MATH 210 PROBLEM SET 3

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This problem set is due on Friday, February 9 at Jarod Alper's office door.

1. Prove that if the Galois group of the splitting field of a cubic over $\mathbb{Q}$ is the cyclic group of order 3 then all the roots of the cubic are real. (Dummit and Foote p. 562, problem 13)

2. Show that $\mathbb{Q}(\sqrt{2+\sqrt{2}})$ is a cyclic quartic field, i.e. is a Galois extension of degree 4 with cyclic Galois group. (Dummit and Foote p. 562, problem 14)

3. Show that every irreducible polynomial in $\mathbb{F}_p[x]$ is a factor of $x^{p^n} - x$ for some n.

4. Suppose E/F is an extension. Define the separable closure F^{sep} of F in E to be the separable elements of E/F. Show that F^{sep} is a subfield of E. If E/F is finite, show that E/F^{sep} is generated by a tower of pth roots. If E/F is algebraic, show that any element of E has some p^kth power in F^{sep}.

5. Suppose the dihedral group with $2n$ elements acts on the field $k(x)$ with generators mapping $x \mapsto 1/x$ and $x \mapsto \zeta x$ (where ζ is a primitive nth root of unity). Find some $y \in k(x)$ such that $k(y)$ is the fixed field of this group action.

6. Show that the elements $\{x_1^{a_1} \cdots x_n^{a_n}\}_{0 \leq a_i < i}$ form a basis for $k(x_1, \dots, x_n)$ over $k(e_1, \dots, e_n)$ (where as in class e_i is the ith symmetric polynomial in $x_1, \dots, x_n$).

Date: Friday, February 2, 2007.