FOUNDATIONS OF ALGEBRAIC GEOMETRY CLASSES 47 AND 48

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CONTENTS

This week: Local criteria for flatness (statement), (relatively) base-point-free, (relatively) ample, very ample, every ample on a proper has a tensor power that is very ample, Serre's criterion for ampleness, Riemann-Roch for generically reduced curves.

1. THE LOCAL CRITERION FOR FLATNESS

I'll end our discussion of flatness with the statement of two results which can be quite useful. (Translation: I've seen them used.) They are both called the local criterion for flatness.

In both situations, assume that $(B, \mathfrak{n}) \to (A, \mathfrak{m})$ is a local morphism of local Noetherian rings (i.e. a ring homomorphism with $\mathfrak{n}A \subset \mathfrak{m}$), and that M is a finitely generated A-module. Of course we picture this in terms of geometry:

$$\begin{array}{c} \tilde{M} \\ \Big| \\ \operatorname{Spec}(A, \mathfrak{m}) \\ \Big\downarrow \\ \operatorname{Spec}(B, \mathfrak{n}). \end{array}$$

The local criteria for flatness are criteria for when M is flat over A. In practice, these are used in two circumstances: to check when a morphism to a locally Noetherian scheme is flat, or when a coherent sheaf on a locally Noetherian scheme is flat.

We've shown that to check if M is flat, we need check if $\operatorname{Tor}_1^B(B/I, M) = 0$ for all ideals I. The (first) local criterion says we need only deal with the maximal ideal.

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1.1. *Theorem (local criterion for flatness). —* M *is* B*-flat if and only if* $\operatorname{Tor}_1^B(B/\mathfrak{n}, M) = 0$*.*

(You can see a proof in Eisenbud, p. 168.)

An even more useful variant is the following. Suppose t is a non-zero-divisor of B in $\mathfrak{m}$ (geometrically: a Cartier divisor on the target passing through the generic point). If M is flat over B, then t is not a zero-divisor of M (we've checked this before: tensor $0 \longrightarrow B \xrightarrow{\times t} B \longrightarrow B/(t) \to 0$ with M). Also, M/tM is a flat B/tB-module (flatness commutes with base change). The next result says that this is a characterization.

1.2. *Theorem (local slicing criterion for flatness). —* M *is* B*-flat if and only if* M/tM *is flat over* $B/(t)$*.*

This is also sometimes called the local criterion for flatness. The proof is short (given the first local criterion). You can read it in Eisenbud (p. 169).

1.3. *Exercise (for those who know what a Cohen-Macaulay scheme is).* Suppose $\pi : X \to Y$ is a map of locally Noetherian schemes, where both X and Y are equidimensional, and Y is nonsingular. Show that if any two of the following hold, then the third does as well:

- π is flat.
- X is Cohen-Macaulay.
- Every fiber X_y is Cohen-Macaulay of the expected dimension.

I concluded the section on flatness by reviewing everything we have learned about flatness, in a good order.

2. Base-point-free, ample, very ample

My goal is to discuss properties of invertible sheaves on schemes (an "absolute" notion), and properties of invertible sheaves on a scheme with a morphism to another scheme (a "relative" notion, meaning that it makes sense in families). The notions fit into this table:

absolute	relative
base-point-free	relatively base-point-free
ample	relatively ample
very ample over a ring	very ample

This is admittedly horrible terminology. Warning: my definitions may have some additional hypotheses not used in EGA. The additional hypotheses exclude some nasty behavior which tends not to come up in nature; indeed, I have only seen these notions used in the circumstances in which I will describe them. There are very few facts to know, and there is fairly little to prove.

2.1. *Definition of base-point-free and relative base-point-free (review from class 22 and class 24, respectively).* Recall that if $\mathcal{F}$ is a quasicoherent sheaf on a scheme X, then $\mathcal{F}$ is generated by global sections if for any $x \in X$, the global sections generate the stalk $\mathcal{F}_x$. Equivalently: $\mathcal{F}$ is the quotient of a free sheaf. If $\mathcal{F}$ is a finite type quasicoherent sheaf, then we just need to check that for any x, the global sections generate the fiber of $\mathcal{F}$, by Nakayama's lemma. If furthermore $\mathcal{F}$ is invertible, we need only check that for any x there is a global section not vanishing there. In the case where $\mathcal{F}$ is invertible, we give "generated by global sections" a special name: *base-point-free*.

2.2. *Exercise (generated $\otimes$ generated = generated for finite type sheaves).* Suppose $\mathcal{F}$ and $\mathcal{G}$ are finite type sheaves on a scheme X that are generated by global sections. Show that $\mathcal{F} \otimes \mathcal{G}$ is also generated by global sections. In particular, if $\mathcal{L}$ and $\mathcal{M}$ are invertible sheaves on a scheme X, and both $\mathcal{L}$ and $\mathcal{M}$ are base-point-free, then so is $\mathcal{L} \otimes \mathcal{M}$. (This is often summarized as "base-point-free + base-point-free = base-point-free". The symbols $+$ is used rather than $\otimes$, because Pic is an abelian group.)

If $\pi : X \to Y$ is a morphism of schemes *that is quasicompact and quasiseparated* (so push-forwards of quasicoherent sheaves are quasicoherent sheaves), and $\mathcal{F}$ is a quasicoherent sheaf on X, we say that $\mathcal{F}$ is *relatively generated by global sections* (or *relatively generated* for short) if $\pi^*\pi_*\mathcal{F} \to \mathcal{F}$ is a surjection of sheaves (class 24). As this is a morphism of quasicoherent sheaves, this can be checked over any affine open subset of the target, and corresponds to "generated by global sections" above each affine. In particular, this notion is affine-local on the target. If $\mathcal{F}$ is locally free, this notion is called *relatively base-point-free*.

2.3. *Definition of very ample.* Suppose $X \to Y$ is a projective morphism. Then $X = \underline{\operatorname{Proj}}\, \mathcal{S}_*$ for some graded algebra, locally generated in degree 1; given this description, X comes with $\mathcal{O}(1)$. Then any invertible sheaf on X of this sort is said to be *very ample* (for the morphism π). The notion of very ample is local on the base. (This is "better" than the notion of projective, which isn't local on the base, as we've seen in class 43/44 p. 4. Recall why: a morphism is projective if there *exists* an $\mathcal{O}(1)$. Thus a morphism $X \to Y \cup Y'$ could be projective over Y and over Y', but not projective over $Y \cup Y'$, as the "$\mathcal{O}(1)$" above Y need not be the same as the "$\mathcal{O}(1)$" above Y'. On the other hand, the notion is "very ample" is precisely the data of "an $\mathcal{O}(1)$".) You'll recall that given such an invertible sheaf, then $X = \underline{\operatorname{Proj}}\, \pi_*\mathcal{L}^{\otimes n}$, where the algebra on the right has the desired form. (It isn't necessarily the same graded algebra as you originally used to construct X.)

Notational remark: If Y is implicit, it is often omitted from the terminology. For example, if X is a complex projective scheme, the phrase "$\mathcal{L}$ is very ample on X" often means that "$\mathcal{L}$ is very ample for the structure morphism $X \to \operatorname{Spec} \mathbb{C}$".

2.4. *Exercise (very ample + very ample = very ample).* If $\mathcal{L}$ and $\mathcal{M}$ are invertible sheaves on a scheme X, and both $\mathcal{L}$ and $\mathcal{M}$ are base-point-free, then so is $\mathcal{L} \otimes \mathcal{M}$. Hint: Segre. In particular, tensor powers of a very ample invertible sheaf are very ample.

2.5. *Tricky exercise+ (very ample + relatively generated = very ample).* Suppose $\mathcal{L}$ is very ample, and $\mathcal{M}$ is relatively generated, both on $X \to Y$. Show that $\mathcal{L} \otimes \mathcal{M}$ is very ample.

(Hint: Reduce to the case where the target is affine. $\mathcal{L}$ induces a map to $\mathbb{P}^n_A$, and this corresponds to $n+1$ sections $s_0, \dots, s_n$ of $\mathcal{L}$. We also have a finite number m of sections $t_1, \dots, t_m$ of $\mathcal{M}$ which generate the stalks. Consider the $(n+1)m$ sections of $\mathcal{L} \otimes \mathcal{M}$ given by $s_i t_j$. Show that these sections are base-point-free, and hence induce a morphism to $\mathbb{P}^{(n+1)m-1}$. Show that it is a closed immersion.)

2.6. *Definition of ample and relatively ample.* Suppose X is a quasicompact scheme. We say an invertible sheaf $\mathcal{L}$ on X is *ample* if for all finite type sheaves $\mathcal{F}$, $\mathcal{F} \otimes \mathcal{L}^n$ is generated by global sections for $n \gg 0$. ("After finite twist, it is generated by global sections.") This is an *absolute* notion, not depending on a morphism.

2.7. *Example.* (a) If X is an affine scheme, and $\mathcal{L}$ is any invertible sheaf on X, then $\mathcal{L}$ is ample.
(b) If $X \to \operatorname{Spec} B$ is a projective morphism and $\mathcal{L}$ is a very ample invertible sheaf on X, then $\mathcal{L}$ is ample (by Serre vanishing, Theorem 4.2(ii), class 29, p. 5). (We may need B Noetherian here.)

We now give the relative version of this notion. Suppose $\pi : X \to Y$ is a morphism, and $\mathcal{L}$ is an invertible sheaf on X. Suppose that for every affine open subset $\operatorname{Spec} B$ of Y there is an n_0 such that $\mathcal{F} \otimes \mathcal{L}^{\otimes n}$ restricted to the preimage of $\operatorname{Spec} B$ is relatively generated by global sections for $n \geq n_0$. (In particular, π is quasicompact and quasiseparated — that was a hypothesis for relatively generated.) Then we say that $\mathcal{L}$ is *relatively ample* (with respect to π; although the reference to the morphism is often suppressed when it is clear from the context). It is also sometimes called π-ample. Warning: the n_0 depends on the affine open; we may not be able to take a single n_0 for all affine opens. We can, however, if Y is quasicompact, and hence we'll see this quasicompactness hypothesis on Y often.

Example. The examples of 2.7 naturally generalize.
(a) If $X \to Y$ is an affine morphism, and $\mathcal{L}$ is any invertible sheaf on X, then $\mathcal{L}$ is relatively ample.
(b) If $X \to Y$ is a projective morphism and $\mathcal{L}$ is a very ample invertible sheaf on X, then $\mathcal{L}$ is relatively ample. (We may need Y locally Noetherian here.)

2.8. *Easy Lemma. — Fix a positive integer* n.

(a) *If $\mathcal{L}$ is an invertible sheaf on a scheme X, then $\mathcal{L}$ is ample if and only if $\mathcal{L}^{\otimes n}$ is ample.*
(b) *If $\pi : X \to Y$ is a morphism, and $\mathcal{L}$ is an invertible sheaf on X, then $\mathcal{L}$ is relatively ample if and only if $\mathcal{L}^{\otimes n}$ is relatively ample.*

In general, statements about ample sheaves (such as (a) above) will have immediate analogues for statements about relatively ample sheaves where the target is quasicompact (such as (b) above), and I won't spell them out in the future. [I'm not sure what I meant by this comment about (b); I'll think about it.]

Proof. We prove (a); (b) is then immediate.

Suppose $\mathcal{L}$ is ample. Then for any finite type sheaf $\mathcal{F}$ on X, there is some m_0 such that for $m \geq m_0$, $\mathcal{F} \otimes \mathcal{L}^{\otimes m_0}$ is generated by global sections. Thus for $m' \geq m_0/n$, $\mathcal{F} \otimes (\mathcal{L}^{\otimes n})^{m'}$ is generated by global sections, so $\mathcal{L}^{\otimes n}$ is ample.

Suppose next that $\mathcal{L}^{\otimes n}$ is ample, and let $\mathcal{F}$ by any finite type sheaf. Then there is some m_0 such that $(\mathcal{F}) \otimes (\mathcal{L}^n)^m$, $(\mathcal{F} \otimes \mathcal{L}) \otimes (\mathcal{L}^n)^m$, $(\mathcal{F} \otimes \mathcal{L}^{\otimes 2}) \otimes (\mathcal{L}^n)^m$, $\dots (\mathcal{F} \otimes \mathcal{L}^{\otimes(m-1)}) \otimes (\mathcal{L}^n)^m$, are all generated by global sections for $m \geq m_0$. In other words, for $m' \geq nm_0$, $\mathcal{F} \otimes \mathcal{L}^{\otimes m'}$ is generated by global sections. Hence $\mathcal{L}$ is ample. □

Example: any positive degree invertible sheaf on a curve is ample. Reason: a high tensor power (such that the degree is at least $2g+1$) is very ample.

2.9. *Proposition. — In each of the following,* X *is a scheme,* $\mathcal{L}$ *is an ample invertible sheaf (hence* X *is quasicompact), and* $\mathcal{M}$ *is an invertible sheaf.*

(a) *(ample + generated = ample) If* $\mathcal{M}$ *is generated by global sections, then* $\mathcal{L} \otimes \mathcal{M}$ *is ample.*
(b) *(ample + ample = ample) If* $\mathcal{M}$ *is ample, then* $\mathcal{L} \otimes \mathcal{M}$ *is ample.*

Similar statements hold for quasicompact and quasiseparated morphisms and relatively ample and relatively generated.

Proof. (a) Suppose $\mathcal{F}$ is any finite type sheaf. Then by ampleness of $\mathcal{L}$, there is an n_0 such that for $n \geq n_0$, $\mathcal{F} \otimes \mathcal{L}^{\otimes n}$ is generated by global sections. Hence $\mathcal{F} \otimes \mathcal{L}^{\otimes n} \otimes \mathcal{M}^{\otimes n}$ is generated by global sections. Thus there is an n_0 such that for $n \geq n_0$, $\mathcal{F} \otimes (\mathcal{L} \otimes \mathcal{M})^{\otimes n}$ is generated by global sections. Hence $\mathcal{L} \otimes \mathcal{M}$ is ample.

(b) As $\mathcal{M}$ is ample, $\mathcal{M}^{\otimes n}$ is base-point-free for some $n > 0$. But $\mathcal{L}^{\otimes n}$ is ample, so by (a) $(\mathcal{L} \otimes \mathcal{M})^{\otimes n}$ is ample, so by Lemma 2.8, $\mathcal{L} \otimes \mathcal{M}$ is ample. □

3. Every ample on a proper has a tensor power that is very ample

We'll spend the rest of our discussion of ampleness considering consequences of the following very useful result.

3.1. *Theorem. — Suppose* $\pi : X \to Y$ *is proper and* $Y = \operatorname{Spec} B$ *is affine. If* $\mathcal{L}$ *is ample, then some tensor power of* $\mathcal{L}$ *is very ample.*

The converse follows from our earlier discussion, that very ample implies ample, Example 2.7(b).

Proof. I hope to type in a short proof at some point. For now, I'll content myself with referring to Hartshorne Theorem II.7.6. (He has more hypotheses, but his argument essentially applies in this more general situation.)

3.2. *Exercise.* Suppose $\pi : X \to Y$ is proper and Y is quasicompact. Show that if $\mathcal{L}$ is relatively ample on X, then some tensor power of $\mathcal{L}$ is very ample.

Serre vanishing holds for any relatively ample invertible sheaf for a proper morphism to a Noetherian base. More precisely:

3.3. *Corollary (Serre vanishing, take two). — Suppose $\pi : X \to Y$ is a proper morphism, Y is quasicompact, and $\mathcal{L}$ is a π-ample invertible sheaf on X. Then for any coherent sheaf $\mathcal{F}$ on X, for $m \gg 0$, $R^i\pi_*\mathcal{F} \otimes \mathcal{L}^{\otimes m} = 0$ for all $i > 0$.*

Proof. By Theorem 3.1, $\mathcal{L}^{\otimes n}$ very ample for some n, so π is projective. Apply Serre vanishing to $\mathcal{F} \otimes \mathcal{L}^{\otimes i}$ for $0 \leq i < n$. □

The converse holds, i.e. this in fact characterizes ampleness. For convenience, we state it for the case of an affine target.

3.4. *Theorem (Serre's criterion for ampleness). — Suppose that $\pi : X \to Y = \operatorname{Spec} B$ is a proper morphism, and $\mathcal{L}$ is an invertible sheaf on X such that for any finite type sheaf $\mathcal{F}$ on X, $\mathcal{F} \otimes \mathcal{L}^{\otimes n}$ is generated by global sections for $n \gg 0$. Then $\mathcal{L}$ is ample.*

Essentially the same statement holds for relatively ample and quasicompact target. *Exercise.* Give and prove the statement. **Whoops! Ziyu and Rob point out that I used Serre's criterion as the *definition* of ampleness (and similarly, relative ampleness). Thus this exercise is nonsense.**

3.5. *Proof of Serre's criterion.* I hope to type in a better proof before long, but for now I'll content myself with referring to Hartshorne, Proposition III.5.3.

3.6. *Exercise.* Use Serre's criterion for ampleness to prove that the pullback of ample sheaf on a projective scheme by a finite morphism is ample. Hence if a base-point-free invertible sheaf on a proper scheme induces a morphism to projective space that is finite onto its image, then it is is ample.

3.7. *Key Corollary. — Suppose $\pi : X \to \operatorname{Spec} B$ is proper, and $\mathcal{L}$ and $\mathcal{M}$ are invertible sheaves on X with $\mathcal{L}$ ample. Then $\mathcal{L}^{\otimes n} \otimes \mathcal{M}$ is very ample for $n \gg 0$.*

3.8. *Exercise.* Give and prove the corresponding statement for a relatively ample invertible sheaf over a quasicompact base.

Proof. The theorem says that $\mathcal{L}^{\otimes n}$ is very ample for $n \gg 0$. By the definition of ampleness, $\mathcal{L}^{\otimes n} \otimes \mathcal{M}$ is generated for $n \gg 0$. Tensor these together, using the above. □

A key implication of the key corollary is:

3.9. *Corollary.* — *Any invertible sheaf on a projective* $X \to \operatorname{Spec} B$ *is a difference of two very ample invertible sheaves.*

Proof. If $\mathcal{M}$ is any invertible sheaf, choose $\mathcal{L}$ very ample. Corollary 3.7 states that $\mathcal{M} \otimes \mathcal{L}^{\otimes n}$ is very ample. As $\mathcal{L}^{\otimes n}$ is very ample (Exercise 2.4), we can write $\mathcal{M}$ as the difference of two very ample sheaves: $\mathcal{M} \cong (\mathcal{M} \otimes \mathcal{L}^{\otimes n}) \otimes (\mathcal{L}^{\otimes n})^*$.

As always, we get a similar statement for relatively ample sheaves over a quasicompact base.

Here are two interesting consequences of Corollary 3.9.

3.10. *Exercise.* Suppose X a projective k-scheme. Show that every invertible sheaf is the difference of two *effective* Cartier divisors. Thus the groupification of the semigroup of effective Cartier divisors is the Picard group. Hence if you want to prove something about Cartier divisors on such a thing, you can study effective Cartier divisors.

(This is false if projective is replaced by proper — ask Sam Payne for an example.)

3.11. *Important exercise.* Suppose C is a generically reduced projective k-curve. Then we can define degree of an invertible sheaf $\mathcal{M}$ as follows. Show that $\mathcal{M}$ has a meromorphic section that is regular at every singular point of C. Thus our old definition (number of zeros minus number of poles, using facts about discrete valuation rings) applies. Prove the Riemann-Roch theorem for generically reduced projective curves. (Hint: our original proof essentially will carry through without change.)

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