MATH 210 PRACTICE FINAL

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Justify all answers!

1. Suppose ω is a primitive cube root of 1 in $\mathbb{C}$. Show that $\mathbb{Q}(\sqrt[3]{3}\omega)/\mathbb{Q}$ is not a normal extension.

2. Suppose $f(x)$ is an irreducible quartic over $\mathbb{Q}$, and $\alpha, \beta, \gamma, \delta \in \mathbb{C}$ are the roots of $f(x) = 0$ If $\mathrm{Gal}(\mathbb{Q}(\alpha, \beta, \gamma, \delta)/\mathbb{Q}) \cong S_4$, how many of β, γ, δ are elements of $\mathbb{Q}(\alpha)$? Repeat the question for each subgroup of S_4 that is a possible Galois group of such an $f(x)$. (Hint: How does $\mathrm{Gal}(\mathbb{Q}(\alpha, \beta, \gamma, \delta)/\mathbb{Q}(\alpha))$ act on β, γ, δ?)

3. Let E be the splitting field of $(x^3 - 3)(x^3 - 2)$ over $\mathbb{Q}$. Desribe the group $\mathrm{Gal}(E/\mathbb{Q})$.

4. (a) For each square-free integer n, describe which roots of unity lie in $\mathbb{Q}(\sqrt{n})$.

(b) As an application, solve the following problem in geometry: for which m can a regular m-gon be found with vertices on lattice points $\{(x, y) : x, y \in \mathbb{Z}\} \subset \mathbb{R}^2$? How about a triangular lattice?

5. In a Noetherian ring, show that a proper ideal I is a radical ideal ($I = \sqrt{I}$) if and only if I is a finite intersection of prime ideals.

6. (In this problem, k is not necessarily algebraically closed.) Show that $I \subset k[x_1, \dots, x_n]$ is a maximal ideal if and only if $k[x_1, \dots, x_n]/I$ is a finite field extension of k.

7. Suppose S is a finitely generated algebra over a field k that is a domain, containing n algebraically independent elements x_1, …, x_n, such that if $R = k[x_1, \dots, x_n]$, then S/R is an integral extension. Show that there exists a chain of $n + 1$ distinct nested prime ideals of S

$$P_0 \subset P_1 \subset \cdots \subset P_n.$$

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