FOUNDATIONS OF ALGEBRAIC GEOMETRY CLASS 3

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CONTENTS

Last day: end of category theory background. Motivation for and definitions of presheaf, sheaf, stalk.

Today: Presheaves and sheaves. Morphisms thereof. Sheafification.

I will be away Wednesday Oct. 5 to Thursday Oct. 13. The *next* class will be Friday, October 14. That means there will be *no class* this Wednesday, or next Monday or Wednesday. If you want to be on the e-mail list (low traffic), and didn't sign up last day, please let me know.

Problem set 1 out today, due Monday Oct. 17.

1. WHERE WE WERE

At this point, you're likely wondering when we're going to get to some algebraic geometry. We'll start that next class. We're currently learning how to think about things correctly. When we define interesting new objects, we'll learn how we want them to behave because we know a little category theory.

1.1. **Category theory.** I think in the heat of the last lecture, I skipped something I shouldn't have. An abelian category has several properties. One of these is that the morphisms form abelian groups: $\mathrm{Hom}(A, B)$ is an abelian group. This behaves well with respect to composition. For example if $f, g : A \to B$, and $h : B \to C$, then $h \circ (f + g) = h \circ f + h \circ g$. There is an obvious dual statement, that I'll leave to you. This implies other things, such as for example $0 \circ f = 0$. I think I forgot to say the above. An abelian category also has

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a 0-object (an object that is both a final object and initial object). An abelian category has finite products. If you stopped there, you'd have the definition of an additive category.

In an additive category, you can define things like kernels, cokernels, images, epimorphisms, monomorphisms, etc. In an abelian category, these things behave just way you expect them to, from your experience with R-modules. I've put the definition in the last day's notes.

2. Presheaves and Sheaves

We then described presheaves and sheaves on a topological space X. I'm going to remind you of two examples, and introduce a third. The **first example** was of a sheaf of nice functions, say differentiable functions, which I will temporarily call $\mathcal{O}_X$. This is an example of a sheaf of rings.

The axioms are as follows. We can have sheaves of rings, groups, abelian groups, and sets.

To each open set, we associate a ring $\mathcal{F}(U)$. Elements of this ring are called *sections of the sheaf over* U. (Notational warning: Several notations are in use, for various good reasons: $\mathcal{F}(U) = \Gamma(\mathcal{F}, U) = H^0(\mathcal{F}, U)$. I will use them all.)

If $U \subset V$ is an inclusion of open sets, we have restriction maps $res_{V,U}; \mathcal{F}(V) \to \mathcal{F}(U)$.

The map $res_{U,U}$ must be the identity for all U.

If you take a section over a big open set, and restrict it to a medium open set, and then restrict that to a small open set, then you get the same thing as if you restrict the section on the big open set to the small open set all at once. In other words, if $U \hookrightarrow V \hookrightarrow W$, then the following diagram commutes:

$$\begin{tikzcd} \mathcal{F}(W) \arrow[rr, "\mathrm{res}_{W,V}"] \arrow[dr, "\mathrm{res}_{W,U}"'] & & \mathcal{F}(V) \arrow[dl, "\mathrm{res}_{V,U}"] \\ & \mathcal{F}(U) & \end{tikzcd}$$

A subtle point that you shouldn't worry about at the start are the sections over the empty set. $\mathcal{F}(\emptyset)$ should be the final object in the category under consideration (sets: a set with one element; abelian groups: 0; rings: the 0-ring). (I'm tentatively going to say that there is a 1-element ring. In other words, I will not assume that rings satisfy $1 \neq 0$. Every ring maps to the 0-ring. But it doesn't map to any other ring, because in a ring morphisms, 0 goes to 0, and 1 goes to 1, but in every ring beside this one, $0 \neq 1$. I think this convention will solve some problems, but it will undoubtedly cause others, and I may eat my words, so only worry about it if you really want to.)

Something satisfying the properties I've described is a *presheaf*. (For experts: a presheaf of rings is the same thing as a contravariant functor from the category of open sets to the category of rings, plus that final object annoyance, see problem set 1.)

Sections of presheaves $\mathcal{F}$ have *germs* at each point $x \in X$ where they are defined, and the set of germs is denoted $\mathcal{F}_x$, and is called the *stalk of $\mathcal{F}$ at* x. Elements of the stalk correspond to sections over some open set containing x. Two of these sections are considered the same if they agree on some smaller open set. If $\mathcal{F}$ is a sheaf of rings, then $\mathcal{F}_x$ is a ring, and ditto for rings replaced by other categories we like.

We add two more axioms to make this into a sheaf.

Identity axiom. If $\{U_i\}_{i \in I}$ is a cover of U, and $f_1, f_2 \in \mathcal{F}(U)$, and $\mathrm{res}_{U,U_i} f_1 = \mathrm{res}_{U,U_i} f_2$, then $f_1 = f_2$.

Gluability axiom. given $f_i \in \mathcal{F}(U_i)$ for all i, such that $\mathrm{res}_{U_i, U_i \cap U_j} f_i = \mathrm{res}_{U_j, U_i \cap U_j} f_j$ for all i, j, then there is some $f \in \mathcal{F}(U)$ such that $\mathrm{res}_{U,U_i} f = f_i$ for all i.

Example 2 (on problem set 1). Suppose we are given a continuous map $f : Y \to X$. The "sections of f" form a sheaf. More precisely, to each open set U of X, associate the set of continuous maps s to Y such that $f \circ s = id|_U$. This forms a sheaf. (Example for those who know this language: a vector bundle.)

Example 3: **Sheaf of $\mathcal{O}_X$-modules.** Suppose $\mathcal{O}_X$ is a sheaf of rings on X. Then we define the notion of a sheaf of $\mathcal{O}_X$-modules. We have a metaphor: rings is to modules, as sheaves of rings is to sheaves of modules.

There is only one possible definition that could go with this name, so let's figure out what it is. For each U, $\mathcal{F}(U)$ should be a $\mathcal{O}_X(U)$-module. Furthermore, this structure should behave well with respect to restriction maps. This means the following. If $U \hookrightarrow V$, then

$$\begin{tikzcd} \mathcal{O}_X(V) \times \mathcal{F}(V) \arrow[r, "\text{action}"] \arrow[d, "\mathrm{res}_{V,U}"] & \mathcal{F}(V) \arrow[d, "\mathrm{res}_{V,U}"] \\ \mathcal{O}_X(U) \times \mathcal{F}(U) \arrow[r, "\text{action}"] & \mathcal{F}(U) \end{tikzcd}$$

commutes. You should think about this later, and convince yourself that I haven't forgotten anything.

For category theorists: the notion of R-module generalizes the notion of abelian group, because an abelian group is the same thing as a $\mathbb{Z}$-module. It is similarly immediate that the notion of $\mathcal{O}_X$-module generalizes the notion of sheaf of abelian groups, because the latter is the same thing as a $\underline{\mathbb{Z}}$-module, where $\underline{\mathbb{Z}}$ is the locally constant sheaf with values in $\mathbb{Z}$. Hence when we are proving things about $\mathcal{O}_X$-modules, we are also proving things about sheaves of abelian groups. For experts: Someone pointed out that we can make the same notion of *presheaf* of $\mathcal{O}_X$-modules, where $\mathcal{O}_X$ is a *presheaf* of rings. In this setting, *presheaves* of abelian groups are the same as modules over the *constant presheaf* $\underline{\mathbb{Z}}^{\text{pre}}$. I doubt we will use this, so feel free to ignore it.

3. Morphisms of presheaves and sheaves

I'll now tell you how to map presheaves to each other; and similarly for sheaves. In other words, I am describing the *category of presheaves* and the *category of sheaves*.

A morphism of presheaves of sets $f : \mathcal{F} \to \mathcal{G}$ is a collection of maps $f_U : \mathcal{F}(U) \to \mathcal{G}(U)$ that commute with the restrictions, in the sense that: if $U \hookrightarrow V$ then

$$\begin{tikzcd} \mathcal{F}(V) \arrow[r, "f_V"] \arrow[d, "\mathrm{res}_{V,U}"] & \mathcal{G}(V) \arrow[d, "\mathrm{res}_{V,U}"] \\ \mathcal{F}(U) \arrow[r, "f_U"] & \mathcal{G}(U) \end{tikzcd}$$

commutes. (Notice: the underlying space remains X!) A morphism of sheaves is defined in the same way. (For category-lovers: a morphism of presheaves on X is a natural transformation of functors. This definition describes the category of sheaves on X as a full subcategory of the category of presheaves on X.)

A morphism of presheaves (or sheaves) of rings (or groups, or abelian groups, or $\mathcal{O}_X$-modules) is defined in the same way.

Exercise. Show morphisms of (pre)sheaves induces morphisms of stalks.

Interesting examples of morphisms of presheaves of abelian groups. Let $X = \mathbb{C}$ with the usual (analytic) topology, and define $\mathcal{O}_X$ to be the sheaf of holomorphic functions, and $\mathcal{O}_X^*$ to be the sheaf of *invertible (= nowhere* 0*) holomorphic functions*. This is a sheaf of abelian groups under multiplication. We have maps of presheaves

$$1 \longrightarrow \underline{\mathbb{Z}}^{\text{pre}} \xrightarrow{\times 2\pi i} \mathcal{O}_X \xrightarrow{\exp} \mathcal{O}_X^* \longrightarrow 1$$

where $\underline{\mathbb{Z}}^{\text{pre}}$ is the constant presheaf. This is *not* an exact sequence of presheaves, and it is worth figuring out why. (Hint: it is not exact at $\mathcal{O}_X$ or $\mathcal{O}_X^*$. Replacing $\underline{\mathbb{Z}}^{\text{pre}}$ with the locally constant sheaf $\underline{\mathbb{Z}}$ remedies the first, but not the second.)

Now abelian groups, and R-modules, form an abelian category — by which I just mean that you are used to taking kernels, images, etc. — and you might hope for the same for sheaves of abelian groups, and sheaves of $\mathcal{O}_X$-modules. That is indeed the case. Presheaves are easier to understand in this way.

The presheaves of abelian groups on X*, or* $\mathcal{O}_X$*-modules on* X*, form an abelian category.* If $f : \mathcal{F} \to \mathcal{G}$ is a morphism of presheaves, then $\ker f$ is a presheaf, with $(\ker f)(U) = \ker f_U$, and $(\operatorname{im} f)(U) = \operatorname{im} f_U$. The resulting things are indeed presheaves. For example, if $U \hookrightarrow V$, there is a natural map $\mathcal{G}(V)/f_V(\mathcal{F}(V)) \to \mathcal{G}(U)/f_U(\mathcal{F}(U))$, as we observe by chasing the following diagram:

$$\begin{tikzcd}
\mathcal{F}(V) \arrow[r] \arrow[d, "\mathrm{res}_{V,U}"] & \mathcal{G}(V) \arrow[r] \arrow[d, "\mathrm{res}_{V,U}"] & \mathcal{G}(V)/\mathcal{F}(V) \arrow[r] \arrow[d, dotted] & 0 \\
\mathcal{F}(U) \arrow[r] & \mathcal{G}(U) \arrow[r] & \mathcal{G}(U)/\mathcal{F}(U) \arrow[r] & 0.
\end{tikzcd}$$

Thus I have defined $\mathcal{G}/\mathcal{F}$, by showing what its sections are, and what its restriction maps are. I have to check that it restriction maps compose — **exercise**. Hence I've defined a presheaf. I still have to convince you that it deserves to be called a cokernel. **Exercise.** Do this. It is less hard than you might think. Here is the definition of cokernel of $g : \mathcal{F} \to \mathcal{G}$. It is a morphism $h : \mathcal{G} \to \mathcal{H}$ such that $h \circ g = 0$, and for any $i : \mathcal{G} \to \mathcal{I}$ such that $i \circ g = 0$, there is a unique morphism $j : \mathcal{H} \to \mathcal{I}$ such that $j \circ h = i$:

$$\begin{tikzcd}
\mathcal{F} \arrow[r] & \mathcal{G} \arrow[r] \arrow[dr] & \mathcal{H} \arrow[d, dotted, "\exists!"] \\
& & \mathcal{I}
\end{tikzcd}$$

(Translation: cokernels in an additive category are defined by a universal property. Hence if they exist, they are unique. We are checking that our construction satisfies the universal property.)

Punchline: The presheaves of $\mathcal{O}_X$-modules is an abelian category, and as nice as can be. We can define terms such as *subpresheaf*, *image presheaf*, *quotient presheaf*, *cokernel presheaf*. You construct kernels, quotients, cokernels, and images open set by open set. (Quotients are special cases of cokernels.)

Exercise. In particular: if $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \cdots \to \mathcal{F}_n \to 0$ is an exact sequence of presheaves, then $0 \to \mathcal{F}_1(U) \to \mathcal{F}_2(U) \to \cdots \to \mathcal{F}_n(U) \to 0$ is also an exact sequence for all U, and vice versa.

However, we are interested in more geometric objects, sheaves, where things are can be understood in terms of their local behavior, thanks to the identity and gluing axioms.

3.1. The category of sheaves of $\mathcal{O}_X$-modules is trickier. It turns out that the kernel of a morphism of sheaves is also sheaf. **Exercise.** Show that this is true. (Confusing translation: this subpresheaf of a sheaf is in fact also a sheaf.) Thus we have the notion of a **subsheaf**.

But other notions behave weirdly.

Example: image sheaf. We don't need an abelian category to talk about images — the notion of image makes sense for a map of sets. And the notion of image is a bit problematic even for sheaves of sets. Let's go back to our example of $\mathcal{O}_X \xrightarrow{\exp} \mathcal{O}_X^*$. What is the image *presheaf*? Well, if U is a *simply connected* open set, then this is surjective: every non-zero holomorphic function on a simply connected set has a logarithm (in fact many). However, this is not true if U is not simply connected — the function $f(z) = z$ on $\mathbb{C} - 0$ does *not* have a logarithm.

However, it locally does.

So what do we do? Answer 1: throw up our hands. Answer 2: Develop a new definition of image. We can't just define anything — we need to figure out what we want the image to be. Answer: category theoretic definition.

The patch: sheafification. Define **sheafification** of a presheaf by universal property: $\mathcal{F} \to \mathcal{F}^{\mathrm{sh}}$. Hence if it is exists, it is unique up to unique isomorphism. (This is analogous to the method of getting a group from a semigroup, see last day's notes.)

(Category-lovers: this says that sheafification is left-adjoint to the forgetful functor. This is just like groupification.)

Theorem (later today): Sheafification exists. (The specific construction will later be useful, but we won't need anything but the universal property right now.)

In class, I attempted to show that the sheafification of the image presheaf satisfies the universal property of the image sheaf, but I realized that I misstated the property. Instead, I will let you show that the sheaf of the cokernel presheaf satisfies the universal property of the cokernel sheaf. See the notes about one page previous for the definition of the cokernel.

Exercise. Do this.

Possible exercise. I'll tell you the definition of the image sheaf, and you can check.

Remark for experts: someone pointed out in class that likely the same arguments apply without change whenever you have an adjoint to a forgetful functor.

In short: $\mathcal{O}_X$-modules form an abelian category. To define image and cokernel (and quotient), you need to sheafify.

3.2. *Exercise.* Suppose $f : \mathcal{F} \to \mathcal{G}$ is a morphism of sheaves. Show that there are natural isomorphisms $\operatorname{im} f \cong \mathcal{F}/\ker f$ and $\operatorname{coker} f \cong \mathcal{G}/\operatorname{im} f$.

Tensor products of $\mathcal{O}_X$-modules: also requires sheafification.

3.3. *Exercise.* Define what we should mean by tensor product of two $\mathcal{O}_X$-modules. Verify that this construction satisfies your definition. (Hint: sheafification is required.)

3.4. *Left-exactness of the global section functor.* Left-exactness of global sections; hints of cohomology. More precisely:

$$0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$$

implies

$$0 \to \mathcal{F}(U) \to \mathcal{G}(U) \to \mathcal{H}(U)$$

is exact. Give example where not right exact, (Hint: $0 \to \mathbb{Z} \to \mathcal{O} \to \mathcal{O}^* \to 0$.)

Caution: The cokernel in the category of sheaves *is* a presheaf, but it is *isn't* the cokernel in the category of presheaves.

3.5. Important Exercise. Show the same thing (3.4) is true for pushforward sheaves. (The previous case is the case of a map from U to a point.)

4. Stalks, and sheafification

4.1. *Important exercise.* Prove that a section of a sheaf is determined by its germs, i.e.

$$\Gamma(U, \mathcal{F}) \to \prod_{x \in U} \mathcal{F}_x$$

is injective. (Hint: you won't use the gluability axiom. So this is true for separated presheaves.) [Answer: Suppose $f, g \in \Gamma(U, \mathcal{F})$, with $f_x = g_x$ in $\mathcal{F}_x$ for all $x \in U$. In terms of the concrete interpretation of stalks, $f_x = (U, f)$ and $g_x = (U, g)$, and $(U, f) = (U, g)$ means that there is an open subset U_x of U, containing x, such that $f|_{U_x} = g|_{U_x}$. The U_x cover U, so by the identity axiom for this cover of U, $f = g$.]

Corollary. In particular, if a sheaf has all stalks 0, then it is the 0-sheaf.

4.2. Morphisms and stalks.

4.3. *Exercise.* Show that morphisms of presheaves induce morphisms of stalks.

4.4. *Exercise.* Show that morphisms of sheaves are determined by morphisms of stalks. Hint # 1: you won't use the gluability axiom. So this is true of morphisms of separated presheaves.) Hint # 2: study the following diagram.

(1)
$$\begin{tikzcd} \mathcal{F}(U) \arrow[r] \arrow[d, hook] & \mathcal{G}(U) \arrow[d, hook] \\ \prod_{x \in U} \mathcal{F}_x \arrow[r] & \prod_{x \in U} \mathcal{G}_x \end{tikzcd}$$

4.5. *Exercise.* Show that a morphism of sheaves is an isomorphism if and only if it induces an isomorphism of all stalks. (Hint: Use (1). Injectivity uses from the previous exercise. Surjectivity will use gluability.)

4.6. *Exercise.* (a) Show that Exercise 4.1 is false for general presheaves. (Hint: take a 2-point space with the discrete topology, i.e. every subset is open.)
(b) Show that Exercise 4.4 is false for general presheaves. (Hint: a 2-point space suffices.)
(c) Show that Exercise 4.5 is false for general presheaves.

4.7. Description of sheafification. Suppose $\mathcal{F}$ is a presheaf on a topological space X. We define $\mathcal{F}^{sh}$ as follows. Sections over $U \subset X$ are stalks at each point, with compatibility conditions (to each element of the stalk, there is a representative (g, U) with g restricting correctly to all stalks in U). More explicitly:

$$\mathcal{F}^{sh}(U) := \{(f_x \in \mathcal{F}_x)_{x \in U} : \forall x \in U, \exists x \subset U_x \subset U, F^x \in \mathcal{F}(U_x) : F^x_y = f_y \forall y \in U_x\}.$$

(Those who want to worry about the empty set are welcome to.)

This is clearly a sheaf: we have restriction maps; they commute; we have identity and gluability.

4.8. For any morphism of presheaves $\phi : \mathcal{F} \to \mathcal{G}$, we get a natural induced morphism of sheaves $\phi^{sh} : \mathcal{F}^{sh} \to \mathcal{G}^{sh}$.

We have a natural presheaf morphism $\mathcal{F} \to \mathcal{F}^{sh}$. This induces a natural morphism of stalks $\mathcal{F}_x \to \mathcal{F}^{sh}_x$ (Exercise 4.3). Hence if $\mathcal{F}$ is a sheaf already, then $\mathcal{F} \to \mathcal{F}^{sh}$ is an isomorphism, by Exercise 4.5. If we knew that $\mathcal{F}^{sh}$ satisfied the universal property of sheafification, this would have been immediate by abstract nonsense, but we don't know that. In fact, we'll show that now. Suppose we have the solid arrows in

$$\begin{tikzcd} \mathcal{F} \arrow[r] \arrow[dr] & \mathcal{F}^{sh} \arrow[d, dotted] \\ & \mathcal{G}. \end{tikzcd}$$

We want to show that there exists a dashed arrow as in the diagram, making the diagram commute, and we want to show that it is unique. By 4.8, $\mathcal{F} \to \mathcal{G}$ induces a morphism $\mathcal{F}^{sh} \to \mathcal{G}^{sh} = \mathcal{G}$, so we have existence.

For uniqueness: as morphisms of sheaves are determined by morphisms of stalks (Exercise 4.4), and for any $x \in X$, we have a commutative diagram

$$\begin{tikzcd} \mathcal{F}_x \arrow[r, "="] \arrow[dr] & \mathcal{F}^{sh}_x \arrow[d] \\ & \mathcal{G}_x, \end{tikzcd}$$

we are done. Thus $\mathcal{F} \to \mathcal{F}^{sh}$ is indeed the sheafification.

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