# FOUNDATIONS OF ALGEBRAIC GEOMETRY CLASS 31

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CONTENTS



## 1. PUSHFORWARDS AND PULLBACKS OF QUASICOHERENT SHEAVES

There are two things you can do with modules and a ring homomorphism $B \to A$. If $M$ is an $A$-module, you can create an $B$-module $M_B$ by simply treating it as an $B$-module. If $N$ is an $B$-module, you can create an $A$-module $N \otimes_B A$.

These notions behave well with respect to localization (in a way that we will soon make precise), and hence work (often) in the category of quasicoherent sheaves (and indeed always in the category of modules over ringed spaces, see Remark 3.7, although this will not concern us here). The two functors are adjoint:

$$\mathrm{Hom}_A(A \otimes_B N, M) \cong \mathrm{Hom}_B(N, M_B)$$

(where this isomorphism of groups is functorial in both arguments), and we will see that this remains true on the scheme level.

One of these constructions will turn into our old friend pushforward. The other will be a relative of pullback, whom I'm reluctant to call an "old friend".

## 2. PUSHFORWARDS OF QUASICOHERENT SHEAVES

The main message of this section is that in "reasonable" situations, the pushforward of a quasicoherent sheaf is quasicoherent, and that this can be understood in terms of one of the module constructions defined above. We begin with a motivating example:

**2.A.** EXERCISE. Let $f : \mathrm{Spec}\, A \to \mathrm{Spec}\, B$ be a morphism of affine schemes, and suppose $M$ is an $A$-module, so $\tilde{M}$ is a quasicoherent sheaf on $\mathrm{Spec}\, A$. Show that $f_* \tilde{M} \cong \widetilde{M_B}$. (Hint: There is only one reasonable way to proceed: look at distinguished open sets!)

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In particular, $f_*\tilde{M}$ is quasicoherent. Perhaps more important, this implies that the pushforward of a quasicoherent sheaf under an affine morphism is also quasicoherent. The following result, proved in an earlier Exercise, generalizes this statement.

**2.1.** *Theorem. — Suppose* $f : X \to Y$ *is a quasicompact quasiseparated morphism, and* $\mathcal{F}$ *is a quasicoherent sheaf on* $X$*. Then* $f_*\mathcal{F}$ *is a quasicoherent sheaf on* $Y$.

**2.B.** EXERCISE. Give an example of a morphism of schemes $\pi : X \to Y$ and a quasicoherent sheaf $\mathcal{F}$ on X such that $\pi_*\mathcal{F}$ is not quasicoherent. (Possible answer: $Y = \mathbb{A}^1$, $X =$ countably many copies of $\mathbb{A}^1$. Then let $f = t$. $X_t$ has a global section $(1/t, 1/t^2, 1/t^3, \dots)$, where the $i$th entry is the function on the $i$th component of X. The key point here is that infinite direct products do not commute with localization.)

Coherent sheaves don't always push forward to coherent sheaves. For example, consider the structure morphism $f : \mathbb{A}^1_k \to \operatorname{Spec} k$, given by $k \mapsto k[t]$. Then $f_*\mathcal{O}_{\mathbb{A}^1_k}$ is the $k[t]$, which is not a finitely generated $k$-module. Under especially good situations, coherent sheaves do push forward. For example:

**2.C.** EXERCISE. Suppose $f : X \to Y$ is a finite morphism of Noetherian schemes. If $\mathcal{F}$ is a coherent sheaf on X, show that $f_*\mathcal{F}$ is a coherent sheaf. (Hint: Show first that $f_*\mathcal{O}_X$ is finite type.)

Once we define cohomology of quasicoherent sheaves, we will quickly prove that if $\mathcal{F}$ is a coherent sheaf on $\mathbb{P}^n_k$, then $\Gamma(\mathbb{P}^n_k, \mathcal{F})$ is a finite-dimensional $k$-module, and more generally if $\mathcal{F}$ is a coherent sheaf on $\operatorname{Proj} S_\bullet$, then $\Gamma(\operatorname{Proj} S_\bullet, \mathcal{F})$ is a coherent A-module (where $S_0 = A$). This is a special case of the fact the "pushforwards of coherent sheaves by projective morphisms are also coherent sheaves". We will first need to define "projective morphism"! This notion is a generalization of $\operatorname{Proj} S_\bullet \to \operatorname{Spec} A$.

More generally, pushforwards of coherent sheaves by proper morphisms are also coherent sheaves. I'd like to give a proof of this, at least in the notes, at some point.

## 3. PULLBACK OF QUASICOHERENT SHEAVES

I find the notion of the pullback of a quasicoherent sheaf to be confusing on first (and second) glance. I will try to introduce it in two ways. One is directly in terms of thinking of quasicoherent sheaves in terms of modules over rings corresponding to affine open sets, and is suitable for direct computation. The other is elegant and functorial in terms of adjoints, and applies to ringed spaces in general. Both perspectives have advantages and disadvantages, and it is worth having some experience working with both.

We note here that pullback to a closed subscheme or an open subscheme is often called **restriction**.

**3.1. Construction/description of the pullback.** Let us now define the pullback functor precisely. Suppose $X \to Y$ is a morphism of schemes, and $\mathcal{G}$ is a quasicoherent sheaf on $Y$. We will describe the pullback quasicoherent sheaf $f^*\mathcal{G}$ on $X$ by describing it as a sheaf on a variant of the distinguished affine base. In our base, we will permit only those affine open sets $U \subset X$ such that $f(U)$ is contained in an affine open set of $Y$. The distinguished restriction map will force this sheaf to be quasicoherent.

Suppose $U \subset X$, $V \subset Y$ are affine open sets, with $f(U) \subset V$, $U \cong \operatorname{Spec} A$, $V \cong \operatorname{Spec} B$. Suppose $\mathcal{F}|_V \cong \tilde{N}$. Then define $\Gamma(U, f_V^*\mathcal{F}) := A \otimes_B N \otimes_B$. Our main goal will be to show that this is independent of our choice of $V$.

We begin as follows: we fix an affine open subset $V \subset Y$, and use it to define sections over any affine open subset $U \subset f^{-1}(V)$. We show that this gives us a quasicoherent sheaf $f_V^*\mathcal{G}$ on $f^{-1}(V)$, by showing that these sections behave well with respect to distinguished restrictions. First, note that if $D(f) \subset U$ is a distinguished open set, then

$$\Gamma(D(f), f_V^*\mathcal{F}) = N \otimes_B A_f \cong (N \otimes_B A) \otimes_A A_f = \Gamma(U, f_V^*\mathcal{F}) \otimes_A A_f.$$

Define the restriction map $\Gamma(U, f_V^*\mathcal{F}) \to \Gamma(D(f), f_V^*\mathcal{F})$ by

$$\Gamma(f_V^*\mathcal{F}, U) \to \Gamma(f_V^*\mathcal{F}, U) \otimes_A A_f \tag{1}$$

(with $\alpha \mapsto \alpha \otimes 1$ of course). Thus on the *distinguished affine topology of* $\operatorname{Spec} A$ we have defined a quasicoherent sheaf.

To sum up: we have defined a quasicoherent sheaf on $f^{-1}(V)$, where $V$ is an affine open subset of $Y$.

We want to show that this construction, as $V$ varies over all affine open subsets of $Y$, glues into a single quasicoherent sheaf on $X$.

**3.A.** EXERCISE. Do this. (Possible hint: possibly use the idea behind the affine covering lemma. Begin by showing that the sheaf on $f^{-1}(\operatorname{Spec} A)$ restricted to the preimage of the distinguished open subset $f^{-1}(\operatorname{Spec} A_g)$ is canonically isomorphic to the sheaf on $f^{-1}(\operatorname{Spec} A_g)$. Another possible hint: figure out what the stalks should be, and define it as a sheaf of compatible germs.)

Hence we have described a quasicoherent sheaf $f^*\mathcal{G}$ on $X$ whose behavior on affines mapping to affines was as promised.

**3.2.** *Theorem. —*

(1) *The pullback of the structure sheaf is the structure sheaf.*
(2) *The pullback of a finite type sheaf is finite type. Hence if $f : X \to Y$ is a morphism of locally Noetherian schemes, then the pullback of a coherent sheaf is coherent. (It is not always true that the pullback of a coherent sheaf is coherent, and the interested reader can think of a counterexample.)*
(3) *The pullback of a locally free sheaf of rank $r$ is another such. (In particular, the pullback of an invertible sheaf is invertible.)*

(4) *(functoriality in the morphism)* $\pi_1^*\pi_2^*\mathcal{F} \cong (\pi_2 \circ \pi_1)^*\mathcal{F}$

(5) *(functoriality in the quasicoherent sheaf) If $\pi : X \to Y$, then $\pi^*$ is a functor from the category of quasicoherent sheaves on $Y$ to the category of quasicoherent sheaves on $Y$. (Hence as a section of a sheaf $\mathcal{F}$ on $Y$ is the data of a map $\mathcal{O}_Y \to \mathcal{F}$, by (1) and (6), if $s : \mathcal{O}_Y \to \mathcal{F}$ is a section of $\mathcal{F}$ then there is a natural section $\pi^* s : \mathcal{O}_X \to \pi^*\mathcal{F}$ of $\pi^*\mathcal{F}$. The pullback of the locus where $s$ vanishes is the locus where the pulled-back section $\pi^* s$ vanishes.)*

(6) *(stalks) If $\pi : X \to Y$, $\pi(x) = y$, then there is an isomorphism* $(\pi^*\mathcal{F})_x \xrightarrow{\sim} \mathcal{F}_y \otimes_{\mathcal{O}_{Y,y}} \mathcal{O}_{X,x}$ .

(7) *(fibers) Pullback of fibers are given as follows: if $\pi : X \to Y$, where $\pi(x) = y$, then*

$$\pi^*\mathcal{F}/\mathfrak{m}_{X,x}\pi^*\mathcal{F} \cong (\mathcal{F}/\mathfrak{m}_{Y,y}\mathcal{F}) \otimes_{\mathcal{O}_{Y,y}/\mathfrak{m}_{Y,y}} \mathcal{O}_{X,x}/\mathfrak{m}_{X,x}$$

(8) *(tensor product)* $\pi^*(\mathcal{F} \otimes \mathcal{G}) = \pi^*\mathcal{F} \otimes \pi^*\mathcal{G}$

(9) *pullback is a right-exact functor*

All of the above are interconnected in obvious ways.

In fact much more is true, that you should be able to prove on a moment's notice, such as for example that the pullback of the symmetric power of a locally free sheaf is naturally isomorphic to the symmetric power of the pullback, and similarly for wedge powers and tensor powers.

Most of these are left to the reader. It is convenient to do right-exactness early; it is related to right-exactness of $\otimes$. For the tensor product fact, show that $(M \otimes_S R) \otimes (N \otimes_S R) \cong (M \otimes N) \otimes_S R$, and that this behaves well with respect to localization. The proof of the fiber fact is as follows. $(S, \mathfrak{n}) \to (R, \mathfrak{m})$.

$$\begin{array}{ccc} S & \longrightarrow & R \\ \downarrow & & \downarrow \\ S/\mathfrak{n} & \longrightarrow & R/\mathfrak{m} \end{array}$$

$(N \otimes_S R) \otimes_R (R/\mathfrak{m}) \cong (N \otimes_S (S/\mathfrak{n})) \otimes_{S/\mathfrak{n}} (R/\mathfrak{m})$ as both sides are isomorphic to $N \otimes_S (R/\mathfrak{m})$.

**3.B.** EXERCISE. Prove the Theorem.

**3.C.** UNIMPORTANT EXERCISE. Verify that the following is a example showing that pullback is not left-exact: consider the exact sequence of sheaves on $\mathbb{A}^1$, where $p$ is the origin:

$$0 \to \mathcal{O}_{\mathbb{A}^1}(-p) \to \mathcal{O}_{\mathbb{A}^1} \to \mathcal{O}_p \to 0.$$

(This is a closed subscheme exact sequence. Algebraically, we have $k[t]$-modules $0 \to tk[t] \to k[t] \to k \to 0$.) Restrict to $p$.

**3.3.** *Remark.* After proving the theorem, you'll see the importance of right-exactness. Given $\pi : X \to Y$, if the functor $\pi^*$ from quasicoherent sheaves on $Y$ to quasicoherent sheaves on $X$ is also left-exact (hence exact), we will say that $\pi$ is a *flat* morphism. This is an incredibly important notion, and we will come back to it later, next quarter.

**3.4. A second definition, that doesn't always apply.** Suppose $\pi : X \to Y$ is a quasicompact quasiseparated morphism, so $\pi_*$ is a functor from quasicoherent sheaves on X to quasicoherent sheaves on Y. Then $\pi^*$ and $\pi_*$ are adjoints. More precisely:

**3.5. Three more "definitions".** Pullback is left-adjoint of the pushforward. If it exists, then it is unique up to unique isomorphism by Yoneda nonsense. One can thus take this as a definition of pullback, at least if $\pi$ is quasicompact and quasiseparated. This defines the pullback up to unique isomorphism. The problem with this is that pullbacks should exist even without these hypotheses on $\pi$. And in any case, any proof by universal property requires an explicit construction as well, so we are led once again to our earlier constructive definition.

**3.6.** *Theorem. — Suppose* $\pi : X \to Y$ *is a quasicompact, quasiseparated morphism. Then pullback is left-adjoint to pushforward. More precisely,* $\mathrm{Hom}(\pi^*\mathcal{G}, \mathcal{F}) \cong \mathrm{Hom}(\mathcal{G}, \pi_*\mathcal{F})$.

More precisely still, we describe natural homomorphisms that are functorial in both arguments. We show that it is a bijection of sets, but it is fairly straightforward to verify that it is an isomorphism of groups. Not surprisingly, we will use adjointness for modules.

*Proof.* Let's unpack the right side. What's an element of $\mathrm{Hom}(\mathcal{G}, f_*\mathcal{F})$? For every affine V in Y, we get an element of $\mathrm{Hom}(\mathcal{G}(V), \mathcal{F}(f^{-1}(V)))$, and this behaves well with respect to distinguished open sets. Equivalently, for every affine V in Y and U in $f^{-1}(V) \subset X$, we have an element $\mathrm{Hom}(\mathcal{G}(V), \mathcal{F}(U))$, that behaves well with respect to localization to distinguished open sets on both affines. By the adjoint property, this corresponds to elements of $\mathrm{Hom}(\mathcal{G}(V) \otimes_{\mathcal{O}_Y(V)} \mathcal{O}_X(U), \mathcal{F}(U))$, which behave well with respect to localization. And that's the left side. □

**3.7.** *Pullback for ringed spaces* $\star$. (This is actually conceptually important but distracting for our exposition; we encourage the reader to skip this, at least on the first reading.) Pullbacks and pushforwards may be defined in the category of $\mathcal{O}$-modules over ringed spaces. We define pushforward in the usual way, and then define the pullback of an $\mathcal{O}$-module using the adjoint property. Then one must show that (i) it exists, and (ii) the pullback of a quasicoherent sheaf is quasicoherent.

Here is a construction that always works. Suppose we have a morphism of ringed spaces $\pi : X \to Y$, and an $\mathcal{O}_Y$-module $\mathcal{G}$. Then define $f^*\mathcal{G} = f^{-1}\mathcal{G} \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X$. We will not show that this definition is equivalent to ours, but the interested reader is welcome to try this as an exercise.

**3.D.** EXERCISE FOR INTERESTED READERS. Show that $\pi^*$ and $\pi_*$ are adjoint functors between the category of $\mathcal{O}_X$-modules and the category of $\mathcal{O}_Y$-modules. Hint: Justify the following.

$$\begin{aligned}\mathrm{Hom}_{\mathcal{O}_X}(f^{-1}\mathcal{G} \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X, \mathcal{F}) &= \mathrm{Hom}_{f^{-1}\mathcal{O}_Y}(f^{-1}\mathcal{G}, \mathcal{F}) \\ &= \mathrm{Hom}_{\mathcal{O}_Y}(\mathcal{G}, f_*\mathcal{F})\end{aligned}$$

The statements of Theorem 3.6 apply in this more general setting.

In particular, by uniqueness of adjointness, this "sheaf-theoretic" definition of pullback agrees with our scheme-theoretic definition of pullback when $\pi$ is quasicompact and quasiseparated. The interested reader may wish to show it in general.

**3.E.** UNIMPORTANT EXERCISE. Show that the scheme-theoretic definition of pullback agrees with the sheaf-theoretic definition in terms of $\mathcal{O}$-modules.

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