MATH 121 PROBLEM SET 6

This set is due at noon on Friday March 9 in Jason Lo's mailbox.

1. Suppose $f(x) = x^3 + e_2 x + e_3$ is a cubic in $\mathbb{Q}[x]$ (i.e. e_2 and e_3 are rational numbers). Show that the discriminant Δ is $-4e_2^3 - 27e_3^2$. (If a, b, and c are the roots of $f(x)$ in its splitting field, then $\Delta = ((a-b)(a-c)(b-c))^2$. As Δ is preserved by the S_3 permuting the roots, it must lie in $\mathbb{Q}$.)

2. If $f(x)$ is an irreducible cubic in $\mathbb{Q}[x]$, show that the Galois group of the splitting field of $f(x)$ is S_3 if and only if Δ is *not* a perfect square in $\mathbb{Q}$. Otherwise, show that the Galois group of $f(x)$ is A_3. What is the Galois group of the splitting field of $f(x) = x^3+2x^2+2x+2$ over $\mathbb{Q}$? (Hint: use problem 1.)

3. Show that the Galois group of the splitting field of $f(x) = x^5 - 4x + 2$ over $\mathbb{Q}$ is S_5. (Make sure to verify that $f(x)$ is irreducible!)

4. Show that $\mathbb{Q}(\sqrt{2+\sqrt{2}})$ is a cyclic quartic field, i.e. is a Galois extension of $\mathbb{Q}$ of degree 4 with cyclic Galois group. Use this to give an example of an irreducible quartic polynomial in $\mathbb{Q}[x]$ whose Galois group (of its splitting field) is $\mathbb{Z}/4$. Make sure to show that this polynomial is irreducible!

Date: Friday, March 2, 2007.