FOUNDATIONS OF ALGEBRAIC GEOMETRY CLASS 6

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CONTENTS

Last day: inverse image sheaf; sheaves on a base; toward schemes; the underlying set of an affine scheme.

1. MORE EXAMPLES OF THE UNDERLYING SETS OF AFFINE SCHEMES

We are in the midst of discussing the underlying set of an affine scheme. We are looking at examples and learning how to draw pictures.

Example 7: $\mathbb{A}^2_{\mathbb{C}} = \operatorname{Spec} \mathbb{C}[x, y]$. (As with Examples 1 and 2, discussion will apply with $\mathbb{C}$ replaced by *any* algebraically closed field.) Sadly, $\mathbb{C}[x, y]$ is not a Principal Ideal Domain: (x, y) is not a principal ideal. We can quickly name *some* prime ideals. One is (0), which has the same flavor as the (0) ideals in the previous examples. $(x - 2, y - 3)$ is prime, and indeed maximal, because $\mathbb{C}[x, y]/(x - 2, y - 3) \cong \mathbb{C}$, where this isomorphism is via $f(x, y) \mapsto f(2, 3)$. More generally, $(x - a, y - b)$ is prime for any $(a, b) \in \mathbb{C}^2$. Also, if $f(x, y)$ is an irreducible polynomial (e.g. $y - x^2$ or $y^2 - x^3$) then $(f(x, y))$ is prime.

1.A. EXERCISE. (Feel free to skip this exercise, as we will see a different proof of this later.) Show that we have identified all the prime ideals of $\mathbb{C}[x, y]$.

We can now attempt to draw a picture of this space. The maximal primes correspond to the old-fashioned points in $\mathbb{C}^2$: $[(x - a, y - b)]$ corresponds to $(a, b) \in \mathbb{C}^2$. We now have to visualize the "bonus points". $[(0)]$ somehow lives behind all of the old-fashioned points; it is somewhere on the plane, but nowhere in particular. So for example, it does not lie on the parabola $y = x^2$. The point $[(y - x^2)]$ lies on the parabola $y = x^2$, but nowhere in particular on it. You can see from this picture that we already want to think about "dimension". The primes $(x - a, y - b)$ are somehow of dimension 0, the primes $(f(x, y))$ are of dimension 1, and (0) is somehow of dimension 2. (All of our dimensions here are *complex* or *algebraic* dimensions. The complex plane $\mathbb{C}^2$ has real dimension 4, but

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complex dimension 2. Complex dimensions are in general half of real dimensions.) We won't define dimension precisely until later, but you should feel free to keep it in mind before then.

Note too that maximal ideals correspond to "smallest" points. Smaller ideals correspond to "bigger" points. "One prime ideal contains another" means that the points "have the opposite containment." All of this will be made precise once we have a topology. This order-reversal is a little confusing, and will remain so even once we have made the notions precise.

We now come to the obvious generalization of Example 7, affine n-space.

Example 8: $\mathbb{A}^n_{\mathbb{C}} := \operatorname{Spec} \mathbb{C}[x_1, \ldots, x_n]$. (More generally, $\mathbb{A}^n_A$ is defined to be $\operatorname{Spec} A[x_1, \ldots, x_n]$, where A is an arbitrary ring.)

For concreteness, let's consider $n = 3$. We now have an interesting question in algebra: What are the prime ideals of $\mathbb{C}[x, y, z]$? Analogously to before, $(x-a, y-b, z-c)$ is a prime ideal. This is a maximal ideal, with residue field $\mathbb{C}$; we think of these as "0-dimensional points". We will often write (a, b, c) for $[(x-a, y-b, z-c)]$ because of our geometric interpretation of these ideals.

There are no more maximal ideals, by Hilbert's Nullstellensatz. (This is sometimes called the "weak version" of the Nullstellensatz.) You may have already seen this result. We will prove it later (in a slightly stronger form), so we will content ourselves by stating it here.

1.1. *Hilbert's Nullstellensatz. — Suppose $A = k[x_1, \ldots, x_n]$, where k is an algebraically closed field. Then the maximal ideals are precisely those of the form $(x_1 - a_1, \ldots, x_n - a_n)$, where $a_i \in k$.*

There are other prime ideals too. We have (0), which is corresponds to a "3-dimensional point". We have $(f(x, y, z))$, where f is irreducible. To this we associate the hypersurface $f = 0$, so this is "2-dimensional" in nature. But we have not found them all! One clue: we have prime ideals of "dimension" 0, 2, and 3 — we are missing "dimension 1". Here is one such prime ideal: (x, y). We picture this as the locus where $x = y = 0$, which is the z-axis. This is a prime ideal, as the corresponding quotient $\mathbb{C}[x, y, z]/(x, y) \cong \mathbb{C}[z]$ is an integral domain (and should be interpreted as the functions on the z-axis). There are lots of one-dimensional primes, and it is not possible to classify them in a reasonable way. It will turn out that they correspond to things that we think of as irreducible curves: the natural answer to this algebraic question is geometric.

1.2. Important fact: Maps of rings induce maps of spectra (as sets). We now make an observation that will later grow up to be morphisms of schemes. If $\phi : B \to A$ is a map of rings, and $\mathfrak{p}$ is a prime ideal of A, then $\phi^{-1}(\mathfrak{p})$ is a prime ideal of B (check this!). Hence a map of rings $\phi : B \to A$ induces a map of sets $\operatorname{Spec} A \to \operatorname{Spec} B$ "in the opposite direction". This gives a contravariant functor from the category of rings to the category of sets: the composition of two maps of rings induces the composition of the corresponding maps of spectra.

We now describe two important cases of this: maps of rings inducing *inclusions* of sets. There are two particularly useful ways of producing new rings from a ring A. One is by taking the quotient by an ideal I. The other is by localizing at a multiplicative set. We'll see how Spec behaves with respect to these operations. In both cases, the new ring has a Spec that is a subset of Spec of the old ring.

First important example (quotients): $\operatorname{Spec} B/I$ **in terms of** $\operatorname{Spec} B$**.** As a motivating example, consider $\operatorname{Spec} B/I$ where $B = \mathbb{C}[x, y]$, $I = (xy)$. We have a picture of $\operatorname{Spec} B$, which is the complex plane, with some mysterious extra "higher-dimensional points". It is an important fact that the primes of B/I are in bijection with the primes of B containing I. (If you do not know why this is true, you should prove it yourself.) Thus we can picture $\operatorname{Spec} B/I$ as a subset of $\operatorname{Spec} B$. We have the "0-dimensional points" $(a, 0)$ and $(0, b)$. We also have two "1-dimensional points" (x) and (y).

We get a bit more: the inclusion structure on the primes of B/I corresponds to the inclusion structure on the primes containing I. More precisely, if $J_1 \subset J_2$ in B/I, and K_i is the ideal of B corresopnding to J_i, then $K_1 \subset K_2$. (Again, prove this yourself if you have not seen it before.)

So the minimal primes of $\mathbb{C}[x, y]/(xy)$ are the "biggest" points we see, and there are two of them: (x) and (y). Thus we have the intuition that will later be made precise: the minimal primes of A correspond to the "components" of $\operatorname{Spec} A$.

As an important motivational special case, you now have a picture of **"complex affine varieties"**. Suppose A is a finitely generated $\mathbb{C}$-algebra, generated by $x_1, \ldots, x_n$, with relations $f_1(x_1, \ldots, x_n) = \cdots = f_r(x_1, \ldots, x_n) = 0$. Then this description in terms of generators and relations naturally gives us an interpretation of $\operatorname{Spec} A$ as a subset of $\mathbb{A}^n_{\mathbb{C}}$, which we think of as "old-fashioned points" (n-tuples of complex numbers) along with some "bonus" points. To see which subsets of the old-fashioned points are in $\operatorname{Spec} A$, we simply solve the equations $f_1 = \cdots = f_r = 0$. For example, $\operatorname{Spec} \mathbb{C}[x, y, z]/(x^2 + y^2 - z^2)$ may be pictured as shown in Figure 1. (Admittedly this is just a "sketch of the $\mathbb{R}$-points", but we will still find it helpful later.) This entire picture carries over (along with the Nullstellensatz) with $\mathbb{C}$ replaced by any algebraically closed field. Indeed, the picture of Figure 1 can be said to represent $k[x, y, z]/(x^2+y^2-z^2)$ for most algebraically closed fields k (although it is misleading in characteristic 2, because of the coincidence $x^2 + y^2 - z^2 = (x + y + z)^2$).

1.B. EXERCISE. Ring elements that have a power that is 0 are called *nilpotents*. If I is an ideal of nilpotents, show that $\operatorname{Spec} B/I \to \operatorname{Spec} B$ is a bijection. Thus nilpotents don't affect the underlying set. (We will soon see in Exercise 2.H that they won't affect the topology either — the difference will be in the structure sheaf.)

Second important example (localization): $\operatorname{Spec} S^{-1}B$ **in terms of** $\operatorname{Spec} B$, where S is a multiplicative subset of B. There are two particularly important flavors of multiplicative subsets. The first is $B \setminus \mathfrak{p}$, where $\mathfrak{p}$ is a prime ideal. This localization $S^{-1}B$ is denoted $B_{\mathfrak{p}}$.

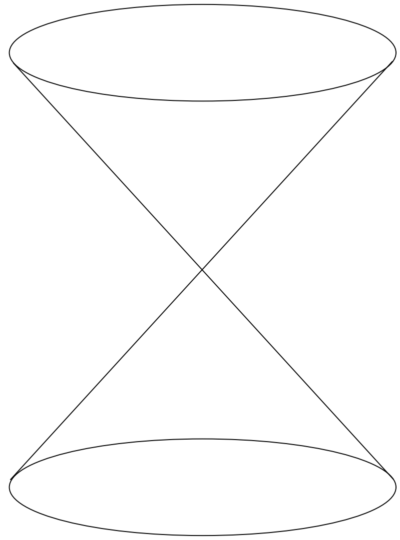

FIGURE 1. A "picture" of $\operatorname{Spec} \mathbb{C}[x, y, z]/(x^2 + y^2 - z^2)$

A motivating example is $B = \mathbb{C}[x, y]$, $S = B - (x, y)$. The second is $\{1, f, f^2, \ldots\}$, where $f \in B$. This localization is denoted B_f. (Notational warning: If $\mathfrak{p}$ is a prime ideal, then $B_\mathfrak{p}$ means you're allowed to divide by elements not in $\mathfrak{p}$. However, if $f \in B$, B_f means you're allowed to divide by f. This can be confusing. For example, if (f) is a prime ideal, then $B_f \neq B_{(f)}$.) A motivating example is $B = \mathbb{C}[x, y]$, $f = x$.

1.3. *Essential algebra fact (to review and know).* The map $\operatorname{Spec} S^{-1}B \to \operatorname{Spec} B$ gives an order-preserving bijection of the primes of $S^{-1}B$ with the primes of B that *don't meet* the multiplicative set S.

So if $S = B - \mathfrak{p}$ where $\mathfrak{p}$ is a prime ideal, the primes of $S^{-1}B$ are just the primes of B *contained in* $\mathfrak{p}$. If $S = \{1, f, f^2, \ldots\}$, the primes of $S^{-1}B$ are just those primes not containing f (the points where "f doesn't vanish" — draw a picture of $\operatorname{Spec} \mathbb{C}[x]_{x^2-x}$ to see how this works).

1.4. *Warning.* sometimes localization is first introduced in the special case where B is an integral domain. In this example, $B \hookrightarrow B_f$, but this isn't true when one inverts zero-divisors. (A **zero-divisor** of a ring B is an element a such that there is a non-zero element b with $ab = 0$. The other elements of B are called **non-zero-divisors**.) One definition of localization is as follows. The elements of $S^{-1}B$ are of the form a/s where $r \in B$ and $s \in S$, and $(a_1/s_1) \times (a_2/s_2) = (a_1 a_2 / s_1 s_2)$, and $(a_1/s_1) + (a_2/s_2) = (a_1 s_2 + s_1 a_2)/(s_1 s_2)$. We say that $a_1/s_1 = a_2/s_2$ if *for some* $s \in S$ $s(a_1 s_2 - a_2 s_1) = 0$. So for example, $B[1/0] \cong 0$.

1.5. Important comment: functions are not determined by their values at points. We are developing machinery that will let us bring our geometric intuition to algebra. There is one point where your intuition will be false, so you should know now, and adjust your intuition appropriately. Suppose we have a function (ring element) vanishing at all points. Then it is not necessarily the zero function! The translation of this question is: is the intersection of all prime ideals necessarily just 0? The answer is no, as is shown by the example of the ring of dual numbers $k[\epsilon]/\epsilon^2$: $\epsilon \neq 0$, but $\epsilon^2 = 0$. (We saw this scheme in an exercise in class 5.) Any function whose power is zero certainly lies in the intersection of all prime ideals. The converse is also true: the intersection of all the prime ideals consists of functions for which some power is zero, otherwise known as the nilradical $\mathfrak{N}$. (You should check that the nilpotents indeed form an ideal. For example, the sum of two nilpotents is always nilpotent.)

1.6. Theorem. *The nilradical $\mathfrak{N}(A)$ is the intersection of all the primes of A.*

1.C. EXERCISE. If you don't know this theorem, then look it up, or even better, prove it yourself. (Hint: one direction is easy. The other will require knowing that any proper ideal of A is contained in a maximal ideal, which requires the axiom of choice.)

In particular, although it is upsetting that functions are not determined by their values at points, we have precisely specified what the failure of this intuition is: two functions have the same values at points if and only if they differ by a nilpotent. And if there are no non-zero nilpotents — if $\mathfrak{N} = 0$ — then functions *are* determined by their values at points.

2. THE ZARISKI TOPOLOGY: THE UNDERLYING TOPOLOGICAL SPACE OF AN AFFINE SCHEME

We next introduce the *Zariski topology* on the spectrum of a ring. At first it seems like an odd definition, but in retrospect it is reasonable. For example, consider $\mathbb{A}^2_{\mathbb{C}} = \operatorname{Spec} \mathbb{C}[x, y]$, the complex plane (with a few extra points). In algebraic geometry, we will only be allowed to consider algebraic functions, i.e. polynomials in x and y. The locus where a polynomial vanishes should reasonably be a closed set, and the Zariski topology is defined by saying that the only sets we should consider closed should be these sets, and other sets forced to be closed by these. In other words, it is the coarsest topology where these sets are closed.

In particular, although topologies are often described using open subsets, it will more convenient for us to define this topology in terms of closed subsets. If S is a subset of a ring A, define the V**anishing set** of S by

$$V(S) := \{[\mathfrak{p}] \in \operatorname{Spec} A : S \subset \mathfrak{p}\}.$$

It is the set of points on which all elements of S are zero. (It should now be second nature to equate "vanishing at a point" with "contained in a prime".) We declare that these (and no other) are the closed subsets.

For example, consider $V(xy, xz) \subset \mathbb{A}^3 = \operatorname{Spec} \mathbb{C}[x, y, z]$. Which points are contained in this locus? We think of this as solving $xy = yz = 0$. Of the "old-fashioned" points (interpreted as ordered triples of complex numbers, thanks to the Hilbert's Nullstellensatz 1.1), we have the points where $y = 0$ or $x = z = 0$: the xz-plane and the y-axis respectively. Of the "new" points, we have the generic point of the xz-plane (also known as the point $[(y)]$), and the generic point of the y-axis (also known as the point $[(x,z)]$). You might imagine that we also have a number of "one-dimensional" points contained in the xz-plane.

2.A. EASIER EXERCISE. Check that the x-axis is contained in this set.

Let's return to the general situation. The following exercise lets us restrict attention to vanishing sets of *ideals*.

2.B. EASIER EXERCISE. Show that if (S) is the ideal generated by S, then $V(S) = V((S))$.

We define the Zariski topology by declaring that $V(S)$ is closed for all S. Let's check that this is a topology. We have to check that the empty set and the total space are open; the union of an arbitrary collection of open sets are open; and the intersection of two open sets are open.

2.C. EXERCISE. (a) Show that $\varnothing$ and $\operatorname{Spec} A$ are both open.
(b) Show that $V(I_1) \cup V(I_2) = V(I_1 I_2)$. Hence show that the intersection of any finite number of open sets is open.
(c) *(The union of any collection of open sets is open.)* If I_i is a collection of ideals (as i runs over some index set), check that $\cap_i V(I_i) = V(\sum_i I_i)$.

2.1. Properties of "vanishing set" function $V(\cdot)$. The function $V(\cdot)$ is obviously inclusion-reversing: If $S_1 \subset S_2$, then $V(S_2) \subset V(S_1)$. Warning: We could have equality in the second inclusion without equality in the first, as the next exercise shows.

2.D. EXERCISE/DEFINITION. If $I \subset R$ is an ideal, then define its **radical** by

$$\sqrt{I} := \{r \in R : r^n \in I \text{ for some } n \in \mathbb{Z}^{\geq 0}\}.$$

For example, the nilradical $\mathfrak{N}$ (§1.5) is $\sqrt{(0)}$. Show that $V(\sqrt{I}) = V(I)$. We say an **ideal is radical** if it equals its own radical.

Here are two useful consequences. As $(I \cap J)^2 \subset IJ \subset I \cap J$, we have that $V(IJ) = V(I \cap J)$ $(= V(I) \cup V(J)$ by Exercise 2.C(b)). Also, combining this with Exercise 2.B, we see $V(S) = V((S)) = V(\sqrt{(S)})$.

2.E. EXERCISE (PRACTICE WITH THE CONCEPT). If $I_1, \dots, I_n$ are ideals of a ring A, show that $\sqrt{\cap_{i=1}^n I_i} = \cap_{i=1}^n \sqrt{I_i}$. (We will use this property without referring back to this exercise.)

2.F. EXERCISE FOR FUTURE USE. Show that $\sqrt{I}$ is the intersection of all the prime ideals containing I. (Hint: Use Theorem 1.6 on an appropriate ring.)

2.2. Examples. Let's see how this meshes with our examples from the previous section.

Recall that $\mathbb{A}^1_{\mathbb{C}}$, as a set, was just the "old-fashioned" points (corresponding to maximal ideals, in bijection with $a \in \mathbb{C}$), and one "new" point (0). The Zariski topology on $\mathbb{A}^1_{\mathbb{C}}$ is not that exciting: the open sets are the empty set, and $\mathbb{A}^1_{\mathbb{C}}$ minus a finite number of maximal ideals. (It "almost" has the cofinite topology. Notice that the open sets are determined by their intersections with the "old-fashioned points". The "new" point (0) comes along for the ride, which is a good sign that it is harmless. Ignoring the "new" point, observe that the topology on $\mathbb{A}^1_{\mathbb{C}}$ is a coarser topology than the analytic topology.)

The case $\operatorname{Spec} \mathbb{Z}$ is similar. The topology is "almost" the cofinite topology in the same way. The open sets are the empty set, and $\operatorname{Spec} \mathbb{Z}$ minus a finite number of "ordinary" ((p) where p is prime) primes.

2.3. *Closed subsets of* $\mathbb{A}^2_{\mathbb{C}}$. The case $\mathbb{A}^2_{\mathbb{C}}$ is more interesting. You should think through where the "one-dimensional primes" fit into the picture. In Exercise 1.A, we identified all the primes of $\mathbb{C}[x, y]$ (i.e. the points of $\mathbb{A}^2_{\mathbb{C}}$) as the maximal ideals $(x-a, y-b)$ $(a, b \in \mathbb{C})$, the "one-dimensional points" $(f(x, y))$ $(f(x, y)$ irreducible), and the "two-dimensional point" (0).

Then the closed subsets are of the following form:

(a) the entire space, and
(b) a finite number (possibly zero) of "curves" (each of which is the closure of a "one-dimensional point") and a finite number (possibly zero) of closed points.

2.4. Important fact: Maps of rings induce continuous maps of topological spaces. We saw in §1.2 that a map of rings $\phi : B \to A$ induces a map of sets $\pi : \operatorname{Spec} A \to \operatorname{Spec} B$.

2.G. IMPORTANT EXERCISE. By showing that closed sets pull back to closed sets, show that π is a *continuous map*.

Not all continuous maps arise in this way. Consider for example the continuous map on $\mathbb{A}^1_{\mathbb{C}}$ that is the identity except 0 and 1 (i.e. $[(x)]$ and $[(x-1)]$ are swapped); there is no polynomial that can manage this.

In §1.2, we saw that $\operatorname{Spec} B/I$ and $\operatorname{Spec} S^{-1}B$ are naturally *subsets* of $\operatorname{Spec} B$. It is natural to ask if the Zariski topology behaves well with respect to these inclusions, and indeed it does.

2.H. IMPORTANT EXERCISE. Suppose that $I, S \subset B$ are an ideal and multiplicative subset respectively. Show that $\operatorname{Spec} B/I$ is naturally a *closed* subset of $\operatorname{Spec} B$. Show that the Zariski topology on $\operatorname{Spec} B/I$ (resp. $\operatorname{Spec} S^{-1}B$) is the subspace topology induced by inclusion in $\operatorname{Spec} B$. (Hint: compare closed subsets.)

In particular, if $I \subset \mathfrak{N}$ is an ideal of nilpotents, the bijection $\operatorname{Spec} B/I \to \operatorname{Spec} B$ (Exercise 1.B) is a homeomorphism. Thus nilpotents don't affect the topological space. (The difference will be in the structure sheaf.)

2.I. USEFUL EXERCISE FOR LATER. Suppose $I \subset B$ is an ideal. Show that f vanishes on $V(I)$ if and only if $f^n \in I$ for some n.

2.J. EXERCISE. Describe the topological space $\operatorname{Spec} k[x]_{(x)}$.

3. TOPOLOGICAL DEFINITIONS

We now describe various properties that it will be useful to have names for.

A topological space is said to be **irreducible** if it is not the union of two proper closed subsets. In other words, X is irreducible if whenever $X = Y \cup Z$ with Y and Z closed, we have $Y = X$ or $Z = X$.

3.A. EASY EXERCISE. Show that on an irreducible topological space, any nonempty open set is dense. (The moral of this is: unlike in the classical topology, in the Zariski topology, non-empty open sets are all "very big".)

3.B. EXERCISE. Show that $\operatorname{Spec} A$ is irreducible if and only if A has only one minimal prime. (Minimality is with respect to inclusion.) In particular, if A is an integral domain, then $\operatorname{Spec} A$ is irreducible.

A point of a topological space $x \in X$ is said to be **closed** if $\{x\}$ is a closed subset. In the old-fashioned topology on $\mathbb{C}^n$, all points are closed.

3.C. EXERCISE. Show that the closed points of $\operatorname{Spec} A$ correspond to the maximal ideals.

Thus Hilbert's Nullstellensatz lets us associate the closed points of $\mathbb{A}^n_{\mathbb{C}}$ with n-tuples of complex numbers. Hence from now on we will say "closed point" instead of "old-fashioned point" and "non-closed point" instead of "bonus" or "new-fangled" point when discussing subsets of $\mathbb{A}^n_{\mathbb{C}}$.

Given two points x, y of a topological space X, we say that x is a **specialization** of y, and y is a **generization** of x, if $x \in \overline{\{y\}}$. This now makes precise our hand-waving about "one point contained another". It is of course nonsense for a point to contain another. But it is not nonsense to say that the closure of a point contains another. For example, in $\mathbb{A}^2_{\mathbb{C}} = \operatorname{Spec} \mathbb{C}[x,y]$, $[(y - x^2)]$ is a generization of $(2,4) = [(x-2, y-4)]$, and $(2,4)$ is a specialization of $[(y - x^2)]$.

3.D. EXERCISE. If $X = \operatorname{Spec} A$, show that $[\mathfrak{p}]$ is a specialization of $[\mathfrak{q}]$ if and only if $\mathfrak{q} \subset \mathfrak{p}$. Verify to your satisfaction that we have made our intuition of "containment of points" precise: it means that the one point is contained in the *closure* of another.

We say that a point $x \in X$ is a **generic point** for a closed subset K if $\overline{\{x\}} = K$.

3.E. EXERCISE. Verify that $[(y - x^2)] \in \mathbb{A}^2$ is a generic point for $V(y - x^2)$.

We will soon see that there is a natural bijection between points of $\operatorname{Spec} A$ and irreducible closed subsets of $\operatorname{Spec} A$. You know enough to show this now, although we'll wait until we have developed some convenient terminology.

3.F. LESS IMPORTANT EXERCISE. (a) Suppose $I = (wz - xy, wy - x^2, xz - y^2) \subset k[w, x, y, z]$. Show that $\operatorname{Spec} k[w, x, y, z]/I$ is irreducible, by showing that I is prime. (One possible approach: Show that the quotient ring is a domain, by showing that it is isomorphic to the subring of $k[a, b]$ including only monomials of degree divisible by 3. There are other approaches as well, some of which we will see later. This is an example of a hard question: how do you tell if an ideal is prime?) We will later see this as the cone over the *twisted cubic curve*.
(b) Note that the ideal of part (a) may be rewritten as

$$\operatorname{rank} \begin{pmatrix} w & x & y \\ x & y & z \end{pmatrix} = 1,$$

i.e. that all determinants of 2×2 submatrices vanish. Generalize this to the ideal of rank 1 $2 \times n$ matrices. This notion will correspond to the cone over the *degree* n *rational normal curve*.

3.1. Noetherian conditions.

In the examples we have considered, the spaces have naturally broken up into some obvious pieces. Let's make that a bit more precise.

A topological space X is called **Noetherian** if it satisfies the **descending chain condition** for closed subsets: any sequence $Z_1 \supseteq Z_2 \supseteq \cdots \supseteq Z_n \supseteq \cdots$ of closed subsets eventually stabilizes: there is an r such that $Z_r = Z_{r+1} = \cdots$.

The following exercise may be enlightening.

3.G. EXERCISE. Show that any decreasing sequence of closed subsets of $\mathbb{A}^2_{\mathbb{C}} = \operatorname{Spec} \mathbb{C}[x, y]$ must eventually stabilize. Note that it can take arbitrarily long to stabilize. (The closed subsets of $\mathbb{A}^2_{\mathbb{C}}$ were described in §2.3.)

3.2. It turns out that all of the spectra we have considered have this property, but that isn't true of the spectra of all rings. The key characteristic all of our examples have had in common is that the rings were *Noetherian*. Recall that a ring is **Noetherian** if every *ascending* sequence $I_1 \subset I_2 \subset \cdots$ of ideals eventually stabilizes: there is an r such that $I_r = I_{r+1} = \cdots$. (This is called the **ascending chain condition** on ideals.)

Here are some quick facts about Noetherian rings. You should be able to prove them all.

- Fields are Noetherian. $\mathbb{Z}$ is Noetherian.
- If A is Noetherian, and I is any ideal, then A/I is Noetherian.
- If A is Noetherian, and S is any multiplicative set, then $S^{-1}A$ is Noetherian.
- In a Noetherian ring, any ideal is finitely generated.
- Any submodule of a finitely generated module over a Noetherian ring is finitely generated. (Hint: prove it for A^n, and use the next exercise.)

3.H. EXERCISE. Suppose $0 \to M' \to M \to M'' \to 0$, and M' and M'' satisfy the ascending chain condition for modules. Show that M does too. (The converse also holds; we won't use this, but you can show it if you wish.)

The next fact is non-trivial.

3.3. *The Hilbert basis theorem. — If A is Noetherian, then so is $A[x]$.*

Using these results, then any polynomial ring over any field, or over the integers, is Noetherian — and also any quotient or localization thereof. Hence for example any finitely-generated algebra over k or $\mathbb{Z}$, or any localization thereof is Noetherian. Most "nice" rings are Noetherian, but not all rings are Noetherian, e.g. $k[x_1, x_2, \ldots]$ because $\mathfrak{m} = (x_1, x_2, \ldots,)$ is not finitely generated.

3.I. EXERCISE. If A is Noetherian, show that $\operatorname{Spec} A$ is a Noetherian topological space.

3.J. LESS IMPORTANT EXERCISE. Show that the converse is not true: if $\operatorname{Spec} A$ is a Noetherian topological space, A need not be Noetherian. Describe a ring A such that $\operatorname{Spec} A$ is not a Noetherian topological space.

I discussed how the finiteness of the game of Chomp is a consequence of the Hilbert basis theorem.

If X is a topological space, and Z is an irreducible closed subset not contained in any larger irreducible closed subset, Z is said to be an *irreducible component* of X. (I drew a picture.)

3.K. EXERCISE. If A is any ring, show that the irreducible components of $\operatorname{Spec} A$ are in bijection with the minimal primes of A.

For example, the only minimal prime of $k[x, y]$ is (0). What are the minimal primes of $k[x, y]/(xy)$?

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