FOUNDATIONS OF ALGEBRAIC GEOMETRY CLASSES 28 AND 29

RAVI VAKIL

CONTENTS

1. INVERTIBLE SHEAVES AND WEIL DIVISORS

In the previous section, we saw a link between line bundles and codimension 1 information. We now continue this theme. The notion of Weil divisors will give a great way of understanding and classifying line bundles, at least on Noetherian normal schemes. Some of what we discuss will apply in more general circumstances, and the expert is invited to consider generalizations.

For the rest of this section, we consider only *Noetherian schemes*. We do this because we want to discuss codimension 1 subsets, and also have decomposition into irreducibles components.

Define a **Weil divisor** as a formal sum of codimension 1 irreducible closed subsets of X. In other words, a Weil divisor is defined to be an object of the form

$$\sum_{Y \subset X \text{ codimension } 1} n_Y [Y]$$

the n_Y are integers, all but a finite number of which are zero. Weil divisors obviously form an abelian group, denoted $\operatorname{Weil} X$.

For example, if X is a curve (such as the Spec of a Dedekind domain), the Weil divisors are linear combination of points.

We say that [Y] is an **irreducible** (Weil) divisor. A Weil divisor is said to be **effective** if $n_Y \geq 0$ for all Y. In this case we say $D \geq 0$, and by $D_1 \geq D_2$ we mean $D_1 - D_2 \geq 0$. The **support** of a Weil divisor D is the subset $\cup_{n_Y \neq 0} Y$. If $U \subset X$ is an open set, there is a natural restriction map $\operatorname{Weil} X \to \operatorname{Weil} U$, where $\sum n_Y [Y] \mapsto \sum_{Y \cap U \neq \varnothing} n_Y [Y \cap U]$.

Suppose now that X is *regular in codimension* 1 (and Noetherian). We add this hypothesis because we will use properties of discrete valuation rings. Suppose that $\mathcal{L}$ is an invertible

Date: Wednesday, January 30 and Friday, February 1, 2008.

sheaf, and s a rational section not vanishing on any irreducible component of X. (Rational sections are given by a section over a dense open subset of X, with the obvious equivalence.) Then s determines a Weil divisor

$$\operatorname{div}(s) := \sum_{Y} \operatorname{val}_Y(s)[Y]$$

called the **divisor of zeros and poles**. To determine the valuation $\operatorname{val}_Y(s)$ of s along Y, take any open set U containing the generic point of Y where $\mathcal{L}$ is trivializable, along with any trivialization over U; under this trivialization, s is a function on U, which thus has a valuation. Any two such trivializations differ by a unit, so this valuation is well-defined. ($\operatorname{val}_Y(s) = 0$ for all but finitely many Y, by an earlier exercise.) This map gives a group homomorphism

$$\operatorname{div} : \{(\mathcal{L}, s)\} \to \operatorname{Weil} X.$$

A unit has no poles or zeros, so this descends to a group homomorphism

$$\operatorname{div} : \{(\mathcal{L}, s)\}/\Gamma(X, \mathcal{O}_X)^* \to \operatorname{Weil} X. \tag{1}$$

1.A. EXERCISE. (a) *(divisors of rational functions)* Verify that on $\mathbb{A}^1_k$, $\operatorname{div}(x^3/(x+1)) = 3[(x)] - [(x+1)] = 3[0] - [-1]$.
(b) *(divisor of a rational sections of a nontrivial invertible sheaf)* On $\mathbb{P}^1_k$, there is a rational section of $\mathcal{O}(1)$ "corresponding to" $x^2/(x+y)$. Figure out what this means, and calculate $\operatorname{div}(x^2/(x+y))$.

We want to classify all invertible sheaves on X, and this homomorphism (1) will be the key. Note that any invertible sheaf will have such a rational section (for each irreducible component, take a non-empty open set not meeting any other irreducible component; then shrink it so that $\mathcal{L}$ is trivial; choose a trivialization; then take the union of all these open sets, and choose the section on this union corresponding to 1 under the trivialization). We will see that in reasonable situations, this map div will be injective, and often even an isomorphism. Thus by forgetting the rational section (taking an appropriate quotient), we will have described the Picard group of all line bundles. Let's put this strategy into action.

1.1. *Proposition. — If X is normal and Noetherian then the map* div *is injective.*

Proof. Suppose $\operatorname{div}(\mathcal{L}, s) = 0$. Then s has no poles. Hence by Hartogs' lemma for invertible sheaves, s is a regular section. Now s vanishes nowhere, so s gives an isomorphism $\mathcal{O}_X \to \mathcal{L}$ (given by $1 \mapsto s$). □

Motivated by this, we try to find the inverse map to div.

1.2. Important Definition. Suppose D is a Weil divisor. If $U \subset X$ is an open subscheme, recall that $\mathsf{FF}(U)$ is the field of total fractions of U, i.e. the product of the stalks at the minimal primes of U (in this case that X is normal). If U is irreducible, this is the function field. Define $\mathsf{FF}(U)^*$ to be those rational functions not vanishing at any generic point of

U, that is, not vanishing on any irreducible component of U. Define the sheaf $\mathcal{O}_X(D)$ by

$$\Gamma(U, \mathcal{O}_X(D)) := \{s \in FF(U)^* : \operatorname{div} s + D|_U \geq 0\}.$$

The subscript will often be omitted when it is clear from the context. Define a rational section s_D of $\mathcal{O}_X(D)$ corresponding to $1 \in FF(U)^*$.

It may seem more reasonable to consider those s such that $\operatorname{div} s \geq D|_U$. The reason for the convention we use is the following exercise.

1.B. IMPORTANT EXERCISE. Show that $\operatorname{div} s_D = D$.

We connect this to the important example of projective space that we have recently studied:

1.C. IMPORTANT EXERCISE. Let $D = \{x_0 = 0\}$ be a hyperplane divisor on $\mathbb{P}^n_k$. Show that $\mathcal{O}(nD) \cong \mathcal{O}(n)$. (For this reason, $\mathcal{O}(1)$ is sometimes called the **hyperplane class** in $\operatorname{Pic} X$.)

1.3. *Proposition. — Suppose $\mathcal{L}$ is an invertible sheaf, and s is a rational section not vanishing on any irreducible component of X. Then there is an isomorphism $(\mathcal{L}, s) \cong (\mathcal{O}(\operatorname{div} s), t)$, where t is the canonical rational section described above.*

Proof. We first describe the isomorphism $\mathcal{O}(\operatorname{div} s) \cong \mathcal{L}$. Over open subscheme $U \subset X$, we have a bijection $\Gamma(U, \mathcal{L}) \to \Gamma(U, \mathcal{O}(\operatorname{div} s))$ given by $s' \mapsto s'/s$, with inverse obviously given by $t' \mapsto st'$. Clearly under this bijection, s corresponds to the section 1 in $FF(U)^*$; this is the section we are calling t. □

We denote the subgroup of $\operatorname{Weil} X$ corresponding to divisors of rational *functions* the subgroup of **principal divisors**, which we denote $\operatorname{Prin} X$. Define the **class group** of X, $\operatorname{Cl} X$, by $\operatorname{Weil} X / \operatorname{Prin} X$. If X is normal, then by taking the quotient of the inclusion (1) by $\operatorname{Prin} X$, we have the inclusion $\operatorname{Pic} X \hookrightarrow \operatorname{Cl} X$. This is summarized in the convenient diagram

$$\begin{array}{ccccc} & & \operatorname{div} : \{(\mathcal{L}, s)\}/\Gamma(X, \mathcal{O}_X)^* & \hookrightarrow & \operatorname{Weil} X \\ & & \downarrow {\scriptstyle /\{(\mathcal{O}_X, s)\}} & & \downarrow {\scriptstyle /\operatorname{Prin} X} \\ \operatorname{Pic} X & = & \{\mathcal{L}\} & \hookrightarrow & \operatorname{Cl} X \end{array}$$

This diagram is very important, and although it is short to say, it takes some time to internalize. (If X is Noetherian and regular in codimension 1 but not necessarily normal, then we have a similar diagram, except the horizontal maps are not necessarily inclusions.)

We can now compute of $\operatorname{Pic} X$ in a number of interesting cases!

1.D. EXERCISE. Suppose that A is a Noetherian domain. Show that A is a Unique Factorization Domain if and only if A is integrally closed and $\operatorname{Cl} \operatorname{Spec} A = 0$. (One direction is easy: we have already shown that Unique Factorization Domains are integrally closed in their fraction fields. Also, an earlier exercise showed that all codimension 1 primes of a Unique Factorization Domain are principal, so that implies that $\operatorname{Cl}\operatorname{Spec} A = 0$. It remains to show that if A is integrally closed and $\operatorname{Cl}\operatorname{Spec} A = 0$, then all codimension 1 prime ideals are principal, as this characterizes Unique Factorization Domains. Hartogs' lemma may arise in your argument.) This is the third important characterization of unique factorization domains promised long ago.

Hence $\operatorname{Cl}(\mathbb{A}^n_k) = 0$, so $\boxed{\operatorname{Pic}(\mathbb{A}^n_k) = 0}$. Geometers will find this believable: "$\mathbb{C}^n$ is a contractible manifold, and hence should have no nontrivial line bundles".

Removing subset of X of codimension greater 1 doesn't change the Class group, as it doesn't change the Weil divisor group or the principal divisors.

Removing a subset of codimension 1 changes the Weil divisor group in a controllable way. For example, suppose Z is an irreducible codimension 1 subset of X. Then we clearly have an exact sequence:

$$0 \longrightarrow \mathbb{Z} \xrightarrow{1 \mapsto [Z]} \operatorname{Weil} X \longrightarrow \operatorname{Weil}(X - Z) \longrightarrow 0.$$

When we take the quotient by principal divisors, we lose exactness on the left, and get:

$$\mathbb{Z} \xrightarrow{1 \mapsto [Z]} \operatorname{Cl} X \longrightarrow \operatorname{Cl}(X - Z) \longrightarrow 0. \tag{2}$$

1.E. EASY EXERCISE. Suppose $X \hookrightarrow \mathbb{A}^n$ is an open subset. Show that $\operatorname{Pic} X = \{0\}$.

For example, let $X = \mathbb{P}^n_k$, and Z be the hyperplane $x_0 = 0$. We have

$$\mathbb{Z} \to \operatorname{Cl} \mathbb{P}^n_k \to \operatorname{Cl} \mathbb{A}^n_k \to 0$$

from which $\operatorname{Cl} \mathbb{P}^n_k = \mathbb{Z}[Z]$ (which is $\mathbb{Z}$ or 0), and $\operatorname{Pic} \mathbb{P}^n_k$ is a subgroup of this.

By Exercise 1.C, $[Z] \to \mathcal{O}(1)$. Hence $\operatorname{Pic} \mathbb{P}^n_k \hookrightarrow \operatorname{Cl} \mathbb{P}^n_k$ is an isomorphism, and $\boxed{\operatorname{Pic} \mathbb{P}^n_k \cong \mathbb{Z}}$, with generator $\mathcal{O}(1)$. The **degree** of an invertible sheaf on $\mathbb{P}^n$ is defined using this: $\deg \mathcal{O}(d) := d$.

More generally:

1.4. *Proposition. — If X is Noetherian and factorial (all stalks are unique factorization domains) then for any Weil divisor D, $\mathcal{O}(D)$ is invertible, and hence the map $\operatorname{Pic} X \to \operatorname{Cl} X$ is an isomorphism.*

Proof. It will suffice to show that $[Y]$ is effective Cartier if Y is any irreducible divisor. Our goal is to cover X by open sets so that on each open set U there is a function whose divisor is $[Y \cap U]$. One open set will be $X - Y$, where we take the function 1. Next, we find an open set U containing an arbitrary $x \in Y$, and a function on U. As $\mathcal{O}_{X,x}$ is a unique factorization domain, the prime corresponding to 1 is codimension 1 and hence principal (by an earlier Exercise). Let $f \in FF(A)$ be a generator. Then f is regular at x. f has a finite number of zeros and poles, and through x there is only one 0, notably $[Y]$. Let U be X minus all the others zeros and poles. $\square$

I will now mention a bunch of other examples of class groups and Picard groups you can calculate.

First, notice that you can restrict invertible sheaves on X to any subscheme Y, and this can be a handy way of checking that an invertible sheaf is not trivial. For example, if X is something crazy, and $Y \cong \mathbb{P}^1$, then we're happy, because we understand invertible sheaves on $\mathbb{P}^1$. Effective Cartier divisors sometimes restrict too: if you have effective Cartier divisor on X, then it restricts to a closed subscheme on Y, locally cut out by one equation. If you are fortunate and this equation doesn't vanish on any associated point of Y, then you get an effective Cartier divisor on Y. You can check that the restriction of effective Cartier divisors corresponds to restriction of invertible sheaves.

1.5. Fun with hypersurface complements.

1.F. EXERCISE: A TORSION PICARD GROUP. Show that Y is an irreducible degree d hypersurface of $\mathbb{P}^n$. Show that $\operatorname{Pic}(\mathbb{P}^n - Y) \cong \mathbb{Z}/d$. (For differential geometers: this is related to the fact that $\pi_1(\mathbb{P}^n - Y) \cong \mathbb{Z}/d$.)

As a very explicit example, we can consider the plane minus a conic ($n = d = 2$).

The next two exercises explore its consequences, and provide us with some examples we have been waiting for.

1.G. EXERCISE. Keeping the same notation, assume $d > 1$ (so $\operatorname{Pic}(\mathbb{P}^n - Y) \neq 0$), and let $H_0, \dots, H_n$ be the $n+1$ coordinate hyperplanes on $\mathbb{P}^n$. Show that $\mathbb{P}^n - Y$ is affine, and $\mathbb{P}^n - Y - H_i$ is a distinguished open subset of it. Show that the $\mathbb{P}^n - Y - H_i$ form an open cover of $\mathbb{P}^n - Y$. Show that $\operatorname{Pic}(\mathbb{P}^n - Y - H_i) = 0$. Then by Exercise 1.D, each $\mathbb{P}^n - Y - H_i$ is the Spec of a unique factorization domain, but $\mathbb{P}^n - Y$ is not. Thus the property of being a unique factorization domain is not an affine-local property — it satisfies only one of the two hypotheses of the affine communication lemma.

1.H. EXERCISE. Keeping the same notation as the previous exercise, show that on $\mathbb{P}^n - Y$, H_i (restricted to this open set) is an effective Cartier divisor that is not cut out by a single equation. (Hint: Otherwise it would give a trivial element of the class group.)

1.6. Quadric surfaces.

1.I. EXERCISE. Let $X = \operatorname{Proj} k[w, x, y, z]/(wz - xy)$, a smooth quadric surface (Figure 1). Show that $\operatorname{Pic} X \cong \mathbb{Z} \oplus \mathbb{Z}$ as follows: Show that if L and M are two lines in different rulings (e.g. $L = V(w, x)$ and $M = V(w, y)$), then $X - L - M \cong \mathbb{A}^2$. This will give you a surjection

$\mathbb{Z} \oplus \mathbb{Z} \twoheadrightarrow \operatorname{Cl} X$. Show that $\mathcal{O}(L)$ restricts to $\mathcal{O}$ on L and $\mathcal{O}(1)$ on M. Show that $\mathcal{O}(M)$ restricts to $\mathcal{O}$ on M and $\mathcal{O}(1)$ on L. (This is a bit longer to do, but enlightening.)

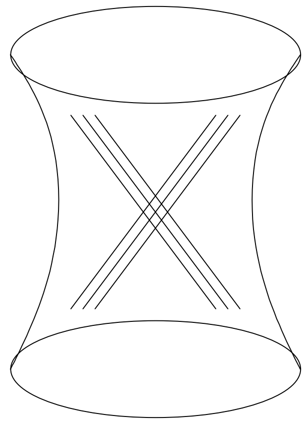

FIGURE 1. Finding all line bundles on the quadric surface

1.J. EXERCISE. Let $X = \operatorname{Spec} k[w, x, y, z]/(xy - z^2)$, a cone. show that $\operatorname{Pic} X = 1$, and $\operatorname{Cl} X \cong \mathbb{Z}/2$. (Hint: show that the ruling $Z = \{x = z = 0\}$ generates $\operatorname{Cl} X$ by showing that its complement is isomorphic to $\mathbb{A}^2_k$. Show that $2[Z] = \operatorname{div}(x)$ (and hence principal), and that Z is not principal (an example we did when learning how to use the Zariski tangent space).

1.7. Nagata's Lemma $\star\star$**.**

I mentioned earlier that I only know a few ways of checking that a ring is a unique factorization domain. Nagata's Lemma is the last, and least useful.

1.K. EXERCISE. Prove Nagata's Lemma: Suppose A is a Noetherian domain, $x \in A$ an element such that (x) is prime and $A[1/x]$ is a unique factorization domain. Then A is a unique factorization domain. (Hint: Exercise 1.D. Use the short exact sequence $[(x)] \to \operatorname{Cl} \operatorname{Spec} A \to \operatorname{Cl} A[1/x] \to 0$ (2) to show that $\operatorname{Cl} \operatorname{Spec} A = 0$. Show that $A[1/x]$ is integrally closed, then show that A is integrally closed as follows. Suppose $T^n + a_{n-1}T^{n-1} + \cdots + a_0 = 0$, where $a_i \in A$, and $T \in FF(A)$. Then by integral closure of $A[1/x]$, we have that $T = r/x^m$, where if $m > 0$, then $r \notin x$. Then we quickly get a contradiction if $m > 0$.)

This leads to a remarkable algebra fact. Suppose k is an algebraically closed field of characteristic not 2. Let $A = k[x_1, \ldots, x_n]/(x_1^2 + \cdots + x_m^2)$ where $m \leq n$. When $m \leq 2$, we get some special behavior. (If $m = 0$, we get affine space; if $m = 1$, we get a non-reduced scheme; if $m = 2$, we get a reducible scheme that is the union of two affine spaces.) If $m \geq 3$, we have verified that $\operatorname{Spec} A$ is normal, in an earlier exercise.

In fact, if $m \geq 3$, then A is a unique factorization domain *unless* $m = 4$. The failure at 4 comes from the geometry of the quadric surface: we have checked that in $\operatorname{Spec} k[w, x, y, z]/(wx-$

$yz)$, there is a codimension 1 prime ideal — the cone over a line in a ruling — that is not principal.

We already understand success at 3: $A = k[x, y, z, w_1, \dots, w_{n-3}]/(x^2+y^2-z^2)$ is a unique factorization domain, as it is normal and has class group 0 (as verified above).

1.L. EXERCISE (THE CASE $m \geq 5$). Suppose that k is algebraically closed of characteristic not 2. Show that if $m \geq 3$, then $A = k[a, b, x_1, \dots, x_n]/(ab - x_1^2 - \cdots - x_m^2)$ is a unique factorization domain, by using the Nagata's Lemma with $x = a$.

E-mail address: vakil@math.stanford.edu