

FOUNDATIONS OF ALGEBRAIC GEOMETRY CLASS 19

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1. DIMENSION AND CODIMENSION

The notion of *dimension* is the first of two algebraically “hard” properties of schemes, the other being smoothness = nonsingularity (coming at the start of next quarter).

1.1. Dimension. One rather basic notion we expect to have of geometric objects is *dimension*, and our goal in this chapter is to define the dimension of schemes. This should agree with, and generalize, our geometric intuition. Keep in mind that although we think of this as a basic notion in geometry, it is a slippery concept, and has been so for historically. (For example, how do we know that there isn't an isomorphism between some 1-dimensional manifold and some 2-dimensional manifold?)

A caution for those thinking over the complex numbers: our dimensions will be algebraic, and hence half that of the “real” picture. For example, $\mathbb{A}_{\mathbb{C}}^1$, which you may picture as the complex numbers (plus one generic point), has dimension 1.

Surprisingly, the right definition is purely topological — it just depends on the topological space, and not on the structure sheaf. We define the **dimension** of a topological space X as the supremum of lengths of chains of closed irreducible sets, starting the indexing with 0. (This dimension may be infinite.) Scholars of the empty set can take the dimension of the empty set to be $-\infty$. Define the **dimension** of a ring as the Krull dimension of its spectrum — the sup of the lengths of the chains of nested prime ideals (where indexing starts at zero). These two definitions of dimension are sometimes called **Krull dimension**. (You might think a Noetherian ring has finite dimension because all chains of prime ideals are finite, but this isn't necessarily true — see Exercise 1.6.)

As we have a natural homeomorphism between $\text{Spec } A$ and $\text{Spec } A/\mathfrak{n}(A)$ (the Zariski topology doesn't care about nilpotents), we have $\dim A = \dim A/\mathfrak{n}(A)$.

Date: Monday, December 4, 2007.

Examples. We have identified all the prime ideals of $k[t]$ (they are 0 , and $(f(t))$ for irreducible polynomials $f(t)$), \mathbb{Z} (0 and (p)), k (only 0), and $k[x]/(x^2)$ (only 0), so we can quickly check that $\dim \mathbb{A}_k^1 = \dim \text{Spec } \mathbb{Z} = 1$, $\dim \text{Spec } k = 0$, $\dim \text{Spec } k[x]/(x^2) = 0$.

We must be careful with the notion of dimension for reducible spaces. If Z is the union of two closed subsets X and Y , then $\dim Z = \max(\dim X, \dim Y)$. In particular, if Z is the disjoint union of something of dimension 2 and something of dimension 1, then it has dimension 2. Thus dimension is not a “local” characteristic of a space. This sometimes bothers us, so we will often talk about dimensions of irreducible topological spaces. If a topological space can be expressed as a finite union of irreducible subsets, then say that it is **equidimensional** or **pure dimensional** (resp. equidimensional of dimension n or pure dimension n) if each of its components has the same dimension (resp. they are all of dimension n).

An equidimensional dimension 1 (resp. 2, n) topological space is said to be a **curve** (resp. **surface**, **n -fold**).

1.2. Codimension. Because dimension behaves oddly for disjoint unions, we need some care when defining codimension, and in using the phrase. For example, if Y is a closed subset of X , we might define the codimension to be $\dim X - \dim Y$, but this behaves badly. For example, if X is the disjoint union of a point Y and a curve Z , then $\dim X - \dim Y = 1$, but the reason for this has nothing to do with the local behavior of X near Y .

A better definition is as follows. In order to avoid excessive pathology, we define the codimension of Y in X *only when Y is irreducible*. We define the **codimension of an irreducible closed subset** $Y \subset X$ of a topological space as the supremum of lengths of *increasing* chains of irreducible closed subsets starting at Y (where indexing starts at 0). The **codimension of a point** is defined to be the codimension of its closure.

We say that a prime ideal \mathfrak{p} in a ring has **codimension** equal to the supremum of lengths of the chains of decreasing prime ideals starting at \mathfrak{p} , with indexing starting at 0. Thus in an integral domain, the ideal (0) has codimension 0; and in \mathbb{Z} , the ideal (23) has codimension 1. Note that the codimension of the prime ideal \mathfrak{p} in A is $\dim A_{\mathfrak{p}}$. (This notion is often called *height*.) Thus the codimension of \mathfrak{p} in A is the codimension of $[\mathfrak{p}]$ in $\text{Spec } A$.

1.A. EXERCISE. Show that if Y is an irreducible subset of a scheme X with generic point y , show that the codimension of Y is the dimension of the local ring $\mathcal{O}_{X,y}$.

Note that Y is codimension 0 in X if it is an irreducible component of X . Similarly, Y is codimension 1 if it is strictly contained in an irreducible component Y' , and there is no irreducible subset strictly between Y and Y' . (See Figure 1 for examples.) An closed subset all of whose irreducible components are codimension 1 in some ambient space X is said to be a **hypersurface** in X .

1.B. EASY EXERCISE. Show that

$$(1) \quad \text{codim}_X Y + \dim Y \leq \dim X.$$

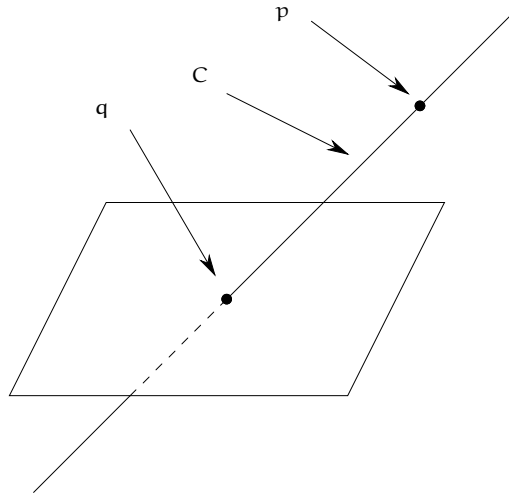


FIGURE 1. Behavior of codimension

We will see next day that equality always holds if X and Y are varieties, but equality doesn't always hold.

Warnings. (1) We have *only* defined codimension for *irreducible* Y in X . Exercise extreme caution in using this word in any other setting. We may use it in the case where the irreducible components of Y *each* have the same codimension.

(2) The notion of codimension still can behave slightly oddly. For example, consider Figure 1. (You should think of this as an intuitive sketch, but once we define dimension correctly, this will be precise.) Here the total space X has dimension 2, but point p is dimension 0, and codimension 1. We also have an example of a codimension 2 subset q contained in a codimension 0 subset C with no codimension 1 subset "in between".

Worse things can happen; we will soon see an example of a closed point in an irreducible surface that is nonetheless codimension 1, not 2. However, for irreducible varieties (finitely generated domains over a field), this can't happen, and the inequality (1) must be an inequality. We'll show this next day.

1.3. What will happen in this chapter.

In this chapter, we'll explore the notions of dimension and codimension, and show that they satisfy properties that we find desirable, and (later) useful. In particular, we'll learn some techniques for computing dimension.

We would certainly want affine n -space to have dimension n . We will indeed show (next day) that $\dim \mathbb{A}_k^n = n$, and show more generally that the dimension of an irreducible variety over k is its transcendence degree. En route, we will see some useful facts, including the Going-Up Theorem, and Noether Normalization. (While proving the Going-Up Theorem, we will see a trick that will let us prove many forms of Nakayama's Lemma,

which will be useful to us in the future.) Related to the Going-Up Theorem is the fact that certain nice (“integral”) morphisms $X \rightarrow Y$ will have the property that $\dim X = \dim Y$ (Exercise 2.H).

Noether Normalization will let us prove Chevalley’s Theorem, stating that the image of a finite type morphism of Noetherian schemes is always constructible. We will also give a short proof of the Nullstellensatz.

We then briefly discuss two useful facts about codimension one. A linear function on a vector space either vanishes in codimension 0 (if it is the 0-function) or else in codimension 1. The same is true much more generally for functions on Noetherian schemes. Informally: a function on a Noetherian scheme also vanishes in pure codimension 0 or 1. More precisely, the irreducible components of its vanishing locus are all codimension at most 1. This is Krull’s Principal Ideal Theorem. A second fact, that we’ll call “Algebraic Hartogs’ Lemma”, informally states that on a normal scheme, any rational function with no poles is in fact a regular function. These two codimension one facts will come in very handy in the future.

We end this introductory section with a first property about codimensions (and hypersurfaces) that we’ll find useful, and a pathology.

1.4. Warm-up proposition. — *In a unique factorization domain A , all codimension 1 prime ideals are principal.*

We will see next day that the converse (in the case where A is Noetherian domain) holds as well.

Proof. Suppose \mathfrak{p} is a codimension 1 prime. Choose any $f \neq 0$ in \mathfrak{p} , and let g be any irreducible/prime factor of f that is in \mathfrak{p} (there is at least one). Then (g) is a prime ideal contained in \mathfrak{p} , so $(0) \subset (g) \subset \mathfrak{p}$. As \mathfrak{p} is codimension 1, we must have $\mathfrak{p} = (g)$, and thus \mathfrak{p} is principal. \square

1.5. A fun but unimportant counterexample. As a Noetherian ring has no infinite chain of prime ideals, you may think that Noetherian rings must have finite dimension. Here is an example of a Noetherian ring with infinite dimension, due to Nagata, the master of counterexamples.

1.6. Exercise * Choose an increasing sequence of positive integers m_1, m_2, \dots whose differences are also increasing ($m_{i+1} - m_i > m_i - m_{i-1}$). Let $P_i = (x_{m_i+1}, \dots, x_{m_{i+1}})$ and $S = A - \cup_i P_i$. Show that S is a multiplicative set. Show that $S^{-1}A$ is Noetherian. Show that each $S^{-1}P$ is the smallest prime ideal in a chain of prime ideals of length $m_{i+1} - m_i$. Hence conclude that $\dim S^{-1}A = \infty$.

2. INTEGRAL EXTENSIONS AND THE GOING-UP THEOREM

A ring homomorphism $\phi : B \rightarrow A$ is **integral** if every element of A is integral over $\phi(B)$. In other words, if a is any element of A , then a satisfies some monic polynomial

$$a^n + \alpha a^{n-1} + \dots + \beta = 0$$

where all the coefficients lie in $\phi(B)$. We call ϕ an **integral extension** if ϕ is an inclusion of rings.

2.A. EXERCISE. Show that if $f : B \rightarrow A$ is a ring homomorphism, and $(b_1, \dots, b_n) = 1$ in B , and $B_{b_i} \rightarrow A_{f(b_i)}$ is integral, then f is integral. Thus we can define the notion of **integral morphism** of schemes.

2.B. EXERCISE. Show that the notion of integral homomorphism is well behaved with respect to localization and quotient of B , and quotient of A , but not localization of A . Show that the notion of integral extension is well behaved with respect to localization and quotient of B , but not quotient of A . If possible, draw pictures of your examples.

2.C. EXERCISE. Show that if B is an integral extension of A , and C is an integral extension of B , then C is an integral extension of A .

2.1. Proposition (finite implies integral). — *If A is a finite B -algebra, then ϕ is an integral homomorphism.*

The converse is false: integral does not imply finite, as $\mathbb{Q} \hookrightarrow \overline{\mathbb{Q}}$ is an integral homomorphism, but $\overline{\mathbb{Q}}$ is not a finite \mathbb{Q} -module.

2.D. UNIMPORTANT EXERCISE: FINITE = INTEGRAL + FINITE TYPE. Show that a morphism is finite if and only if it is integral and finite type.

Proof. The proof involves a useful trick.

Choose a finite generating set m_1, \dots, m_n of A as a B -module. Then $am_i = \sum b_{ij}m_j$, for some $b_{ij} \in B$. Thus

$$(2) \quad (aI_{n \times n} - [b_{ij}]_{ij}) \begin{pmatrix} m_1 \\ \vdots \\ m_n \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}.$$

We can't quite invert this matrix $(aI_{n \times n} - [b_{ij}]_{ij})$, but we almost can. Recall that any $n \times n$ matrix M has an *adjoint matrix* $\text{adj}(M)$ such that $\text{adj}(M)M = \det(M)\text{Id}_n$. (The ij th entry of $\text{adj}(M)$ is the determinant of the matrix obtained from M by deleting the i th column and j th row, times $(-1)^{i+j}$.) The coefficients of $\text{adj}(M)$ are polynomials in the coefficients of M . (You've likely seen this in the form of a formula for M^{-1} when there is an inverse.)

Multiplying both sides of (3) on the left by $\text{adj}(\text{Id}_n - A)$, we obtain

$$\det(\text{Id}_n - A) \begin{pmatrix} m_1 \\ \vdots \\ m_n \end{pmatrix} = 0.$$

Multiplying (2) by the adjoint of $(aI_{n \times n} - [b_{ij}]_{ij})$, we get

$$\det(aI_{n \times n} - [b_{ij}]_{ij}) \begin{pmatrix} m_1 \\ \vdots \\ m_n \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}.$$

So $\det(aI - M)$ annihilates A , i.e. $\det(aI - M) = 0$. But expanding the determinant yields an integral equation for a with coefficients in B . \square

We now state the Going-up theorem.

2.2. The Cohen-Seidenberg Going up theorem. — Suppose $\phi : B \rightarrow A$ is an integral extension. Then for any prime ideal $\mathfrak{q} \subset B$, there is a prime ideal $\mathfrak{p} \subset A$ such that $\mathfrak{p} \cap B = \mathfrak{q}$.

Although this is a theorem in algebra, the name can be interpreted geometrically: the theorem asserts that the corresponding morphism of schemes is surjective, and that “above” every prime \mathfrak{q} “downstairs”, there is a prime \mathfrak{p} “upstairs”, see Figure 2. (For this reason, it is often said that \mathfrak{p} is “above” \mathfrak{q} if $\mathfrak{p} \cap B = \mathfrak{q}$.)

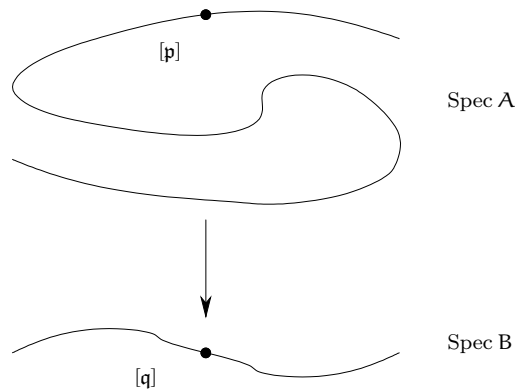


FIGURE 2. A picture of the Going-up theorem

2.E. EXERCISE (REALITY CHECK). The morphism $k[t] \rightarrow k[t]_{(t)}$ is not integral, as $1/t$ satisfies no monic polynomial with coefficients in $k[t]$. Show that the conclusion of the Going-up theorem 2.2 fails.

*Proof of the Cohen-Seidenberg Going-Up theorem 2.2 **. This proof is eminently readable, but could be skipped on first reading. We start with an exercise.

2.F. EXERCISE. Show that the special case where A is a field translates to: if $B \subset A$ is a subring with A integral over B , then B is a field. Prove this. (Hint: all you need to do is show that all nonzero elements in B have inverses in B . Here is the start: If $b \in B$, then $1/b \in A$, and this satisfies some integral equation over B .)

Proof of the Going-Up Theorem 2.2. We first make a reduction: by localizing at \mathfrak{q} , so we can assume that (B, \mathfrak{q}) is a local ring.

Then let \mathfrak{p} be any *maximal* ideal of A . We will see that $\mathfrak{p} \cap B = \mathfrak{q}$. Consider the following diagram.

$$\begin{array}{ccc}
 A & \longrightarrow & A/\mathfrak{p} & \text{field} \\
 \uparrow & & \uparrow & \\
 B & \longrightarrow & B/(\mathfrak{p} \cap B) &
 \end{array}$$

By the Exercise above, the lower right is a field too, so $\mathfrak{p} \cap B$ is a maximal ideal, hence \mathfrak{q} . □

2.G. IMPORTANT BUT STRAIGHTFORWARD EXERCISE (SOMETIMES ALSO CALLED THE GOING-UP THEOREM). Show that if $\mathfrak{q}_1 \subset \mathfrak{q}_2 \subset \cdots \subset \mathfrak{q}_n$ is a chain of prime ideals of B , and $\mathfrak{p}_1 \subset \cdots \subset \mathfrak{p}_m$ is a chain of prime ideals of A such that \mathfrak{p}_i “lies over” \mathfrak{q}_i (and $m < n$), then the second chain can be extended to $\mathfrak{p}_1 \subset \cdots \subset \mathfrak{p}_n$ so that this remains true.

This version of the Going-up Theorem has an important consequence.

2.H. IMPORTANT EXERCISE. Show that if $f : \text{Spec } A \rightarrow \text{Spec } B$ corresponds to an integral extension of rings, then $\dim \text{Spec } A = \dim \text{Spec } B$. (Hint: show that a chain of prime ideals downstairs gives a chain upstairs, by the previous exercise, of the same length. Conversely, a chain upstairs gives a chain downstairs. We need to check that no two elements of the chain upstairs goes to the same element $[\mathfrak{q}] \in \text{Spec } B$ of the chain downstairs. As integral extensions are well-behaved by localization and quotients of $\text{Spec } B$ (Exercise 2.B), we can replace B by $B_{\mathfrak{q}}/\mathfrak{q}B_{\mathfrak{q}}$ (and A by $A \otimes_B (B_{\mathfrak{q}}/\mathfrak{q}B_{\mathfrak{q}})$). Thus we can assume B is a field. Hence we must show that if $\phi : k \rightarrow A$ is an integral extension, then $\dim A = 0$. Outline of proof: Suppose $\mathfrak{p} \subset \mathfrak{m}$ are two prime ideals of \mathfrak{p} . Mod out by \mathfrak{p} , so we can assume that A is a domain. I claim that any non-zero element is invertible: Say $x \in A$, and $x \neq 0$. Then the minimal monic polynomial for x has non-zero constant term. But then x is invertible — recall the coefficients are in a field.)

The trick in the proof of Proposition 2.1 is very handy, and can be used to quickly prove Nakayama's lemma. This name is used for several different but related results. Nakayama isn't especially closely related to dimension, but we may as well prove it while the trick is fresh in our minds.

3.1. Nakayama's Lemma version 1. — Suppose A is a ring, I an ideal of A , and M is a finitely-generated A -module. Suppose $M = IM$. Then there exists an $a \in A$ with $a \equiv 1 \pmod{I}$ with $aM = 0$.

Proof. Say M is generated by m_1, \dots, m_n . Then as $M = IM$, we have $m_i = \sum_j a_{ij}m_j$ for some $a_{ij} \in I$. Thus

$$(3) \quad (\text{Id}_n - A) \begin{pmatrix} m_1 \\ \vdots \\ m_n \end{pmatrix} = 0$$

where Id_n is the $n \times n$ identity matrix in A , and $A = (a_{ij})$. Multiplying both sides of (3) on the left by $\text{adj}(\text{Id}_n - A)$, we obtain

$$\det(\text{Id}_n - A) \begin{pmatrix} m_1 \\ \vdots \\ m_n \end{pmatrix} = 0.$$

But when you expand out $\det(\text{Id}_n - A)$, you get something that is $1 \pmod{I}$. □

Here is why you care: Suppose I is contained in all maximal ideals of A . (The intersection of all the maximal ideals is called the *Jacobson radical*, but we won't use this phrase. For comparison, recall that the nilradical was the intersection of the *prime ideals* of A .) Then I claim that any $a \equiv 1 \pmod{I}$ is invertible. For otherwise $(a) \neq A$, so the ideal (a) is contained in some maximal ideal \mathfrak{m} — but $a \equiv 1 \pmod{\mathfrak{m}}$, contradiction. Then as a is invertible, we have the following.

3.2. Nakayama's Lemma version 2. — Suppose A is a ring, I an ideal of A contained in all maximal ideals, and M is a finitely-generated A -module. (The most interesting case is when A is a local ring, and I is the maximal ideal.) Suppose $M = IM$. Then $M = 0$.

3.A. EXERCISE (NAKAYAMA'S LEMMA VERSION 3). Suppose A is a ring, and I is an ideal of A contained in all maximal ideals. Suppose M is a finitely generated A -module, and $N \subset M$ is a submodule. If $N/IN \rightarrow M/IM$ an isomorphism, then $M = N$. (This can be useful, although it won't come up again for us.)

3.B. IMPORTANT EXERCISE (NAKAYAMA'S LEMMA VERSION 4). Suppose (A, \mathfrak{m}) is a local ring. Suppose M is a finitely-generated A -module, and $f_1, \dots, f_n \in M$, with (the images

of f_1, \dots, f_n generating M/mM . Then f_1, \dots, f_n generate M . (In particular, taking $M = m$, if we have generators of m/m^2 , they also generate m .)

3.C. UNIMPORTANT EXERCISE (NAKAYAMA'S LEMMA VERSION 5). Prove Nakayama version 1 (Lemma 3.1) without the hypothesis that M is finitely generated, but with the hypothesis that $I^n = 0$ for some n . (This argument does *not* use the trick.) This result is quite useful, although we won't use it.

3.D. IMPORTANT EXERCISE THAT WE WILL USE SOON. Suppose S is a subring of a ring A , and $r \in A$. Suppose there is a faithful $S[r]$ -module M that is finitely generated as an S -module. Show that r is integral over S . (Hint: look carefully at the proof of Nakayama's Lemma version 1, and change a few words.)

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