FOUNDATIONS OF ALGEBRAIC GEOMETRY CLASS 23

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CONTENTS

1. VALUATIVE CRITERIA FOR SEPARATEDNESS AND PROPERNESS

We now come to a topic that I regret bringing up. It is useful in practice, although to be honest, I've never used it myself in any meaningful way, and we will not use it later in this course. In fairness, I should say that many people love this fact, and the reason I felt compelled to discuss is was that I feared I would be cast out of the algebraic geometric if I didn't talk about it. But in retrospect I think you shouldn't see it soon after seeing separatedness the first time. In particular, you probably should just ignore this section.

In good circumstances, it is possible to verify separatedness by checking only maps from spectra of discrete valuations rings.

There are two reasons you might like it (even if you never use it). First, it gives useful intuition for what separated morphisms look like. Second, given that we understand schemes by maps to them (the Yoneda philosophy), we might expect to understand morphisms by mapping certain maps of schemes to them, and this is how you can interpret the diagram you'll see soon.

We begin with a valuative criterion that applies in a case that will suffice for the interests of most people, that of finite type morphisms of Noetherian schemes. We'll then give a more general version for more general readers.

1.1. *Theorem (Valuative criterion for separatedness for morphisms of finite type of Noetherian schemes). — Suppose $f : X \to Y$ is a morphism of finite type of Noetherian schemes. Then f is separated if and only if the following condition holds. For any discrete valuation ring A with function field K, and any diagram of the form*

(1)
$$\begin{tikzcd} \operatorname{Spec} K \arrow[r] \arrow[d, hook, "\textit{open imm.}"'] & X \arrow[d, "f"] \\ \operatorname{Spec} A \arrow[r] & Y \end{tikzcd}$$

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(where the vertical morphism on the left corresponds to the inclusion $A \hookrightarrow K$), there is at most one morphism $\operatorname{Spec} A \to X$ such that the diagram

$$(2) \qquad \begin{tikzcd} \operatorname{Spec} K \arrow[r] \arrow[d, hook, "\text{open imm.}"'] & X \arrow[d, "f"] \\ \operatorname{Spec} A \arrow[r] \arrow[ur, dashed, "\leq 1"] & Y \end{tikzcd}$$

commutes.

A useful thing to take away from this statement is the intuition behind it. We think of $\operatorname{Spec} A$ as a "germ of a curve", and $\operatorname{Spec} K$ as the "germ minus the origin". Then this says that if we have a map from a germ of a curve to Y, and have a lift of the map away from the origin to X, then there is at most one way to lift the map from the entire germ. In the case where Y is a field, you can think of this as saying that limits of one-parameter families are unique (if they exist).

For example, this captures the idea of what is wrong with the map of the line with the doubled origin over k: we take $\operatorname{Spec} A$ to be the germ of the affine line at the origin, and consider the map of the germ minus the origin to the line with doubled origin. Then we have two choices for how the map can extend over the origin. (I drew pictures here, which I have not yet latexed up: the map of the line with doubled origin to a point; the map of the line with the doubled origin to a line; and the map of the line with doubled origin to itself. In the first two cases, we could see the valuative criterion failing. In the last case, it did not fail.)

1.A. EXERCISE. Make this precise: show that map of the line with doubled origin over k to $\operatorname{Spec} k$ fails the valuative criterion for separatedness.

1.2. Note on moduli spaces and the valuative criterion of separatedness. I said a little more about separatedness of moduli spaces, for those familiar such objects. Suppose we are interested in a moduli space of a certain kind of object. That means that there is a scheme M with a "universal family" of such objects over M, such that there is a bijection between families of such objects over an arbitrary scheme S, and morphisms $S \to B$. (One direction of this map is as follows: given a morphism $S \to B$, we get a family of objects over S by pulling back the universal family over B.) The separatedness of the moduli space (over the base field, for example, if there is one) can be interpreted as follows. Fix a valuation ring A (or even discrete valuation ring, if our moduli space of of finite type) with fraction field K. We interpret Spec intuitively as a germ of a curve, and we interpret $\operatorname{Spec} K$ as the germ minus the "origin" (an analogue of a small punctured disk). Then we have a family of objects over $\operatorname{Spec} K$ (or over the punctured disk), or equivalently a map $\operatorname{Spec} K \to M$, and the moduli space is separated if there is *at most one way* to fill in the family over the origin, i.e. a family over $\operatorname{Spec} A$.

$\star$ The rest of this section should be ignored upon first reading.

Proof. (This proof is more telegraphic than I'd like. I may fill it out more later. Because we won't be using this result later in the course, you should feel free to skip it, but you may want to skim it.) One direction is fairly straightforward. Suppose $f : X \to Y$ is separated, and such a diagram (1) were given. suppose g_1 and g_2 were two morphisms $\operatorname{Spec} A \to X$ making (2) commute. Then $g = (g_1, g_2) : \operatorname{Spec} A \to X \times_Y X$ is a morphism, with $g(\operatorname{Spec} K)$ contained in the diagonal. Hence as $\operatorname{Spec} K$ is dense in $\operatorname{Spec} A$, and g is continuous, $g(\operatorname{Spec} A)$ is contained in the closure of the diagonal. As the diagonal is closed (the separated hypotheses), $g(\operatorname{Spec} A)$ is also contained *set-theoretically* in the diagonal. As $\operatorname{Spec} A$ is reduced, g factors through the induced reduced subscheme structure of the diagonal. Hence g factors through the diagonal:

$$\operatorname{Spec} A \longrightarrow X \xrightarrow{\delta} X \times_Y X,$$

which means $g_1 = g_2$ by our earlier exercise about maps from a reduced schemed to a separated scheme.

Suppose conversely that f is not separated, i.e. that the diagonal $\Delta \subset X \times_Y X$ is not closed. Note that $X \times_Y X$ is Noetherian (X is Noetherian, and $X \times_Y X \to X$ is finite type as it is obtained by base change from the finite type $X \to Y$), As Δ isn't a closed subset, there is a point in $\overline{\Delta} - \Delta$ and another point in Δ so that the first (say z) is in the closure of the second (say a). (I believe we checked earlier in our discussion of Chevalley's theorem that for Noetherian schemes, a subset is closed if and only if it is closed under specialization.) By the Noetherian condition, there is a maximal chain of closed subsets

$$\overline{a} \subset \overline{b} \subset \cdots \subset \overline{z}$$

(where $a, \dots, z$ are the generic points). Thus we can find two "adjacent" points (say p and q, so $\operatorname{codim}_{\overline{q}} p = 1$) such that $q \in \Delta$ and $p \notin \Delta$. Let Q be the scheme obtained by giving the induced reduced subscheme structure to $\overline{q}$. Then p is a codimension 1 point on Q; let $A' = \mathcal{O}_{Q,p}$ be the local ring of Q at p. Then A' is a Noetherian local domain of dimension 1. Let A'' be the normalization of A. Choose any point p'' of $\operatorname{Spec} A''$ mapping to p; such a point exists because the normalization morphism $\operatorname{Spec} A \to \operatorname{Spec} A'$ is surjective (normalization is an integral extension, hence surjective by the Going-up theorem). Now A'' is Noetherian (I need to explain why... if $R \hookrightarrow R'$ is an integral extension of rings, then R is Noetherian if and only if R'' is Noetherian, by the going down theorem...). Let A be the localization of A'' at p''. Then A is a normal Noetherian local domain of dimension 1, and hence a discrete valuation ring. Let K be its fraction field. Then $\operatorname{Spec} A \to X \times_Y X$ does not factor through the diagonal, but $\operatorname{Spec} K \to X \times_Y X$ does, and we are done. $\square$

With a more powerful invocation of commutative algebra, we can prove a valuative criterion with much less restrictive hypotheses.

1.3. *Theorem: Valuative criterion of separatedness. — Suppose $f : X \to Y$ is a quasiseparated morphism. Then f is separated if and only if the following condition holds. For any valuation ring A with function field K, and any diagram of the form* (1)*, there is at most one morphism $\operatorname{Spec} A \to X$ such that the diagram* (2) *commutes.*

Because I've already proved something useful that we'll never use, I feel no urge to prove this harder fact. The proof of one direction, that separated implies that the criterion holds, is identical. The other direction is similar: get P and Q. Then use an algebra fact.

There is a valuative criterion for properness too. I've never used it personally, but it *is* useful, both directly, and also philosophically. I'll make statements, and then discuss some philosophy.

1.4. *Theorem (Valuative criterion for properness for morphisms of finite type of Noetherian schemes). — Suppose* $f : X \to Y$ *is a morphism of finite type of locally Noetherian schemes. Then* f *is proper if and only if the following condition holds. For any discrete valuation ring* A *with function field* K*, and or any diagram of the form*

(3)
$$\begin{tikzcd} \operatorname{Spec} K \arrow[r] \arrow[d, hook] & X \arrow[d, "f"] \\ \operatorname{Spec} A \arrow[r] & Y \end{tikzcd}$$

(where the vertical morphism on the left corresponds to the inclusion $A \hookrightarrow K$*), there is* exactly *one morphism* $\operatorname{Spec} A \to X$ *such that the diagram*

(4)
$$\begin{tikzcd} \operatorname{Spec} K \arrow[r] \arrow[d, hook] & X \arrow[d, "f"] \\ \operatorname{Spec} A \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

commutes.

Recall that the valuative criterion for separatedness was the same, except that *exact* was replaced by *at most*.

In the case where Y is a field, you can think of this as saying that limits of one-parameter families always exist, and are unique.

I discussed the moduli interpretation of this criterion.

1.B. EXERCISE. Use the valuative criterion of properness to prove that $\mathbb{P}^n_A \to \operatorname{Spec} A$ is proper if A is Noetherian. (This is a difficult way to prove this fact!)

1.5. *Theorem (Valuative criterion of properness). — Suppose* $f : X \to Y$ *is a quasiseparated, finite type (hence quasicompact) morphism. Then* f *is proper if and only if the following condition holds. For any valuation ring* R *with function field* K*, and or any diagram of the form* (3)*, there is* exactly *one morphism* $\operatorname{Spec} R \to X$ *such that the diagram* (4) *commutes.*

Uses: (1) intuition. (2) moduli idea: exactly one way to fill it in (stable curves). (3) motivates the definition of properness for stacks.

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