FOUNDATIONS OF ALGEBRAIC GEOMETRY CLASS 28

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CONTENTS

Last day: More on properness. Rational maps.

Today: Curves.

(I also discussed rational maps a touch more, but I've included that in the class 27 notes for the sake of continuity.)

1. CURVES

Let's now use our technology to study something explicit! For our discussion here, we will temporarily define a *curve* to be an integral variety over k of dimension 1. (In particular, curves are reduced, irreducible, separated, and finite type over k.)

I gave an incomplete proof to the following proposition. Because I don't think I'll use it, I haven't tried to patch it. But if there is interest, I'll include the proof with the hole, in case one of you can figure out how to make it work. (We showed that each closed point gives a discrete valuation, and we showed that each discrete valuation gives a morphism from the Spec corresponding discrete valuation ring to the curve, but we didn't show that it was the local ring of the corresponding closed point. I would like to do this without invoking any algebra that we haven't yet proved.)

1.1. *Proposition. — Suppose C is a projective nonsingular curve. Then each closed point of C yields a discrete valuation ring, and hence a discrete valuation on $FF(C)$. This gives a bijection from closed points of C and discrete valuations on $FF(C)$.*

Thus a projective nonsingular curve is a convenient way of seeing all the discrete valuations at once, in a nice geometric package.

I had wanted to ask you the following exercise (for those with arithmetic proclivities), but I won't now: Suppose A is the ring of integers in a number field (i.e. the integral

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closure of $\mathbb{Z}$ in a finite field extension $K/\mathbb{Q}$ — $K = FF(A)$). Show that there is a natural bijection between discrete valuations on K are in bijection with the maximal ideals of A.

1.2. *Key Proposition. — Suppose C is a dimension 1 finite type k-scheme, and p is a nonsingular point of it. Suppose Y is a projective k-scheme. Then any morphism $C - p \to Y$ extends to $C \to Y$.*

Note: if such an extension exists, then it is unique: The non-reduced locus of C is a closed subset (we checked this earlier for any Noetherian scheme), not including p, so by replacing C by an open neighborhood of p that is reduced, we can use our recently-proved theorem that maps from reduced schemes to separated schemes are determined by their behavior on a dense open set (Important Theorem 3.3 in last day's notes).

I'd like to give two proofs, which are enlightening in different ways.

Proof 1. By restricting to an affine neighborhood of C, we can reduce to the case where C is affine.

We next reduce to the case where $Y = \mathbb{P}^n_k$. Here is how. Choose a closed immersion $Y \to \mathbb{P}^n_k$. If the result holds for $\mathbb{P}^n$, and we have a morphism $C \to \mathbb{P}^n$ with $C - p$ mapping to Y, then C must map to Y as well. Reason: we can reduce to the case where the source is an affine open subset, and the target is $\mathbb{A}^n_k \subset \mathbb{P}^n_k$ (and hence affine). Then the functions vanishing on $Y \cap \mathbb{A}^n_k$ pull back to functions that vanish at the generic point of C and hence vanish everywhere on C, i.e. C maps to Y.

Choose a uniformizer $t \in \mathfrak{m} - \mathfrak{m}^2$ in the local ring. By discarding the points of the vanishing set $V(t)$ aside from p, and taking an affine open subset of p in the remainder we reduce to the case where t cuts out precisely $\mathfrak{m}$ (i.e. $\mathfrak{m} = (y)$). Choose a dense open subset U of $C - p$ where the pullback of $\mathcal{O}(1)$ is trivial. Take an affine open neighborhood $\operatorname{Spec} A$ of p in $U \cup \{p\}$. Then the map $\operatorname{Spec} A - p \to \mathbb{P}^n$ corresponds to $n+1$ functions, say $f_0, \dots, f_n \in A_\mathfrak{m}$, not all zero. Let m be the smallest valuation of all the f_i. Then $[t^{-m} f_0; \dots; t^{-m} f_n]$ has all entries in A, and not all in the maximal ideal, and thus is defined at p as well. □

Proof 2. We extend the map $\operatorname{Spec} FF(C) \to Y$ to $\operatorname{Spec} \mathcal{O}_{C,p} \to Y$ as follows. Note that $\mathcal{O}_{C,p}$ is a discrete valuation ring. We show first that there is a morphism $\operatorname{Spec} \mathcal{O}_{C,p} \to \mathbb{P}^n$. The rational map can be described as $[a_0; a_1; \cdots; a_n]$ where $a_i \in \mathcal{O}_{C,p}$. Let m be the minimum valuation of the a_i, and let t be a uniformizer of $\mathcal{O}_{C,p}$ (an element of valuation 1). Then $[t^{-m} a_0; t^{-m} a_1; \dots t^{-m} a_n]$ is another description of the morphism $\operatorname{Spec} FF(\mathcal{O}_{C,p}) \to \mathbb{P}^n$, and each of the entries lie in $\mathcal{O}_{C,p}$, and not all entries lie in $\mathfrak{m}$ (as one of the entries has valuation 0). This same expression gives a morphism $\operatorname{Spec} \mathcal{O}_{C,p} \to \mathbb{P}^n$.

Our intuition now is that we want to glue the maps $\operatorname{Spec} \mathcal{O}_{C,p} \to Y$ (which we picture as a map from a germ of a curve) and $C - p \to Y$ (which we picture as the rest of the curve). Let $\operatorname{Spec} R \subset Y$ be an affine open subset of Y containing the image of $\operatorname{Spec} \mathcal{O}_{C,p}$. Let $\operatorname{Spec} A \subset C$ be an affine open of C containing p, and such that the image of $\operatorname{Spec} A - p$ in Y lies in $\operatorname{Spec} R$, and such that p is cut out scheme-theoretically by a single equation (i.e.

there is an element $t \in A$ such that (t) is the maximal ideal corresponding to $\mathfrak{p}$. Then R and A are domains, and we have two maps $R \to A_{(t)}$ (corresponding to $\operatorname{Spec} \mathcal{O}_{C,\mathfrak{p}} \to \operatorname{Spec} R$) and $R \to A_t$ (corresponding to $\operatorname{Spec} A - \mathfrak{p} \to \operatorname{Spec} R$) that agree "at the generic point", i.e. that give the same map $R \to FF(A)$. But $A_t \cap A_{(t)} = A$ in $FF(A)$ (e.g. by Hartogs' theorem — elements of the fraction field of A that don't have any poles away from t, nor at t, must lie in A), so we indeed have a map $R \to A$ agreeing with both morphisms. □

1.3. *Exercise (Useful practice!).* Suppose X is a Noetherian k-scheme, and Z is an irreducible codimension 1 subvariety whose generic point is a nonsingular point of X (so the local ring $\mathcal{O}_{X,Z}$ is a discrete valuation ring). Suppose $X \dashrightarrow Y$ is a rational map to a projective k-scheme. Show that the domain of definition of the rational map includes a dense open subset of Z. In other words, rational maps from Noetherian k-schemes to projective k-schemes can be extended over nonsingular codimension 1 sets. We saw this principle in action with the Cremona transformation, in Class 27 Exercise 4.6. (By the easy direction of the valuative criterion of separatedness, or the theorem of uniqueness of extensions of maps from reduced schemes to separated schemes — Theorem 3.3 of Class 27 — this map is unique.)

1.4. *Theorem. — If C is a nonsingular curve, then there is some projective nonsingular curve C' and an open immersion $C \hookrightarrow C'$.*

This proof has a bit of a different flavor than proofs we've seen before. We'll use make particular use of the fact that one-dimensional Noetherian schemes have a boring topology.

Proof. Given a nonsingular curve C, take a non-empty=dense affine open set, and take any non-constant function f on that affine open set to get a rational map $C \dashrightarrow \mathbb{P}^1$ given by $[1; f]$. As a dense open set of a dimension 1 scheme consists of everything but a finite number of points, by Proposition 1.2, this extends to a morphism $C \to \mathbb{P}^1$.

We now take the normalization of $\mathbb{P}^1$ in the function field $FF(C)$ of C (a finite extension of $FF(\mathbb{P}^1)$), to obtain $C' \to \mathbb{P}^1$. Now C' is normal, hence nonsingular (as nonsingular = normal in dimension 1). By the finiteness of integral closure, $C' \to \mathbb{P}^1$ is a finite morphism. Moreover, finite morphisms are projective, so by considering the composition of projective morphisms $C' \to \mathbb{P}^1 \to \operatorname{Spec} k$, we see that C' is projective over k. Thus we have an isomorphism $FF(C) \to FF(C')$, hence a rational map $C \dashrightarrow C'$, which extends to a morphism $C \to C'$ by Key Proposition 1.2.

Finally, I claim that $C \to C'$ is an open immersion. If we can prove this, then we are done. I note first that this is an injection of sets:

- the generic point goes to the generic point
- the closed points of C correspond to distinct valuations on $FF(C)$ (as C is separated, by the easy direction of the valuative criterion of separatedness)

Thus as sets, C is C' minus a finite number of points. As the topology on C and C' is the "cofinite topology" (i.e. the open sets include the empty set, plus everything minus a finite number of closed points), the map $C \to C'$ of topological spaces expresses C as a homeomorphism of C onto its image $\mathrm{im}(C)$. Let $f : C \to \mathrm{im}(C)$ be this morphism of schemes. Then the morphism $\mathcal{O}_{\mathrm{im}(C)} \to f_*\mathcal{O}_C$ can be interpreted as $\mathcal{O}_{\mathrm{im}(C)} \to \mathcal{O}_C$ (where we are identifying C and $\mathrm{im}(C)$ via the homeomorphism f). This morphism of sheaves is an isomorphism of stalks at all points $p \in \mathrm{im}(C)$ (it is the isomorphism the discrete valuation ring corresponding to $p \in C'$), and is hence an isomorphism. Thus $C \to \mathrm{im}(C)$ is an isomorphism of schemes, and thus $C \to C'$ is an open immersion. □

We now come to the big theorem of today (although the Key Proposition 1.2 above was also pretty big).

1.5. *Theorem. — The following categories are equivalent.*

(i) *nonsingular projective curves, and surjective morphisms.*
(ii) *nonsingular projective curves, and dominant morphisms.*
(iii) *nonsingular projective curves, and dominant rational maps*
(iv) *quasiprojective reduced curves, and dominant rational maps*
(v) *function fields of dimension* 1 *over* k*, and* k*-homomorphisms.*

(All morphisms and maps are assumed to be k-morphisms and k-rational maps, i.e. they are all over k. Remember that today we are working in the category of k-schemes.)

This has a lot of implications. For example, each quasiprojective reduced curve is isomorphic to precisely one projective nonsingular curve.

This leads to a motivating question that I mentioned informally last day (and that isn't in the notes). Suppose k is algebraically closed (such as $\mathbb{C}$). Is it true that all nonsingular projective curves are isomorphic to $\mathbb{P}^1_k$? Equivalently, are all quasiprojective reduced curves birational to $\mathbb{A}^1_k$? Equivalently, are all transcendence degree 1 extensions of k generated (as a field) by a single element? The answer (as most of you know) is *no*, but we can't yet see that.

1.6. *Exercise.* Show that all nonsingular proper curves are projective.

(We may eventually see that all reduced proper curves over k are projective, but I'm not sure; this will use the Riemann-Roch theorem, and I may just prove it for projective curves.)

Before we get to the proof, I want to mention a sticky point that came up in class. If $k = \mathbb{R}$, then we are allowing curves such as $\mathbb{P}^1_{\mathbb{C}}$ that "we don't want". One way of making this precise is noting that they are not geometrically irreducible (as $\mathbb{C}(t)_{\otimes_{\mathbb{R}}}\mathbb{C} \cong \mathbb{C}(t) \oplus \mathbb{C}(t)$). Another way is to note that this function field K does not satisfy $\overline{k} \cap K = k$ in $\overline{K}$. If this bothers you, then add it to each of the 5 categories. (For example, in (i)–(iii), we consider only nonsingular projective curves whose function field K satisfies $\overline{k} \cap K = k$ in $\overline{K}$.) If this doesn't bother you, please ignore this paragraph!

Proof. Any surjective morphism is a dominant morphism, and any dominant morphism is a dominant rational map, and each nonsingular projective curve is a quasiprojective curve, so we've shown (i) $\to$ (ii) $\to$ (iii) $\to$ (iv). To get from (iv) to (i), we first note that the nonsingular points on a quasiprojective reduced curve are dense. (One method, suggested by Joe: we know that normalization is an isomorphism away from a closed subset.) Given a dominant rational map between quasiprojective reduced curves $C \to C'$, we get a dominant rational map between their normalizations, which in turn gives a dominant rational map between their projective models $D \dashrightarrow D'$. The the dominant rational map is necessarily a morphism by Proposition 1.2, and then this morphism is necessarily projective and hence closed, and hence surjective (as the image contains the generic point of D', and hence its closure). Thus we have established (iv) $\to$ (i).

It remains to connect (i). Each dominant rational map of quasiprojective reduced curves indeed yields a map of function fields of dimension 1 (their fraction fields). Each function field of dimension 1 yields a reduced affine (hence quasiprojective) curve over k, and each map of two such yields a dominant rational map of the curves. □

1.7. Degree of a morphism between projective nonsingular curves.

We conclude with a useful fact: Any non-constant morphism from one projective nonsingular curve to another has a well-behaved degree, in a sense that we will now make precise. We will also show that any non-constant finite morphism from one nonsingular curve to another has a well-behaved degree in the same sense.

Suppose $f : C \to C'$ is a surjective (or equivalently, dominant) map of nonsingular projective curves.

It is a finite morphism. Here is why. (If we had already proved that quasifinite projective or proper morphisms with finite fibers were finite, we would know this. Once we *do* know this, the contents of this section would extend to the case where C is not necessarily non-singular.) Let C'' be the normalization of C' in the function field of C. Then we have an isomorphism $FF(C) \cong FF(C'')$ which leads to birational maps $C \dashleftarrow\dashrightarrow C''$ which extend to morphisms as both C and C'' are nonsingular and projective. Thus this yields an isomorphism of C and C''. But $C'' \to C$ is a finite morphism by the finiteness of integral closure.

We can then use the following proposition, which applies in more general situations.

1.8. *Proposition. — Suppose that $\pi : C \to C'$ is a surjective finite morphism, where C is an integral curve, and C' is an integral nonsingular curve. Then $\pi_* \mathcal{O}_C$ is locally free of finite rank.*

As π is finite, $\pi_* \mathcal{O}_C$ is a finite type sheaf on $\mathcal{O}'_C$. In case you care, the hypothesis "integral" on C' is redundant.

Before proving the proposition. I want to remind you what this means. Suppose d is the rank of this allegedly locally free sheaf. Then the fiber over any point of C with residue field K is the Spec of an algebra of dimension d over K. This means that the number of points in the fiber, counted with appropriate multiplicity, is always d.

As a motivating example, consider the map $\mathbb{Q}[y] \to \mathbb{Q}[x]$ given by $x \mapsto y^2$. (We've seen this example before.) I picture this as the projection of the parabola $x = y^2$ to the x-axis. (i) The fiber over $x = 1$ is $\mathbb{Q}[y]/(y^2 - 1)$, so we get 2 points. (ii) The fiber over $x = 0$ is $\mathbb{Q}[y]/(y^2)$ — we get one point, with multiplicity 2, arising because of the nonreducedness. (iii) The fiber over $x = -1$ is $\mathbb{Q}[y]/(y^2 + 1) \cong \mathbb{Q}[i]$ — we get one point, with multiplicity 2, arising because of the field extension. (iv) Finally, the fiber over the generic point $\operatorname{Spec} \mathbb{Q}(x)$ is $\operatorname{Spec} \mathbb{Q}(y)$, which is one point, with multiplicity 2, arising again because of the field extension (as $\mathbb{Q}(y)/\mathbb{Q}(x)$ is a degree 2 extension). We thus see three sorts of behaviors (as (iii) and (iv) are the same behavior). Note that even if you only work with algebraically closed fields, you will still be forced to this third type of behavior, because residue fields at generic points tend not to be algebraically closed (witness case (iv) above).

Note that we need C' to be nonsingular for this to be true. (I gave a picture of the normalization of a nodal curve as an example. A picture would help here.)

We will see the proof next day.

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