FOUNDATIONS OF ALGEBRAIC GEOMETRY CLASS 30

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Today, we will discuss the relationship between quasicoherent sheaves on projective A-schemes and graded modules.

1. THE QUASICOHERENT SHEAF CORRESPONDING TO A GRADED MODULE

We now describe quasicoherent sheaves on a projective A-scheme. Recall that a projective A-scheme is produced from the data of $\mathbb{Z}^{\geq 0}$-graded ring $S_\bullet$, with $S_0 = A$, and S_+ finitely generated as an A-module. The resulting scheme is denoted $\operatorname{Proj} S_\bullet$.

Let $X = \operatorname{Proj} S_\bullet$. Suppose $M_\bullet$ is a graded $S_\bullet$ module, *graded by* $\mathbb{Z}$. (While reading the next section, you may wonder why we don't grade by $\mathbb{Z}^+$. You'll see that it doesn't really matter either way. The reason to prefer a $\mathbb{Z}$-grading is when we produce an $M_\bullet$ from a quasicoherent sheaf on $\operatorname{Proj} S_\bullet$.) We define the quasicoherent sheaf $\widetilde{M_\bullet}$ as follows. For each f of positive degree, we define a quasicoherent sheaf $\widetilde{M_\bullet(f)}$ on the distinguished open $D(f) = \{p : f(p) \neq 0\}$ by

$$\widetilde{M_\bullet}(f) := \widetilde{(M_f)_0}.$$

The subscript 0 here means "the 0-graded piece". We have obvious isomorphisms of the restriction of $\widetilde{M_\bullet(f)}$ and $\widetilde{M_\bullet(g)}$ to $D(fg)$, satisfying the cocycle conditions. (Think through this yourself, to be sure you agree with the word "obvious"!) Then by an earlier problem set problem telling how to glue sheaves, these sheaves glue together to a single sheaf on $\widetilde{M_\bullet}$ on X. We then discard the temporary notation $\widetilde{M_\bullet(f)}$.

This is clearly quasicoherent, because it is quasicoherent on each $D(f)$, and quasicoherence is local.

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1.A. EXERCISE. Show that the stalk of $\widetilde{M_\bullet}$ at a point corresponding to homogeneous prime $\mathfrak{p} \subset S_\bullet$ is isomorphic to the 0th graded piece of $(M_\bullet)_{\mathfrak{p}}$.

1.B. UNIMPORTANT EXERCISE. Use the previous exercise to give an alternate definition of $\widetilde{M_\bullet}$ in terms of "compatible stalks".

Given a map of graded modules $\phi : M_\bullet \to N_\bullet$, we we get an induced map of sheaves $\widetilde{M_\bullet} \to \widetilde{N_\bullet}$. Explicitly, over $D(f)$, the map $M_\bullet \to N_\bullet$ induces $M_\bullet[1/f] \to N_\bullet[1/f]$, which induces $\phi_f : (M_\bullet[1/f])_0 \to (N_\bullet[1/f])_0$; and this behaves well with respect to restriction to smaller distinguished open sets, i.e. the following diagram commutes.

$$\begin{array}{ccc} (M_\bullet[1/f])_0 & \xrightarrow{\phi_f} & (N_\bullet[1/f])_0 \\ \downarrow & & \downarrow \\ (M_\bullet[1/(fg)])_0 & \xrightarrow{\phi_{fg}} & (N_\bullet[1/(fg)])_0. \end{array}$$

Thus $\sim$ is a functor from the category of graded $S_\bullet$-modules to the category of quasicoherent sheaves on $\operatorname{Proj} S_\bullet$. We shall see that this isn't quite an isomorphism, but it is close. The relationship is akin to that between presheaves and sheaves, and the sheafification functor.

1.C. EASY EXERCISE. Show that $\sim$ is an exact functor.

1.D. EXERCISE. Show that if $M_\bullet$ and $M'_\bullet$ agree in high enough degrees, then $\widetilde{M_\bullet} \cong \widetilde{M'_\bullet}$. Thus the map from graded $S_\bullet$-modules to quasicoherent sheaves on $\operatorname{Proj} S_\bullet$ is not a bijection.

1.E. EXERCISE. Describe a map of S_0-modules $M_0 \to \Gamma(\widetilde{M_\bullet}, X)$. (This foreshadows the "saturation map" that takes a graded module to its saturation.)

1.F. EXERCISE. Show that $\widetilde{M_\bullet} \otimes \widetilde{N_\bullet} \cong \widetilde{M_\bullet \otimes_{S_\bullet} N_\bullet}$. (Hint: describe the isomorphism of sections over each $D(f)$, and show that this isomorphism behaves well with respect to smaller distinguished open sets.)

1.1. Graded ideals of $S_\bullet$ give closed subschemes of $\operatorname{Proj} S_\bullet$. Recall that a graded ideal $I_\bullet \subset S_\bullet$ yields a closed subscheme. $\operatorname{Proj} S_\bullet/I_\bullet \hookrightarrow \operatorname{Proj} S_\bullet$.

For example, suppose $S_\bullet = k[w,x,y,z]$, so $\operatorname{Proj} S_\bullet \cong \mathbb{P}^3$. The ideal $I_\bullet = (wz - xy, x^2 - wy, y^2 - xz)$ yields our old friend, the twisted cubic.

1.G. EXERCISE. Show that if the functor $\sim$ is applied to the exact sequence of graded $S_\bullet$-modules

$$0 \to I_\bullet \to S_\bullet \to S_\bullet/I_\bullet \to 0$$

we obtain the closed subscheme exact sequence for $\operatorname{Proj} S_\bullet/I_\bullet \hookrightarrow \operatorname{Proj} S_\bullet$.

We will soon see (§4.E) that all closed subschemes of $\operatorname{Proj} S_\bullet$ arise in this way.

2. Invertible sheaves (line bundles) on projective A-schemes

Suppose that $S_\bullet$ is generated in degree 1. By an earlier exercise, this is not a huge assumption, as we can change the grading by some multiple to arrange that this is the case. Suppose $M_\bullet$ is a graded $S_\bullet$-module. Define the graded module $M(n)_\bullet$ so that $M(n)_m := M_{n+m}$. Thus the quasicoherent sheaf $\widetilde{M(n)_\bullet}$ satisfies

$$\Gamma(D(f), \widetilde{M(n)_\bullet}) = (\widetilde{M_f})_n$$

where here the subscript means we take the nth graded piece. (These subscripts are admittedly confusing!)

2.A. Important exercise. If $S_\bullet$ is generated in degree 1, show that $\mathcal{O}_{\operatorname{Proj} S_\bullet}(n)$ is an invertible sheaf.

2.B. Exercise. If $S_\bullet = k[x_0, \ldots, x_m]$, so $\operatorname{Proj} S_\bullet = \mathbb{P}^m_k$, show that this definition of $\mathcal{O}(n)$ agrees with our earlier definition involving transition functions.

If $\mathcal{F}$ is a quasicoherent sheaf on $\operatorname{Proj} S_\bullet$, define $\mathcal{F}(n) := \mathcal{F} \otimes \mathcal{O}(n)$. This is often called *twisting $\mathcal{F}$ by $\mathcal{O}(n)$*. More generally, if $\mathcal{L}$ is an invertible sheaf, then $\mathcal{F} \otimes \mathcal{L}$ is often called "twisting $\mathcal{F}$ by $\mathcal{L}$".

2.C. Exercise. Show that $\widetilde{M_\bullet}(n) \cong \widetilde{M(n)_\bullet}$.

2.D. Exercise. Show that $\mathcal{O}(m+n) \cong \mathcal{O}(m) \otimes \mathcal{O}(n)$.

2.1. *Unimportant remark.* Even if $S_\bullet$ is not generated in degree 1, then by Exercise , $S_{d\bullet}$ is generated in degree 1 for some d. In this case, we may define the invertible sheaves $\mathcal{O}(dn)$ for $n \in \mathbb{Z}$. This does *not* mean that we *can't* define $\mathcal{O}(1)$; this depends on $S_\bullet$. For example, if $S_\bullet$ is the polynomial ring $k[x, y]$ with the usual grading, except without linear terms, then $S_{2\bullet}$ and $S_{3\bullet}$ are both generated in degree 1, meaning that we may define $\mathcal{O}(2)$ and $\mathcal{O}(3)$. There is good reason to call their "difference" $\mathcal{O}(1)$.

3. Generation by global sections, and Serre's Theorem

3.1. Generated by global sections. Suppose X is a scheme, and $\mathcal{F}$ is a $\mathcal{O}_X$-module. We say that $\mathcal{F}$ is *generated by global sections at a point* p if we can find $\phi : \mathcal{O}^{\oplus I} \to \mathcal{F}$ that is surjective at the stalk of p: $\phi_p : \mathcal{O}_p^{\oplus I} \to \mathcal{F}_p$ is surjective. (Some what more precisely, the stalk of $\mathcal{F}$ at p is generated by global sections of $\mathcal{F}$. The global sections in question are the images of the 1's in $|I|$ factors of $\mathcal{O}_p^{\oplus I}$.) We say that $\mathcal{F}$ is *generated by global sections* or *globally generated* if it is generated by global sections at all p, or equivalently, if we can we can find $\mathcal{O}^{\oplus I} \to \mathcal{F}$ that is surjective. (By our earlier result that we can check surjectivity at stalks, so this is the same as saying that it is surjective at all stalks.) If I can be taken to be finite, we say that $\mathcal{F}$ is generated by a finite number of global sections. We'll see soon why we care.

3.A. EASY EXERCISE. If quasicoherent sheaves $\mathcal{F}$ and $\mathcal{G}$ are generated by global sections at a point p, then so is $\mathcal{F} \otimes \mathcal{G}$. (This exercise is less important, but is good practice.)

3.B. EASY EXERCISE. If $\mathcal{F}$ is a finite type sheaf, show that $\mathcal{F}$ is generated by global sections at p if and only if "the fiber of $\mathcal{F}$ is generated by global sections at p", i.e. the map from global sections to the fiber $\mathcal{F}_p/\mathfrak{m}\mathcal{F}_p$ is surjective, where $\mathfrak{m}$ is the maximal ideal of $\mathcal{O}_{X,p}$. (Hint: Geometric Nakayama.)

3.C. EASY EXERCISE. An invertible sheaf $\mathcal{L}$ on X is generated by global sections if and only if for any point $x \in X$, there is a section of $\mathcal{L}$ not vanishing at x. We'll soon discuss classifying maps to projective space in terms of invertible sheaves generated by global sections, and we'll see then why we care about such notions.

3.D. EASY EXERCISE. If $\mathcal{F}$ is finite type, and X is quasicompact, show that $\mathcal{F}$ is generated by global sections if and only if it is generated by a *finite number* of global sections.

3.2. *Lemma. — Suppose $\mathcal{F}$ is a finite type sheaf on X. Then the set of points where $\mathcal{F}$ is generated by global sections is an open set.*

Proof. Suppose $\mathcal{F}$ is generated by global sections at a point p. Then it is generated by a finite number of global sections, say m. This gives a morphism $\phi : \mathcal{O}^{\oplus m} \to \mathcal{F}$, hence $\operatorname{im}\phi \hookrightarrow \mathcal{F}$. The support of the (finite type) cokernel sheaf is a closed subset not containing p. ☐

3.E. IMPORTANT EXERCISE (AN IMPORTANT THEOREM OF SERRE). Suppose S_0 is a Noetherian ring, and $S_\bullet$ is generated in degree 1. Let $\mathcal{F}$ be any finite type sheaf on $\operatorname{Proj} S_\bullet$. Show that for some integer n_0, for all $n \geq n_0$, $\mathcal{F}(n)$ can be generated by a finite number of global sections.

I'm going to sketch how you should tackle this exercise, after first telling you the reason we will care.

3.3. *Corollary. — Any coherent sheaf $\mathcal{F}$ on $\operatorname{Proj} S_\bullet$ can be presented as:*

$$\oplus_{finite}\mathcal{O}(-n) \to \mathcal{F} \to 0.$$

We're going to use this a lot! One clue of how this might be useful: we can use this to build a resolution of $\mathcal{F}$:

$$\cdots \to \oplus\mathcal{O}(-n_2) \to \oplus\mathcal{O}(-n_1) \to \mathcal{F} \to 0.$$

We understand the $\mathcal{O}(n)$'s pretty well, so we can use this to prove things about coherent sheaves (such as vector bundles) in general.

This Corollary is false for quasicoherent sheaves in general; consider $\oplus_{m\leq 0}\mathcal{O}(m)$.

Proof. Suppose we have m global sections $s_1, \ldots, s_m$ of $\mathcal{F}(n)$ that generate $\mathcal{F}(n)$. This gives a map

$$\oplus_m \mathcal{O} \longrightarrow \mathcal{F}(n)$$

given by $(f_1, \ldots, f_m) \mapsto f_1 s_1 + \cdots + f_m s_m$ on any open set. Because these global sections generate $\mathcal{F}$, this is a surjection. Tensoring with $\mathcal{O}(-n)$ (which is exact, as tensoring with any locally free is exact) gives the desired result. □

Here is now a hint/sketch for the Serre exercise 3.E.

Suppose $\deg f = 1$. Say $\mathcal{F}|_{D(f)} = \tilde{M}$, where M is a $(S_\bullet[1/f])_0$-module, generated by m_1, $\ldots$, m_n. As these elements generate the module, they clearly generate all the stalks over all the points of $D(f)$. They are sections over this ("big") distinguished open set $D(f)$. It would be wonderful if we knew that they had to be restrictions of global sections, i.e. that there was a global section m_i' that restricted to m_i on $D(f)$. If that were always true, then we would cover X with a finite number of each of these $D(f)$'s, and for each of them, we would take the finite number of generators of the corresponding module. Sadly this is not true.

However, we will see that $f^N m$ "extends", where m is any of the m_i's, and N is sufficiently large. We will see this by (easily) checking first that $f^N m$ extends over another distinguished open $D(g)$ (i.e. that there is a section of $\mathcal{F}(N)$ over $D(g)$ that restricts to $f^N m$ on $D(g) \cap D(f) = D(fg)$).

So we're done, right? Wrong — we still don't that these extensions on various open sets glue together, and in fact they might not! More precisely: we don't know that the extension over $D(g)$ and over some other $D(g')$ agree on the overlap $D(g) \cap D(g') = D(gg')$. But after multiplying both extensions by $f^{N'}$ for large enough N', we will see that they agree on the overlap. By quasicompactness, we need to to extend over only a finite number of $D(g)$'s, and to make sure extensions agree over the finite number of pairs of such $D(g)$'s, so we will be done.

Let's now begin to make this precise. We first investigate what happens on $D(g) = \operatorname{Spec} A$, where the degree of g is also 1. Say $\mathcal{F}|_{D(g)} \cong \tilde{N}$. Let $f' = f/g$ be "the function corresponding to f on $D(g)$". We have a section over $D(f')$ on the affine scheme $D(g)$, i.e. an element of $N_{f'}$, i.e. something of the form $n/(f')^N$ for some $n \in N$. So then if we multiply it by f'^N, we can certainly extend it! So if we multiply by a big enough power of f, m certainly extends over any $D(g)$.

As described earlier, the only problem is, we can't guarantee that the extensions over $D(g)$ and $D(g')$ agree on the overlap (and hence glue to a single extensions). Let's check on the intersection $D(g) \cap D(g') = D(gg')$. Say $m = n/(f')^N = n'/(f')^{N'}$ where we can take $N = N'$ (by increasing N or N' if necessary). We certainly may not have $n = n'$, but by the (concrete) definition of localization, after multiplying with enough f''s, they become the same.

In conclusion: after multiplying with enough f's, our sections over $D(f)$ extend over each $D(g)$. After multiplying by even more, they will all agree on the overlaps of any two such distinguished affine. Exercise 3.E is to make this precise.

4. Every quasicoherent sheaf on a projective A-scheme arises from a graded module

We have gotten lots of quasicoherent sheaves on $\operatorname{Proj} S_\bullet$ from graded $S_\bullet$-modules. We'll now see that we can get them all in this way.

We want to figure out how to "undo" the $\sim$ construction. When you do the Exercise computing the space of global sections of $\mathcal{O}(m)$ on $\mathbb{P}^n_k$, you will suspect that in good situations,

$$M_n \cong \Gamma(\operatorname{Proj} S_\bullet, \tilde{M}(n)).$$

Motivated by this, we define

$$\Gamma_n(\mathcal{F}) := \Gamma(\operatorname{Proj} S_\bullet, \mathcal{F}(n)).$$

Then $\Gamma_\bullet(\mathcal{F})$ is a graded $S_\bullet$-module, and we can dream that $\Gamma_\bullet(\mathcal{F})^\sim \cong \mathcal{F}$. We will see that this is indeed the case!

4.A. Exercise. Show that $\Gamma_\bullet$ gives a functor from the category of quasicoherent sheaves on $\operatorname{Proj} S_\bullet$ to the category of graded $S_\bullet$-modules. In other words, show that if $\mathcal{F} \to \mathcal{G}$ is a morphism of quasicoherent sheaves on $\operatorname{Proj} S_\bullet$, describe the natural map $\Gamma_\bullet(\mathcal{F}) \to \Gamma_\bullet(\mathcal{G})$, and show that such natural maps respect the identity and composition.

Note that $\sim$ and $\Gamma_\bullet$ cannot be inverses, as $\sim$ can turn two different graded modules into the same quasicoherent sheaf (see for example Exercise 1.D).

Our initial goal is to show that there is a natural isomorphism $\widetilde{\Gamma_\bullet(\mathcal{F})} \to \mathcal{F}$, and that there is a natural map $M_\bullet \to \Gamma_\bullet(\widetilde{M_\bullet})$. The latter map is called the **saturation map**, although this language isn't important to us. We will show something better: that $\sim$ and $\Gamma_\bullet$ are adjoint.

We start by describing the saturation map $M_\bullet \to \Gamma_\bullet(\widetilde{M_\bullet})$. We describe it in degree n. Given an element m_n, we seek an element of $\Gamma(\operatorname{Proj} S_\bullet, \widetilde{M_\bullet}(n)) = \Gamma(\operatorname{Proj} S_\bullet, \widetilde{M_{(n+\bullet)}})$. By shifting the grading of $M_\bullet$ by n, we can assume $n = 0$. For each $D(f)$, we certainly have an element of $(M[1/f])_0$ (namely m), and they agree on overlaps, so the map is clear.

4.B. EXERCISE. Show that this canonical map need not be injective, nor need it be surjective. (Hint: $S_\bullet = k[x]$, $M_\bullet = k[x]/x^2$ or $M_\bullet = \{$ polynomials with no constant terms $\}$.)

The natural map $\widetilde{\Gamma_\bullet \mathcal{F}} \to \mathcal{F}$ is more subtle, but will have the advantage of being an isomorphism.

4.C. EXERCISE. Describe the natural map $\widetilde{\Gamma_\bullet \mathcal{F}} \to \mathcal{F}$ as follows. First describe it over $D(f)$. Note that sections of the left side are of the form m/f^n where $m \in \Gamma_{n \deg f} \mathcal{F}$, and $m/f^n = m'/f^{n'}$ if there is some N with $f^N(f^{n'}m - f^n m') = 0$. Show that your map behaves well on overlaps $D(f) \cap D(g) = D(fg)$.

4.D. LONGER EXERCISE. Show that the natural map $\widetilde{\Gamma_\bullet \mathcal{F}} \to \mathcal{F}$ is an isomorphism, by showing that it is an isomorphism over $D(f)$ for any f. Do this by first showing that it is surjective. This will require following some of the steps of the proof of Serre's theorem (Exercise 3.E). Then show that it is injective.

4.1. *Corollary. — Every quasicoherent sheaf arises from this tilde construction.*

4.E. EXERCISE. Show that each closed subscheme of $\operatorname{Proj} S_\bullet$ arises from a graded ideal $I_\bullet \subset S_\bullet$. (Hint: Suppose Z is a closed subscheme of $\operatorname{Proj} S_\bullet$. Consider the exact sequence $0 \to \mathcal{I}_Z \to \mathcal{O}_{\operatorname{Proj} S_\bullet} \to \mathcal{O}_Z \to 0$. Apply $\Gamma_\bullet$, and then $\sim$.)

4.F. EXERCISE ($\Gamma_\bullet$ AND $\sim$ ARE ADJOINT FUNCTORS, PART 1). Prove part of the statement that $\Gamma_\bullet$ and $\sim$ are adjoint functors, by describing a natural bijection $\operatorname{Hom}(M_\bullet, \Gamma_\bullet(\mathcal{F})) \cong \operatorname{Hom}(\widetilde{M_\bullet}, \mathcal{F})$. For the map from left to right, start with a morphism $M_\bullet \to \Gamma_\bullet(\mathcal{F})$. Apply $\sim$, and postcompose with the isomorphism $\widetilde{\Gamma_\bullet \mathcal{F}} \to \mathcal{F}$, to obtain

$$\widetilde{M_\bullet} \to \widetilde{\Gamma_\bullet \mathcal{F}} \to \mathcal{F}.$$

Do something similar to get from right to left. Show that "both compositions are the identity in the appropriate category".

4.G. EXERCISE ($\Gamma_\bullet$ AND $\sim$ ARE ADJOINT FUNCTORS, PART 2) $\star$. Show that $\Gamma_\bullet$ and $\sim$ are adjoint.

4.2. Saturated $S_\bullet$-modules. We end with a remark: different graded $S_\bullet$-modules give the same quasicoherent sheaf on $\operatorname{Proj} S_\bullet$, but the results of this section show that there is a "best" (saturated) graded module for each quasicoherent sheaf, and there is a map from each graded module to its "best" version, $M_\bullet \to \Gamma_\bullet(\widetilde{M_\bullet})$. A module for which this is an isomorphism (a "best" module) is called *saturated*. We won't use this term later.

This "saturation" map $M_\bullet \to \Gamma_\bullet(\widetilde{M_\bullet})$ is analogous to the sheafification map, taking presheaves to sheaves. For example, the saturation of the saturation equals the saturation.

There is a bijection between saturated quasicoherent sheaves of ideals on $\operatorname{Proj} S_\bullet$ and closed subschemes of $\operatorname{Proj} S_\bullet$.

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