FOUNDATIONS OF ALGEBRAIC GEOMETRY CLASSES 33 AND 34

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In these two lectures, we will use universal properties to define two more useful constructions, $\underline{\mathrm{Spec}}$ of a sheaf of algebras $\mathcal{A}$, and $\underline{\mathrm{Proj}}$ of a sheaf of graded algebras $\mathcal{A}_\bullet$ on a scheme X. These will both generalize (globalize) our constructions of Spec and Proj of A-algebras and graded A-algebras. We'll see that affine morphisms are precisely those of the form $\underline{\mathrm{Spec}}\,\mathcal{A} \to X$, and so we'll *define* projective morphisms to be those of the form $\underline{\mathrm{Proj}}\,\mathcal{A}_\bullet \to X$.

1. Relative Spec of a (quasicoherent) sheaf of algebras

Given an A-algebra, B, we can take its Spec to get an affine scheme over $\mathrm{Spec}\,A$: $\mathrm{Spec}\,B \to \mathrm{Spec}\,A$. We will now see universal property description of a globalization of that notation. Consider an arbitrary scheme X, and a quasicoherent sheaf of algebras $\mathcal{A}$ on it. We will define how to take Spec of this sheaf of algebras, and we will get a scheme $\underline{\mathrm{Spec}}\,\mathcal{A} \to X$ that is "affine over X", i.e. the structure morphism is an affine morphism.

You can think of this in two ways. First, and most concretely, for any affine open set $\mathrm{Spec}\,A \subset X$, $\Gamma(\mathrm{Spec}\,A, \mathcal{A})$ is some A-algebra; call it B. Then above $\mathrm{Spec}\,A$, $\underline{\mathrm{Spec}}\,A$ will be $\mathrm{Spec}\,B$.

Second, it will satisfy a universal property. We could define the A-scheme $\mathrm{Spec}\,B$ by the fact that maps to $\mathrm{Spec}\,B$ (from an A-scheme Y, over $\mathrm{Spec}\,A$) correspond to maps of A-algebras $B \to \Gamma(Y, \mathcal{O}_Y)$. The universal property for $\underline{\mathrm{Spec}}\,\mathcal{A}$ is similar. More precisely, we describe a universal property for the morphism $\beta : \underline{\mathrm{Spec}}\,\mathcal{A} \to X$ along with an isomorphism $\phi : \mathcal{A} \to \beta_* \mathcal{O}_{\underline{\mathrm{Spec}}\,\mathcal{A}}$: to each morphism $\pi : Y \to X$ along with a morphism of $\mathcal{O}_X$-modules

$$\alpha : \mathcal{A} \to \pi_* \mathcal{O}_Y,$$

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there is a unique map $f : Y \to \underline{\mathrm{Spec}}\,\mathcal{A}$ factoring π, i.e. so that the following diagram commutes

$$\begin{array}{ccc} Y & \xrightarrow{\exists! f} & \underline{\mathrm{Spec}}\,\mathcal{A} \\ & {\scriptstyle\pi}\searrow \quad \swarrow{\scriptstyle\beta} & \\ & X & \end{array}$$

where α is the composition

$$\mathcal{A} \xrightarrow{\phi} \beta_* \mathcal{O}_{\underline{\mathrm{Spec}}\,\mathcal{A}} \longrightarrow \beta_* f_* \mathcal{O}_Y = \pi_* \mathcal{O}_Y$$

(For experts: we need to work with $\mathcal{O}_X$-modules, and to leave our category of quasicoherent sheaves on X, because we only showed that the pushforward of quasicoherent sheaves are quasicoherent for quasicompact quasiseparated morphisms, and we don't need such hypotheses here.) This bijection $\mathrm{Hom}(\mathcal{A} \to \pi_* \mathcal{O}_Y) \leftrightarrow \mathrm{Mor}_X(Y, \underline{\mathrm{Spec}}\,\mathcal{A})$ is natural in Y, i.e. given $Y' \to Y$ the diagram

$$\begin{array}{ccc} \mathrm{Hom}(\mathcal{A} \to \pi_* \mathcal{O}_Y) & \longleftrightarrow & \mathrm{Mor}_X(Y, \underline{\mathrm{Spec}}\,\mathcal{A}) \\ \downarrow & & \downarrow \\ \mathrm{Hom}(\mathcal{A} \to \pi_* \mathcal{O}_{Y'}) & \longleftrightarrow & \mathrm{Mor}_X(Y', \underline{\mathrm{Spec}}\,\mathcal{A}) \end{array}$$

commutes. By universal property nonsense, this determines $\underline{\mathrm{Spec}}\,\mathcal{A}$ up to unique isomorphism, assuming of course that it exists.

1.A. EXERCISE. Show that if X is affine, say $\mathrm{Spec}\,A$, and $\mathcal{A} = \tilde{B}$, where B is an A-algebra, then $\mathrm{Spec}\,B \to \mathrm{Spec}\,A$ satisfies this universal property. (Hint: recall that maps to an affine scheme correspond to maps of rings of functions in the opposite direction.) Show that this affine construction behaves well with respect to "affine base change": given a map $g : \mathrm{Spec}\,A' \to \mathrm{Spec}\,A$, then describe a canonical isomorphism $\underline{\mathrm{Spec}}\,g^*\mathcal{A} \cong \mathrm{Spec}\,A' \otimes_A B$.

1.1. *Remark.* In particular, if p is a point of $\mathrm{Spec}\,A$, $k(p)$ is the residue field at p, and $\mathrm{Spec}\,k(p) \to \mathrm{Spec}\,A$ is the inclusion, then the fiber of $\mathrm{Spec}\,B \to \mathrm{Spec}\,A$ is canonically identified (as a scheme) with $\mathrm{Spec}\,B \otimes_A k(p)$. This is the motivation for our construction below.

We define $\underline{\mathrm{Spec}}\,\mathcal{A}$ by describing the points, then the topology, and then the structure sheaf. (Experts: where does the quasicoherence of $\mathcal{A}$ come in?)

First the points: above the point $p \in X$, the points of $\underline{\mathrm{Spec}}\,\mathcal{A}$ are defined to be the points of $\mathrm{Spec}(\mathcal{A} \otimes k(p))$. (For example, take the stalk, and mod out by the maximal ideal. Or take any affine open neighborhood of p, and apply the construction of Remark 1.1.

We topologize this set as follows. Above the affine open subset $\mathrm{Spec}\,A \subset X$, the points are identified with the points of $\mathrm{Spec}\,\Gamma(\mathrm{Spec}\,A, \mathcal{A})$, by Remark 1.1. We impose that this be an open subset of $\underline{\mathrm{Spec}}\,A$, and the topology restricted to this open set is required to be the Zariski topology on $\mathrm{Spec}\,\Gamma(\mathrm{Spec}\,A, \mathcal{A})$.

1.B. EXERCISE. Show that this topology is well-defined. In other words, show that if $\operatorname{Spec} A$ and $\operatorname{Spec} A'$ are affine open subsets of X, then the topology imposed on $\beta^{-1}(\operatorname{Spec} A \cap \operatorname{Spec} A')$ by the construction using $\operatorname{Spec} A$ agrees with the topology imposed by $\operatorname{Spec} A'$. (Some ideas behind the Affine Communication Lemma may be helpful. For example, this question is much easier if $\operatorname{Spec} A'$ is a distinguished open subset of $\operatorname{Spec} A$.)

Next, we describe the structure sheaf, and the description is precisely what you might expect: on $\beta^{-1}(\operatorname{Spec} A) \subset \underline{\operatorname{Spec}}\, \mathcal{A}$, the sheaf is isomorphic to the structure sheaf on $\operatorname{Spec} \Gamma(\operatorname{Spec} A, \mathcal{A})$.

1.C. EXERCISE. Rigorously define the structure sheaf. How do you glue these sheaves on small open sets together? Once again, the ideas behind the Affine Communication Lemma may help.

1.D. EXERCISE. Describe the isomorphism $\phi : \mathcal{A} \to \beta_* \mathcal{O}_{\underline{\operatorname{Spec}}\, \mathcal{A}}$. Show that given any $\pi : Y \to X$, this construction yields the isomorphism $\operatorname{Mor}_X(Y, \underline{\operatorname{Spec}}\, \mathcal{A}) \to \operatorname{Hom}(\mathcal{A} \to \pi_* \mathcal{O}_Y)$ via the composition

$$\mathcal{A} \xrightarrow{\phi} \beta_* \mathcal{O}_{\underline{\operatorname{Spec}}\, \mathcal{A}} \longrightarrow \beta_* f_* \mathcal{O}_Y = \pi_* \mathcal{O}_Y.$$

1.E. EXERCISE. Show that $\underline{\operatorname{Spec}}\, \mathcal{A}$ satisfies the desired universal property. (Hint: figure out how to reduce to the case X affine, Exercise 1.A.)

We make some quick observations, some verified in exercises. First $\underline{\operatorname{Spec}}\, \mathcal{A}$ can be "computed affine-locally on X".

Second, this gives an important way to understand affine morphisms. Note that $\underline{\operatorname{Spec}}\, \mathcal{A} \to X$ is an affine morphism. The "converse" is also true:

1.F. EXERCISE. Show that if $f : Z \to X$ is an affine morphism, then we have a natural isomorphism $Z \cong \underline{\operatorname{Spec}}\, f_* \mathcal{O}_Z$ of X-schemes.

Hence we can recover any affine morphism in this way. More precisely, a morphism is affine if and only if it is of the form $\underline{\operatorname{Spec}}\, \mathcal{A} \to X$.

1.G. EXERCISE ($\underline{\operatorname{Spec}}$ BEHAVES WELL WITH RESPECT TO BASE CHANGE). Suppose $f : Z \to X$ is any morphism, and $\mathcal{A}$ is a quasicoherent sheaf of algebras on X. Show that there is a natural isomorphism $Z \times_X \underline{\operatorname{Spec}}\, \mathcal{A} \cong \underline{\operatorname{Spec}}\, f^* \mathcal{A}$.

An important example of this $\underline{\operatorname{Spec}}$ construction is the **total space of a finite rank locally free sheaf** $\mathcal{F}$, which we define to be $\underline{\operatorname{Spec}} \operatorname{Sym}^\bullet \mathcal{F}^\vee$.

1.H. EXERCISE. Show that this is a vector bundle, i.e. that given any point $p \in X$, there is a neighborhood $p \in U \subset X$ such that $\underline{\mathrm{Spec}}\, \mathrm{Sym}^\bullet \mathcal{F}^\vee|_U \cong \mathbb{A}^n_U$. Show that $\mathcal{F}$ is isomorphic to the sheaf of sections of it.

In particular, if $\mathcal{F}$ is a *free* sheaf of rank n, then $\underline{\mathrm{Spec}}\, \mathrm{Sym}^* \mathcal{F}^\vee$ is called $\mathbb{A}^n_X$, generalizing our earlier notions of $\mathbb{A}^n_A$. As the notion of a free sheave behaves well with respect to base change, so does the notion of $\mathbb{A}^n_X$, i.e. given $X \to Y$, $\mathbb{A}^n_Y \times_Y X \cong \mathbb{A}^n_X$.

Here is one last fact that can be useful.

1.I. EXERCISE. Suppose $f : \underline{\mathrm{Spec}}\, \mathcal{A} \to X$ is a morphism. Show that the category of quasicoherent sheaves on $\underline{\mathrm{Spec}}\, A$ is "essentially the same as" (i.e. equivalent to) the category of quasicoherent sheaves on X with the structure of $\mathcal{A}$-modules (quasicoherent $\mathcal{A}$-modules on X).

The reason you could imagine caring is when X is quite simple, and $\underline{\mathrm{Spec}}\, \mathcal{A}$ is complicated. We'll use this before long when $X \cong \mathbb{P}^1$, and $\underline{\mathrm{Spec}}\, \mathcal{A}$ is a more complicated curve.

1.J. IMPORTANT EXERCISE: THE TAUTOLOGICAL BUNDLE ON $\mathbb{P}^n$ IS $\mathcal{O}(-1)$. Define the subset $X \subset \mathbb{A}^{n+1}_k \times \mathbb{P}^n_k$ corresponding to "points of $\mathbb{A}^{n+1}_k$ on the corresponding line of $\mathbb{P}^n_k$", so that the fiber of the map $\pi : X \to \mathbb{P}^n$ corresponding to a point $l = [x_0; \cdots; x_n]$ is the line in $\mathbb{A}^{n+1}$ corresponding to l, i.e. the scalar multiples of $(x_0, \ldots, x_n)$. Show that $\pi : X \to \mathbb{P}^n$ is (the line bundle corresponding to) the invertible sheaf $\mathcal{O}(-1)$. (Possible hint: work first over the usual affine open sets of $\mathbb{P}^n$, and figure out transition functions.) (For this reason, $\mathcal{O}(-1)$ is often called the **tautological bundle** of $\mathbb{P}^n$.)

2. RELATIVE PROJ OF A SHEAF OF GRADED ALGEBRAS

In parallel with $\underline{\mathrm{Spec}}$, we will define a relative version of Proj, denoted $\underline{\mathrm{Proj}}$.

Suppose now that $\mathcal{S}_\bullet$ is a quasicoherent sheaf of graded algebras of X. We require that $\mathcal{S}_\bullet$ is *locally generated in degree* 1 (i.e. there is a cover by small affine open sets, where for each affine open set, the corresponding algebra is generated in degree 1), and $\mathcal{S}_1$ is finite type. We will define $\underline{\mathrm{Proj}}\, \mathcal{S}_\bullet$ by describing a universal property, and the constructing it.

In order to understand the universal property, let's revisit maps to $\mathrm{Proj}\, S_\bullet$ (over a base ring A), satisfying the analogous assumptions. Suppose S_1 is generated by $x_1, \ldots, x_n$. Recall that maps from an A-scheme to projective space

$$\begin{array}{ccc} Y & \longrightarrow & \mathrm{Proj}\, S_\bullet \\ & \searrow \quad \swarrow & \\ & \mathrm{Spec}\, A & \end{array}$$

correspond to invertible sheaves $\mathcal{L}$ on Y and sections $s_1, \ldots, s_n$,

(i) with no common zeros (they are a *base-point-free linear system*),
(ii) satisfying "the same relations as $x_1, \dots, x_n$".

It is helpful to write this map as

$$Y \xrightarrow{[s_1;\cdots;s_n]} \operatorname{Proj} S_\bullet.$$

The condition that $s_1, \dots, s_n$ satisfy the same conditions as $x_1, \dots, x_n$ can be formalized to say that there is a map of graded A-algebras

$$\Gamma_\bullet(Y, \mathcal{L}) := \oplus_{i=0}^{\infty} \Gamma(Y, \mathcal{L}^{\otimes i}) \longleftarrow S_\bullet$$

given by $x_i \mapsto s_i$. This will yield a "relative" version of (ii).

We now describe a relative version of (i).

2.1. Definition. Given a morphism $\pi : Y \to X$, an invertible sheaf $\mathcal{L}$ on Y is **relatively base-point-free** (with respect to π) if for every point of $y \in Y$, there is an open subset $U \subset X$ and a section s of $\mathcal{L}$ above U ($s \in \Gamma(\pi^{-1}(U), \mathcal{L})$) such that $s(y) \neq 0$.

2.A. EASY EXERCISE. If $X = \operatorname{Spec} A$, and $\mathcal{L}$ is base-point-free, show that $\mathcal{L}$ is relatively base-point-free.

Thus $\mathcal{L}$ is relatively base-point-free if it is "base-point-free over an affine cover X".

2.B. EXERCISE. Suppose π is quasicompact and quasiseparated (so π_* sends quasicoherent sheaves to quasicoherent sheaves). Show that $\mathcal{L}$ is basepoint free if the canonical map $\pi^*\pi_*\mathcal{L} \to \mathcal{L}$ is surjective.

More generally, if $\mathcal{F}$ is a quasicoherent and quasiseparated, we say that a quasicoherent sheaf $\mathcal{F}$ on X is **relatively generated** (with respect to π) if the canonical map $\pi^*\pi_*\mathcal{F} \to \mathcal{F}$ is surjective. We won't be using this notion.

2.C. EXERCISE. Describe why this is the relative version of *generated by global sections*.

Having defined relative versions of (i) and (ii) above, we are now ready to define $\underline{\operatorname{Proj}}$.

2.2. Definition. Suppose $\mathcal{S}_\bullet$ is a graded quasicoherent sheaf of algebras on a scheme X, locally generated in degree 1. In analogy with $\underline{\operatorname{Spec}}$, we define

$$(\beta : \operatorname{Proj} \mathcal{S}_\bullet \to X, \mathcal{O}(1), \phi : \mathcal{S}_\bullet \to \oplus_n \beta_* \mathcal{O}(n))$$

by the following universal property. (Here ϕ is a map of graded sheaves, and is *not* required to be an isomorphism.)

Maps

$$\begin{array}{ccc} Y & \xrightarrow{f} & \underline{\mathrm{Proj}}\,\mathcal{S}_\bullet \\ & \searrow^{\pi} \quad {}^{\beta}\swarrow & \\ & X & \end{array}$$

correspond to maps $\alpha : \mathcal{S}_\bullet \to \oplus_{n=0}^{\infty} \pi_* \mathcal{L}^{\otimes n}$, where $\mathcal{L}$ is an invertible sheaf on Y, α factors as

$$\mathcal{S}_\bullet \xrightarrow{\phi} \oplus \beta_* \mathcal{O}(n) \longrightarrow \oplus \beta_* f_* \mathcal{L}^{\otimes n} = \oplus \pi_* \mathcal{L}^{\otimes n},$$

and the image of S_1 is relatively base-point free. (You might be worried about what happens if π is not quasicompact and quasiseparated, in which case we don't know that π_* is a quasicoherent sheaf. This isn't a problem: we can work with $\mathcal{O}_X$-modules. This won't cause any complication.)

As usual, if $(\beta : \mathrm{Proj}\,\mathcal{S}_\bullet \to X, \mathcal{O}(1), \phi : \mathcal{S}_\bullet \to \oplus_n \beta_* \mathcal{O}(n))$ exists, it is unique up to unique isomorphism. We now show that it exists, in analogy with $\underline{\mathrm{Spec}}$.

2.D. EXERCISE. Show the result if X is affine by restating what we know about the Proj construction.

Note that this construction behaves well with respect to affine base change.

Motivated by this, we define the points of $\mathrm{Proj}\,\mathcal{S}_\bullet$ over a point $p \in X$ as the points of $\mathrm{Proj}(\mathcal{S}_\bullet \otimes k(p))$.

2.E. EXERCISE. Define a topology on this set as follows: above each affine open subset of $\mathrm{Spec}\,A \subset X$, take the Zariski topology on $\mathrm{Proj}\,\Gamma(\mathrm{Spec}\,A, \mathcal{S}_\bullet)$. Be sure to show this is well-defined.

2.F. EXERCISE. Define the structure sheaf on this topological space as follows: above each affine open subset of $\mathrm{Spec}\,A \subset X$, take the structure sheaf of $\mathrm{Proj}\,\Gamma(\mathrm{Spec}\,A, \mathcal{S}_\bullet)$. Be sure to show this is well-defined.

2.G. EXERCISE. Define the map $\phi : \mathcal{S}_\bullet \to \oplus \mathcal{O}(n)$.

2.H. EXERCISE. Show that your construction satisfies the universal property.

2.I. EXERCISE ($\underline{\mathrm{Proj}}$ BEHAVES WELL WITH RESPECT TO BASE CHANGE). Suppose $\mathcal{S}_\bullet$ is a quasicoherent sheaf of graded algebras on X satisfying the required hypotheses above for $\underline{\mathrm{Proj}}\,\mathcal{S}_\bullet$ to exist. Let $f : Y \to X$ be any morphism. Give a natural isomorphism

$$(\underline{\mathrm{Proj}}\, f^* \mathcal{S}_\bullet, \mathcal{O}_{\underline{\mathrm{Proj}}\, f^* \mathcal{S}_\bullet}(1)) \cong (Y \times_X \underline{\mathrm{Proj}}\,\mathcal{S}_\bullet, g^* \mathcal{O}_{\underline{\mathrm{Proj}}\,\mathcal{S}_\bullet}(1))$$

where g is the natural morphism in the base change diagram

$$\begin{array}{ccc} Y \times_X \underline{\mathrm{Proj}}\, \mathcal{S}_\bullet & \xrightarrow{g} & \underline{\mathrm{Proj}}\, \mathcal{S}_\bullet \\ \downarrow & & \downarrow \\ Y & \longrightarrow & X. \end{array}$$

2.3. *Definition.* If $\mathcal{F}$ is a finite rank locally free sheaf on X. Then $\underline{\mathrm{Proj}}\, \mathrm{Sym}^\bullet \mathcal{F}$ is called its *projectivization*. If $\mathcal{F}$ is a free sheaf of rank $n+1$, then we define $\mathbb{P}^n_X := \underline{\mathrm{Proj}}\, \mathrm{Sym}^\bullet \mathcal{F}$. (Then $\mathbb{P}^n_{\mathrm{Spec}\, A}$ agrees with our earlier definition of $\mathbb{P}^n_A$.) Clearly this notion behaves well with respect to base change.

This "relative $\mathcal{O}(1)$" we have constructed is a little subtle. Here are couple of exercises to give you practice with the concept.

2.J. EXERCISE. $\underline{\mathrm{Proj}}(\mathcal{S}_\bullet[t]) \cong \underline{\mathrm{Spec}}\, \mathcal{S}_\bullet \coprod \underline{\mathrm{Proj}}\, \mathcal{S}_\bullet$, where $\underline{\mathrm{Spec}}\, \mathcal{S}_\bullet$ is an open subscheme, and $\underline{\mathrm{Proj}}\, \mathcal{S}_\bullet$ is a closed subscheme. Show that $\underline{\mathrm{Proj}}\, \mathcal{S}*$ is an effective Cartier divisor, corresponding to the invertible sheaf $\mathcal{O}_{\underline{\mathrm{Proj}}\, N}(1)$. (This is the generalization of the projective and affine cone.)

2.K. EXERCISE. Suppose $\mathcal{L}$ is an invertible sheaf on X, and $\mathcal{S}_\bullet$ is a quasicoherent sheaf of graded algebras on X satisfying the required hypotheses above for $\underline{\mathrm{Proj}}\, \mathcal{S}_\bullet$ to exist. Define $\mathcal{S}'_* = \oplus_{n=0} \mathcal{S}_n \otimes \mathcal{L}_n$. Give a natural isomorphism of X-schemes

$$(\underline{\mathrm{Proj}}\, \mathcal{S}'_*, \mathcal{O}_{\underline{\mathrm{Proj}}\, \mathcal{S}'_*}(1)) \cong (\underline{\mathrm{Proj}}\, \mathcal{S}_\bullet, \mathcal{O}_{\underline{\mathrm{Proj}}\, \mathcal{S}_\bullet}(1) \otimes \pi^* \mathcal{L}),$$

where $\pi : \underline{\mathrm{Proj}}\, \mathcal{S}_\bullet \to X$ is the structure morphism. In other words, informally speaking, the $\underline{\mathrm{Proj}}$ is the same, but the $\mathcal{O}(1)$ is twisted by $\mathcal{L}$.

3. PROJECTIVE MORPHISMS

In §1, that we reinterpreted affine morphisms: $X \to Y$ is an affine morphism if there is an isomorphism $X \cong \underline{\mathrm{Spec}}\, \mathcal{A}$ of Y-schemes for some quasicoherent sheaf of algebras $\mathcal{A}$ on Y. We now *define* the notion of a projective morphism similarly.

3.1. *Definition.* A morphism $X \to Y$ is **projective** if there is an isomorphism

$$\begin{array}{ccc} X & \xrightarrow{\sim} & \underline{\mathrm{Proj}}\, \mathcal{S}_\bullet \\ & \searrow \quad \swarrow & \\ & Y & \end{array}$$

for a quasicoherent sheaf of algebras $\mathcal{S}_\bullet$ on Y. X is said to be a **projective Y-scheme**, or **projective over Y**. This generalizes the notion of a projective A-scheme.

3.2. *Warnings.* First, notice that $\mathcal{O}(1)$, an important part of the definition of $\underline{\text{Proj}}$, is not mentioned. As a result, the notion of affine morphism is affine-local on the target, but this notion is not affine-local on the target. (In nice circumstances it is, as we'll see later. We'll also see an example where this is not.)

Second, Hartshorne gives a different definition; we are following the more general definition of Grothendieck. These definitions turn out to be the same in nice circumstances.

We now establish a number of properties of projective morphisms.

Note first that projective morphisms are proper. (Reason: properness is local on the base, and we've seen earlier that projective A-schemes are proper over A.) Equivalently (by definition of properness!) they are separated, finite type, and universally closed.

3.A. IMPORTANT EXERCISE: FINITE MORPHISMS ARE PROJECTIVE. Show that finite morphisms are projective as follows. Suppose $Y \to X$ is finite, and that $Y = \underline{\text{Spec}}\, \mathcal{A}$ where $\mathcal{A}$ is a finite type quasicoherent sheaf on X. Describe a sheaf of graded algebras $\mathcal{S}_\bullet$ where $\mathcal{S}_0 \cong \mathcal{O}_X$ and $\mathcal{S}_n \cong \mathcal{A}$ for $n > 0$. (What is the multiplication in this algebra?) Describe an X-isomorphism $Y \cong \underline{\text{Proj}}\, \mathcal{S}_\bullet$.

In particular, closed immersions are projective. We have the sequence of implications for morphisms

$$\text{closed immersion} \Rightarrow \text{finite} \Rightarrow \text{projective} \Rightarrow \text{proper}.$$

3.B. EXERCISE. Show that a morphism (over $\text{Spec}\, k$) from a projective k-scheme to a separated k-scheme is always projective. (Hint: the Cancellation Theorem for properties of morphisms.)

3.C. EXERCISE. Show that the property of a morphism being projective is preserved by base change.

3.D. HARDER EXERCISE. Show that the property of being projective is preserved by composition. (Ask me for a hint. The main thing is to figure out a candidate $\mathcal{O}(1)$.)

The previous two exercises imply that the property of being projective is preserved by products: if $f : X \to Y$ and $f' : X' \to Y$ are projective, then so is $f \times f' : X \times X' \to Y \times Y'$.

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