FOUNDATIONS OF ALGEBRAIC GEOMETRY CLASS 13

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CONTENTS

We now define a bunch of types of morphisms. (These notes include some topics discussed the previous class.)

1. SOME TYPES OF MORPHISMS: QUASICOMPACT AND QUASISEPARATED; OPEN IMMERSION; AFFINE, FINITE, CLOSED IMMERSION; LOCALLY CLOSED IMMERSION

In this section, we'll give some analogues of open subsets, closed subsets, and locally closed subsets. This will also give us an excuse to define affine and finite morphisms (closed immersions are a special case). It will also give us an excuse to define some important special closed immersions, in the next section. In section after that, we'll define some more types of morphisms.

1.1. Quasicompact and quasiseparated morphisms.

A morphism $f : X \to Y$ is **quasicompact** if for every open affine subset U of Y, $f^{-1}(U)$ is quasicompact. Equivalently, the preimage of any quasicompact open subset is quasicompact. We will like this notion because (i) we know how to take the maximum of a finite set of numbers, and (ii) most reasonable schemes will be quasicompact.

1.A. EASY EXERCISE. Show that the composition of two quasicompact morphisms is quasicompact.

1.B. EXERCISE. Show that any morphism from a Noetherian scheme is quasicompact.

1.C. EXERCISE (QUASICOMPACTNESS IS AFFINE-LOCAL ON THE TARGET). Show that a morphism $f : X \to Y$ is quasicompact if there is cover of Y by open affine sets U_i such that $f^{-1}(U_i)$ is quasicompact. (Hint: easy application of the affine communication lemma!)

Along with quasicompactness comes the weird notion of quasiseparatedness. A morphism $f : X \to Y$ is **quasiseparated** if for every open affine subset U of Y, $f^{-1}(U)$ is a quasiseparated scheme. This will be a useful hypothesis in theorems (in conjunction with quasicompactness), and that various interesting kinds of morphisms (locally Noetherian source, affine, separated, see Exercise 1.D, Exercise 1.J, and an exercise next quarter resp.) are quasiseparated, and this will allow us to state theorems more succinctly.

1.D. EXERCISE. Show that any morphism from a locally Noetherian scheme is quasiseparated. (Hint: locally Noetherian schemes are quasiseparated.) Thus those readers working only with Noetherian schemes may take this as a standing hypothesis.

1.E. EASY EXERCISE. Show that the composition of two quasiseparated morphisms is quasiseparated.

1.F. EXERCISE (QUASISEPARATEDNESS IS AFFINE-LOCAL ON THE TARGET). Show that a morphism $f : X \to Y$ is quasiseparated if there is cover of Y by open affine sets U_i such that $f^{-1}(U_i)$ is quasiseparated. (Hint: easy application of the affine communication lemma!)

Following Grothendieck's philosophy of thinking that the important notions are properties of morphisms, not of objects, we can restate the definition of quasicompact (resp. quasiseparated) scheme as a scheme that is quasicompact (resp. quasiseparated) over the final object $\operatorname{Spec} \mathbb{Z}$ in the category of schemes.

1.2. Open immersions.

An **open immersion of schemes** is defined to be an open immersion as ringed spaces. In other words, a morphism $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ of schemes is an open immersion if f factors as

$$(X, \mathcal{O}_X) \xrightarrow[\sim]{g} (U, \mathcal{O}_Y|_U) \xrightarrow{h} (Y, \mathcal{O}_Y)$$

where g is an isomorphism, and $U \hookrightarrow Y$ is an inclusion of an open set. It is immediate that isomorphisms are open immersions. We say that $(U, \mathcal{O}_Y|_U)$ is an **open subscheme** of $(Y, \mathcal{O}_Y)$, and often sloppily say that $(X, \mathcal{O}_X)$ is an open subscheme of $(Y, \mathcal{O}_Y)$.

1.G. EXERCISE. Suppose $i : U \to Z$ is an open immersion, and $f : Y \to Z$ is any morphism. Show that $U \times_Z Y$ exists. (Hint: I'll even tell you what it is: $(f^{-1}(U), \mathcal{O}_Y|_{f^{-1}(U)})$.)

1.H. EASY EXERCISE. Show that open immersions are monomorphisms.

1.I. EASY EXERCISE. Suppose $f : X \to Y$ is an open immersion. Show that if Y is locally Noetherian, then X is too. Show that if Y is Noetherian, then X is too. However, show that if Y is quasicompact, X need not be. (Hint: let Y be affine but not Noetherian.)

"Open immersions" are scheme-theoretic analogues of open subsets. "Closed immersions" are scheme-theoretic analogues of closed subsets, but they are of a quite different flavor, as we'll see soon.

1.3. Affine morphisms.

A morphism $f : X \to Y$ is **affine** if for every affine open set U of Y, $f^{-1}(U)$ is an affine scheme. We have immediately that affine morphisms are quasicompact.

1.J. FAST EXERCISE. Show that affine morphisms are quasiseparated. (Hint: affine schemes are quasiseparated, an earlier exercise.)

1.4. *Proposition (the property of "affineness" is affine-local on the target).* — *A morphism* $f : X \to Y$ *is affine if there is a cover of* Y *by affine open sets* U *such that* $f^{-1}(U)$ *is affine.*

For part of the proof, it will be handy to have a lemma.

1.5. *Lemma.* — *If* X *is a quasicompact quasiseparated scheme and* $s \in \Gamma(X, \mathcal{O}_X)$, *then the natural map* $\Gamma(X, \mathcal{O}_X)_s \to \Gamma(X_s, \mathcal{O}_X)$ *is an isomorphism.*

A brief reassuring comment on the "quasicompact quasiseparated" hypothesis: This just means that X can be covered by a finite number of affine open subsets, any two of which have intersection also covered by a finite number of affine open subsets. The hypothesis applies in lots of interesting situations, such as if X is affine or Noetherian.

Proof. Cover X with finitely many affine open sets $U_i = \operatorname{Spec} A_i$. Let $U_{ij} = U_i \cap U_j$. Then

$$\Gamma(X, \mathcal{O}_X) \to \prod_i A_i \rightrightarrows \prod_{i,j} \Gamma(U_{ij}, \mathcal{O}_X)$$

is exact. Localizing at s gives

$$\Gamma(X, \mathcal{O}_X)_s \to \left(\prod_i A_i \right)_s \rightrightarrows \left(\prod_{i,j} \Gamma(U_{ij}, \mathcal{O}_X) \right)_s$$

As localization commutes with finite products,

$$\Gamma(X, \mathcal{O}_X)_s \to \prod_i (A_i)_{s_i} \rightrightarrows \prod_{i,j} \Gamma(U_{ij}, \mathcal{O}_X)_s$$

is exact, where the global function s induces functions $s_i \in A_i$. If $\Gamma(U_{ij}, \mathcal{O}_X)_s \cong \Gamma((U_{ij})_s, \mathcal{O}_X)$, then it is clear that $\Gamma(X, \mathcal{O}_X)_s$ are the sections over X_s. Note that U_{ij} are quasicompact, by the quasiseparatedness hypothesis, and also quasiseparated, as open subsets of quasiseparated schemes are quasiseparated. Therefore we can reduce to the case where $X \subseteq \operatorname{Spec} A$

is a (quasicompact quasiseparated) open subset of an affine scheme. Then $U_{ij} = \operatorname{Spec} A_{f_i f_j}$ is affine and $\Gamma(U_{ij}, \mathcal{O}_x)_s = \Gamma((U_{ij})_s, \mathcal{O}_X)$ so the same exact sequence implies the result. $\quad\square$

Proof of Proposition 1.4. As usual, we use the Affine Communication Lemma. We check our two criteria. First, suppose $f : X \to Y$ is affine over $\operatorname{Spec} B$, i.e. $f^{-1}(\operatorname{Spec} B) = \operatorname{Spec} A$. Then $f^{-1}(\operatorname{Spec} B_s) = \operatorname{Spec} A_{f^\# s}$.

Second, suppose we are given $f : X \to \operatorname{Spec} B$ and $(f_1, \ldots, f_n) = B$ with X_{f_i} affine $(\operatorname{Spec} A_i$, say). We wish to show that X is affine too. X is quasi-compact (as it is covered by n affine open sets). Let $t_i \in \Gamma(X, \mathcal{O}_X)$ be the pullback of the sections $s_i \in B$. The morphism f factors as $h \circ g$ where $g : X \to \operatorname{Spec} \Gamma(X, \mathcal{O}_X)$ and $h : \operatorname{Spec} \Gamma(X, \mathcal{O}_X) \to \operatorname{Spec} B$ are the natural maps. Then Lemma 1.5 implies that $g|_{f^{-1}(\operatorname{Spec} B_{s_i})} : X_{t_i} \to \operatorname{Spec} \Gamma(X, \mathcal{O}_X)_{t_i}$ are isomorphisms. Therefore, g is an isomorphism and X is affine. $\quad\square$

1.6. Finite morphisms.

An affine morphism $f : X \to Y$ is **finite** if for every affine open set $\operatorname{Spec} B$ of Y, $f^{-1}(\operatorname{Spec} B)$ is the spectrum of an B-algebra that is a finitely-generated B-*module*. Warning about terminology (finite vs. finitely-generated): Recall that if we have a ring homomorphism $A \to B$ such that B is a finitely-generated A-*module* then we say that B is a **finite** A-algebra. This is stronger than being a finitely-generated A-*algebra*.

By definition, finite morphisms are affine.

1.K. EXERCISE (THE PROPERTY OF FINITENESS IS AFFINE-LOCAL ON THE TARGET). Show that a morphism $f : X \to Y$ is finite if there is a cover of Y by affine open sets $\operatorname{Spec} A$ such that $f^{-1}(\operatorname{Spec} A)$ is the spectrum of a finite A-algebra.

1.L. EASY EXERCISE. Show that the composition of two finite morphisms is also finite.

We now give four examples of finite morphisms, to give you some feeling for how finite morphisms behave. In each example, you'll notice two things. In each case, the maps are always finite-to-one. We'll verify this in Exercise 3.E. You'll also notice that the morphisms are closed, i.e. the image of closed sets are closed. This argument uses the going-up theorem, and we'll verify this when we discuss that. Intuitively, you should think of finite as being closed plus finite fibers, although this isn't quite true. We'll make this precise later.

Example 1: Branched covers. If $p(t) \in k[t]$ is a non-zero polynomial, then $\operatorname{Spec} k[t] \to \operatorname{Spec}[u]$ given by $u \mapsto p(t)$ is a finite morphism, see Figure 1.

Example 2: Closed immersions (to be defined soon, in §1.8). The morphism $\operatorname{Spec} k \to \operatorname{Spec} k[t]$ given by $t \mapsto 0$ is a finite morphism, see Figure 2.
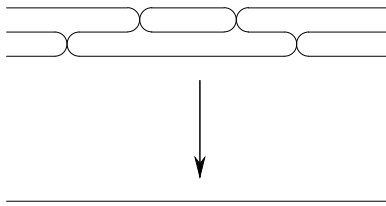
FIGURE 1. The "branched cover" of $\mathbb{A}^1$ given by $u \mapsto p(t)$ is finite
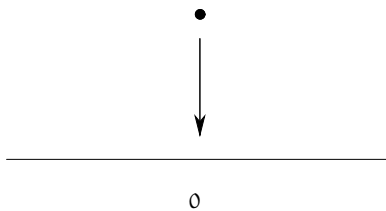

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FIGURE 2. The "closed immersion" $\operatorname{Spec} k \to \operatorname{Spec} k[t]$ is finite

Example 3: Normalization (to be defined later). The morphism $\operatorname{Spec} k[t] \to \operatorname{Spec} k[x,y]/(y^2 - x^2 - x^3)$ given by $(x,y) \mapsto (t^2 - 1, t^3 - t)$ (check that this is a well-defined ring map!) is a finite morphism, see Figure 3.

1.M. IMPORTANT EXERCISE (EXAMPLE 4, FINITE MORPHISMS TO $\operatorname{Spec} k$). Show that if $X \to \operatorname{Spec} k$ is a finite morphism, then X is a discrete finite union of points, each with residue field a finite extension of k, see Figure 4. (An example is $\operatorname{Spec} \mathbb{F}_8 \times \mathbb{F}_4[x,y]/(x^2, y^4) \times \mathbb{F}_4[t]/t^9 \times \mathbb{F}_2 \to \operatorname{Spec} F_2$.)

1.7. *Example.* The natural map $\mathbb{A}^2 - \{(0,0)\} \to \mathbb{A}^2$ is an open immersion, and has finite fibers, but is not affine (as $\mathbb{A}^2 - \{(0,0)\}$ isn't affine) and hence not finite.

1.8. Closed immersions and closed subschemes.

Just as open immersions (the scheme-theoretic version of open set) are locally modeled on open sets $U \subset Y$, the analogue of closed subsets also has a local model. This was foreshadowed by our understanding of closed subsets of $\operatorname{Spec} B$ as roughly corresponding to ideals. If $I \subset B$ is an ideal, then $\operatorname{Spec} B/I \hookrightarrow \operatorname{Spec} B$ is a morphism of schemes, and this is our prototypical example of a closed immersion.

A morphism $f : X \to Y$ is a **closed immersion** if it is an affine morphism, and for each open subset $\operatorname{Spec} B \subset Y$, with $f^{-1}(\operatorname{Spec} B) \cong \operatorname{Spec} A$, $B \to A$ is a surjective map (i.e. of the form $B \to B/I$, our desired local model). We often say that X is a **closed subscheme** of Y.
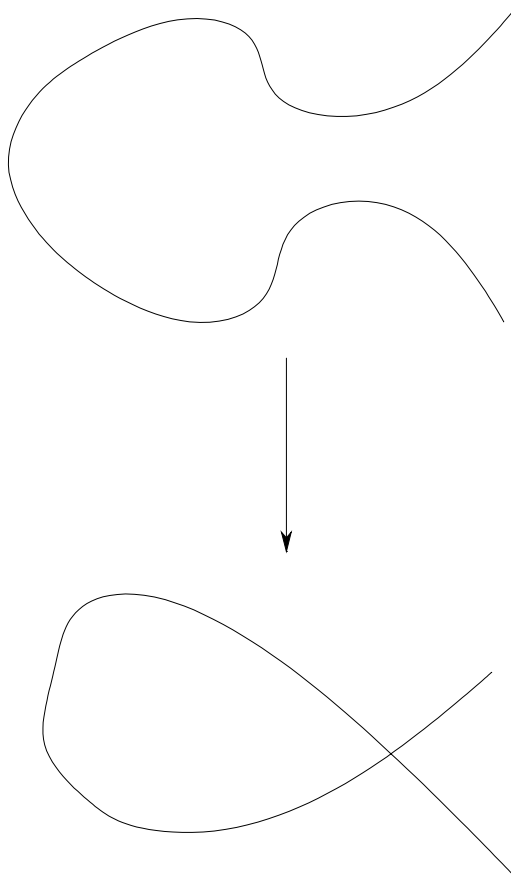
FIGURE 3. The "normalization" $\operatorname{Spec} k[t] \to \operatorname{Spec} k[x, y]/(y^2 - x^2 - x^3)$ given by $(x, y) \mapsto (t^2 - 1, t^3 - t)$ is finite

FIGURE 4. A picture of a finite morphism to $\operatorname{Spec} k$. Notice that bigger fields are written as bigger dots. [I'd like to add some fuzz to some of these points at some point.]

1.N. EASY EXERCISE. Show that closed immersions are finite.

1.O. EXERCISE. Show that the property of being a closed immersion is affine-local on the target.

A closed immersion $f : X \hookrightarrow Y$ determines an *ideal sheaf* on Y, as the kernel $\mathcal{I}_{X/Y}$ of the map of $\mathcal{O}_Y$-modules

$$\mathcal{O}_Y \to f_* \mathcal{O}_X$$

(An **ideal sheaf** on Y is what it sounds like: it is a sheaf of ideals. It is a sub-$\mathcal{O}_Y$-module $\mathcal{I} \hookrightarrow \mathcal{O}_Y$. On each open subset, it gives an ideal $\mathcal{I}(U) \hookrightarrow \mathcal{O}_Y(U)$.) We thus have an exact sequence $0 \to \mathcal{I}_{X/Y} \to \mathcal{O}_Y \to f_* \mathcal{O}_X \to 0$.

1.P. IMPORTANT EXERCISE: A USEFUL CRITERION FOR WHEN IDEALS IN AFFINE OPEN SETS DEFINE A CLOSED SUBSCHEME. It will be convenient (for example in §2) to define certain closed subschemes of Y by defining on any affine open subset $\operatorname{Spec} B$ of Y an ideal $I_B \subset B$. Show that these $\operatorname{Spec} B/I_B \hookrightarrow \operatorname{Spec} B$ glue together to form a closed subscheme precisely if for each affine open subset $\operatorname{Spec} B \hookrightarrow Y$ and each $f \in B$, $I_{(B_f)} = (I_B)_f$.

Warning: you might hope that closed subschemes correspond to ideal sheaves of $\mathcal{O}_Y$. Sadly not every ideal sheaf arises in this way. Here is an example.

1.Q. UNIMPORTANT EXERCISE. Let $X = \operatorname{Spec} k[x]_{(x)}$, the germ of the affine line at the origin, which has two points, the closed point and the generic point η. Define $\mathcal{I}(X) = \{0\} \subset \mathcal{O}_X(X) = k[x]_{(x)}$, and $\mathcal{I}(\eta) = k(x) = \mathcal{O}_X(\eta)$. Show that this sheaf of ideals does not correspond to a closed subscheme (see Exercise 1.P).

We will see later that closed subschemes correspond to *quasicoherent* sheaves of ideals; the mathematical content of this statement will turn out to be precisely Exercise 1.P.

1.R. IMPORTANT EXERCISE. (a) In analogy with closed subsets, define the notion of a finite union of closed subschemes of X, and an arbitrary intersection of closed subschemes. (b) Describe the scheme-theoretic intersection of $(y - x^2)$ and y in $\mathbb{A}^2$. See Figure 5 for a picture. (For example, explain informally how this corresponds to two curves meeting at a single point with multiplicity 2 — notice how the 2 is visible in your answer. Alternatively, what is the non-reducedness telling you — both its "size" and its "direction"?) Describe the scheme-theoretic union.
(c) Describe the scheme-theoretic intersection of $(y^2 - x^2)$ and y in $\mathbb{A}^2$. Draw a picture. (Are you surprised? Did you expect the intersection to be multiplicity one or multiplicity two?) Hence show that if X, Y, and Z are closed subschemes of W, then $(X \cap Z) \cup (Y \cap Z) \neq (X \cup Y) \cap Z$ in general.
(d) Show that the underlying set of a finite union of closed subschemes is the finite union of the underlying sets, and similarly for arbitrary intersections.

1.S. IMPORTANT EXAMPLE THAT SHOULD HAVE BEEN DONE EARLIER. We now make a preliminary definition of projective n-space $\mathbb{P}^n_k$, by gluing together $n + 1$ open sets each isomorphic to $\mathbb{A}^n_k$. Judicious choice of notation for these open sets will make our life easier. Our motivation is as follows. In the construction of $\mathbb{P}^1$ above, we thought of points of projective space as $[x_0; x_1]$, where (x_0, x_1) are only determined up to scalars, i.e. (x_0, x_1)
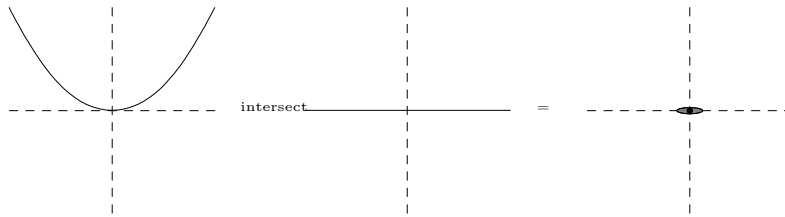
FIGURE 5. The scheme-theoretic intersection of the parabola $y = x^2$ and the x-axis is a non-reduced scheme (with fuzz in the x-direction)

is considered the same as $(\lambda x_0, \lambda x_1)$. Then the first patch can be interpreted by taking the locus where $x_0 \neq 0$, and then we consider the points $[1; t]$, and we think of t as x_1/x_0; even though x_0 and x_1 are not well-defined, x_1/x_0 *is*. The second corresponds to where $x_1 \neq 0$, and we consider the points $[u; 1]$, and we think of u as x_0/x_1. It will be useful to instead use the notation $x_{1/0}$ for t and $x_{0/1}$ for u.

For $\mathbb{P}^n$, we glue together $n+1$ open sets, one for each of $i = 0, \ldots, n+1$. The ith open set U_i will have co-ordinates $x_{0/i}, \ldots, x_{(i-1)/i}, x_{(i+1)/i}, \ldots, x_{n/i}$. It will be convenient to write this as

$$\operatorname{Spec} k[x_{0/i}, x_{1/i}, \ldots, x_{n/i}]/(x_{i/i} - 1)$$

(so we have introduced a "dummy variable" $x_{i/i}$ which we set to 1). We glue the distinguished open set $D(x_{j/i})$ of U_i to the distinguished open set $D(x_{i/j})$ of U_j, by identifying these two schemes by describing the identification of rings

$$\operatorname{Spec} k[x_{0/i}, x_{1/i}, \ldots, x_{n/i}, 1/x_{j/i}]/(x_{i/i} - 1) \cong$$

$$\operatorname{Spec} k[x_{0/j}, x_{1/j}, \ldots, x_{n/j}, 1/x_{i/j}]/(x_{j/j} - 1)$$

via $x_{k/i} = x_{k/j}/x_{i/j}$ and $x_{k/j} = x_{k/i}/x_{j/i}$ (which implies $x_{i/j}x_{j/i} = 1$). We need to check that this gluing information agrees over triple overlaps.

1.T. EXERCISE. Check this, as painlessly as possible. (Possible hint: the triple intersection is affine; describe the corresponding ring.)

Note that our definition doesn't use the fact that k is a field. Hence we may as well define $\mathbb{P}^n_A$ for any *ring* A. This will be useful later.

1.9. *Example: Closed immersions of projective space* $\mathbb{P}^n_A$. Consider the definition of projective space $\mathbb{P}^n_A$ given above. Any *homogeneous* polynomial f in $x_0, \ldots, x_n$ defines a closed subscheme. (Thus even though $x_0, \ldots, x_n$ don't make sense as functions, their vanishing locus still makes sense.) On open set U_i, the closed subscheme is $f(x_{0/i}, \ldots, x_{n/i})$, which we think of as $f(x_0, \ldots, x_n)/x_i^{\deg f}$. On the overlap

$$U_i \cap U_j = \operatorname{Spec} A[x_{0/i}, \ldots, x_{n/i}, x_{j/i}^{-1}]/(x_{i/i} - 1),$$

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these functions on U_i and U_j don't exactly agree, but they agree up to a non-vanishing scalar, and hence cut out the same subscheme of $U_i \cap U_j$:

$$f(x_{0/i}, \ldots, f_{n/i}) = x_{j/i}^{\deg f} f(x_{0/j}, \ldots, x_{n/j}).$$

Thus by intersecting such closed subschemes, we see that any collection of homogeneous polynomials in $A[x_0, \ldots, x_n]$ cut out a closed subscheme of $\mathbb{P}_A^n$. We could take this as a provisional definition of a *projective A-scheme* (or a projective scheme over A). (We'll give a better definition in the next Chapter.)

Notice: piggybacking on the annoying calculation that $\mathbb{P}^n$ consists of $n+1$ pieces glued together nicely is the fact that any closed subscheme of $\mathbb{P}^n$ cut out by a bunch of homogeneous polynomials consists of $n+1$ pieces glued together nicely.

Notice also that this subscheme is not in general cut out by a single global function on $\mathbb{P}_A^n$. For example, if $A = k$, there *are* no nonconstant global functions. We take this opportunity to introduce some related terminology. A closed subscheme is **locally principal** if on each open set in a small enough open cover it is cut out by a single equation. Thus each homogeneous polynomial in $n + 1$ variables defines a locally principal closed subscheme. (Warning: one can check this on a fine enough affine open cover, but this is not an affine-local condition! We will see an example in the next day's notes — one $\mathbb{P}^2$ minus a conic, consider a line.) A case that will be important later is when the ideal sheaf is not just locally generated by a function, but is generated by a function that is not a zero-divisor. In this case (once we have defined our terms) we will call this an *invertible ideal sheaf*, and the closed subscheme will be an *effective Cartier divisor*.

A closed subscheme cut out by a single (homogeneous) equation is called a **hypersurface** in $\mathbb{P}_k^n$. The **degree of a hypersurface** is the degree of the polynomial. (Implicit in this is that this notion can be determined from the subscheme itself; we haven't yet checked this.) A hypersurface of degree 1 (resp. degree 2, 3, …) is called a **hyperplane** (resp. **quadric, cubic, quartic, quintic, sextic, septic, octic, …hypersurface**). If $n = 2$, a degree 1 hypersurface is called a **line**, and a degree 2 hypersurface is called a **conic curve**, or a **conic** for short. If $n = 3$, a hypersurface is called a **surface**.) (In a couple of weeks, we will justify the terms *curve* and *surface*.)

1.U. EXERCISE. (a) Show that $wz = xy, x^2 = wy, y^2 = xz$ describes an irreducible curve in $\mathbb{P}_k^3$. This curve is called the *twisted cubic*. The twisted cubic is a good non-trivial example of many things, so it you should make friends with it as soon as possible. (b) Show that the twisted cubic is isomorphic to $\mathbb{P}_k^1$.

1.V. UNIMPORTANT EXERCISE. The usual definition of a closed immersion is a morphism $f : X \to Y$ such that f induces a homeomorphism of the underlying topological space of Y onto a closed subset of the topological space of X, and the induced map $f^\# : \mathcal{O}_X \to f_* \mathcal{O}_Y$ of sheaves on X is surjective. Show that this definition agrees with the one given above. (To show that our definition involving surjectivity on the level of affine open sets implies this definition, you can use the fact that surjectivity of a morphism of sheaves can be checked on a base, which you can verify yourself.)

1.10. ⋆ **A fun example.** The affine-locality of affine morphisms (Proposition 1.4) has some non-obvious consequences, as shown in the next exercise.

1.W. EXERCISE. Suppose X is an affine scheme, and Y is a closed subscheme locally cut out by one equation (e.g. if Y is an effective Cartier divisor). Show that $X - Y$ is affine. (This is clear if Y is globally cut out by one equation f; then if $X = \operatorname{Spec} A$ then $Y = \operatorname{Spec} A_f$. However, Y is not always of this form.)

1.11. *Example.* Here is an explicit consequence. We showed earlier that on the cone over the smooth quadric surface $\operatorname{Spec} k[w, x, y, z]/(wz - xy)$, the cone over a ruling $w = x = 0$ is not cut out scheme-theoretically by a single equation, by considering Zariski-tangent spaces. We now show that it isn't even cut out set-theoretically by a single equation. For if it were, its complement would be affine. But then the closed subscheme of the complement cut out by $y = z = 0$ would be affine. But this is the scheme $y = z = 0$ (also known as the wx-plane) minus the point $w = x = 0$, which we've seen is non-affine. (For comparison, on the cone $\operatorname{Spec} k[x, y, z]/(xy - z^2)$, see Figure 6, the ruling $x = z = 0$ is not cut out scheme-theoretically by a single equation, but it *is* cut out set-theoretically by $x = 0$.) Verify all this! (Hint: Use Exercise 1.4.)
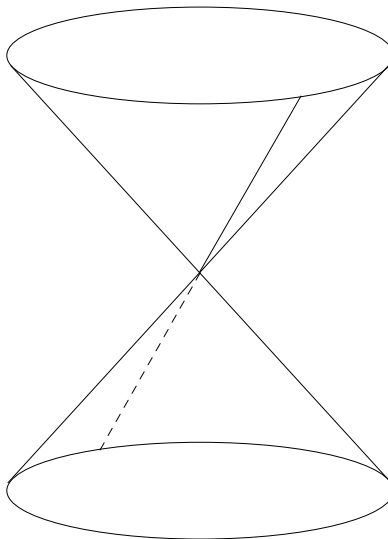

FIGURE 6. $V(x, z) \subset \operatorname{Spec} k[x, y, z]/(xy - z^2)$ is a ruling on a cone

We have now defined the analog of open subsets and closed subsets in the land of schemes. Their definition is slightly less "symmetric" than in the usual topological setting: the "complement" of a closed subscheme is a unique open subscheme, but there are many "complementary" closed subschemes to a given open subscheme in general. (We'll soon define one that is "best", that has a reduced structure, §2.6.)

1.12. Locally closed immersions and locally closed subschemes.

Now that we have defined analogs of open and closed subsets, it is natural to define the analog of locally closed subsets. Recall that locally closed subsets are intersections of open subsets and closed subsets. Hence they are closed subsets of open subsets, or equivalently open subsets of closed subsets. That equivalence will be a little subtle in the land of schemes.

We say a morphism $X \to Y$ is a **locally closed immersion** if it can factored into $X \xrightarrow{f} Z \xrightarrow{g} Y$ where f is a closed immersion and g is an open immersion. (Warning: The term *immersion* is often used instead of *locally closed immersion*, but this is unwise terminology, as the differential geometric notion of immersion is closer to what algebraic geometers call *unramified*, which we'll define next quarter. The algebro-geometric notion of locally closed immersion is closest to the differential geometric notion of *embedding*.) It is often said that X is a **locally closed subscheme** of Y.

For example, $\operatorname{Spec} k[t, t^{-1}] \to \operatorname{Spec} k[x, y]$ where $(x, y) \mapsto (t, 0)$ is a locally closed immersion (see Figure 7).
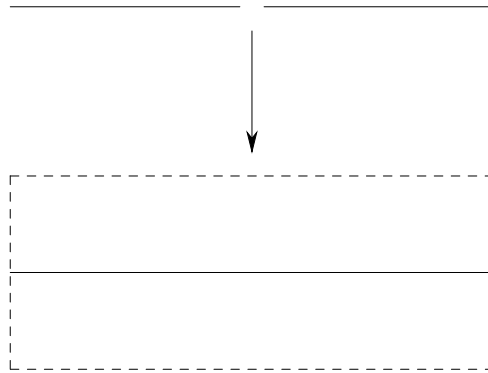

FIGURE 7. The locally closed immersion $\operatorname{Spec} k[t, t^{-1}] \to k[x, y]$ ($t \mapsto (t, 0) = (x, y)$, i.e. $(x, y) \to (t, 0)$)

We can make sense of finite intersections of locally closed immersions.

Clearly a open subscheme U of a closed subscheme V of X can be interpreted as a closed subscheme of an open subscheme: as the topology on V is induced from the topology on X, the underlying set of U is the intersection of some open subset U' on X with V. We can take $V' = V \cap U$, and then $V' \to U'$ is a closed immersion, and $U' \to X$ is an open immersion.

It is less clear that a closed subscheme V' of an open subscheme U' can be expressed as an open subscheme U of a closed subscheme V. In the category of topological spaces, we would take V as the closure of V', so we are now motivated to define the analogous construction, which will give us an excuse to introduce several related ideas, in the next section. We will then resolve this issue in good cases (e.g. if X is Noetherian) in Exercise 2.D.

2. Constructions related to "smallest closed subschemes": scheme-theoretic image, scheme-theoretic closure, induced reduced subscheme, and the reduction of a scheme

We now define a series of notions that are all of the form "the smallest closed subscheme such that something or other is true". One example will be the notion of scheme-theoretic closure of a locally closed immersion, which will allow us to interpret locally closed immersions in three equivalent ways (open subscheme intersect closed subscheme; open subscheme of closed subscheme; and closed subscheme of open subscheme).

2.1. Scheme-theoretic image.

We start with the notion of scheme-theoretic image. If $f : X \to Y$ is a morphism of schemes, the notion of the image of f as *sets* is clear: we just take the points in Y that are the image of points in X. But if we would like the image as a scheme, then the notion becomes more problematic. (For example, what is the image of $\mathbb{A}^2 \to \mathbb{A}^2$ given by $(x, y) \mapsto (x, xy)$?) We will come back to the notion of image later, but for now we will define the "scheme-theoretic image". This will incorporate the notion that the image of something with non-reduced structure ("fuzz") can also have non-reduced structure.

Definition. Suppose $i : Z \hookrightarrow Y$ is a closed subscheme, giving an exact sequence $0 \to \mathcal{I}_{Z/Y} \to \mathcal{O}_Y \to i_*\mathcal{O}_Z \to 0$. We say that the image of $f : X \to Y$ lies in Z if the composition $\mathcal{I}_{Z/Y} \to \mathcal{O})_Y \to f_*\mathcal{O}_X$ is zero. Informally, locally functions vanishing on Z pull back to the zero function on X. If the image of f lies in two subschemes Z_1 and Z_2, it clearly lies in their intersection $Z_1 \cap Z_2$. We then define the **scheme-theoretic image of** f of f, a closed subscheme on Y, as the "smallest closed subscheme containing the image", i.e. the intersection of all closed subschemes containing the image.

Example 1. Consider $\operatorname{Spec} k[\epsilon]/\epsilon^2 \to \operatorname{Spec} k[x] = \mathbb{A}_k^1$ given by $x \mapsto \epsilon$. Then the scheme-theoretic image is given by $k[x]/x^2$ (the polynomials pulling back to 0 are precisely multiples of x^2). Thus the image of the fuzzy point still has some fuzz.

Example 2. Consider $f : \operatorname{Spec} k[\epsilon]/\epsilon^2 \to \operatorname{Spec} k[x] = \mathbb{A}_k^1$ given by $x \mapsto 0$. Then the scheme-theoretic image is given by $k[x]/x$: the image is reduced. In this picture, the fuzz is "collapsed" by f.

Example 3. Consider $f : \operatorname{Spec} k[t, t^{-1}] = \mathbb{A}^1 - \{0\} \to \mathbb{A}^1 = \operatorname{Spec} k[u]$ given by $u \mapsto t$. Any function $g(u)$ which pulls back to 0 as a function of t must be the zero-function. Thus the scheme-theoretic image is everything. The set-theoretic image, on the other hand, is the distinguished open set $\mathbb{A}^1 - \{0\}$. Thus in not-too-pathological cases, the underlying set of the scheme-theoretic image is not the set-theoretic image. But the situation isn't terrible: the underlying set of the scheme-theoretic image must be closed, and indeed it is the closure of the set-theoretic image. We might imagine that in reasonable cases this will be true, and in even nicer cases, the underlying set of the scheme-theoretic image will be set-theoretic image. We will later see that this is indeed the case.

But we feel obliged to show that pathologies can happen.

Example 4. Let $X = \coprod k[\epsilon_n]/(\epsilon_n^n)$ and $Y = \operatorname{Spec} k[x]$, and define $X \to Y$ by $x \to \epsilon_n$ on the nth component of X. Then if a function $g(x)$ on Y pulls back to 0 on X, then its Taylor expansion is 0 to order n (by examining the pullback to the nth component of X, so $g(x)$ must be 0. Thus the scheme-theoretic image is Y, while the set-theoretic image is easily seen to be just the origin.

This example clearly is weird though, and we can show that in "reasonable circumstances" such pathology doesn't occur. It would be great to compute the scheme-theoretic image affine-locally. On affine open set $\operatorname{Spec} B \subset Y$, define the ideal $I_B \subset B$ of functions which pullback to 0 on X. (Formally, $I_B := \ker(B \to \Gamma(f_*(\mathcal{O}_X), \operatorname{Spec} B)$.) Then if for each such B, and each $g \in B$, $I_B \otimes_B B_g \to I_{B_g}$ is an isomorphism, then we will have defined the pushforward subscheme (see Exercise 1.P). Clearly each function on $\operatorname{Spec} B$ that vanishes when pulled back to $f^{-1}(\operatorname{Spec} B)$ also vanishes when restricted to $D(g)$ and then pulled back to $f^{-1}(D(g))$. So the question is: given a function r/g^n on $D(g)$ that pulls back to $f^{-1}D(g)$, is it true that for some m, $rg^m = 0$ when pulled back to $f^{-1}(\operatorname{Spec} B)$? (i) The answer is clearly yes if $f^{-1}(\operatorname{Spec} B)$ is reduced: we simply take rg. (ii) The answer is also yes if $f^{-1}(\operatorname{Spec} B)$ is affine, say $\operatorname{Spec} A$: if $r' = f^\# r$ and $g' = f^\# g$ in A, then if $r' = 0$ on $D(g')$, then there is an m such that $r'(g')^m = 0$: $r' = 0$ in $D(g')$, which means precisely this fact. (iii) Furthermore, the answer is yes if $f^{-1}(\operatorname{Spec} B)$ is quasicompact: cover $f^{-1}(\operatorname{Spec} B)$ with finitely many affine open sets. For each one there will be some m_i so that $rg^{m_i} = 0$ when pulled back to this open set. Then let $m = \max(m_i)$. (We now see why quasicompactness is our friend!)

In conclusion, we have proved the following theorem.

2.2. *Theorem.* — *Suppose* $f : X \to Y$ *is a morphism of schemes. If X is reduced or f is quasicompact (e.g. if X is Noetherian, Exercise 1.B), then the scheme-theoretic image of f may be computed affine-locally.*

2.3. *Corollary.* — *Under the hypotheses of the previous theorem, the closure of the set-theoretic image of f is the underlying set of the scheme-theoretic image.*

Example 4 above shows that we cannot excise these hypotheses.

Proof. The set-theoretic image is clearly in the underlying set of the scheme-theoretic image. The underlying set of the scheme-theoretic image is closed, so the closure of the set-theoretic image is contained in underlying set of the scheme-theoretic image. On the other hand, if U is the complement of the closure of the set-theoretic image, $f^{-1}(U) = \varnothing$. As under these hypotheses, the scheme theoretic image can be computed locally, the scheme-theoretic image is the empty set on U. $\qquad \square$

We conclude with a few stray remarks.

2.A. EASY EXERCISE. If X is reduced, show that the scheme-theoretic image of $f : X \to Y$ is also reduced.

More generally, you might expect there to be no unnecessary non-reduced structure on the image not forced by non-reduced structure on the source. We make this precise in the locally Noetherian case, when we can talk about associated points.

2.B. ⋆ UNIMPORTANT EXERCISE. If $f : X \to Y$ is a morphism of locally Noetherian schemes, show that the associated points of the image subscheme are a subset of the image of the associated points of X.

2.4. Aside: set-theoretic images can be nice too. I want to say a little more on what the set-theoretic image of a morphism can look like, although we'll hold off before proving these statements. We know that the set-theoretic image can be open (open immersion), and closed (closed immersions), and locally closed (locally closed immersions). But it can be weirder still: consider the example $\mathbb{A}^2 \to \mathbb{A}^2$ given by $(x, y) \mapsto (x, xy)$ mentioned earlier. The image is the plane, minus the y-axis, plus the origin. The image can be stranger still, and indeed if S is *any* subset of a scheme Y, it can be the image of a morphism: let X be the disjoint union of spectra of the residue fields of all the points of S, and let $f : X \to Y$ be the natural map. This is quite pathological, and in fact that if we are in any reasonable situation, the image is essentially no worse than arose in the previous example.

We define a **constructible subset of a Noetherian scheme** to be a subset which belongs to the smallest family of subsets such that (i) every open set is in the family, (ii) a finite intersection of family members is in the family, and (iii) the complement of a family member is also in the family. So for example the image of $(x, y) \mapsto (x, xy)$ is constructible.

Note that if $X \to Y$ is a morphism of schemes, then the preimage of a constructible set is a constructible set.

2.C. EXERCISE. Suppose X is a Noetherian scheme. Show that a subset of X is constructible if and only if it is the finite disjoint union of locally closed subsets.

Then if $f : X \to Y$ is a finite type morphism of Noetherian schemes, the image of any constructible set is constructible. This is *Chevalley's Theorem*, and we will prove it later. We will also have reasonable criteria for when the image is closed.

(For hardened experts only: [EGA 0_{III}.9.1] gives a definition of constructible in more generality. A *constructible subset of a topological space* X is a member of the Boolean algebra generated by open subsets U of X such that the inclusion $U \hookrightarrow X$ is quasicompact. If X is an affine scheme, or more generally quasicompact and quasiseparated, this is equivalent to U being quasicompact. A subset $Z \subset X$ is *locally constructible* if X admits an open covering $\{V_i\}$ such that $Z \cap V_i \subset V_i$ is constructible for each i. If X is quasicompact and quasiseparated, this is the same as $Z \subset X$ being constructible, so if X is a scheme, then it is equivalent to say that $Z \cap V$ is constructible for every affine open set V. The general form of Chevalley's constructibility theorem [EGA IV_1.1.8.4] is that the image of a locally constructible set under a finitely presented map, is also locally constructibility. Thanks to Brian Conrad for this!)

2.5. Scheme-theoretic closure of a locally closed subscheme.

We define the **scheme-theoretic closure** of a locally closed immersion $f : X \to Y$ as the scheme-theoretic image of X.

2.D. EXERCISE. If $X \to Y$ is quasicompact (e.g. if X is Noetherian, Exercise 1.B) or if X is reduced, show that the following three notions are the same. (Hint: Theorem niceschemetheoreticimage.)

   (a) V is an open subscheme of X intersect a closed subscheme of X
   (b) V is an open subscheme of a closed subscheme of X
   (c) V is a closed subscheme of an open subscheme of X.

(Hint: it will be helpful to note that the scheme-theoretic image may be computed on each open subset of the base.)

2.E. UNIMPORTANT EXERCISE USEFUL FOR INTUITION. If $f : X \to Y$ is a locally closed immersion into a locally Noetherian scheme (so X is also locally Noetherian), then the associated points of the scheme-theoretic image are (naturally in bijection with) the associated points of X. (Hint: Exercise 2.B.) Informally, we get no non-reduced structure on the scheme-theoretic closure not "forced by" that on X.

2.6. The induced reduced subscheme structure on a closed subset.

Suppose X^{set} is a closed sub*set* of a scheme Y. Then we can define a canonical scheme structure X on X^{set}, that is reduced. We could describe it as being cut out by those functions whose values are zero at all the points of X^{set}. On affine open subset $\text{Spec } B$ of Y, if the set X^{set} corresponds to the radical ideal $I = I(X^{\text{set}})$, the scheme X corresponds to $\text{Spec } B/I$. We could also consider this construction as an example of a scheme-theoretic image in the following crazy way: let W be the scheme that is a disjoint union of all the points of X^{set}, where the point corresponding to p in X^{set} is Spec of the residue field of $\mathcal{O}_{Y,p}$. Let $f : W \to Y$ be the "canonical" map sending "p to p", and giving an isomorphism on residue fields. Then the scheme structure on X is the scheme-theoretic image of f. A third definition: it is the smallest closed subscheme whose underlying set contains X^{set}.

This construction is called the **induced reduced subscheme structure** on the closed subset X^{set}. (Vague exercise: Make a definition of the induced reduced subscheme structure precise and rigorous to your satisfaction.)

2.F. EXERCISE. Show that the underlying set of the induced reduced subscheme $X \to Y$ is indeed the closed subset X^{set}. Show that X is reduced.

2.7. Reduced version of a scheme.

In the special case where X^{set} all of Y, we obtain a *reduced closed subscheme* $Y^{red} \to Y$, called the **reduction** of Y. On affine open subset $\mathrm{Spec}\,B \hookrightarrow Y$, $Y^{red} \hookrightarrow Y$ corresponds to the nilradical $\mathfrak{N}(B)$ of B. The *reduction* of a scheme is the "reduced version" of the scheme, and informally corresponds to "shearing off the fuzz".

An alternative equivalent definition: on the affine open subset $\mathrm{Spec}\,B \hookrightarrow Y$, the reduction of Y corresponds to the ideal $\mathcal{N}(B) \subset Y$. As for any $f \in B$, $\mathcal{N}(B)_f = \mathcal{N}(B_f)$, by Exercise 1.P this defines a closed subscheme.

2.G. UNIMPORTANT EXERCISE (BUT USEFUL FOR VISUALIZATION). Show that if Y is a locally Noetherian scheme, the "reduced locus" of Y (where $Y^{red} \to Y$ is an isomorphism) is an open subset of Y. (In fact the non-reduced locus is a closure of certain associated points.)

3. MORE FINITENESS CONDITIONS ON MORPHISMS: (LOCALLY) OF FINITE TYPE, QUASIFINITE, (LOCALLY) OF FINITE PRESENTATION

3.1. Morphisms (locally of) finite type.

A morphism $f : X \to Y$ is **locally of finite type** if for every affine open set $\mathrm{Spec}\,B$ of Y, $f^{-1}(\mathrm{Spec}\,B)$ can be covered with open sets $\mathrm{Spec}\,A_i$ so that the induced morphism $B \to A_i$ expresses A_i as a finitely generated B-algebra. By the affine-locality of finite-typeness of B-schemes, this is equivalent to: for *every* affine open set $\mathrm{Spec}\,A_i$ in X, A_i is a finitely generated B-algebra.

A morphism is **of finite type** if it is locally of finite type and quasicompact. Translation: for every affine open set $\mathrm{Spec}\,B$ of Y, $f^{-1}(\mathrm{Spec}\,B)$ can be covered with *a finite number of* open sets $\mathrm{Spec}\,A_i$ so that the induced morphism $B \to A_i$ expresses A_i as a finitely generated B-algebra.

3.A. EXERCISE (THE NOTIONS "LOCALLY OF FINITE TYPE" AND "FINITE TYPE" ARE AFFINE-LOCAL ON THE TARGET). Show that a morphism $f : X \to Y$ is locally of finite type if there is a cover of Y by affine open sets $\mathrm{Spec}\,B_i$ such that $f^{-1}(\mathrm{Spec}\,B_i)$ is locally of finite type over B_i.

3.B. EXERCISE. Show that a morphism $f : X \to Y$ is locally of finite type if for *every* affine open subsets $\mathrm{Spec}\,A \subset X$, $\mathrm{Spec}\,B \subset Y$, with $f(\mathrm{Spec}\,A) \subset \mathrm{Spec}\,B$, A is a finitely generated B-algebra. (Hint: use the affine communication lemma on $f^{-1}(\mathrm{Spec}\,B)$.)

Example: the "structure morphism" $\mathbb{P}^n_A \to \mathrm{Spec}\,A$ is of finite type, as $\mathbb{P}^n_A$ is covered by $n + 1$ open sets of the form $\mathrm{Spec}\,A[x_1, \ldots, x_n]$. More generally, $\mathrm{Proj}\,S_* \to \mathrm{Spec}\,A$ (where $S_0 = A$) is of finite type.

More generally still: our earlier definition of schemes of "finite type over k" (or "finite type k-schemes") is now a special case of this more general notion: a scheme X is of finite type over k means that we are given a morphism $X \to \operatorname{Spec} k$ (the "structure morphism") that is of finite type.

Here are some properties enjoyed by morphisms of finite type.

3.C. EASY EXERCISE. Show that finite morphisms are of finite type. Hence closed immersions are of finite type.

3.D. EXERCISES (NOT HARD, BUT IMPORTANT).

(a) Show that an open immersion is locally of finite type. Show that an open immersion into a locally Noetherian scheme is of finite type. More generally, show that a quasicompact open immersion is of finite type.
(b) Show that the composition of two morphisms of locally finite type is locally of finite type. (Hence as quasicompact morphisms also compose, the composition of two morphisms of finite type is also of finite type.)
(c) Suppose we have morphisms $X \xrightarrow{f} Y \xrightarrow{g} Z$, with f quasicompact, and $g \circ f$ of finite type. Show that f is finite type.
(d) Suppose $f : X \to Y$ is finite type, and Y is Noetherian. Show that X is also Noetherian.

A morphism f is **quasifinite** if it is of finite type, and for all $y \in Y$, $f^{-1}(y)$ is a finite set. The main point of this definition is the "finite fiber" part; the "finite type" part is there so this notion is "preserved by fibered product" (an exercise in the class on fiber products next week).

3.E. EXERCISE. Show that finite morphisms are quasifinite. (This is a useful exercise, because you will have to figure out how to figure out how to get at points in a fiber of a morphism: given $f : X \to Y$, and $y \in Y$, what are the points of $f^{-1}(y)$? Here is a hint: if $X = \operatorname{Spec} A$ and $Y = \operatorname{Spec} B$ are both affine, and $y = [\mathfrak{p}]$, then we can throw out everything in A outside $\bar{y}$ by modding out by $\mathfrak{p}$; you can show that the preimage is $A/\mathfrak{p}$. Then we have reduced to the case where Y is the Spec of an integral domain, and $[\mathfrak{p}] = [0]$ is the generic point. We can throw out the rest of the points by localizing at 0. You can show that the preimage is $(A_{\mathfrak{p}})/\mathfrak{p}A_{\mathfrak{p}}$. Then, once you have shown that finiteness behaves well with respect to the operations you made done, you have reduced the problem to Exercise 1.M.)

There are quasifinite morphisms which are not finite, for example $\mathbb{A}^2 - \{(0,0)\} \to \mathbb{A}^2$ (Example 1.7). The key example of a morphism with finite fibers that is not quasifinite is $\operatorname{Spec} \overline{\mathbb{Q}} \to \operatorname{Spec} \mathbb{Q}$.

How to picture quasifinite morphisms, thanks go Brian Conrad. If $X \to Y$ is a finite morphism, then quasi-compact open subset $U \subset X$ is quasi-finite over Y. In fact *every* reasonable quasifinite morphism arises in this way. Thus the right way to visualize quasifiniteness is as a finite map with some (closed locus of) points removed.

3.2. $\star$ *Morphisms (locally) of finite presentation.* There is a variant often of use to non-Noetherian people. A morphism $f : X \to Y$ is **locally of finite presentation** (or **locally finitely presented**) if for each affine open subset $\operatorname{Spec} B$ of Y, $f^{-1}(\operatorname{Spec} B) == \cup_i \operatorname{Spec} A_i$ with $B \to A_i$ finitely presented (finitely generated with a finite number of relations). A morphism is of **finite presentation** (or **finitely presented**) if it is locally of finite presentation and quasicompact.

If X is locally Noetherian, then locally of finite presentation is the same as locally of finite type, and finite presentation is the same as finite type. So if you are a Noetherian person, you needn't worry about this notion.

3.F. EXERCISE. Show that the notion of "locally finite presentation" is affine-local.

3.G. $\star\star$ EXERCISE: LOCALLY OF FINITE PRESENTATION IS A PURELY CATEGORICAL NOTION. Show that "locally of finite presentation" is equivalent to the following. If $F : (\mathcal{S}ch/Y) \to (\mathcal{S}ets)$, $S \mapsto \operatorname{Hom}_Y(S, X)$, we require F to commute with direct limits, i.e. if $\{A_i\}$ is a direct system, then $F(\varinjlim A_i) = \varinjlim F(A_i)$.

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