal{R}} \to \mathrm{Tor}_i^A(M, N')_{\mathcal{R}'}.$$

We say two maps of complexes $f, g : C_* \to C'_*$ are **homotopic** if there is a sequence of maps $w : C_i \to C'_{i+1}$ such that $f - g = dw + wd$. Two homotopic maps give the same map on homology. (Exercise: verify this if you haven't seen this before.)

1.A. CRUCIAL EXERCISE. Show that any two lifts $\mathcal{R} \to \mathcal{R}'$ are homotopic.

We now pull these observations together.

(1) We get a covariant functor from $\mathrm{Tor}_i^A(M, N)_{\mathcal{R}} \to \mathrm{Tor}_i^A(M, N')_{\mathcal{R}'}$, independent of the lift $\mathcal{R} \to \mathcal{R}'$.
(2) Hence for any two resolutions $\mathcal{R}$ and $\mathcal{R}'$ we get a canonical isomorphism $\mathrm{Tor}_i^A(M, N)_{\mathcal{R}} \cong \mathrm{Tor}_i^1(M, N)_{\mathcal{R}'}$. Here's why. Choose lifts $\mathcal{R} \to \mathcal{R}'$ and $\mathcal{R}' \to \mathcal{R}$. The composition $\mathcal{R} \to \mathcal{R}' \to \mathcal{R}$ is homotopic to the identity (as it is a lift of the identity map $N \to N$). Thus if $f_{\mathcal{R} \to \mathcal{R}'} : \mathrm{Tor}_i^A(M, N)_{\mathcal{R}} \to \mathrm{Tor}_i^1(M, N)_{\mathcal{R}'}$ is the map induced by $\mathcal{R} \to \mathcal{R}'$, and similarly $f_{\mathcal{R}' \to \mathcal{R}}$ is the map induced by $\mathcal{R} \to \mathcal{R}'$, then $f_{\mathcal{R}' \to \mathcal{R}} \circ f_{\mathcal{R} \to \mathcal{R}'}$ is the identity, and similarly $f_{\mathcal{R} \to \mathcal{R}'} \circ f_{\mathcal{R}' \to \mathcal{R}}$ is the identity.
(3) Hence the covariant functor doesn't depend on the resolutions!

Finally:
(4) For any short exact sequence we get a long exact sequence of Tor's (1). Here's why: given a short exact sequence, choose resolutions of N' and N''. Then use these to get a resolution for N in the obvious way (see below; the map $A^{\oplus(n'_0 \to n''_0)} \to N$ is the composition $A^{\oplus n'_0} \to N' \to N$ along with any lift of $A^{n''_0} \to N''$ to N) so that we have a short exact

sequence of resolutions

$$\begin{tikzcd}
& 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
\cdots \arrow[r] & A^{\oplus n_1'} \arrow[r] \arrow[d] & A^{\oplus n_0'} \arrow[r] \arrow[d] & N' \arrow[r] \arrow[d] & 0 \\
\cdots \arrow[r] & A^{\oplus (n_1'+n_1'')} \arrow[r] \arrow[d] & A^{\oplus (n_0'+n_0'')} \arrow[r] \arrow[d] & N \arrow[r] \arrow[d] & 0 \\
\cdots \arrow[r] & A^{\oplus n_1''} \arrow[r] \arrow[d] & A^{\oplus n_0''} \arrow[r] \arrow[d] & N'' \arrow[r] \arrow[d] & 0 \\
& 0 & 0 & 0 &
\end{tikzcd}$$

Then truncate (removing the right column $0 \to N' \to N \to N'' \to 0$), tensor with M (obtaining a short exact sequence of complexes) and take cohomology, yielding a long exact sequence.

We have thus established the foundations of Tor!

Note that if N is a free module, then $\mathrm{Tor}_i^A(M, N) = 0$ for all M and all $i > 0$, as N has the trivial resolution $0 \to N \to N \to 0$ (it is "its own resolution").

1.B. EXERCISE. Show that the following are equivalent conditions on an A-module M.

(i) M is flat
(ii) $\mathrm{Tor}_i^A(M, N) = 0$ for all $i > 0$ and all A-modules N,
(iii) $\mathrm{Tor}_1^A(M, N) = 0$ for all A-modules N.

2. FROM TOR TO DERIVED FUNCTORS IN GENERAL

2.1. *Projective resolutions.* We used very little about free modules in the above construction of Tor; in fact we used only that free modules are **projective**, i.e. those modules M such that for any surjection $M' \to M''$, it is possible to lift any morphism $M \to M''$ to $M \to M'$. This is summarized in the following diagram.

$$\begin{tikzcd}
M \arrow[d, dashed, "\text{exists}"'] \arrow[dr] & \\
M' \arrow[r, two heads] & M''
\end{tikzcd}$$

Equivalently, $\mathrm{Hom}(M, \cdot)$ is an *exact functor* ($\mathrm{Hom}(M, \cdot)$ is always left-exact for any M). More generally, the same idea yields the definition of a **projective object in any abelian category**. Hence (i) we can compute $\mathrm{Tor}_i^A(M, N)$ by taking any projective resolution of N, and (ii) $\mathrm{Tor}_i^A(M, N) = 0$ for any projective A-module N.

2.A. INTERESTING EXERCISE: DERIVED FUNCTORS CAN BE COMPUTED USING ACYCLIC RESOLUTIONS. Show that you can also compute derived functor cohomology using *flat resolutions*, i.e. by a resolution

$$\cdots \to F_2 \to F_1 \to F_0 \to N \to 0$$

by flat A-modules. Hint: show that you can construct a double complex

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
\cdots \arrow[r] & A^{\oplus n_{2,1}} \arrow[r] \arrow[d] & A^{\oplus n_{1,1}} \arrow[r] \arrow[d] & A^{\oplus n_{0,1}} \arrow[r] \arrow[d] & A^{\oplus n_1} \arrow[r] \arrow[d] & 0 \\
\cdots \arrow[r] & A^{\oplus n_{2,0}} \arrow[r] \arrow[d] & A^{\oplus n_{1,0}} \arrow[r] \arrow[d] & A^{\oplus n_{0,0}} \arrow[r] \arrow[d] & A^{\oplus n_0} \arrow[r] \arrow[d] & 0 \\
\cdots \arrow[r] & F_2 \arrow[r] & F_1 \arrow[r] & F_0 \arrow[r] & N \arrow[r] & 0
\end{tikzcd}$$

where the rows and columns are exact. Do this by constructing the $A^{\oplus ??}$ inductively from the bottom left. Tensor the double complex with M, to obtain a new double complex. Remove the bottom row, and the right-most nonzero column. Use a spectral sequence argument to show that (i) the double complex has homology equal to Tor, and (ii) the homology of the double complex agrees with the homology of the free resolution (truncated) tensored with M.

You will notice in the solution to the above exercise that what mattered was that flat modules had no higher Tor's (Exercise 1.B). This will later directly generalize to the statements that *derived functors can be computed with acyclic resolutions* ("acyclic" means "no higher (co)homology").

2.2. Derived functors of right-exact functors.

The above description was low-tech, but immediately generalizes drastically. All we are using is that $M \otimes_A$ is a right-exact functor. In general, if F is *any* right-exact covariant functor from the category of A-modules to any abelian category, this construction will define a sequence of functors $L_i F$ (called left-derived functors of F) such that $L_0 F = F$ and the L_i's give a long-exact sequence. We can make this more general still. We say that an abelian category **has enough projectives** if for any object N there is a surjection onto it from a projective object. Then if F is any right-exact functor from an abelian category with enough projectives to any abelian category, then F has left-derived functors.

2.B. UNIMPORTANT EXERCISE. Show that an object P is projective if and only if every short exact sequence $0 \to A \to B \to P \to 0$ splits.

2.C. EXERCISE. The notion of an **injective object** in an abelian category is dual to the notion of a projective object. (a) State precisely the definition of an injective object. (b) Define derived functors for (i) covariant left-exact functors (these are called **right-derived**

functors), (ii) contravariant left-exact functors (also called **right-derived functors**), and (iii) contravariant right-exact functors (these are called **left-derived functors**), making explicit the necessary assumptions of the category having enough injectives or projectives.

2.3. *Notation.* If F is a right-exact functor, its (left-)derived functors are denoted L_iF ($i \geq 0$, with $L_0F = F$). If F is a left-exact functor, its (right-) derived functors are denoted R^iF.

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