FOUNDATIONS OF ALGEBRAIC GEOMETRY CLASS 1

CONTENTS

Today: About this course. Why algebraic geometry? Motivation and program. Crash course in category theory: universal properties, Yoneda's lemma.

1. WELCOME

Welcome! This is Math 216A, Foundations of Algebraic Geometry, the first of a three-quarter sequence on the topic. I'd like to tell you a little about what I intend with this course.

Algebraic geometry is a subject that somehow connects and unifies several parts of mathematics, including obviously algebra and geometry, but also number theory, and depending on your point of view many other things, including topology, string theory, etc. As a result, it can be a handy thing to know if you are in a variety of subjects, notably number theory, symplectic geometry, and certain kinds of topology. The power of the field arises from a point of view that was developed in the 1960's in Paris, by the group led by Alexandre Grothendieck. The power comes from rather heavy formal and technical machinery, in which it is easy to lose sight of the intuitive nature of the objects under consideration. This is one reason why it used to strike fear into the hearts of the uninitiated.

The rough edges have been softened over the ensuing decades, but there is an inescapable need to understand the subject on its own terms.

This class is intended to be an experiment. I hope to try several things, which are mutually incompatible. Over the year, I want to cover the foundations of the subject fairly completely: the idea of varieties and schemes, the morphisms between them, their properties, cohomology theories, and more. I would like to do this rigorously, while

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trying hard to keep track of the geometric intuition behind it. This is the second time I will have taught such a class, and the first time I'm going to try to do this without working from a text. So in particular, I may find that I talk myself into a corner, and may tell you about something, and then realize I'll have to go backwards and say a little more about an earlier something.

Some of you have asked what background will be required, and how fast this class will move. In terms of background, I'm going to try to assume as little as possible, ideally just commutative ring theory, and some comfort with things like prime ideals and localization. *(All my rings will be commutative, and have unit!)* The more you know, the better, of course. But if I say things that you don't understand, please slow me down in class, and also talk to me after class. Given the amount of material that there is in the foundations of the subject, I'm afraid I'm going to move faster than I would like, which means that for you it will be like drinking from a firehose, as one of you put it. If it helps, I'm very happy to do my part to make it easier for you, and I'm happy to talk about things outside of class. I also intend to post notes for as many classes as I can. They will usually appear before the next class, but not always.

In particular, this will not be the type of class where you can sit back and hope to pick up things casually. The only way to avoid losing yourself in a sea of definitions is to become comfortable with the ideas by playing with examples.

To this end, I intend to give problem sets, to be handed in. They aren't intended to be onerous, and if they become so, please tell me. But they *are* intended to force you to become familiar with the ideas we'll be using.

Okay, I think I've said enough to scare most of you away from coming back, so I want to emphasize that I'd like to do everything in my power to make it better, short of covering less material. The best way to get comfortable with the material is to talk to me on a regular basis about it.

One other technical detail: you'll undoubtedly have noticed that this class is schedule for Mondays, Wednesdays, and Fridays, 9–10:30, $4\frac{1}{2}$ hours per week, not the usual 3. That's not because I'm psychotic; it was presumably a mistake. So I'm going to take advantage of it, and most weeks just meet two days a week, and I'll propose usually meeting on Mondays and Wednesday. I'll be away for some days, and so I'll make up for it by meeting on Fridays as well some weeks. I'll warn you well in advance.

Office hours: I haven't decided if it will be useful to have formal office hours rather than being available to talk after class, and also on many days by appointment. One possibility would be to have office hours on the 3rd day of the week during the time scheduled for class. Another is to have it some afternoon. I'm open to suggestions.

Okay, let's get down to business. I'd like to say a few words about what algebraic geometry is about, and then to start discussing the machinery.

Texts: Here are some books to have handy. Hartshorne's *Algebraic Geometry* has most of the material that I'll be discussing. It isn't a book that you should sit down and read, but you might find it handy to flip through for certain results. It should be at the bookstore, and is on 2-day reserve at the library. Mumford's *Red Book of Varieties and Schemes* has a good deal of the material I'll be discussing, and with a lot of motivation too. That is also on 2-day reserve in the library. The second edition is strictly worse than the 1st, because someone at Springer retyped it without understanding the math, introducing an irritating number of errors. If you would like something gentler, I would suggest Shafarevich's books on algebraic geometry. Another excellent foundational reference is Eisenbud and Harris' book *The geometry of schemes*, and Harris' earlier book *Algebraic geometry* is a beautiful tour of the subject.

For background, it will be handy to have your favorite commutative algebra book around. Good examples are Eisenbud's *Commutative Algebra with a View to Algebraic Geometry*, or Atiyah and Macdonald's *Commutative Algebra*. If you'd like something with homological algebra, category theory, and abstract nonsense, I'd suggest Weibel's book *Introduction to Homological Algebra*.

2. WHY ALGEBRAIC GEOMETRY?

It is hard to define algebraic geometry in its vast generality in a couple of sentences. So I'll talk around it a bit.

As a motivation, consider the study of manifolds. Real manifolds are things that locally look like bits of real n-space, and they are glued together to make interesting shapes. There is already some subtlety here — when you glue things together, you have to specify what kind of gluing is allowed. For example, if the transition functions are required to be differentiable, then you get the notion of a differentiable manifold.

A great example of a manifold is a submanifold of $\mathbb{R}^n$ (consider a picture of a torus). In fact, any compact manifold can be described in such a way. You could even make this your definition, and not worry about gluing. This is a good way to think about manifolds, but not the best way. There is something arbitrary and inessential about defining manifolds in this way. Much cleaner is the notion of an *abstract manifold*, which is the current definition used by the mathematical community.

There is an even more sophisticated way of thinking about manifolds. A differentiable manifold is obviously a topological space, but it is a little bit more. There is a very clever way of summarizing what additional information is there, basically by declaring what functions on this topological space are differentiable. The right notion is that of a sheaf, which is a simple idea, that I'll soon define for you. It is true, but non-obvious, that this ring of functions that we are declaring to be differentiable determines the differentiable manifold structure.

Very roughly, algebraic geometry, at least in its geometric guise, is the kind of geometry you can describe with polynomials. So you are allowed to talk about things like $y^2 = x^3 + x$, but not $y = \sin x$. So some of the fundamental geometric objects under consideration are things in n-space cut out by polynomials. Depending on how you define them, they are called *affine varieties* or *affine schemes*. They are the analogues of the patches on a manifold. Then you can glue these things together, using things that you can describe with polynomials, to obtain more general varieties and schemes. So then we'll have these algebraic objects, that we call varieties or schemes, and we can talk about maps between them, and things like that.

In comparison with manifold theory, we've really restricted ourselves by only letting ourselves use polynomials. But on the other hand, we have gained a huge amount too. First of all, we can now talk about things that aren't smooth (that are *singular*), and we can work with these things. (One thing we'll have to do is to define what we mean by smooth and singular!) Also, we needn't work over the real or complex numbers, so we can talk about arithmetic questions, such as: what are the rational points on $y^2 = x^3 + x^2$? (Here, we work over the field $\mathbb{Q}$.) More generally, the recipe by which we make geometric objects out of things to do with polynomials can generalize drastically, and we can make a geometric object out of rings. This ends up being surprisingly useful — all sorts of old facts in algebra can be interpreted geometrically, and indeed progress in the field of commutative algebra these days usually requires a strong geometric background.

Let me give you some examples that will show you some surprising links between geometry and number theory. To the ring of integers $\mathbb{Z}$, we will associate a smooth curve $\operatorname{Spec}\mathbb{Z}$. In fact, to the ring of integers in a number field, there is always a smooth curve, and to its orders (subrings), we have singular = non-smooth curves.

An old flavor of Diophantine question is something like this. Given an equation in two variables, $y^2 = x^3 + x^2$, how many rational solutions are there? So we're looking to solve this equation over the field $\mathbb{Q}$. Instead, let's look at the equation over the field $\mathbb{C}$. It turns out that we get a complex surface, perhaps singular, and certainly non-compact. So let me separate all the singular points, and compactify, by adding in points. The resulting thing turns out to be a compact oriented surface, so (assuming it is connected) it has a genus g, which is the number of holes it has. For example, $y^2 = x^3 + x^2$ turns out to have genus 0. Then Mordell conjectured that if the genus is at least 2, then there are at most a finite number of rational solutions. The set of complex solutions somehow tells you about the number of rational solutions! Mordell's conjecture was proved by Faltings, and earned him a Fields Medal in 1986. As an application, consider Fermat's Last Theorem. We're looking for integer solutions to $x^n + y^n = z^n$. If you think about it, we are basically looking for rational solutions to $X^n + Y^n = 1$. Well, it turns out that this has genus $\binom{n-1}{2}$ — we'll verify something close to this at some point in the future. Thus if n is at least 4, there are only a finite number of solutions. Thus Falting's Theorem implies that for each $n \geq 4$, there are only a finite number of counterexamples to Fermat's last theorem. Of course, we now know that Fermat is true — but Falting's theorem applies much more widely — for example, in more variables. The equations $x^3 + y^2 + z^14 + xy + 17 = 0$ and $3x^{14} + x^{34}y + \cdots = 0$, assuming their complex solutions form a surface of genus at least 2, which they probably do, have only a finite number of solutions.

So here is where we are going. Algebraic geometry involves a new kind of "space", which will allow both singularities, and arithmetic interpretations. We are going to define these spaces, and define maps between them, and other geometric constructions such as vector bundles and sheaves, and pretty soon, cohomology groups.

In order to think about these notions clearly and cleanly, it really helps to use the language of categories. There is not much to know about categories to get started; it is just a very useful language.

Here is an informal definition. I won't give you the precise definition unless you really want me to. A category has some *objects*, and some maps, or *morphisms*, between them. (For the pedants, I won't worry about sets and classes. And I'm going to accept the axiom of choice.) *The prototypical example to keep in mind is the category of sets.* The objects are sets, and the morphisms are maps of sets. Another good example is that of vector spaces over your favorite filed k. The objects are k-vector spaces, and the morphisms are linear transformations.

For each object, there is always an *identity morphism* from from the object to itself. There is a way of composing morphisms: if you have a morphism $f : A \to B$ and another $g : B \to C$, then there is a composed morphism $g \circ f : A \to C$. I could be pedantic and say that we have a map of sets $\mathrm{Mor}(A, B) \times \mathrm{Mor}(B, C) \to \mathrm{Mor}(A, C)$. Composition is associative: $(h \circ g) \circ f = h \circ (g \circ f)$. When you compose with the identity, you get the same thing.

Exercise. A category in which each morphism is an isomorphism is called a *groupoid*. (a) A perverse definition of a group is: a groupoid with one element. Make sense of this. (b) Describe a groupoid that is not a group. (This isn't an important notion for this course. The point of this exercise is to give you some practice with categories, by relating them to an object you know well.)

Here are a couple of other important categories. If R is a ring, then R-modules form a category. In the special case where R is a field, we get the category of vector spaces. There is a category of rings, where the objects are rings, and the morphisms are morphisms of rings (which I'll assume send 1 to 1).

If we have a category, then we have a notion of isomorphism between two objects (if we have two morphisms $f : A \to B$ and $g : B \to A$, *both* of whose compositions are the identity on the appropriate object), and a notion of automorphism.

3.1. **Functors.**

A *covariant functor* is a map from one category to another, sending objects to objects, and morphisms to morphisms, such that everything behaves the way you want it to; if $F : \mathcal{A} \to \mathcal{B}$, and $a_1, a_2 \in \mathcal{A}$, and $m : a_1 \to a_2$ is a morphism in $\mathcal{A}$, then $F(m)$ is a morphism from $F(a_1) \to F(a_2)$ in B. Everything composes the way it should.

Example: If $\mathcal{A}$ is the category of complex vector spaces, and $\mathcal{B}$ is the category of sets, then there is a forgetful functor where to a complex vector space, we associate the set of its elements. Then linear transformations certainly can be interpreted as set maps.

A *contravariant functor* is just the same, except the arrows switch directions: in the above language, $F(m)$ is now an arrow from $F(a_2)$ to $F(a_1)$.

Example: If $\mathcal{A}$ is the category complex vector spaces, then taking duals gives a contravariant functor $\mathcal{A} \to \mathcal{A}$. Indeed, to each linear transformation $V \to W$, we have a dual transformation $W^* \to V^*$.

3.2. **Universal properties.** Given some category that we come up with, we often will have ways of producing new objects from old. In good circumstances, such a definition can be made using the notion of a *universal property*. Informally, we wish that there is an object with some property. We first show that if it exists, then it is essentially unique, or more precisely, is unique up to unique isomorphism. Then we go about constructing an example of such an object.

A good example of this, that you may well have seen, is the notion of a tensor product of R-modules. The way in which it is often defined is as follows. Suppose you have two R-modules M and N. Then the tensor product $M \otimes_R N$ is often first defined for people as follows: elements are of the form $m \otimes n$ ($m \in M$, $n \in N$), subject to relations $(m_1+m_2)\otimes n = m_1 \otimes n + m_2 \otimes n$, $m \otimes (n_1+n_2) = m\otimes n_1 + m \otimes n_2$, $r(m\otimes n) = (rm)\otimes n = m \otimes n$ (where $r \in R$).

Special case: if R is a field k, we get the tensor product of vector spaces.

Exercise (if you haven't seen tensor products before). Calculate $\mathbb{Z}/10 \otimes_{\mathbb{Z}} \mathbb{Z}/12$. (The point of this exercise is to give you a very little hands-on practice with tensor products.)

This is a weird definition!! And this is a clue that it is a "wrong" definition. A better definition: notice that there is a natural R-bilinear map $M \times N \to M \otimes_R N$. *Any* R-bilinear map $M \times N \to C$ factors through the tensor product uniquely: $M \times N \to M \otimes_R N \to C$. This is kind of clear when you think of it.

I could almost take this as the *definition* of the tensor product. Because if I could create something satisfying this property, $(M \otimes_R N)'$, and you were to create something else $(M \otimes_R N)''$, then by my universal property for $C = (M \otimes_R N)''$, there would be a unique map $(M \otimes_R N)' \to (M \otimes_R N)''$ interpolating $M \times N \to (M \otimes_R N)''$, and similarly by your universal property there would be a unique universal map $(M \otimes_R N)'' \to (M \otimes_R N)'$. The composition of these two maps in one order

$$(M \otimes_R N)' \to (M \otimes_R N)'' \to (M \otimes_R N)'$$

has to be the identity, by the universal property for $C = (M \otimes_R N)'$, and similarly for the other composition. Thus we have shown that these two maps are inverses, and our two spaces are isomorphic. In short: our two definitions may not be the *same*, but there is a canonical isomorphism between them. Then the "usual" construction works, but someone else may have another construction which works just as well.

I want to make three remarks. First, if you have never seen this sort of argument before, then you might think you get it, but you don't. So you should go back over the notes, and think about it some more, because it is rather amazing. Second, the language I would use to describe this is as follows: There is an R-bilinear map $t : M \times N \to M \otimes_R N$, unique up to unique isomorphism, defined by the following universal property: for any R-bilinear map $s : M \times N \to C$ there is a unique $f : M \otimes_R N \to C$ such that $s = f \circ t$. Third, you might notice that I didn't use much about the R-module structure, and indeed I can vary this to get a very general statement. This takes us to a powerful fact, that is very zen: it is very deep, but also very shallow. It's hard, but easy. It is black, but white. I'm going to tell you about it, and it will be mysterious, but then I'll show you some concrete examples.

Here is a motivational example: the notion of **product**. You have likely seen product defined in many cases, for example the notion of a product of manifolds. In each case, the definition agreed with your intuition of what a product should be. We can now make this precise. I'll describe product in the category of sets, in a categorical manner. Given two sets M and N, there is a unique set $M \times N$, along with maps to M and N, such that for *any other set* S *with maps to* M *and* N, this map must factor *uniquely* through $M \times N$:

$$\begin{array}{ccc} S & & \\ & \overset{\exists!}{\searrow} & \\ & M \times N & \longrightarrow N \\ & \downarrow & \\ & M. & \end{array}$$

You can immediately check that this agrees with the usual definition. But it has the advantage that we now have a definition in any category! The product may not exist, but if it does, then we know that it is unique up to unique isomorphism! (Explain.) This is handy even in cases that you understand. For example, one way of defining the product of two manifolds M and N is to cut them both up in to charts, then take products of charts, then glue them together. But if I cut up the manifolds in one way, and you cut them up in another, how do we know our resulting manifolds are the "same"? We could wave our hands, or make an annoying argument about refining covers, but instead, we should just show that they are indeed products, and hence the "same" (aka isomorphic).

3.3. **Yoneda's Lemma.** I want to begin with an easy fact that I'll state in a complicated way. Suppose we have a category $\mathcal{C}$. This isn't scary — just pick your favorite friendly low-brow category. Pick an object in your category $A \in \mathcal{C}$. Then for any object $C \in \mathcal{C}$, we have a set of morphisms $\mathrm{Mor}(C, A)$. If we have a morphism $f : B \to C$, we get a map of sets

$$\mathrm{Mor}(C, A) \to \mathrm{Mor}(B, A), \tag{1}$$

just by composition: given a map from C to A, we immediately get a map from B to A by precomposing with f. In fancy language, we have a contravariant functor from the category $\mathcal{C}$ to the category of sets $\underline{\mathtt{Sets}}$. Yoneda's lemma, or at least part of it, says that this functor determines A up to unique isomorphism. Translation: If we have two objects A and A', and isomorphisms

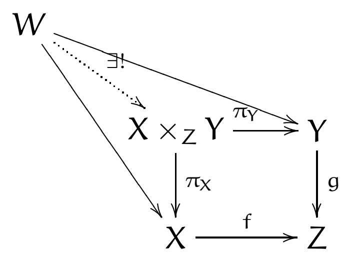
$$i_C : \mathrm{Mor}(C, A) \to \mathrm{Mor}(C, A') \tag{2}$$

that commute with the maps (1), then the i_C must be induced from a unique morphism $A \to A'$.

Important Exercise. Prove this. This sounds hard, but it really is not. This statement is so general that there are really only a couple of things that you could possibly try. For example, if you're hoping to find an isomorphism $A \to A'$, where will you find it? Well, you're looking for an element $\mathrm{Mor}(A, A')$. So just plug in $C = A$ to (2), and see where the identity goes. (Everyone should prove Yoneda's Lemma once in their life. This is your chance.)

Remark. There is an analogous statement with the arrows reversed, where instead of maps into A, you think of maps *from* A.

Example: Fibered products. Suppose we have morphisms $X, Y \to Z$. Then the *fibered product* is an object $X \times_Z Y$ along with morphisms to X and Y, where the two compositions $X \times_Z Y \to Z$ agree, such that given any other object W with maps to X and Y (whose compositions to Z agree), these maps factor through some unique $W \to X \times_Z Y$:



The right way to interpret this is first to think about what it means in the category of sets. I'll tell you it, and let you figure out why I'm right: $X \times_Z Y = \{(x \in X, y \in Y) : f(x) = g(y)\}$.

In any category, we can make this definition, and we know thanks to Yoneda that if it exists, then it is unique up to unique isomorphism, and so we should reasonably be allowed to give it the name $X \times_Z Y$. We know what maps to it are: they are precisely maps to X and maps to Y that agree on maps to Z.

(Remark for experts: if our category has a final object, then the fibered product over the final object is just the product.)

The notion of fibered product will be important for us later.

Exercises on fibered product. (a) Interpret fibered product in the category of sets: If we are given maps from sets X and Y to the set Z, interpret $X \times_Z Y$. (This will help you build intuition about this concept.)
(b) A morphism $f : X \to Y$ is said to be a a **monomorphism** if any two morphisms $g_1, g_2 : Z \to X$ such that $f \circ g_1 = f \circ g_2$ must satisfy $g_1 = g_2$. This is the generalization of an injection of sets. Prove that a morphism is a monomorphism if and only if the natural morphism $X \to X \times_Y X$ is an isomorphism. (We may then take this as the definition of monomorphism.) (Monomorphisms aren't very central to future discussions, although they will come up again. This exercise is just good practice.)
(c) Suppose $X \to Y$ is a monomorphism, and $W, Z \to X$ are two morphisms. Show that $W \times_X Z$ and $W \times_Y Z$ are canonically isomorphic. (We will use this later when talking about fibered products.)
(d) Given $X \to Y \to Z$, show that there is a natural morphism $X \times_Y X \to X \times_Z X$. (This is trivial once you figure out what it is saying. The point of this exercise is to see why it is trivial.)

Important Exercise. Suppose $T \to R, S$ are two ring morphisms. Let I be an ideal of R. We get a morphism $R \to R \otimes_T S$ by definition. Let I^e be the extension of I to $R \otimes_T S$. (These are the elements $\sum_j i_j \otimes s_j$ where $i_j \in I$, $s_j \in S$. But it is more elegant to solve this exercise using the universal property.) Show that there is a natural isomorphism

$$R/I \otimes_T S \cong (R \otimes_T S)/I^e.$$

Hence the natural morphism $S \otimes_T R \to S \otimes_T R/I$ is a surjection. As an application, we can compute tensor products of finitely generated k algebras over k. For example,

$$k[x_1, x_2]/(x_1^2 - x_2) \otimes_k k[y_1, y_2]/(y_1^3 + y_2^3) \cong k[x_1, x_2, y_1, y_2]/(x_1^2 - x_2, y_1^3 + y_2^3).$$

Exercise. Define coproduct in a category by reversing all the arrows in the definition of product. Show that coproduct for sets is disjoint union.

I then discussed adjoint functors briefly. I will describe them again briefly next day.

Next day: more examples of universal properties, including direct and inverse limits. Groupification. Sheaves!

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FOUNDATIONS OF ALGEBRAIC GEOMETRY CLASS 2

RAVI VAKIL

CONTENTS

Last day: What is algebraic geometry? Crash course in category theory, ending with Yoneda's Lemma.

Today: more examples of things defined using universal properties: inverse limits, direct limits, adjoint functors, groupification. Sheaves: the motivating example of differentiable functions. Definition of presheaves and sheaves.

At the start of the class, everyone filled out a sign-up sheet, giving their name, e-mail address, mathematical interests, and odds of attending. You're certainly not committing yourself to anything by signing it; it will just give me a good sense of who is in the class. Also, I'll use this for an e-mail list. If you didn't fill out the sheet, but want to, e-mail me.

I'll give out some homework problems on Monday, that will be based on this week's lectures. Most or all of the problems will already have been asked in class. Most likely I will give something like ten problems, and ask you to do five of them.

If you have any questions, please ask me, both in and out of class. Other people you can ask include the other algebraic geometers in the class, including Rob Easton, Andy Schultz, Jarod Alper, Joe Rabinoff, Nikola Penev, and others.

1. SOME CONSTRUCTIONS USING UNIVERSAL PROPERTIES, OLD AND NEW

Last day, I defined categories and functors. I hope I convinced you that you already have a feeling for categories, because you know so many examples.

A key point was **Yoneda's Lemma**, which says informally that you can essentially recover an object in a category by knowing the maps into it. For example, the data of maps to $X \times Y$ are precisely the data of maps to X and to Y. The data of maps to $X \times_Z Y$ are

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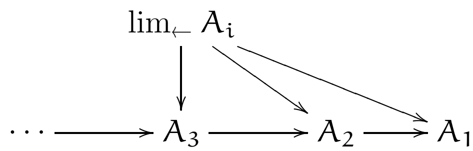
precisely the data of maps to X and Y that commute with maps to Z. (Explain how the universal property says this.)

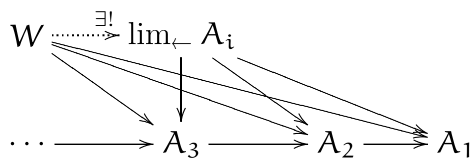
1.1. Example: Inverse limits. Here is another example of something defined by universal properties. Suppose you have a sequence

$$\cdots \longrightarrow A_3 \longrightarrow A_2 \longrightarrow A_1$$

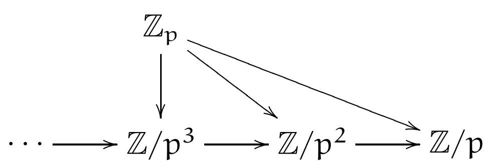
of morphisms in your category. Then the **inverse limit** is an object $\lim_{\leftarrow} A_i$ along with commuting morphisms to all the A_i



so that any other object along with maps to the A_i factors through $\lim_{\leftarrow} A_i$



You've likely seen such a thing before. How many of you have seen the p-adics ($\mathbb{Z}_p = ? + ?p + ?p^2 + ?p^3 + \cdots$)? Here's an example in the categories of rings.



You can check using your universal property experience that if it exists, then it is unique, up to unique isomorphism. It will boil down to the following fact: we know precisely what the maps to $\lim_{\leftarrow} A_i$ are: they are the same as maps to all the A_i's.

A few quick comments.

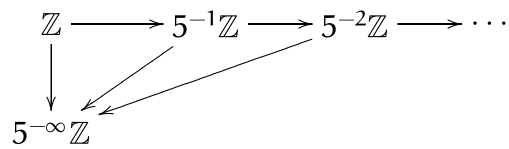
(1) We don't know in general that they have to exist.

(2) Often you can see that it exists. If these objects in your categories are all sets, as they are in this case of $\mathbb{Z}_p$, you can interpret the elements of the inverse limit as an element of $a_i \in A_i$ for each i, satisfying $f(a_i) = a_{i-1}$. From this point of view, $2 + 3p + 2p^2 + \cdots$ should be understood as the sequence $(2, 2+3p, 2+3p+2p^2, \ldots)$.

(3) We could generalize the system in any different ways. We could basically replace it with any category, although this is way too general. (Most often this will be a *partially ordered set*, often called *poset* for short.) If you wanted to say it in a fancy way, you could say the system could be indexed by an arbitrary category. Example: the product is an example of an inverse limit. Another example: the fibered product is an example of an inverse limit. (What are the partially ordered sets in each case?) Infinite products, or indeed products in general, are examples of inverse limits.

1.2. Example: Direct limits. More immediately relevant for us will be the dual of this notion. We just flip all the arrows, and get the notion of a *direct limit*. Again, if it exists, it is unique up to unique isomorphism.

Here is an example. $5^{-\infty}\mathbb{Z} = \lim_{\rightarrow} 5^{-i}\mathbb{Z}$ is an example. (These are the rational numbers whose denominators are required to be powers of 5.)



Even though we have just flipped the arrows, somehow it behaves quite differently from the inverse limit.

Some observations:

(1) In this example, each element of the direct limit is an element of something upstairs, but you can't say in advance what it is an element of. For example, 17/125 is an element of the $5^{-3}\mathbb{Z}$ (or $5^{-4}\mathbb{Z}$, or later ones), but not $5^{-2}\mathbb{Z}$.

(2) We can index this by any partially ordered set (or *poset*). (Or even any category, although I don't know if we care about this generality.)

1.3. Remark. (3) That first remark applies in some generality for the category of A-modules, where A is a ring. (See Atiyah-Macdonald p. 32, Exercise 14.) We say a partially ordered set I is a *directed set* if for $i, j \in I$, there is some $k \in I$ with $i, j \leq k$. We can show that the direct limit of any system of R-modules indexed by I exists, by constructing it. Say the system is given by M_i ($i \in I$), and $f_{ij} : M_i \to M_j$ ($i \leq j$ in I). Let $M = \oplus_i M_i$, where each M_i is associated with its image in M, and let R be the submodule generated by all elements of the form $m_i - f_{ij}(m_i)$ where $m_i \in M_i$ and $i \leq j$. **Exercise.** Show that M/R (with the inclusion maps from the M_i) is $\lim_{\rightarrow} M_i$. (This example will come up soon.) You will notice that the same argument works in other interesting categories, such as: sets; groups; and abelian groups. (Less important question for the experts: what hypotheses do we need for this to work more generally?)

(4) (Infinite) sums are examples of direct limits.

2. Adjoint functors

Let me re-define adjoint functors (Weibel Definition 2.3.9). Two *covariant* functors $L : \mathcal{A} \to \mathcal{B}$ and $R : \mathcal{B} \to \mathcal{A}$ are *adjoint* if there is a natural bijection for all $A \in \mathcal{A}$ and $B \in \mathcal{B}$

$$\tau_{AB} : \mathrm{Hom}_{\mathcal{B}}(L(A), B) \to \mathrm{Hom}_{\mathcal{A}}(A, R(B)).$$

In this instance, let me make precise what "natural" means, which will also let us see why the functors here are covariant. For all $f : A \to A'$ in $\mathcal{A}$, we require

$$\begin{array}{ccc}
\mathrm{Hom}_{\mathcal{B}}(L(A'), B) & \xrightarrow{Lf^*} & \mathrm{Hom}_{\mathcal{B}}(L(A), B) \\
\downarrow \tau & & \downarrow \tau \\
\mathrm{Hom}_{\mathcal{A}}(A', R(B)) & \xrightarrow{f^*} & \mathrm{Hom}_{\mathcal{A}}(A, R(B))
\end{array}$$

to commute, and for all $g : B \to B'$ in $\mathcal{B}$ we want a similar commutative diagram to commute. (Here f^* is the map induced by $f : A \to A'$, and Lf^* is the map induced by $Lf : L(A) \to L(A')$.)

Exercise. Write down what this diagram should be.

We could figure out what this should mean if the functors were both contravariant. I haven't tried to see if this could make sense.

You've actually seen this before, in linear algebra, when you have seen adjoint matrices. But I've long forgotten how they work, so let me show you another example. (Question for the audience: is there a very nice example out there?)

2.1. Example: groupification. Motivating example: getting a group from a semigroup. A semigroup is just like a group, except you don't require an inverse. Examples: the non-negative integers $0, 1, 2, \ldots$ under addition, or the positive integers under multiplication $1, 2, \ldots$. From a semigroup, you can create a group, and this could be called groupification. Here is a formalization of that notion. If S is a semigroup, then its groupification is a map of semigroups $\pi : S \to G$ such that G is a group, and any other map of semigroups from S to a *group* G' factors *uniquely* through G.

$$\begin{array}{ccc}
S & \longrightarrow & G \\
& \overset{\pi}{\searrow} & \vdots\ \exists! \\
& & G'
\end{array}$$

(Thanks Jack for explaining how to make dashed arrows in `\xymatrix`.)

(General idea for experts: We have a full subcategory of a category. We want to "project" from the category to the subcategory. We have $\mathrm{Hom}_{\text{category}}(S, H) = \mathrm{Hom}_{\text{subcategory}}(G, H)$ automatically; thus we are describing the left adjoint to the forgetful functor. How the argument worked: we constructed something which was in the small category, which automatically satisfies the universal property.)

Example of a universal property argument: If a semigroup is *already* a group then groupification is the identity morphism, by the universal property.

Exercise to get practice with this. Suppose R is a ring, and S is a multiplicative subset. Then $S^{-1}R$-modules are a full subcategory of the category of R-modules. Show that $M \to$

$S^{-1}M$ satisfies a universal property. Translation: Figure out what the universal property is.

2.2. Additive and abelian categories. There is one last concept that we will use later. It is convenient to give a name to categories with some additional structure. Here are some definitions.

Initial object of a category. It is an object with a unique map to any other object. (By a universal property argument, if it exists, it is unique up to unique isomorphism.) Example: the empty set, in the category of sets.

Final object of a category. It is an object with a unique map from any other object. (By a universal property argument, if it exists, it is unique up to unique isomorphism.) Question: does the category of sets have a final object?

Exercise. If Z is the final object in a category $\mathcal{C}$, and $X, Y \in \mathcal{C}$, then "$X \times_Z Y = X \times Y$" ("the" fibered product over Z is canonically isomorphic to "the" product). (This is an exercise about unwinding the definition.)

Additive categories (Weibel, p. 5). (I think I forgot to say part of this definition in class.) A category $\mathcal{C}$ is said to be *additive* if it has the following properties. For each $A, B \in \mathcal{C}$, $\mathrm{Hom}_{\mathcal{C}}(A, B)$ is an abelian group, such that composition of morphisms distributes over addition (think about what this could mean). It has a 0-object (= simultaneously initial object and final object), and products (a product $A \times B$ for any pair of objects). (Why is the 0-object called the 0-object?)

Yiannis points out that Banach spaces form an additive category. Another example are R-modules for a ring R, but they have even more structure.

Abelian categories (Weibel, p. 6). I deliberately didn't give a precise definition in class, as you should first get used to this concept before reading the technical definition.

But here it is. Let $\mathcal{C}$ be an additive category. First, a *kernel* of a morphism $f : B \to C$ is a map $i : A \to B$ such that $f \circ i = 0$, and that is universal with respect to this property. (Hence it is unique up to unique isomorphism by universal property nonsense. Note that we said "a" kernel, not "the" kernel.) A *cokernel* is defined dually by reversing the arrows — do this yourself. We say a morphism i in $\mathcal{C}$ is *monic* if $i \circ g = 0$, where the source of g is the target of i, implies $g = 0$. Dually, there is the notion of *epi* — reverse the arrows to find out what that is.

An *abelian category* is an additive category satisfying three properties. 1. Every map has a kernel and cokernel. 2. Every monic is the kernel of its cokernel. 3. Every epi is the cokernel of its kernel.

It is a non-obvious and imprecise fact that every property you want to be true about kernels, cokernels, etc. follows from these three.

An abelian category has kernels and cokernels and images, and they behave the way you expect them to. So you can have exact sequences: we say

$$A \xrightarrow{f} B \xrightarrow{g} C$$

is exact if $\ker g = \operatorname{im} f$.

The key example of an abelian category is the category of R-modules (where R is a ring).

3. SHEAVES

I now want to discuss an important new concept, the notion of a *sheaf*. A sheaf is the kind of object you will automatically consider if you are interested in something like the continuous functions on a space X, or the differentiable functions on a space X, or things like that. Basically, you want to consider all continuous functions on all open sets all at once, and see what properties this sort of collection of information has. I'm going to motivate it for you, and tell you the definition. I find this part quite intuitive. Then I will do things with this concept (for example talking about cokernels of maps of sheaves), and things become less intuitive.

3.1. Motivating example: sheaf of differentiable functions.

We'll consider differentiable functions on $X = \mathbb{R}^n$, or a more general manifold X. To each open set $U \subset X$, we have a ring of differentiable functions. I will denote this ring $\mathcal{O}(U)$.

If you take a differentiable function on an open set, you can restrict it to a smaller open set, and you'll get a differentiable function there. In other words, if $U \subset V$ is an inclusion of open sets, we have a map $\mathrm{res}_{V,U}; \mathcal{O}(V) \to \mathcal{O}(U)$.

If you take a differentiable function on a big open set, and restrict it to a medium open set, and then restrict that to a small open set, then you get the same thing as if you restrict the differentiable function on the big open set to the small open set all at once. In other words, if $U \hookrightarrow V \hookrightarrow W$, then the following diagram commutes:

$$\begin{array}{ccc} \mathcal{O}(W) & \xrightarrow{\mathrm{res}_{W,V}} & \mathcal{O}(V) \\ & {}_{\mathrm{res}_{W,U}}\searrow \quad \swarrow_{\mathrm{res}_{V,U}} & \\ & \mathcal{O}(U) & \end{array}$$

Now say you have two differentiable functions f_1 and f_2 on a big open set U, and you have an open cover of U by some U_i. Suppose that f_1 and f_2 agree on each of these U_i. Then they must have been the same function to begin with. Right? In other words, if $\{U_i\}_{i \in I}$ is a cover of U, and $f_1, f_2 \in \mathcal{O}(U)$, and $\mathrm{res}_{U,U_i} f_1 = \mathrm{res}_{U,U_i} f_2$, then $f_1 = f_2$. In other words, I can *identify* functions locally.

Finally, suppose I still have my U, and my cover U_i of U. Suppose I've got a differentiable function on each of the U_i — a function on U_1, a function on U_2, and so on — and they agree on the overlaps. Then I can glue all of them together to make one function on all of U. Right? In other words: given $f_i \in \mathcal{O}(U_i)$ for all i, such that $\mathrm{res}_{U_i, U_i \cap U_j} f_i = \mathrm{res}_{U_j, U_i \cap U_j} f_j$ for all i, j, then there is some $f \in \mathcal{O}(U)$ such that $\mathrm{res}_{U, U_i} f = f_i$ for all i.

Great. Now I could have done all this with continuous functions. [Go back over it all, with differentiable replaced by continuous.] Or smooth functions. Or just functions. That's the idea that we'll formalize soon into a sheaf.

3.2. Motivating example continued: the germ of a differentiable function. Before we do, I want to point out another definition, that of the germ of a differentiable function at a point $x \in X$. Intuitively, it is a shred of a differentiable function at x. Germs are objects of the form $\{(f, \text{open } U) : x \in U, f \in \mathcal{O}(U)\}$ modulo the relation that $(f, U) \sim (g, V)$ if there is some open set $W \subset U, V$ where $f|_W = g|_W$ (or in our earlier language, $\mathrm{res}_{U,W} f = \mathrm{res}_{V,W} g$). In other words, two functions that are the same here near x but differ way over there have the same germ. Let me call this set of germs $\mathcal{O}_x$. Notice that this forms a ring: you can add two germs, and get another germ: if you have a function f defined on U, and a function g defined on V, then $f + g$ is defined on $U \cap V$. Notice also that if $x \in U$, you get a map

$$\mathcal{O}(U) \to \{\text{germs at } x\}.$$

Aside for the experts: this is another example of a direct limit, and I'll tell you why in a bit.

Fact: $\mathcal{O}_x$ is a local ring. Reason: Consider those germs vanishing at x. That certainly is an ideal: it is closed under addition, and when you multiply something vanishing at x by any other function, you'll get something else vanishing at x. Anything not in this ideal is invertible: given a germ of a function f not vanishing at x, then f is non-zero near x by continuity, so $1/f$ is defined near x. The residue map should map onto a field, and in this case it does: we have an exact sequence:

$$0 \longrightarrow \mathfrak{m} := \text{ideal of germs vanishing at } x \longrightarrow \mathcal{O}_x \xrightarrow{f \mapsto f(x)} \mathbb{R} \longrightarrow 0$$

If you have never seen exact sequences before, this is a good chance to figure out how they work. This is what is called a short exact sequence. **Exercise.** Check that this is an exact sequence, i.e. that the image of each map is the kernel of the next. Show that this implies that the map on the left is an injection, and the one on the right is a surjection.

(Interesting fact, for people with a little experience with a little geometry: $\mathfrak{m}/\mathfrak{m}^2$ is a module over $\mathcal{O}_x/\mathfrak{m} \cong \mathbb{R}$, i.e. it is a real vector space. It turns out to be "naturally" — whatever that means — the cotangent space to the manifold at x. This will turn out to be handy later on, when we define tangent and cotangent spaces of schemes.)

Conclusion: We can interpret the value of a function at a point, or the value of a germ at a point, as an element of the local ring modulo the maximal ideal. (However, this can be a bit more problematic for more general sheaves.)

3.3. Definition of sheaf and presheaf.

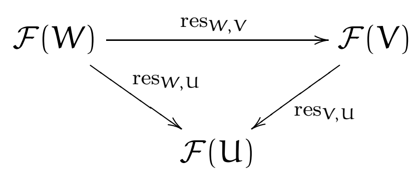
We are now ready to formalize these notions.

Definition: Sheaf on a topological space X. (A note on language: this is called a sheaf because of an earlier, different perspective on the definition, see Serre's *Faisceaux Algébriques Cohérents*. I'm not going to discuss this earlier definition, so you'll have to take this word without any motivation.)

I will define a sheaf of sets , just to be concrete. But you can have sheaves of groups, rings, modules, etc. without changing the definitions at all. Indeed, if you want to be fancy, you can say that you can have a sheaf with values in any category.

A *presheaf* $\mathcal{F}$ is the following data. To each open set $U \subset X$, we have a set $\mathcal{F}(U)$ (e.g. the set of differentiable functions). (Notational warning: Several notations are in use, for various good reasons: $\mathcal{F}(U) = \Gamma(U, \mathcal{F}) = H^0(U, \mathcal{F})$. I will use them all. I forgot to say this in class, but will say it next day.) The elements of $\mathcal{F}(U)$ are called *sections of $\mathcal{F}$ over* U.

For each inclusion $U \hookrightarrow V$, we have a restriction map $\mathrm{res}_{V,U} : \mathcal{F}(V) \to \mathcal{F}(U)$ (just as we did for differentiable functions). The map $\mathrm{res}_{U,U}$ should be the identity. If $U \hookrightarrow V \hookrightarrow W$, then the restriction maps commute, i.e. the following diagram commutes.



That ends the definition of a presheaf.

3.4. Useful exercise for experts liking category theory: "A presheaf is the same as a contravariant functor". Given any topological space X, we can get a category, which I will call the "category of open sets". The objects are the open sets. The morphisms are the inclusions $U \hookrightarrow V$. (What is the initial object? What is the final object?) Verify that the data of a presheaf is precisely the data of a contravariant functor from the category of open sets of X to the category of sets.

Note for pedants, which can be ignored by everyone else. An annoying question is: what is $\mathcal{F}(\emptyset)$. We will see that it can be convenient to have $\mathcal{F}(\emptyset) = \{1\}$, or more generally, if we are having sheaves with value in some category $\mathcal{C}$ (such as Groups), we would like $\mathcal{F}(\emptyset)$ to be the final object in the category. This should probably be part of the definition of presheaf. (For example, Weibel, p. 26, takes it as such; Weibel seems to define sheaves with values only in an abelian category.) I hope to be fairly scrupulous in this course, so I hope people who care keep me honest on issues like this.

We add two more axioms to make this into a sheaf.

Identity axiom. If $\{U_i\}_{i \in I}$ is a open cover of U, and $f_1, f_2 \in \mathcal{F}(U)$, and $\mathrm{res}_{U,U_i} f_1 = \mathrm{res}_{U,U_i} f_2$, then $f_1 = f_2$.

(A presheaf + identity axiom is sometimes called a *separated sheaf*, but we will not use that notation here.)

Gluability axiom. If $\{U_i\}_{i \in I}$ is a open cover of U, then given $f_i \in \mathcal{F}(U_i)$ for all i, such that $\mathrm{res}_{U_i, U_i \cap U_j} f_i = \mathrm{res}_{U_j, U_i \cap U_j} f_j$ for all i, j, then there is some $f \in \mathcal{F}(U)$ such that $\mathrm{res}_{U, U_i} f = f_i$ for all i.

(Philosophical note: identity means there is at most one way to glue. Gluability means that there is at least one way to glue.)

Remark for people enjoying category theory for the first time — as opposed to learning it for the first time. The gluability axiom may be interpreted as saying that $\mathcal{F}(\cup_{i \in I} U_i)$ is a certain inverse limit.

Example. If U and V are disjoint, then $\mathcal{F}(U \cup V) = \mathcal{F}(U) \times \mathcal{F}(V)$. (Here we use the fact that $F(\emptyset)$ is the final object, from the "note for pedants" above.)

3.5. Exercise. Suppose Y is a topological space. Show that "continuous maps to Y" form a sheaf of sets on X. More precisely, to each open set U of X, we associate the set of continuous maps to Y. Show that this forms a sheaf.
(Fancier versions that you can try:
(b) Suppose we are given a continuous map $f : Y \to X$. Show that "sections of f" form a sheaf. More precisely, to each open set U of X, associate the set of continuous maps s to Y such that $f \circ s = \mathrm{id}|_U$. Show that this forms a sheaf.
(c) (If you know what a topological group is.) Suppose that Y is a topological group. Show that maps to Y form a sheaf of *groups*. (If you don't know what a topological group is, you might be able to guess.)

Example: skyscraper sheaf. Suppose X is a topological space, with $x \in X$, and G is a group. Then $\mathcal{F}$ defined by $\mathcal{F}(U) = G$ if $x \in U$ and $\mathcal{F}(U) = \{e\}$ if $x \notin U$ forms a sheaf. (Check this if you don't see how.) This is called a *skyscraper sheaf*, because the informal picture of it looks like a skyscraper at x.

Important example/exercise: the pushforward . Suppose $f : X \to Y$ is a continuous map, and $\mathcal{F}$ is a sheaf on X. Then define $f_*\mathcal{F}$ by $f_*\mathcal{F}(V) = \mathcal{F}(f^{-1}(V))$, where V is an open subset of Y. Show that $f_*\mathcal{F}$ is a sheaf. This is called a *pushforward sheaf*. More precisely, $f_*\mathcal{F}$ is called the *pushforward of $\mathcal{F}$ by* f.

Example / exercise. (a) Let X be a topological space, and S a set with more than one element, and define $\mathcal{F}(U) = S$ for all open sets U. Show that this forms a presheaf (with the obvious restriction maps), and even satisfies the identity axiom. Show that this needn't form a sheaf. (Actually, for this to work, here we need $\mathcal{F}(\emptyset)$ to be the final object, not S. Without this patch, the constant presheaf *is* a sheaf. You can already see how the empty set is giving me a headache.) This is called the *constant presheaf with values in* S. We will denote this presheaf by $\underline{S}^{\mathrm{pre}}$.
(b) Now let $\mathcal{F}(U)$ be the maps to S that are *locally constant*, i.e. for any point x in U, there is a neighborhood of x where the function is constant. A better description is this: endow S with the discrete topology, and let $\mathcal{F}(U)$ be the continuous maps $U \to S$. Show that this is a *sheaf*. (Here we need $\mathcal{F}(\emptyset)$ to be the final object again, *not* S.) Using the "better description", this follows immediately from Exercise 3.5. We will try to call this the *locally*

constant sheaf. (Unfortunately, in the real world, this is stupidly called the *constant sheaf*.) We will denote this sheaf by $\underline{S}$.

3.6. Stalks. We define **stalk** = set of germs of a (pre)sheaf $\mathcal{F}$ in just the same way as before: Elements are $\{(f, \text{open } U) : x \in U, f \in \mathcal{O}(U)\}$ modulo the relation that $(f, U) \sim (g, V)$ if there is some open set $W \subset U, V$ where $\operatorname{res}_{U,W} f = \operatorname{res}_{V,W} g$. In other words, two section that are the same near x but differ far away have the same germ. This set of germs is denoted $\mathcal{F}_x$.

A useful equivalent definition is as a direct limit, of all $\mathcal{F}(U)$ where $x \in U$:

$$\mathcal{F}_x := \{\text{germs at } x\} = \varinjlim \mathcal{F}(U).$$

(All such U into a partially ordered set using inclusion. People having thought about the category of open sets, §3.4, will have a warm feeling in their stomachs.) This poset is a directed set (§1.3: given any two such sets, there is a third such set contained in both), so these two definitions are the same by Remark/Exercise 1.3. It would be good for you to think this through. Hence by that Remark/Exercise, we can have stalks for sheaves of sets, groups, rings, and other things for which direct limits exist for directed sets.

Let me repeat: it is useful to think of stalks in both ways, as direct limits, and also as something extremely explicit: an element of a stalk at p has as a representative a section over an open set near p.

Caution: Value at a point doesn't yet make sense.

3.7. Exercise. Show that pushforward induces maps of stalks.

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FOUNDATIONS OF ALGEBRAIC GEOMETRY CLASS 3

RAVI VAKIL

CONTENTS

Last day: end of category theory background. Motivation for and definitions of presheaf, sheaf, stalk.

Today: Presheaves and sheaves. Morphisms thereof. Sheafification.

I will be away Wednesday Oct. 5 to Thursday Oct. 13. The *next* class will be Friday, October 14. That means there will be *no class* this Wednesday, or next Monday or Wednesday. If you want to be on the e-mail list (low traffic), and didn't sign up last day, please let me know.

Problem set 1 out today, due Monday Oct. 17.

1. WHERE WE WERE

At this point, you're likely wondering when we're going to get to some algebraic geometry. We'll start that next class. We're currently learning how to think about things correctly. When we define interesting new objects, we'll learn how we want them to behave because we know a little category theory.

1.1. **Category theory.** I think in the heat of the last lecture, I skipped something I shouldn't have. An abelian category has several properties. One of these is that the morphisms form abelian groups: $\mathrm{Hom}(A, B)$ is an abelian group. This behaves well with respect to composition. For example if $f, g : A \to B$, and $h : B \to C$, then $h \circ (f + g) = h \circ f + h \circ g$. There is an obvious dual statement, that I'll leave to you. This implies other things, such as for example $0 \circ f = 0$. I think I forgot to say the above. An abelian category also has

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a 0-object (an object that is both a final object and initial object). An abelian category has finite products. If you stopped there, you'd have the definition of an additive category.

In an additive category, you can define things like kernels, cokernels, images, epimorphisms, monomorphisms, etc. In an abelian category, these things behave just way you expect them to, from your experience with R-modules. I've put the definition in the last day's notes.

2. PRESHEAVES AND SHEAVES

We then described presheaves and sheaves on a topological space X. I'm going to remind you of two examples, and introduce a third. The **first example** was of a sheaf of nice functions, say differentiable functions, which I will temporarily call $\mathcal{O}_X$. This is an example of a sheaf of rings.

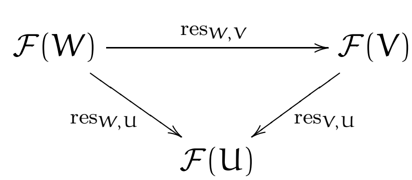
The axioms are as follows. We can have sheaves of rings, groups, abelian groups, and sets.

To each open set, we associate a ring $\mathcal{F}(U)$. Elements of this ring are called *sections of the sheaf over* U. (Notational warning: Several notations are in use, for various good reasons: $\mathcal{F}(U) = \Gamma(\mathcal{F}, U) = H^0(\mathcal{F}, U)$. I will use them all.)

If $U \subset V$ is an inclusion of open sets, we have restriction maps $res_{V,U}; \mathcal{F}(V) \to \mathcal{F}(U)$.

The map $res_{U,U}$ must be the identity for all U.

If you take a section over a big open set, and restrict it to a medium open set, and then restrict that to a small open set, then you get the same thing as if you restrict the section on the big open set to the small open set all at once. In other words, if $U \hookrightarrow V \hookrightarrow W$, then the following diagram commutes:



A subtle point that you shouldn't worry about at the start are the sections over the empty set. $\mathcal{F}(\emptyset)$ should be the final object in the category under consideration (sets: a set with one element; abelian groups: 0; rings: the 0-ring). (I'm tentatively going to say that there is a 1-element ring. In other words, I will not assume that rings satisfy $1 \neq 0$. Every ring maps to the 0-ring. But it doesn't map to any other ring, because in a ring morphisms, 0 goes to 0, and 1 goes to 1, but in every ring beside this one, $0 \neq 1$. I think this convention will solve some problems, but it will undoubtedly cause others, and I may eat my words, so only worry about it if you really want to.)

Something satisfying the properties I've described is a *presheaf*. (For experts: a presheaf of rings is the same thing as a contravariant functor from the category of open sets to the category of rings, plus that final object annoyance, see problem set 1.)

Sections of presheaves $\mathcal{F}$ have *germs* at each point $x \in X$ where they are defined, and the set of germs is denoted $\mathcal{F}_x$, and is called the *stalk of $\mathcal{F}$ at* x. Elements of the stalk correspond to sections over some open set containing x. Two of these sections are considered the same if they agree on some smaller open set. If $\mathcal{F}$ is a sheaf of rings, then $\mathcal{F}_x$ is a ring, and ditto for rings replaced by other categories we like.

We add two more axioms to make this into a sheaf.

Identity axiom. If $\{U_i\}_{i \in I}$ is a cover of U, and $f_1, f_2 \in \mathcal{F}(U)$, and $\operatorname{res}_{U,U_i} f_1 = \operatorname{res}_{U,U_i} f_2$, then $f_1 = f_2$.

Gluability axiom. given $f_i \in \mathcal{F}(U_i)$ for all i, such that $\operatorname{res}_{U_i, U_i \cap U_j} f_i = \operatorname{res}_{U_j, U_i \cap U_j} f_j$ for all i, j, then there is some $f \in \mathcal{F}(U)$ such that $\operatorname{res}_{U,U_i} f = f_i$ for all i.

Example 2 (on problem set 1). Suppose we are given a continuous map $f : Y \to X$. The "sections of f" form a sheaf. More precisely, to each open set U of X, associate the set of continuous maps s to Y such that $f \circ s = id|_U$. This forms a sheaf. (Example for those who know this language: a vector bundle.)

Example 3: **Sheaf of $\mathcal{O}_X$-modules.** Suppose $\mathcal{O}_X$ is a sheaf of rings on X. Then we define the notion of a sheaf of $\mathcal{O}_X$-modules. We have a metaphor: rings is to modules, as sheaves of rings is to sheaves of modules.

There is only one possible definition that could go with this name, so let's figure out what it is. For each U, $\mathcal{F}(U)$ should be a $\mathcal{O}_X(U)$-module. Furthermore, this structure should behave well with respect to restriction maps. This means the following. If $U \hookrightarrow V$, then

$$\begin{array}{ccc} \mathcal{O}_X(V) \times \mathcal{F}(V) & \xrightarrow{\text{action}} & \mathcal{F}(V) \\ \downarrow \operatorname{res}_{V,U} & & \downarrow \operatorname{res}_{V,U} \\ \mathcal{O}_X(U) \times \mathcal{F}(U) & \xrightarrow{\text{action}} & \mathcal{F}(U) \end{array}$$

commutes. You should think about this later, and convince yourself that I haven't forgotten anything.

For category theorists: the notion of R-module generalizes the notion of abelian group, because an abelian group is the same thing as a $\mathbb{Z}$-module. It is similarly immediate that the notion of $\mathcal{O}_X$-module generalizes the notion of sheaf of abelian groups, because the latter is the same thing as a $\underline{\mathbb{Z}}$-module, where $\underline{\mathbb{Z}}$ is the locally constant sheaf with values in $\mathbb{Z}$. Hence when we are proving things about $\mathcal{O}_X$-modules, we are also proving things about sheaves of abelian groups. For experts: Someone pointed out that we can make the same notion of *presheaf* of $\mathcal{O}_X$-modules, where $\mathcal{O}_X$ is a *presheaf* of rings. In this setting, *presheaves* of abelian groups are the same as modules over the *constant presheaf* $\underline{\mathbb{Z}}^{\text{pre}}$. I doubt we will use this, so feel free to ignore it.

3. Morphisms of presheaves and sheaves

I'll now tell you how to map presheaves to each other; and similarly for sheaves. In other words, I am describing the *category of presheaves* and the *category of sheaves*.

A morphism of presheaves of sets $f : \mathcal{F} \to \mathcal{G}$ is a collection of maps $f_U : \mathcal{F}(U) \to \mathcal{G}(U)$ that commute with the restrictions, in the sense that: if $U \hookrightarrow V$ then

$$\begin{array}{ccc} \mathcal{F}(V) & \xrightarrow{f_V} & \mathcal{G}(V) \\ \downarrow{\scriptstyle \operatorname{res}_{V,U}} & & \downarrow{\scriptstyle \operatorname{res}_{V,U}} \\ \mathcal{F}(U) & \xrightarrow{f_U} & \mathcal{G}(U) \end{array}$$

commutes. (Notice: the underlying space remains X!) A morphism of sheaves is defined in the same way. (For category-lovers: a morphism of presheaves on X is a natural transformation of functors. This definition describes the category of sheaves on X as a full subcategory of the category of presheaves on X.)

A morphism of presheaves (or sheaves) of rings (or groups, or abelian groups, or $\mathcal{O}_X$-modules) is defined in the same way.

Exercise. Show morphisms of (pre)sheaves induces morphisms of stalks.

Interesting examples of morphisms of presheaves of abelian groups. Let $X = \mathbb{C}$ with the usual (analytic) topology, and define $\mathcal{O}_X$ to be the sheaf of holomorphic functions, and $\mathcal{O}_X^*$ to be the sheaf of *invertible (= nowhere* 0*) holomorphic functions*. This is a sheaf of abelian groups under multiplication. We have maps of presheaves

$$1 \longrightarrow \underline{\mathbb{Z}}^{\text{pre}} \xrightarrow{\times 2\pi i} \mathcal{O}_X \xrightarrow{\exp} \mathcal{O}_X^* \longrightarrow 1$$

where $\underline{\mathbb{Z}}^{\text{pre}}$ is the constant presheaf. This is *not* an exact sequence of presheaves, and it is worth figuring out why. (Hint: it is not exact at $\mathcal{O}_X$ or $\mathcal{O}_X^*$. Replacing $\underline{\mathbb{Z}}^{\text{pre}}$ with the locally constant sheaf $\underline{\mathbb{Z}}$ remedies the first, but not the second.)

Now abelian groups, and R-modules, form an abelian category — by which I just mean that you are used to taking kernels, images, etc. — and you might hope for the same for sheaves of abelian groups, and sheaves of $\mathcal{O}_X$-modules. That is indeed the case. Presheaves are easier to understand in this way.

The presheaves of abelian groups on X*, or* $\mathcal{O}_X$*-modules on* X*, form an abelian category.* If $f : \mathcal{F} \to \mathcal{G}$ is a morphism of presheaves, then $\ker f$ is a presheaf, with $(\ker f)(U) = \ker f_U$, and $(\operatorname{im} f)(U) = \operatorname{im} f_U$. The resulting things are indeed presheaves. For example, if $U \hookrightarrow V$, there is a natural map $\mathcal{G}(V)/f_V(\mathcal{F}(V)) \to \mathcal{G}(U)/f_U(\mathcal{F}(U))$, as we observe by chasing the following diagram:

$$\begin{array}{ccccccc}
\mathcal{F}(V) & \longrightarrow & \mathcal{G}(V) & \longrightarrow & \mathcal{G}(V)/\mathcal{F}(V) & \longrightarrow & 0 \\
\downarrow{\scriptstyle \mathrm{res}_{V,U}} & & \downarrow{\scriptstyle \mathrm{res}_{V,U}} & & \vdots & & \\
\mathcal{F}(U) & \longrightarrow & \mathcal{G}(U) & \longrightarrow & \mathcal{G}(U)/\mathcal{F}(U) & \longrightarrow & 0.
\end{array}$$

Thus I have defined $\mathcal{G}/\mathcal{F}$, by showing what its sections are, and what its restriction maps are. I have to check that it restriction maps compose — **exercise**. Hence I've defined a presheaf. I still have to convince you that it deserves to be called a cokernel. **Exercise.** Do this. It is less hard than you might think. Here is the definition of cokernel of $g : \mathcal{F} \to \mathcal{G}$. It is a morphism $h : \mathcal{G} \to \mathcal{H}$ such that $h \circ g = 0$, and for any $i : \mathcal{G} \to \mathcal{I}$ such that $i \circ g = 0$, there is a unique morphism $j : \mathcal{H} \to \mathcal{I}$ such that $j \circ h = i$:

$$\begin{array}{ccccc}
\mathcal{F} & \longrightarrow & \mathcal{G} & \longrightarrow & \mathcal{H} \\
 & & & \searrow & \vdots\ {\scriptstyle \exists!} \\
 & & & & \mathcal{I}
\end{array}$$

(Translation: cokernels in an additive category are defined by a universal property. Hence if they exist, they are unique. We are checking that our construction satisfies the universal property.)

Punchline: The presheaves of $\mathcal{O}_X$-modules is an abelian category, and as nice as can be. We can define terms such as *subpresheaf*, *image presheaf*, *quotient presheaf*, *cokernel presheaf*. You construct kernels, quotients, cokernels, and images open set by open set. (Quotients are special cases of cokernels.)

Exercise. In particular: if $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \cdots \to \mathcal{F}_n \to 0$ is an exact sequence of presheaves, then $0 \to \mathcal{F}_1(U) \to \mathcal{F}_2(U) \to \cdots \to \mathcal{F}_n(U) \to 0$ is also an exact sequence for all U, and vice versa.

However, we are interested in more geometric objects, sheaves, where things are can be understood in terms of their local behavior, thanks to the identity and gluing axioms.

3.1. The category of sheaves of $\mathcal{O}_X$-modules is trickier. It turns out that the kernel of a morphism of sheaves is also sheaf. **Exercise.** Show that this is true. (Confusing translation: this subpresheaf of a sheaf is in fact also a sheaf.) Thus we have the notion of a **subsheaf**.

But other notions behave weirdly.

Example: image sheaf. We don't need an abelian category to talk about images — the notion of image makes sense for a map of sets. And the notion of image is a bit problematic even for sheaves of sets. Let's go back to our example of $\mathcal{O}_X \xrightarrow{\exp} \mathcal{O}_X^*$. What is the image *presheaf*? Well, if U is a *simply connected* open set, then this is surjective: every non-zero holomorphic function on a simply connected set has a logarithm (in fact many). However, this is not true if U is not simply connected — the function $f(z) = z$ on $\mathbb{C} - 0$ does *not* have a logarithm.

However, it locally does.

So what do we do? Answer 1: throw up our hands. Answer 2: Develop a new definition of image. We can't just define anything — we need to figure out what we want the image to be. Answer: category theoretic definition.

The patch: sheafification. Define **sheafification** of a presheaf by universal property: $\mathcal{F} \to \mathcal{F}^{\mathrm{sh}}$. Hence if it is exists, it is unique up to unique isomorphism. (This is analogous to the method of getting a group from a semigroup, see last day's notes.)

(Category-lovers: this says that sheafification is left-adjoint to the forgetful functor. This is just like groupification.)

Theorem (later today): Sheafification exists. (The specific construction will later be useful, but we won't need anything but the universal property right now.)

In class, I attempted to show that the sheafification of the image presheaf satisfies the universal property of the image sheaf, but I realized that I misstated the property. Instead, I will let you show that the sheaf of the cokernel presheaf satisfies the universal property of the cokernel sheaf. See the notes about one page previous for the definition of the cokernel.

Exercise. Do this.

Possible exercise. I'll tell you the definition of the image sheaf, and you can check.

Remark for experts: someone pointed out in class that likely the same arguments apply without change whenever you have an adjoint to a forgetful functor.

In short: $\mathcal{O}_X$-modules form an abelian category. To define image and cokernel (and quotient), you need to sheafify.

3.2. *Exercise.* Suppose $f : \mathcal{F} \to \mathcal{G}$ is a morphism of sheaves. Show that there are natural isomorphisms $\operatorname{im} f \cong \mathcal{F}/\ker f$ and $\operatorname{coker} f \cong \mathcal{G}/\operatorname{im} f$.

Tensor products of $\mathcal{O}_X$-modules: also requires sheafification.

3.3. *Exercise.* Define what we should mean by tensor product of two $\mathcal{O}_X$-modules. Verify that this construction satisfies your definition. (Hint: sheafification is required.)

3.4. *Left-exactness of the global section functor.* Left-exactness of global sections; hints of cohomology. More precisely:

$$0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$$

implies

$$0 \to \mathcal{F}(U) \to \mathcal{G}(U) \to \mathcal{H}(U)$$

is exact. Give example where not right exact, (Hint: $0 \to \mathbb{Z} \to \mathcal{O} \to \mathcal{O}^* \to 0$.)

Caution: The cokernel in the category of sheaves *is* a presheaf, but it is *isn't* the cokernel in the category of presheaves.

3.5. Important Exercise. Show the same thing (3.4) is true for pushforward sheaves. (The previous case is the case of a map from U to a point.)

4. Stalks, and sheafification

4.1. *Important exercise.* Prove that a section of a sheaf is determined by its germs, i.e.

$$\Gamma(U, \mathcal{F}) \to \prod_{x \in U} \mathcal{F}_x$$

is injective. (Hint: you won't use the gluability axiom. So this is true for separated presheaves.) [Answer: Suppose $f, g \in \Gamma(U, \mathcal{F})$, with $f_x = g_x$ in $\mathcal{F}_x$ for all $x \in U$. In terms of the concrete interpretation of stalks, $f_x = (U, f)$ and $g_x = (U, g)$, and $(U, f) = (U, g)$ means that there is an open subset U_x of U, containing x, such that $f|_{U_x} = g|_{U_x}$. The U_x cover U, so by the identity axiom for this cover of U, $f = g$.]

Corollary. In particular, if a sheaf has all stalks 0, then it is the 0-sheaf.

4.2. Morphisms and stalks.

4.3. *Exercise.* Show that morphisms of presheaves induce morphisms of stalks.

4.4. *Exercise.* Show that morphisms of sheaves are determined by morphisms of stalks. Hint # 1: you won't use the gluability axiom. So this is true of morphisms of separated presheaves.) Hint # 2: study the following diagram.

(1)
$$\begin{array}{ccc} \mathcal{F}(U) & \longrightarrow & \mathcal{G}(U) \\ \downarrow & & \downarrow \\ \prod_{x \in U} \mathcal{F}_x & \longrightarrow & \prod_{x \in U} \mathcal{G}_x \end{array}$$

4.5. *Exercise.* Show that a morphism of sheaves is an isomorphism if and only if it induces an isomorphism of all stalks. (Hint: Use (1). Injectivity uses from the previous exercise. Surjectivity will use gluability.)

4.6. *Exercise.* (a) Show that Exercise 4.1 is false for general presheaves. (Hint: take a 2-point space with the discrete topology, i.e. every subset is open.)
(b) Show that Exercise 4.4 is false for general presheaves. (Hint: a 2-point space suffices.)
(c) Show that Exercise 4.5 is false for general presheaves.

4.7. Description of sheafification. Suppose $\mathcal{F}$ is a presheaf on a topological space X. We define $\mathcal{F}^{sh}$ as follows. Sections over $U \subset X$ are stalks at each point, with compatibility conditions (to each element of the stalk, there is a representative (g, U) with g restricting correctly to all stalks in U). More explicitly:

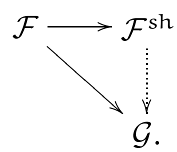
$$\mathcal{F}^{sh}(U) := \{(f_x \in \mathcal{F}_x)_{x \in U} : \forall x \in U, \exists x \subset U_x \subset U, F^x \in \mathcal{F}(U_x) : F^x_y = f_y \forall y \in U_x\}.$$

(Those who want to worry about the empty set are welcome to.)

This is clearly a sheaf: we have restriction maps; they commute; we have identity and gluability.

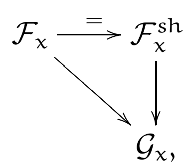
4.8. For any morphism of presheaves $\phi : \mathcal{F} \to \mathcal{G}$, we get a natural induced morphism of sheaves $\phi^{sh} : \mathcal{F}^{sh} \to \mathcal{G}^{sh}$.

We have a natural presheaf morphism $\mathcal{F} \to \mathcal{F}^{sh}$. This induces a natural morphism of stalks $\mathcal{F}_x \to \mathcal{F}^{sh}_x$ (Exercise 4.3). Hence if $\mathcal{F}$ is a sheaf already, then $\mathcal{F} \to \mathcal{F}^{sh}$ is an isomorphism, by Exercise 4.5. If we knew that $\mathcal{F}^{sh}$ satisfied the universal property of sheafification, this would have been immediate by abstract nonsense, but we don't know that. In fact, we'll show that now. Suppose we have the solid arrows in



We want to show that there exists a dashed arrow as in the diagram, making the diagram commute, and we want to show that it is unique. By 4.8, $\mathcal{F} \to \mathcal{G}$ induces a morphism $\mathcal{F}^{sh} \to \mathcal{G}^{sh} = \mathcal{G}$, so we have existence.

For uniqueness: as morphisms of sheaves are determined by morphisms of stalks (Exercise 4.4), and for any $x \in X$, we have a commutative diagram



we are done. Thus $\mathcal{F} \to \mathcal{F}^{sh}$ is indeed the sheafification.

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FOUNDATIONS OF ALGEBRAIC GEOMETRY CLASS 4

RAVI VAKIL

CONTENTS

Last day: Presheaves and sheaves. Morphisms thereof. Sheafification.

Today: Understanding sheaves via stalks. Understanding sheaves via "sheaves on a nice base of a topology". Affine schemes $\operatorname{Spec} R$**: the set.**

Here's where we are. I introduced you to some of the notions of category theory. Our motivation is as follows. We will be creating some new mathematical objects, and we expect them to act like object we have seen before. We could try to nail down precisely what we mean, and what minimal set of things we have to check in order to verify that they act the way we expect. Fortunately, we don't have to — other people have done this before us, by defining key notions, like abelian categories, which behave like modules over a ring.

We then defined presheaves and sheaves. We have seen sheaves of sets and rings. We have also seen sheaves of abelian groups and of $\mathcal{O}_X$-modules, which form an abelian category. Let me contrast again presheaves and sheaves. Presheaves are simpler to define, and notions such as kernel and cokernel are straightforward, and are defined open set by open set. Sheaves are more complicated to define, and some notions such as cokernel require the notion of sheafification. But we like sheaves because they are in some sense geometric; you can get information about a sheaf locally. Today, I'd like to go over some of the things we talked about last day in more detail. I'm going to talk again about stalks, and how information about sheaves are contained in stalks.

First, a small comment I should have said earlier. Suppose we have an exact sequence of sheaves of abelian groups (or $\mathcal{O}_X$-modules) on X

$$0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H}.$$

Date: Friday, October 14, 2005. Small updates January 31, 2007.

If $U \subset X$ is any open set, then

$$0 \to \mathcal{F}(U) \to \mathcal{G}(U) \to \mathcal{H}(U)$$

is exact. Translation: taking sections over U is a *left-exact functor*. Reason: the kernel sheaf of $\mathcal{G} \to \mathcal{H}$ is in fact the kernel presheaf (see the previous lectures). Note that $\mathcal{G}(U) \to \mathcal{H}(U)$ is not necessarily surjective (the functor is not exact); a counterexample is given by our old friend

$$0 \to \underline{\mathbb{Z}} \to \mathcal{O}_X \to \mathcal{O}_X^* \to 0.$$

(By now you should be able to guess what U to use.)

1. Stalks, and sheafification

1.1. *Important exercise.* Prove that a section of a sheaf is determined by its germs, i.e.

$$\mathcal{F}(U) \to \prod_{x \in U} \mathcal{F}_x$$

is injective. (Hint: you won't use the gluability axiom. So this is true for separated presheaves.)

Corollary. In particular, if a sheaf has all stalks 0, then it is the 0-sheaf.

1.2. Morphisms and stalks.

1.3. *Exercise.* Show that morphisms of presheaves (and sheaves) induce morphisms of stalks.

1.4. *Exercise.* Show that morphisms of sheaves are determined by morphisms of stalks. Hint # 1: you won't use the gluability axiom. So this is true of morphisms of separated presheaves. Hint # 2: study the following diagram.

(1)
$$\begin{array}{ccc} \mathcal{F}(U) & \longrightarrow & \mathcal{G}(U) \\ \big\downarrow & & \big\downarrow \\ \prod_{x \in U} \mathcal{F}_x & \longrightarrow & \prod_{x \in U} \mathcal{G}_x \end{array}$$

1.5. *Exercise.* Show that a morphism of sheaves is an isomorphism if and only if it induces an isomorphism of all stalks. (Hint: Use (1). Injectivity uses the previous exercise. Surjectivity will use gluability.)

1.6. *Exercise.* (a) Show that Exercise 1.1 is false for general presheaves. (Hint: take a 2-point space with the discrete topology, i.e. every subset is open.)
(b) Show that Exercise 1.4 is false for general presheaves. (Hint: a 2-point space suffices.)
(c) Show that Exercise 1.5 is false for general presheaves.

1.7. Description of sheafification. I described sheafification a bit quickly last time. I will do it again now.

Suppose $\mathcal{F}$ is a presheaf on a topological space X. We define $\mathcal{F}^{sh}$ as follows. Sections over $U \subset X$ are stalks at each point, with compatibility conditions (to each element of the stalk, there is a representative (g, U) with g restricting correctly to all stalks in U). More explicitly:

$$\mathcal{F}^{sh}(U) := \{(f_x \in \mathcal{F}_x)_{x \in U} : \forall x \in U, \exists U_X \text{ with } x \subset U_x \subset U, F^x \in \mathcal{F}(U_x) : F^x_y = f_y \forall y \in U_x\}.$$

(Those who want to worry about the empty set are welcome to.)

This is less confusing than it seems. $\mathcal{F}^{sh}(U)$ is clearly a sheaf: we have restriction maps; they commute; we have identity and gluability. It would be good to know that it satisfies the universal property of sheafification.

1.8. *Exercise.* The stalks of $\mathcal{F}^{sh}$ are the same as the stalks of $\mathcal{F}$. Reason: Use the concrete description of the stalks.

1.9. *Exercise.* For any morphism of presheaves $\phi : \mathcal{F} \to \mathcal{G}$, we get a natural induced morphism of sheaves $\phi^{sh} : \mathcal{F}^{sh} \to \mathcal{G}^{sh}$.

We have a natural presheaf morphism $\mathcal{F} \to \mathcal{F}^{sh}$. This induces a natural morphism of stalks $\mathcal{F}_x \to \mathcal{F}^{sh}_x$ (Exercise 1.3). This is an isomorphism by remark a couple of paragraphs previous. Hence if $\mathcal{F}$ is a sheaf already, then $\mathcal{F} \to \mathcal{F}^{sh}$ is an isomorphism, by Exercise 1.5. If we knew that $\mathcal{F}^{sh}$ satisfied the universal property of sheafification, this would have been immediate by abstract nonsense, but we don't know that yet. In fact, we'll show that now. Suppose we have the solid arrows in

$$\begin{array}{ccc} \mathcal{F} & \longrightarrow & \mathcal{F}^{sh} \\ & \searrow & \vdots \\ & & \mathcal{G}. \end{array}$$

We want to show that there exists a dashed arrow as in the diagram, making the diagram commute, and we want to show that it is unique. By 1.9, $\mathcal{F} \to \mathcal{G}$ induces a morphism $\mathcal{F}^{sh} \to \mathcal{G}^{sh} = \mathcal{G}$, so we have existence.

For uniqueness: as morphisms of sheaves are determined by morphisms of stalks (Exercise 1.4), and for any $x \in X$, we have a commutative diagram

$$\begin{array}{ccc} \mathcal{F}_x & \xrightarrow{=} & \mathcal{F}^{sh}_x \\ & \searrow & \downarrow \\ & & \mathcal{G}_x, \end{array}$$

we are done. Thus $\mathcal{F} \to \mathcal{F}^{sh}$ is indeed the sheafification.

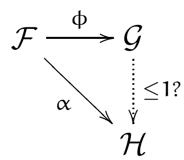
Four properties of morphisms of sheaves that you can check on stalks.

You can verify the following.

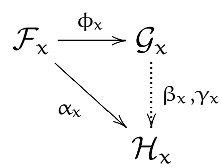
- A morphism of sheaves of sets is injective (monomorphism) if and only if it is injective on all stalks.
- Same with surjective (epimorphism).
- Same with isomorphic — we've already seen this.
- Suppose $\mathcal{F} \to \mathcal{G} \to \mathcal{H}$ is a complex of sheaves of abelian groups (or $\mathcal{O}_X$-modules). Then it is exact if and only if it is on stalks.

I'll prove one of these, to show you how it works: surjectivity.

Suppose first that we have surjectivity on all stalks for a morphism $\phi : \mathcal{F} \to \mathcal{G}$. We want to check the definition of epimorphism. Suppose we have $\alpha : \mathcal{F} \to \mathcal{H}$, and $\beta, \gamma : \mathcal{G} \to \mathcal{H}$ such that $\alpha = \beta \circ \phi = \gamma \circ \phi$.

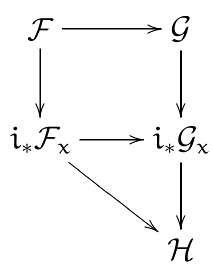


Then by taking stalks at x, we have



By surjectivity (epimorphism-ness) of the morphisms of stalks, $\beta_x = \gamma_x$. But as morphisms are determined by morphisms at stalks (Exercise 1.4), we must have $\beta = \gamma$.

Next assume that $\phi : \mathcal{F} \to \mathcal{G}$ is an epimorphism of sheaves, and $x \in X$. We will show that $\phi_x : \mathcal{F}_x \to \mathcal{G}_x$ is a epimorphism for any given $x \in X$. Choose for $\mathcal{H}$ any skyscraper sheaf supported at x. (the stalk of a skyscraper sheaf at the skyscraper point is just the skyscraper set/group/ring). Then the maps α, β, γ factor through the stalk maps:



skyscrapers

and then we are basically done.

2. Recovering sheaves from a "sheaf on a base"

Sheaves are natural things to want to think about, but hard to get one's hands on. We like the identity and gluability axioms, but they make proving things trickier than for presheaves. We've just talked about how we can understand sheaves using stalks. I now want to introduce a second way of getting a hold of sheaves, by introducing the notion of a *sheaf on a nice base*.

First, let me define the notion of a *base of a topology*. Suppose we have a topological space X, i.e. we know which subsets of X are open $\{U_i\}$. Then a base of a topology is a subcollection of the open sets $\{B_j\} \subset \{U_i\}$, such that each U_i is a union of the B_j. There is one example that you have seen early in your mathematical life. Suppose $X = \mathbb{R}^n$. Then the way the usual topology is often first defined is by defining *open balls* $B_r(x) = \{y \in \mathbb{R}^n : |y - x| < r\}$ *are open sets*, and declaring that any union of balls is open. So the balls form a base of the usual topology. Equivalently, we often say that they *generate* the usual topology. As an application of how we use them, if you want to check continuity of some map $f : X \to \mathbb{R}^n$ for example, you need only think about the pullback of balls on $\mathbb{R}^n$.

There is a slightly nicer notion I want to use. A base is particularly pleasant if the intersection of any two elements is also an element of the base. (Does this have a name?) I will call this a *nice base*. For example if $X = \mathbb{R}^n$, then a base would be *convex open sets*. Certainly the intersection of two convex open sets is another convex open set. Also, this certainly forms a base, because it includes the balls.

Now suppose we have a sheaf $\mathcal{F}$ on X, and a nice base $\{B_i\}$ on X. Then consider the information $(\{\mathcal{F}(B_i)\}, \{\phi_{ij} : \mathcal{F}(B_i) \to \mathcal{F}(B_j)\}$, which is a subset of the information contained in the sheaf — we are only paying attention to the information involving elements of the base, not all open sets.

Observation. We can recover the entire sheaf from this information. Proof:

$$\mathcal{F}(U) = \{(f_i \in \mathcal{F}(B_i))_{B_i \subset U} : \phi_{ij}(f_i) = f_j\}.$$

The map from the left side to the right side is clear. We get a map from the right side to the left side as follows. By gluability, each element gives at least one element of the left side. By identity, it gives a unique element.

Conclusion: we can recover a sheaf from less information. This even suggests a notion, that of a *sheaf on a nice base*.

A sheaf of sets (rings etc.) on a nice base $\{B_i\}$ is the following. For each B_i in the base, we have a set $\mathcal{F}(B_i)$. If $B_i \subset B_j$, we have maps $\mathrm{res}_{ji} : \mathcal{F}(B_j) \to \mathcal{F}(B_i)$. (Everywhere things called B are assumed to be in the base.) If $B_i \subset B_j \subset B_k$, then $\mathrm{res}_{B_k,B_i} = \mathrm{res}_{B_j,B_i} \circ \mathrm{res}_{B_k,B_j}$. For the pedants, $\mathcal{F}(\emptyset)$ is a one-element set (a final object). So far we have defined a *presheaf on a nice base*.

We also have base identity: If $B = \cup B_i$, then if $f, g \in \mathcal{F}(B)$ such that $\mathrm{res}_{B,B_i} f = \mathrm{res}_{B,B_i} g$ for all i, then $f = g$.

And base gluability: If $B = \cup B_i$, and we have $f_i \in \mathcal{F}(B_i)$ such that f_i agrees with f_j on basic open set $B_i \cap B_j$ (i.e. $\mathrm{res}_{B_i,B_i \cap B_j} f_i = \mathrm{res}_{B_j,B_i \cap B_j} f_j$) then there exist $f \in \mathcal{F}(B)$ such that $\mathrm{res}_{B,B_i} = f_i$ for all i.

2.1. *Theorem. — Suppose we have data* $F(U_i)$, ϕ_{ij}, *satisfying "base presheaf", "base identity" and "base gluability". Then (if the base is nice) this uniquely determines a sheaf of sets (or rings, etc.)* $\mathcal{F}$, *extending this.*

This argument will later get trumped by one given in Class 13.

Proof. Step 1: define the sections over an arbitrary U. For $U \neq \emptyset$, define

$$\mathcal{F}(U) = \{f_i \in F(B_i) \text{ for all } B_i \subset U : \operatorname{res}_{B_i, B_i \cap B_j} f_i = \operatorname{res}_{B_j, B_i \cap B_j} f_j \text{ in } F(B_i \cap B_j)\}$$

where if the set is empty, then we use the final object in our category; this is the only place where we needed to determine our category in advance. We get $\operatorname{res}_{U,V}$ in the obvious way. We get a presheaf.

$\mathcal{F}(B_i) = F(B_i)$ and $\operatorname{res}_{B_i, B_j}$ is as expected; both are clear.

Step 2: check the identity axiom. Take $f, g \in \mathcal{F}(U)$ restricting to $f_i \in \mathcal{F}(U_i)$. Then f, g agree on any base element contained in some U_i. We'll show that for each $B_j \subset U$, they agree. Take a cover of B_j by base elements each contained entirely in some U_i. The intersection of any two is also contained some U_i; they agree there too. Hence by "base identity" we get identity.

Step 3: check the gluability axiom. Suppose we have some $f_i \in \mathcal{F}(U_i)$ that agree on overlaps. Take any $B_j \subset \cup U_i$. Take a cover by basic opens that each lie in some U_i. Then they agree on overlaps. By "base gluability", we get a section over B_j. (Unique by "base identity".) Any two of the f_j's agree on the overlap. □

2.2. *Remark.* In practice, to find a section of such a sheaf over some open set U we may choose a smaller (finite if possible) cover of U.

Eventually, we will define a sheaf on a base in general, not just on a nice base. Experts may want to ponder the definition, and how to prove the above theorem in that case.

2.3. *Important Exercise.* (a) Verify that a morphism of sheaves is determined by a morphism on the base. (b) Show that a "morphism of sheaves on the base" (i.e. such that the diagram

$$\begin{array}{ccc} \Gamma(B_i, \mathcal{F}) & \longrightarrow & \Gamma(B_i, \mathcal{G}) \\ \downarrow & & \downarrow \\ \Gamma(B_j, \mathcal{F}) & \longrightarrow & \Gamma(B_j, \mathcal{G}) \end{array}$$

commutes) gives a morphism of sheaves.

2.4. *Remark.* Suppose you have a presheaf you want to sheafify, and when restricted to a base it is already a sheaf. Then the sheafification is obtained by taking this process.

Example: Let $X = \mathbb{C}$, and consider the sequence

$$1 \longrightarrow \underline{\mathbb{Z}} \xrightarrow{\times 2\pi i} \mathcal{O}_X \xrightarrow{\exp} \mathcal{O}_X^* \longrightarrow 1.$$

Let's check that it is exact, using our new knowledge. We instead work on the nice base of convex open sets. then on these open sets, this is indeed exact. The key fact here is that on any convex open set B, every element of $\mathcal{O}_X^*(B)$ has a logarithm, so we have surjectivity here.

3. Toward schemes

We're now ready to define schemes! Here is where we are going. After some more motivation for what kind of objects affine schemes are, I'll define affine schemes, which are like balls in the analytic topology. We'll generalize in three transverse directions. I'll define schemes in general, including projective schemes. I'll define morphisms between schemes. And I'll define sheaves on schemes. These notions will take up the rest of the quarter.

We will define schemes as a *topological space* along with a *sheaf of "algebraic functions"* (that we'll call the *structure sheaf*). Thus our construction will have three steps: we'll describe the *set*, then the *topology*, and then the *sheaf*.

We will try to draw pictures throughout; geometric intuition can guide algebra (and vice versa). Pictures develop geometric intuition. We learn to draw them; the algebra tells how to think about them geometrically. So these comments are saying: "this is a good way to think". Eventually the picture tells you some algebra.

4. Motivating examples

As motivation for why this is a good foundation for a kind of "space", we'll reinterpret differentiable manifolds in this way. We will feel free to be informal in this section.

Usual definition of differentiable manifold: atlas, and gluing functions. (There is also a Hausdorff axiom, which I'm going to neglect for now.)

A fancier definition is as follows: as a topological space, with a sheaf of differentiable functions. (Some observations: Functions are determined by values at points. This is an obvious statement, but won't be true for schemes in general. Note: Stalks are local rings $(\mathcal{O}_x, \mathfrak{m}_x)$; the residue map is "value at a point" $0 \to \mathfrak{m}_x \to \mathcal{O}_x \to \mathbb{R} \to 0$, as I described in an earlier class, probably class 1 or class 2.)

There is an interesting fact that I'd like to mention now, but that you're not quite ready for. So don't write this down, but hopefully let some of it subconsciously sink into your head. The tangent space at a point x can be naturally identified with $(\mathfrak{m}_x/\mathfrak{m}_x^2)^*$. Let's make this a bit explicit. Every function vanishing at p canonically gives a functional on the tangent space to X at p. If $X = \mathbb{R}^2$, the function $\sin x - y + y^2$ gives the functional $x - y$.

Morphisms $X \to Y$: these are certain continuous maps — but which ones? We can pull back functions along continuous maps. Differentiable functions pull back to differentiable functions. We haven't defined the inverse image of sheaves yet — if you're curious, that will be in the second problem set — but if we had, we would have a map $f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$. (I don't want to call it "pullback" because that word is used for a slightly different concept.) Inverse image is left-adjoint to pushforward, which we *have* seen, so we get map $f^\# : \mathcal{O}_Y \to f_*\mathcal{O}_X$.

Interesting question: which continuous maps are differentiable? Answer: Precisely those for which the induced map of functions sends differentiable functions to differentiable functions. (Check on local patches.)

4.1. *Unimportant Exercise.* Show that a morphism of differentiable manifolds $f : X \to Y$ with $f(p) = q$ induces a morphism of stalks $f^\# : \mathcal{O}_{Y,q} \to \mathcal{O}_{X,p}$. Show that $f^\#(\mathfrak{m}_{Y,q}) \subset \mathfrak{m}_{X,p}$. (In other words, if you pull back a function that vanishes at q, you get a function that vanishes at p.)

More for experts: Notice that this induces a map on tangent spaces $(\mathfrak{m}_{X,p}/\mathfrak{m}_{X,p}^2)^* \to (\mathfrak{m}_{Y,q}/\mathfrak{m}_{Y,q}^2)^*$. This is the tangent map you would geometrically expect. Interesting fact: the cotangent space, and cotangent map, is somehow more algebraically natural, despite the fact that tangent spaces, and tangent maps, are more geometrically natural. Rhetorical questions: How to check for submersion ("smooth morphism")? How to check for inclusion, but not just set-theoretically? Answer: differential information.

[Then we have a normal exact sequence.

Vector bundle can be rewritten in terms of sheaves; explain how.]

Side Remark. Manifolds are covered by disks that are all isomorphic. Schemes will not have isomorphic open sets, even varieties won't. An example will be given later.

5. Affine schemes I: the underlying set

For any ring R, we are going to define something called $\operatorname{Spec} R$. First I'll define it as a set, then I'll tell you its topology, and finally I'll give you a sheaf of rings on it, which I'll call the sheaf of functions. Such an object is called an affine scheme. In the future, $\operatorname{Spec} R$ will denote the set, the topology, and the structure sheaf, and I might use the notation $\operatorname{sp}(\operatorname{Spec} R)$ to mean the underlying topological space. But for now, as there is no possibility of confusion, $\operatorname{Spec} R$ will just be the set.

The set $\operatorname{sp}(\operatorname{Spec} R)$ is the set of prime ideals of R.

Let's do some examples. Along with the examples, I'll say a few things that aren't yet rigorously defined. But I hope they will motivate the topological space we'll eventually define, and also the structure sheaf.

Example 1. $\mathbb{A}^1_{\mathbb{C}} = \operatorname{Spec} \mathbb{C}[x]$. "The affine line", "the affine line over $\mathbb{C}$". What are the prime ideals? 0. $(x - a)$ where $a \in \mathbb{C}$. There are no others. Proof: $\mathbb{C}[x]$ is a Unique Factorization Domain. Suppose $\mathfrak{p}$ were prime. If $\mathfrak{p} \neq 0$, then suppose $f(x) \in \mathfrak{p}$ is an element of smallest degree. If $f(x)$ is not linear, then factor $f(x) = g(x)h(x)$, where $g(x)$ and $h(x)$ have positive degree. Then $g(x)$ or $h(x) \in \mathfrak{p}$, contradicting the minimality of the degree of f. Hence there is a linear element $(x - a)$ of $\mathfrak{p}$. Then I claim that $\mathfrak{p} = (x - a)$. Suppose $f(x) \in \mathfrak{p}$. Then the division algorithm would give $f(x) = g(x)(x - a) + m$ where $m \in \mathbb{C}$. Then $m = f(x) - g(x)(x - a) \in \mathfrak{p}$. If $m \neq 0$, then $1 \in \mathfrak{p}$, giving a contradiction: prime ideals can't contain 1.

Thus we have a picture of $\operatorname{Spec} \mathbb{C}[x]$. There is one point for each complex number, plus one extra point. Where is this point? How do we think of it? We'll soon see; but it is a special kind of point. Because (0) is contained in all of these primes, I'm going to somehow identify it with this line passing through all the other points. Here is one way to think of it. You can ask me: is it on the line? I'd answer yes. You'd say: is it here? I'd answer no. This is kind of zen.

To give you an idea of this space, let me make some statements that are currently undefined. The functions on $\mathbb{A}^1_{\mathbb{C}}$ are the polynomials. So $f(x) = x^2 - 3x + 1$ is a function. What is its value at $(x - 1)$ = "1"? Plug in 1! Or evaluate mod $x - 1$ — same thing by division algorithm! (What is its value at (0)? We'll see later. In general, values at maximal ideals are immediate, and we'll have to think a bit more when primes aren't maximal.)

Here is a "rational function": $(x - 3)^3/(x - 2)$. This function is defined everywhere but $x = 2$; it is an element of the structure sheaf on the open set $\mathbb{A}^1_{\mathbb{C}} - \{2\}$. It has a triple zero at 3, and a single pole at 2.

Example 2. Let k be an algebraically closed field. $\mathbb{A}^1_k = \operatorname{Spec} k[x]$. The same thing works, without change.

Example 3. $\operatorname{Spec} \mathbb{Z}$. One amazing fact is that from our perspective, this will look a lot like the affine line. Another unique factorization domain. Prime ideals: (0), (p) where p is prime. (Do this if you don't know it!!) Hence we have a picture of this Spec (omitted from notes).

Fun facts: 100 is a function on this space. It's value at (3) is "$1 \pmod 3$". It's value at (2) is "$0 \pmod 2$", and in fact it has a double 0. We will have to think a little bit to make sense of its value at (0).

$27/4$ is a rational function on $\operatorname{Spec} \mathbb{Z}$. It has a double pole at (2), a triple zero at (3). What is its value at (5)? Answer

$$27 \times 4^{-1} \equiv 2 \times (-1) \equiv 3 \pmod 5.$$

Example 4: stupid examples. $\operatorname{Spec} k$ where k is any field is boring: only one point. $\operatorname{Spec} 0$, where 0 is the zero-ring: the empty set, as 0 has no prime ideals.

Exercise. Describe the set $\operatorname{Spec} k[x]/x^2$. The ring $k[x]/x^2$ is called the *ring of dual numbers* (over k).

Example 5: $\mathbb{R}[x]$. The primes are (0), $(x-a)$ where $a \in \mathbb{R}$, and (x^2+ax+b) where x^2+ax+b is an irreducible quadratic **(exercise)**. The latter two are maximal ideals, i.e. their quotients are fields. Example: $\mathbb{R}[x]/(x-3) \cong \mathbb{R}$, $\mathbb{R}[x]/(x^2+1) \cong \mathbb{C}$. So things are a bit more complicated: we have points corresponding to real numbers, and points corresponding to *conjugate pairs* of complex numbers. So consider the "function" x^3-1 at the point $(x-2)$. We get 7. How about at (x^2+1)? We get

$$x^3-1 \equiv x-1 \pmod{x^2+1}.$$

This is profitably thought of as $i-1$.

One moral of this example is that we can work over a non-algebraically closed field if we wish. (i) It is more complicated, (ii) but we can recover much of the information we wanted.

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FOUNDATIONS OF ALGEBRAIC GEOMETRY CLASS 5

CONTENTS

Last day: Understanding sheaves via stalks, and via "nice base" of topology. $\operatorname{Spec} R$**: the set.**

Today: $\operatorname{Spec} R$**: the set, and the Zariski topology.**

Here is a reminder of where we are going. Affine schemes $\operatorname{Spec} R$ will be defined as a topological space with a sheaf of rings, that we will refer to as the sheaf of functions, called the structure sheaf. A scheme in general will be such a thing (a topological space with a sheaf of rings) that locally looks like $\operatorname{Spec} R$'s. Last day we defined the set: it is the set of primes of R.

We're in the process of doing lots of examples. In the course of doing these examples, we are saying things that we aren't allowed to say yet, because we're using words that we haven't defined. We're doing this because it will motivate where we're going.

We discussed the example $R = k[x]$ where k is algebraically closed (notation: $\mathbb{A}^1_k = \operatorname{Spec} k[x]$). This has old-fashioned points $(x - a)$ corresponding to $a \in k$. (Such a point is often just called a, rather than $(x - a)$.) But we have a new point, (0). (Notational caution: $0 \neq (0)$.) This is a "smooth irreducible curve". (We don't know what any of these words mean!)

We then discussed $R = \mathbb{Z}$. This has points (p) where p is an old-fashioned prime, and the point (0). This is also a smooth irreducible curve (whatever that means).

We discussed the case $R = k[x]$ where k is not necessarily algebraically closed, in particular $k = \mathbb{R}$. The maximal ideals here correspond to unions of Galois conjugates of points in $\mathbb{A}^1_{\mathbb{C}}$.

Example 6 for more arithmetic people: $\mathbb{F}_p[x]$. As in the previous examples, this is a Unique Factorization Domain, so we can figure out its primes in a hands-on way. The points are (0), and irreducible polynomials, which come in any degree. Irreducible polynomials correspond to sets of Galois conjugates in $\overline{\mathbb{F}}_p$. You should think about this, even if you are

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a geometric person — there is some arithmetic intuition that later turns into geometric intuition.

Example 7. $\mathbb{A}^2_{\mathbb{C}} = \operatorname{Spec} \mathbb{C}[x, y]$. (This discussion will apply with $\mathbb{C}$ replaced by *any* algebraically closed field.) Sadly, $\mathbb{C}[x, y]$ is not a Principal Ideal Domain: (x, y) is not a principal ideal. We can quickly name *some* prime ideals. One is (0), which has the same flavor as the (0) ideals in the previous examples. $(x - 2, y - 3)$ is prime, because $\mathbb{C}[x, y]/(x-2, y-3) \cong \mathbb{C}$, where this isomorphism is via $f(x, y) \mapsto f(2, 3)$. More generally, $(x-a, y-b)$ is prime for any $(a, b) \in \mathbb{C}^2$. Also, if $f(x, y)$ is an irreducible polynomial (e.g. $y - x^2$ or $y^2 - x^3$) then $(f(x, y))$ is prime. We will later prove that we have identified all these primes. Here is a picture: the "maximal primes" correspond to the old-fashioned points in $\mathbb{C}^2$ (I drew it). (0) somehow lives behind all of these points (I drew it). $(y - x^2)$ somehow is associated to this parabola (I drew it). Etc. You can see from this picture that we already want to think about "dimension". The primes $(x - a, y - b)$ are somehow of dimension 0, the primes $(f(x, y))$ are of dimension 1, and (0) is somehow of dimension 2. I won't define dimension today, so every time I say it, you should imagine that I am waving my hands wildly.

(This paragraph will not be comprehensible in the notes.) Let's try to picture this. Where is the prime $(y - x^2)$? Well, is it in the plane? Yes. Is it at $(2, 4)$? No. Is it in the set cut out by $y - x^2$? Yes. Is it in the set cut out by $(y^2 - x^3)$? No. Is it in the set cut out by $xy(y - x^2)$? Yes.

Note: maximal ideals correspond to "smallest" points. Smaller ideals correspond to "bigger" points. "One prime ideal contains another" means that the points "have the opposite containment." All of this will be made precise once we have a topology. This order-reversal can be a little confusing (and will remain so even once we have made the notions precise).

Example: $\mathbb{A}^n_{\mathbb{C}} = \operatorname{Spec} \mathbb{C}[x_1, \dots, x_n]$. (More generally, $\mathbb{A}^n_k = \operatorname{Spec} k[x_1, \dots, x_n]$, and even $A^n_R = \operatorname{Spec} R[x_1, \dots, x_n]$ where R is an arbitrary ring.)

For concreteness, let's consider $n = 3$. We now have an interesting question in algebra: What are the prime ideals of $\mathbb{C}[x, y, z]$? We have $(x-a, y-b, z-c)$. This is a maximal ideal, with residue field $\mathbb{C}$; we think of these as "0-dimensional points". Have we discovered all the maximal ideals? The answer is yes, by Hilbert's Nullstellensatz, which is covered in Math 210.

Hilbert's Nullstellensatz, Version 1. (This is sometimes called the "weak version" of the Nullstellensatz.) Suppose $R = k[x_1, \dots, x_n]$, where k is an algebraically closed field. Then the maximal ideals are precisely those of the form $(x_1 - a_1, \dots, x_n - a_n)$, where $a_i \in k$.

There are other prime ideals too. We have (0), which is a "3-dimensional point". We have $(f(x, y, z))$, where f is irreducible. To this we associate the hypersurface $f = 0$, so this is "2-dimensional" in nature. Do we have them all? No! One clue: we're missing dimension 1 things. Here is a "one-dimensional" prime ideal: (x, y). (Picture: it is the z-axis, which is cut out by $x = y = 0$.) How do we check that this is prime? The easiest way is to check that the quotient is an integral domain, and indeed $\mathbb{C}[x, y, z]/(x, y) \cong \mathbb{C}[z]$ is an integral domain (and visibly the functions on the z-axis). There are lots of one-dimensional primes, and it is not possible to classify them in a reasonable way. It will turn out that they correspond to things that we think of as "irreducible" curves: the natural answer to this algebraic question is geometric.

0.1. We now come to two more general flavors of affine schemes that will be useful in the future. There are two nice ways of producing new rings from a ring R. One is by taking the quotient by an ideal I. The other is by localizing at a multiplicative set. We'll see how Spec behaves with respect to these operations. In both cases, the new ring has a Spec that is a subset of the Spec of the old ring.

Important example: $\operatorname{Spec} R/I$ *in terms of* $\operatorname{Spec} R$. As a motivating example, consider $\operatorname{Spec} R/I$ where $R = \mathbb{C}[x, y]$, $I = (xy)$. We have a picture of $\operatorname{Spec} R$, which is the complex plane, with some mysterious extra "higher-dimensional points". Important algebra fact: The primes of R/I are in bijection with the primes of R containing I. (Here I'm using a prerequisite from Math 210. You should review this fact! This is not a result that you should memorize — you should know why it is true. If you don't remember why it is true, or didn't know this fact, then treat this as an **exercise** and do it yourself.) Thus we can picture $\operatorname{Spec} R/I$ as a subset of $\operatorname{Spec} R$. We have the "0-dimensional points" $(a, 0)$ and $(0, b)$. We also have two "1-dimensional points" (x) and (y).

We get a bit more: the inclusion structure on the primes of R/I corresponds to the inclusion structure on the primes containing I. More precisely, if $J_1 \subset J_2$ in R/I, and K_i is the ideal of R corresopnding to J_i, then $K_1 \subset K_2$.

So the minimal primes of $\mathbb{C}/(xy)$ are the "biggest" points we see, and there are two of them: (x) and (y). Thus we have the intuition that will later be precise: the minimal primes correspond to the "components" of $\operatorname{Spec} R$.

Important example: $\operatorname{Spec} S^{-1}R$ *in terms of* $\operatorname{Spec} R$, where S is a multiplicative subset of R. There are two particularly important flavors of multiplicative subsets. The first is $R - \mathfrak{p}$, where $\mathfrak{p}$ is a prime ideal. (Check that this is a multiplicative set!) The localization $S^{-1}R$ is denoted $R_\mathfrak{p}$. Here is a motivating example: $R = \mathbb{C}[x, y]$, $S = R - (x, y)$. The second is $\{1, f, f^2, \dots\}$, where $f \in R$. The localization is denoted R_f. (Notational warning: If (f) is a prime ideal, then $R_f \neq R_{(f)}$.) Here is an example: $R = \mathbb{C}[x, y]$, $f = x$.

Important algebra fact (to review and know): The primes of $S^{-1}R$ are in bijection with the primes of R that *don't meet* the multiplicative set S. So if $S = R - \mathfrak{p}$ where $\mathfrak{p}$ is a prime ideal, the primes of $S^{-1}R$ are just the primes of R *contained in* $\mathfrak{p}$. If $S = \{1, f, f^2, \dots\}$, the primes of $S^{-1}R$ are just those primes not containing f (the points where "f doesn't vanish"). A bit more is true: the inclusion structure on the primes of $S^{-1}R$ corresponds to the inclusion structure on the primes not meeting S. (If you didn't know it, take this as an **exercise** and prove it yourself!)

In each of these two cases, a picture is worth a thousand words. In these notes, I'm not making pictures unfortunately. But I'll try to describe them in less than a thousand words.

The case of $S = \{1, f, f^2, \dots\}$ is easier: we just throw out those points where f vanishes. (We will soon call this a *distinguished open set*, once we know what open sets are.) In our example of $R = k[x,y]$, $f = x$, we throw out the y-axis.

Warning: sometimes people are first introduced to localizations in the special case that R is an integral domain. In this case, $R \hookrightarrow R_f$, but this isn't true in general. Here's the definition of localization (which you should be familiar with). The elements of $S^{-1}R$ are of the form r/s where $r \in R$ and $s \in S$, and $(r_1/s_1) \times (r_2/s_2) = (r_1 r_2/s_1 s_2)$, and $(r_1/s_1) + (r_2/s_2) = (r_1 s_2 + s_1 r_2)/(s_1 s_2)$. We say that $r_1/s_1 = r_2/s_2$ if **for some** $s \in S$ $s(r_1 s_2 - r_2 s_1) = 0$.

Example/warning: $R[1/0] = 0$. Everything in $R[1/0]$ is 0. (Geometrically, this is good: the locus of points where 0 doesn't vanish is the empty set, so certainly $D(0) = \operatorname{Spec} R_0$.)

In general, inverting zero-divisors can make things behave weirdly. Example: $R = k[x,y]/(xy)$. $f = x$. What do you get? It's actually a straightforward ring, and we'll use some geometric intuition to figure out what it is. $\operatorname{Spec} k[x,y]/(xy)$ "is" the union of the two axes in the plane. Localizing means throwing out the locus where x vanishes. So we're left with the x-axis, minus the origin, so we expect $\operatorname{Spec} k[x]_x$. So there should be some natural isomorphism $(k[x,y]/(xy))_x \cong k[x]_x$. **Exercise.** Figure out why these two rings are isomorphic. (You'll see that y on the left goes to 0 on the right.)

In the case of $S = R - \mathfrak{p}$, we keep only those primes contained in $\mathfrak{p}$. In our example $R = k[x,y]$, $\mathfrak{p} = (x,y)$, we keep all those points corresponding to "things through the origin", i.e. the 0-dimensional point (x,y), the 2-dimensional point (0), and those 1-dimensional points $(f(x,y))$ where $f(x,y)$ is irreducible and $f(0,0) = 0$, i.e. those "irreducible curves through the origin". (There is a picture of this in Mumford's Red Book: Example F, Ch. 2, §1, p. 140.)

Caution with notation: If $\mathfrak{p}$ is a prime ideal, then $R_{\mathfrak{p}}$ means you're allowed to divide by elements not in $\mathfrak{p}$. However, if $f \in R$, R_f means you're allowed to divide by f. I find this a bit confusing. Especially when (f) is a prime ideal, and then $R_{(f)} \neq R_f$.

1. Affine schemes II: the underlying topological space

We now define the **Zariski topology** on $\operatorname{Spec} R$. Topologies are often described using open subsets, but it will more convenient for us to define this topology in terms of their complements, closed subsets. If S is a subset of R, define

$$V(S) := \{\mathfrak{p} \in \operatorname{Spec} R : S \subset \mathfrak{p}\}.$$

We interpret this as the **vanishing set** of S; it is the set of points on which all elements of S are zero. We declare that these (and no others) are the closed subsets.

1.1. Exercise. Show that if (S) is the ideal generated by S, then $V(S) = V((S))$. This lets us restrict attention to vanishing sets of ideals.

Let's check that this is a topology. Remember the requirements: the empty set and the total space should be open; the union of an arbitrary collection of open sets should be open; and the intersection of two open sets should be open.

1.2. Exercise. (a) Show that $\emptyset$ and $\operatorname{Spec} R$ are both open.
(b) (The intersection of two open sets is open.) Check that $V(I_1 I_2) = V(I_1) \cup V(I_2)$.
(c) (The union of any collection of open sets is open.) If I_i is a collection of ideals (as i runs over some index set), check that $V(\sum_i I_i) = \cap_i V(I_i)$.

1.3. Properties of "vanishing set" function $V(\cdot)$. The function $V(\cdot)$ is obviously inclusion-reversing: If $S_1 \subset S_2$, then $V(S_2) \subset V(S_1)$. (Warning: We could have equality in the second inclusion without equality in the first, as the next exercise shows.)

1.4. Exercise. If $I \subset R$ is an ideal, then define its *radical* by

$$\sqrt{I} := \{r \in R : r^n \in I \text{ for some } n \in \mathbb{Z}^{\geq 0}\}.$$

Show that $V(\sqrt{I}) = V(I)$. (We say an *ideal is radical* if it equals its own radical.)

Hence: $V(IJ) = V(I \cap J)$. (Reason: $(I \cap J)^2 \subset IJ \subset I \cap J$.) Combining this with Exercise 1.1, we see

$$V(S) = V((S)) = V(\sqrt{(S)}).$$

1.5. Examples. Let's see how this meshes with our examples from earlier.

Recall that $\mathbb{A}^1_{\mathbb{C}}$, as a set, was just the "old-fashioned" points (corresponding to maximal ideals, in bijection with $a \in \mathbb{C}$), and one "weird" point (0). The Zariski topology on $\mathbb{A}^1_{\mathbb{C}}$ is not that exciting. The open sets are the empty set, and $\mathbb{A}^1_{\mathbb{C}}$ minus a finite number of maximal ideals. (It "almost" has the cofinite topology. Notice that the open sets are determined by their intersections with the "old-fashioned points". The "weird" point (0) comes along for the ride, which is a good sign that it is harmless. Ignoring the "weird" point, observe that the topology on $\mathbb{A}^1_{\mathbb{C}}$ is a coarser topology than the analytic topology.)

The case $\operatorname{Spec} \mathbb{Z}$ is similar. The topology is "almost" the cofinite topology in the same way. The open sets are the empty set, and $\operatorname{Spec} \mathbb{Z}$ minus a finite number of "ordinary" ((p) where p is prime) primes.

The case $\mathbb{A}^2_{\mathbb{C}}$ is more interesting. I discussed it in a bit of detail in class, using pictures.

1.6. Topological definitions. We'll now define some words to do with the topology.

A topological space is said to be *irreducible* if it is not the union of two proper closed subsets. In other words, X if irreducible if whenever $X = Y \cup Z$ with Y and Z closed, we have $Y = X$ or $Z = X$.

1.7. Exercise. Show that if R is an integral domain, then $\operatorname{Spec} R$ is an irreducible topological space. (Hint: look at the point $[(0)]$.)

A point of a topological space $x \in X$ is said to be *closed* if $\overline{\{x\}} = \{x\}$.

1.8. Exercise. Show that the closed points of $\operatorname{Spec} R$ correspond to the maximal ideals.

Given two points x, y of a topological space X, we say that x is a *specialization* of y, and y is a *generization* of x, if $x \in \overline{\{y\}}$. This now makes precise our hand-waving about "one point contained another". It is of course nonsense for a point to contain another. But it is no longer nonsense to say that the closure of a point contains another.

1.9. Exercise. If $X = \operatorname{Spec} R$, show that $[\mathfrak{p}]$ is a specialization of $[\mathfrak{q}]$ if and only if $\mathfrak{q} \subset \mathfrak{p}$. Verify to your satisfaction that this is precisely the intuition of "containment of points" that we were talking about before.

We say that a point $x \in X$ is a *generic point* for a closed subset K if $\overline{\{x\}} = K$.

1.10. Exercise. Verify that $[(y - x^2)] \in \mathbb{A}^2$ is a generic point for $V(y - x^2)$.

A topological space X is *quasicompact* if given any cover $X = \cup_{i \in I} U_i$ by open sets, there is a finite subset S of the index set I such that $X = \cup_{i \in S} U_i$. Informally: every cover has a finite subcover. This is "half of the definition of quasicompactness". We will like this condition, because we are afraid of infinity.

1.11. Exercise. Show that $\operatorname{Spec} R$ is quasicompact. (Warning: it can have nonquasicompact open sets.)

1.12. Exercise. If X is a finite union of quasicompact spaces, show that X is quasicompact.

Earlier today, we explained that $\operatorname{Spec} R/I$ and $\operatorname{Spec} S^{-1}R$ are naturally subsets of $\operatorname{Spec} R$. All of these have Zariski topologies, and it is natural to ask if the topology behaves well with respect to these inclusions, and indeed it does.

1.13. Exercise. Suppose that $I, S \subset R$ are an ideal and multiplicative subset respectively. Show that $\operatorname{Spec} R/I$ is naturally a closed subset of $\operatorname{Spec} R$. Show that the Zariski topology on $\operatorname{Spec} R/I$ (resp. $\operatorname{Spec} S^{-1}R$) is the subspace topology induced by inclusion in $\operatorname{Spec} R$. (Hint: compare closed subsets.)

1.14. The function $I(\cdot)$, taking subsets of $\operatorname{Spec} R$ to ideals of R. Here is another notion, that is in some sense "opposite" to the vanishing set function $V(\cdot)$. Given a subset $S \subset \operatorname{Spec} R$, $I(S)$ is the ideal of functions vanishing on S. Three quick points: it is clearly an ideal. $I(\overline{S}) = I(S)$. And $I(\cdot)$ is inclusion-reversing: if $S_1 \subset S_2$, then $I(S_2) \subset I(S_1)$.

1.15. Exercise/Example. Let $R = k[x, y]$. If $S = \{(x), (x - 1, y)\}$ (draw this!), then $I(S)$ consists of those polynomials vanishing on the y axis, and at the point $(1, 0)$. Give generators for this ideal.

More generally:

1.16. Exercise. Show that $V(I(S)) = \overline{S}$. Hence $V(I(S)) = S$ for a closed set S.

1.17. Exercise. Suppose $X \subset \mathbb{A}^3$ is the union of the three axes. Give generators for the ideal $I(X)$.

Note that $I(S)$ is always a radical ideal — if $f \in \sqrt{I(S)}$, then f^n vanishes on S for some $n > 0$, so then f vanishes on S, so $f \in I(S)$.

Here is a handy algebraic fact to know. The *nilradical* $\mathfrak{N} = \mathfrak{N}(R)$ of a ring R is defined as $\sqrt{0}$ — it consists of all functions that have a power that is zero. (Checked that this set is indeed an ideal, for example that it is closed under addition!)

1.18. Theorem. *The nilradical $\mathfrak{N}(R)$ is the intersection of all the primes of R.*

If you don't know it, then look it up, or even better, prove it yourself. (Hint: one direction is easy. The other will require knowing that any proper ideal of R is contained in a maximal ideal, which requires the axiom of choice.) As a corollary, $\sqrt{I}$ is the intersection of all the prime ideals containing I. (Hint of proof: consider the ring R/I, and use the previous theorem.)

1.19. Exercise. Prove that if $I \subset R$ is an ideal, then $I(V(I)) = \sqrt{I}$.

Hence in combination with Exercise 1.16, we get the following:

1.20. *Theorem.* — *$V(\cdot)$ and $I(\cdot)$ give a bijection between closed subsets of $\operatorname{Spec} R$ and radical ideals of R (where a closed subset gives a radical ideal by $I(\cdot)$, and a radical ideal gives a closed subset by $V(\cdot)$).*

1.21. Important Exercise. Show that $V(\cdot)$ and $I(\cdot)$ give a bijection between *irreducible closed subsets* of $\operatorname{Spec} R$ and *prime* ideals of R. From this conclude that in $\operatorname{Spec} R$ there is a bijection between points of $\operatorname{Spec} R$ and irreducible closed subsets of $\operatorname{Spec} R$ (where a point determines an irreducible closed subset by taking the closure). Hence each irreducible closed subset has precisely one generic point.

To drive this point home: Suppose Z is an irreducible closed subset of $\operatorname{Spec} R$. Then there is one and only one $z \in Z$ such that $Z = \overline{\{z\}}$.

2. Distinguished open sets

If $f \in R$, define the **distinguished open set** $D(f) = \{\mathfrak{p} \in \operatorname{Spec} R : f \notin \mathfrak{p}\}$. It is the locus where f doesn't vanish. (I often privately write this as $D(f \neq 0)$ to remind myself of this. I also private call this a D*oesn't vanish set* in analogy with $V(f)$ being the Vanishing set.) We have already seen this set when discussing $\operatorname{Spec} R_f$ as a subset of $\operatorname{Spec} R$.

2.1. Important exercise. Show that the distinguished opens form a base for the Zariski topology.

2.2. Easy important exercise. Suppose $f_i \in R$ for $i \in I$. Show that $\cup_{i \in I} D(f_i) = \operatorname{Spec} R$ if and only if $(f_i) = R$.

2.3. Easy important exercise. Show that $D(f) \cap D(g) = D(fg)$. Hence the distinguished base is a *nice* base.

2.4. Easy important exercise. Show that if $D(f) \subset D(g)$, then $f^n \in (g)$ for some n.

2.5. Easy important exercise. Show that $f \in \mathfrak{N}$ if and only if $D(f) = \emptyset$.

We have already observed that the Zariski topology on the distinguished open $D(f) \subset \operatorname{Spec} R$ coincides with the Zariski topology on $\operatorname{Spec} R_f$.

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FOUNDATIONS OF ALGEBRAIC GEOMETRY CLASS 6

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CONTENTS

Last day: $\operatorname{Spec} R$**: the set, and the topology**

Today: The structure sheaf, and schemes in general.

Announcements: on problem set 2, there was a serious typo in # 10. $\operatorname{Hom}(\mathcal{F}(U), \mathcal{G}(U))$ should read $\operatorname{Hom}(\mathcal{F}|_U, \mathcal{G}|_U)$. The notation is new, but will likely be clear to you after you think about it a little. If $\mathcal{F}$ is a sheaf on X, and U is an open subset, then we can define the sheaf $\mathcal{F}|_U$ on U in the obvious way. This is sometimes called *the restriction of the sheaf* $\mathcal{F}$ *to the open set* U (not to be confused with restriction maps!). This homomorphism $\operatorname{Hom}(\mathcal{F}|_U, \mathcal{G}|_U)$ is the set of all *sheaf homomorphisms* from $\mathcal{F}|_U$ to $\mathcal{G}|_U$. The revised version is posted on the website.

Also, the final problem set this quarter will be due no later than Monday, December 12, the Monday after the last class.

1. RECAP OF LAST DAY, AND FURTHER DISCUSSION

Last day, we saw lots of examples of the underlying sets of affine schemes, which correspond to primes in a ring. In this dictionary, "an element r of the ring lying in a prime ideal $\mathfrak{p}$" translates to "an element r of the ring vanishing at the point $[\mathfrak{p}]$, and I will use these phrases interchangeably.

There was some language I was using informally, and I've decided to make it more formal: elements $r \in R$ will officially be called "global functions", and their value at the point $[\mathfrak{p}]$ will be $r \pmod{\mathfrak{p}}$. This language will be "justified" by the end of today.

I then defined the Zariski topology. The closed subsets were just those points where some set of ring elements all vanish.

Date: Wednesday, October 19, 2005. Minor update January 30, 2007.

As a reminder, here are the key words that we learned about topological spaces: irreducible; generic point; closed points (points p such that $\overline{\{p\}} = \{p\}$; did I forget to say this last time?); specialization/generization; quasicompact. All of these words can be used on any topological spaces, but they tend to be boring (or highly improbable) on spaces that you knew and loved before.

On $\operatorname{Spec} R$, closed points correspond to maximal ideals of R. Also, I described a bijection between closed subsets and radical ideals. The two maps of this bijection use the "vanishing set" function V and the "ideal of functions vanishing" function I. I also described a bijection between points and closed subsets; one direction involved taking closures, and the other involved taking generic points. Some of this was left to you in the form of **exercises**.

As an example, consider the prime (or point) $(y - x^2)$ in $k[x, y]$ (or $\operatorname{Spec} k[x, y]$). What is its closure? We look at all functions vanishing at this point, and see at what other points they all vanish. In other words, we look for all prime ideals containing all elements of this one. In other words, we look at all prime ideals containing this one. Picture: we get all the closed points on the parabola. We get the closed set corresponding to this point. (Caveat: I haven't proved that I've described all the primes in $k[x, y]$.)

In class, I spontaneously showed you that the Zariski closure of the countable set (n, n^2) in $\mathbb{A}^2_{\mathbb{C}}$ was the parabola. The Zariski closure of a finite set of points will just be itself: a finite union of closed sets is closed.

Last day I showed that the Zariski topology behaves well with respect to taking quotients, and localizing. I said a little more today.

About taking quotients: suppose you have a ring R, and an ideal I. Then there is a bijection of the points of $\operatorname{Spec} R/I$ with the points of $V(I)$ in $\operatorname{Spec} R$. (Just unwind the algebraic definition! Both correspond to primes $\mathfrak{p}$ of R containing I.) My comments of last day showed that this is in fact a homeomorphism: $\operatorname{Spec} R/I$ may be identified with the closed subset of $\operatorname{Spec} R$ as a topological space: the subspace topology induced from $\operatorname{Spec} R$ is indeed the topology of $\operatorname{Spec} R$. The reason was not sophisticated: there is a natural correspondence of closed subsets.

About localizing: this is quite a general procedure, so in general you can't say much besides the fact that $\operatorname{Spec} S^{-1}R$ is naturally a subset of $\operatorname{Spec} R$, with the induced topology. Last day I discussed the important case where $S = R - \mathfrak{p}$, the complement of a prime ideal, so then $S^{-1}R = R_{\mathfrak{p}}$.

But there is a second important example of localization, when $S = \{1, f, f^2, \dots\}$ for some $f \in R$. In this case we get the ring denoted R_f. In this case, $\operatorname{Spec} R_f$ is $D(f)$, again just by unwinding the definitions: both consist of the primes not containing f (= the points where f doesn't vanish). The Zariski topology on $D(f)$ agrees with the Zariski topology on $\operatorname{Spec} R_f$.

Here is an **exercise** from last day. Show that $(f_1, \dots, f_n) = (1)$ if and only if $\cup D(f_i) = \operatorname{Spec} R$. I want to do it for you, to show you how it can be interpreted simultaneously in both algebra and geometry. Here is one direction. Suppose $[\mathfrak{p}] \notin \cup D(f_i)$. You can unwind this to get an algebraic statement. I think of it as follows. All of the f_i vanish at $[\mathfrak{p}]$, i.e. all $f_i \in \mathfrak{p}$, so then $(f_1, \dots) \subset \mathfrak{p}$ and hence this ideal can't be all of R. Conversely, consider the ideal $(f_1, \dots)$. If it isn't R, then it is contained in a maximal ideal. (For logic-lovers: we're using the axiom of choice, which I said I'd assume at the very start of this class.) But then there is some prime ideal containing all the f_i. Translation: $[\mathfrak{m}] \notin D(f_i)$ for any i. (As an added bonus: this argument shows that if $\operatorname{Spec} R$ is an infinite union $\cup_{i \in I} D(f_i)$, then in fact it is a union of a finite number of these. This is one way of proving quasicompactness.)

Important comment: This machinery will let us bring our geometric intuition to algebra. There is one point where your intuition will be false, and I want to tell you now, so you can adjust your intuition appropriately. Suppose we have a function (ring element) vanishing at all points. Is it zero? Not necessarily! Translation: is intersection of all prime ideals necessarily just 0? No: $k[\epsilon]/\epsilon^2$ is a good example, as $\epsilon \neq 0$, but $\epsilon^2 = 0$. This is called the *ring of dual numbers* (over the field k). Any function whose power is zero certainly lies in the intersection of all prime ideals. And the converse is true (algebraic fact): the intersection of all the prime ideals consists of functions for which some power is zero, otherwise known as the nilradical $\mathfrak{N}$. (Ring elements that have a power that is 0 are called *nilpotents*.) Summary: "functions on affine schemes" will not be determined by their values at points. (For example: $\operatorname{Spec} k[\epsilon]/\epsilon^2$ has one point. $3 + 4\epsilon$ has value 3 at that point, but the function isn't 3.) In particular, any function vanishing at all points might not be zero, but some power of it is zero. This takes some getting used to.

1.1. *Easy fun unimportant exercise.* Suppose we have a polynomial $f(x) \in k[x]$. Instead, we work in $k[x, \epsilon]/\epsilon^2$. What then is $f(x + \epsilon)$? (Do a couple of examples, and you will see the pattern. For example, if $f(x) = 3x^3 + 2x$, we get $f(x + \epsilon) = (3x^3 + 2x) + \epsilon(9x^2 + 2)$. Prove the pattern!) Useful tip: the dual numbers are a good source of (counter)examples, being the "smallest ring with nilpotents". They will also end up being important in defining differential information.

Here is one more (important!) algebraic fact: suppose $D(f) \subset D(g)$. Then $f^n \in (g)$ for some n. I'm going to let you prove this (**Exercise** from last day), but I want to tell you how I think of it geometrically. Draw a picture of $\operatorname{Spec} R$. Draw the closed subset $V(g) = \operatorname{Spec} R/(g)$. That's where g vanishes, and the complement is $D(g)$, where g doesn't vanish. Then f is a function on this closed subset, and it vanishes at all points of the closed subset. (Translation: Consider f as an element of the ring $R/(g)$.) Now any function vanishing at every point of Spec a ring must have some power which is 0. Translation: there is some n such that $f^n = 0$ in $R/(g)$, i.e. $f^n \equiv 0 \pmod{g}$ in R, i.e. $f^n \in (g)$.

2. THE FINAL INGREDIENT IN THE DEFINITION OF AFFINE SCHEMES: THE STRUCTURE SHEAF

The final ingredient in the definition of an affine scheme is the **structure sheaf** $\mathcal{O}_{\operatorname{Spec} R}$, which we think of as the "sheaf of algebraic functions". These functions will have values at points, but won't be determined by their values at points. Like all sheaves, they will indeed be determined by their germs.

It suffices to describe it as a sheaf on the nice base of distinguished open sets. We define the sections on the base by

$$\mathcal{O}_{\operatorname{Spec} R}(D(f)) = R_f \tag{1}$$

We define the restriction maps $\operatorname{res}_{D(g),D(f)}$ as follows. If $D(f) \subset D(g)$, then we have shown that $f^n \in (g)$, i.e. we can write $f^n = ag$. There is a natural map $R_g \to R_f$ given by $r/g^m \mapsto (ra^m)/(f^{mn})$, and we define

$$\operatorname{res}_{D(g),D(f)} : \mathcal{O}_{\operatorname{Spec} R}(D(g)) \to \mathcal{O}_{\operatorname{Spec} R}(D(f))$$

to be this map.

2.1. *Exercise.* (a) Verify that (1) is well-defined, i.e. if $D(f) = D(f')$ then R_f is canonically isomorphic to $R_{f'}$. (b) Show that $\operatorname{res}_{D(g),D(f)}$ is well-defined, i.e. that it is independent of the choice of a and n, and if $D(f) = D(f')$ and $D(g) = D(g')$, then

$$\begin{array}{ccc} R_g & \xrightarrow{\operatorname{res}_{D(g),D(f)}} & R_f \\ \downarrow \sim & & \downarrow \sim \\ R_{g'} & \xrightarrow{\operatorname{res}_{D(g'),D(f')}} & R_{f'} \end{array}$$

commutes.

We now come to the big theorem of today.

2.2. *Theorem.* — *The data just described gives a sheaf on the (nice) distinguished base, and hence determines a sheaf on the topological space* $\operatorname{Spec} R$*.*

This sheaf is called the **structure sheaf**, and will be denoted $\mathcal{O}_{\operatorname{Spec} R}$, or sometimes $\mathcal{O}$ if the scheme in question is clear from the context. Such a topological space, with sheaf, will be called an **affine scheme**. The notation $\operatorname{Spec} R$ will hereafter be a topological space, with a structure sheaf.

Proof. Clearly this is a presheaf on the base: if $D(f) \subset D(g) \subset D(h)$ then the following diagram commutes:

$$\begin{array}{ccc} R_h & \xrightarrow{\operatorname{res}_{D(h),D(g)}} & R_g \\ & {\scriptstyle \operatorname{res}_{D(h),D(f)}} \searrow \quad \swarrow {\scriptstyle \operatorname{res}_{D(g),D(f)}} & \\ & R_f. & \end{array} \tag{2}$$

You can check this directly. Here is a trick which helps (and may help you with Exercise 2.1 above). As $D(g) \subset D(h)$, $D(gh) = D(g)$. (Translation: The locus where g doesn't vanish is a subset of where h doesn't vanish, so the locus where gh doesn't vanish is the same as the locus where g doesn't vanish.) So we can replace R_g by R_{gh}, and R_f by R_{fgh}, in (2). The restriction maps are $\operatorname{res}_{D(h),D(gh)} : a/h \mapsto ag/gh$, $\operatorname{res}_{D(gh),D(fgh)} : b/gh \mapsto bf/fgh$, and $\operatorname{res}_{D(h),D(fgh)} : a/h \to afg/fgh$, so they clearly commute as desired.

We next check identity on the base. We deal with the case of a cover of the entire space R, and let the reader verify that essentially the same argument holds for a cover of some R_f. Suppose that $\operatorname{Spec} R = \cup_{i \in A} D(f_i)$ where i runs over some index set I. By quasicompactness, there is some finite subset of I, which we name $\{1, \ldots, n\}$, such that $\operatorname{Spec} R = \cup_{i=1}^n D(f_i)$, i.e. $(f_1, \ldots, f_n) = R$. (Now you see why we like quasicompactness!) Suppose we are given $s \in R$ such that $\operatorname{res}_{\operatorname{Spec} R, D(f_i)} s = 0$ in R_{f_i} for all i. Hence there is some m such that for each $i \in \{1, \ldots, n\}$, $f_i^m s = 0$. (Reminder: $R \to R_f$. What goes to 0? Precisely things killed by some power of f.) Now $(f_1^m, \ldots, f_n^m) = R$ (do you know why?), so there are $r_i \in R$ with $\sum_{i=1}^n r_i f_i^m = 1$ in R, from which

$$s = \left(\sum r_i f_i^m \right) s = 0.$$

Thus we have checked the "base identity" axiom for $\operatorname{Spec} R$.

Remark. Serre has described this as a "partition of unity" argument, and if you look at it in the right way, his insight is very enlightening.

2.3. *Exercise.* Make the tiny changes to the above argument to show base identity for any distinguished open $D(f)$.

We next show base gluability. As with base identity, we deal with the case where we wish to glue sections to produce a section over $\operatorname{Spec} R$. As before, we leave the general case where we wish to glue sections to produce a section over $D(f)$ to the reader (Exercise 2.4).

Suppose $\cup_{i \in I} D(f_i) = \operatorname{Spec} R$, where I is a index set (possibly horribly uncountably infinite). Suppose we are given

$$\frac{a_i}{f_i^{l_i}} \in R_{f_i} \quad (i \in I)$$

such that for all $i, j \in I$, there is some $m_{ij} \geq l_i, l_j$ with

$$(f_i f_j)^{m_{ij}} \left(f_j^{l_j} a_i - f_i^{l_i} a_j \right) = 0 \tag{3}$$

in R. We wish to show that there is some $r \in R$ such that $r = a_i / f_i^{l_i}$ in R_{f_i} for all $i \in I$.

Choose a finite subset $\{1, \ldots, n\} \subset I$ with $(f_1, \ldots, f_n) = R$.

To save ourself some notation, we may take the l_i to all be 1, by replacing f_i with $f_i^{l_i}$ (as $D(f_i) = D(f_i^{l_i})$). We may take m_{ij} ($1 \leq i, j \leq n$) to be the same, say m — take $m = \max m_{ij}$.) The only reason to do this is to have fewer variables.

Let $a_i' = a_i f_i^m$. Then $a_i / f_i = a_i' / f_i^{m+1}$, and (3) becomes

$$f_j^{m+1} a_i' - f_i^{m+1} a_j' = 0 \tag{4}$$

As $(f_1, \ldots, f_n) = R$, we have $(f_1^{m+1}, \ldots, f_n^{m+1}) = R$, from which $1 = \sum b_i f_i^{m+1}$ for some $b_i \in R$. Define

$$r = b_1 a_1' + \cdots + b_n a_n'.$$

This will be the r that we seek. For each $i \in \{1, \ldots, n\}$, we will show that $r - a_i'/f_i^{m+1} = 0$ in D_{f_i}. Indeed,

$$\begin{aligned} rf_i^{m+1} - a_i' &= \sum_{j=1}^{n} a_j' b_j f_i^{m+1} - \sum_{j=1}^{n} a_i' b_j f_j^{m+1} \\ &= \sum_{j \neq i} b_j (a_j' f_i^{m+1} - a_i' f_j^{m+1}) \\ &= 0 \quad \text{(by (4))} \end{aligned}$$

So are we done? No! We are supposed to have something that restricts to $a_i/f_i^{l_i}$ for *all* $i \in I$, not just $i = 1, \ldots, n$. But a short trick takes care of this. We now show that for any $\alpha \in I - \{1, \ldots, n\}$, $r = a_\alpha/f_\alpha^{l_\alpha}$ in R_{f_α}. Repeat the entire process above with $\{1, \ldots, n, \alpha\}$ in place of $\{1, \ldots, n\}$, to obtain $r' \in R$ which restricts to $a_i/f_i^{l_i}$ for $i \in \{1, \ldots, n, \alpha\}$. Then by base identity, $r' = r$. Hence r restricts to $a_\alpha/f_\alpha^{l_\alpha}$ as desired.

2.4. *Exercise.* Alter this argument appropriately to show base gluability for any distinguished open $D(f)$.

□

So now you know what an affine scheme is!

We can even define a scheme in general: it is a topological space X, along with a structure sheaf $\mathcal{O}_X$, that locally looks like an affine scheme: for any $x \in X$, there is an open neighborhood U of x such that $(U, \mathcal{O}_X|_U)$ is an affine scheme.

On Friday, I'll discuss some of the ramifications of this definition. In particular, you'll see that stalks of this sheaf are something familiar, and I'll show you that constructing the sheaf by looking at this nice distinguished base isn't just a kluge, it's something very natural — we'll do this by finding sections of $\mathcal{O}_{\mathbb{A}^2}$ over the open set $\mathbb{A}^2 - (0, 0)$.

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FOUNDATIONS OF ALGEBRAIC GEOMETRY CLASS 7

CONTENTS

Last day: The structure sheaf.

Today: $\tilde{M}$, sheaf associated to R-module M; Chinese remainder theorem; germs and value at a point of the structure sheaf; non-affine schemes $\mathbb{A}^2 - (0,0)$, line with doubled origin, $\mathbb{P}^n$.

Another problem set 2 issue, about the pullback sheaf. First, I think I'd like to call it the inverse image sheaf, because I don't want to confuse it with something that I'll also call the pullback. Second, and more importantly, I didn't give the correct definition.

Here is what I should have said (and what is now in the problem set). Define $f^{-1}\mathcal{G}^{\text{pre}}(U) = \lim_{\to V \supset f(U)} \mathcal{G}(V)$. Then show that this is a *presheaf*. Then the sheafification of this is said to be the *inverse image sheaf* (sometimes called the pullback sheaf) $f^{-1}\mathcal{G} := (f^{-1}\mathcal{G}^{\text{pre}})^{\text{sh}}$. Thanks to Kate for pointing out this important patch!

1. PLAYING WITH THE STRUCTURE SHEAF

Here's where we were last day. We defined the *structure sheaf* $\mathcal{O}_{\operatorname{Spec} R}$ on an affine scheme $\operatorname{Spec} R$. We did this by describing it as a *sheaf on the distinguished base*.

An immediate consequence is that we can recover our ring R from the scheme $\operatorname{Spec} R$ by taking global sections, as the entire scheme is $D(1)$:

$$\begin{aligned} \Gamma(\operatorname{Spec} R, \mathcal{O}_{\operatorname{Spec} R}) &= \Gamma(D(1), \mathcal{O}_{\operatorname{Spec} R}) \quad \text{as } D(1) = \operatorname{Spec} R \\ &= R_1 \quad \text{(i.e. allow 1's in the denominator) by definition} \\ &= R \end{aligned}$$

Another easy consequence is that the restriction of the sheaf $\mathcal{O}_{\operatorname{Spec} R}$ to the distinguished open set $D(f)$ gives us the affine scheme $\operatorname{Spec} R_f$. Thus not only does the set restrict, but also the topology (as we've seen), and the structure sheaf (as this exercise shows).

Date: Friday, October 21, 2005. Minor updates January 28, 2007.

1.1. *Important but easy exercise.* Suppose $f \in R$. Show that under the identification of $D(f)$ in $\operatorname{Spec} R$ with $\operatorname{Spec} R_f$, there is a natural isomorphism of sheaves $(D(f), \mathcal{O}_{\operatorname{Spec} R}|_{D(f)}) \cong (\operatorname{Spec} R_f, \mathcal{O}_{\operatorname{Spec} R_f})$.

The proof of Big Theorem of last time (that the object $\mathcal{O}_{\operatorname{Spec} R}$ defined by $\Gamma(D(f), \mathcal{O}_{\operatorname{Spec} R}) = R_f$ forms a sheaf on the distinguished base, and hence a sheaf) immediately generalizes, as the following exercise shows. This exercise will be essential for the definition of a quasicoherent sheaf later on.

1.2. *Important but easy exercise.* Suppose M is an R-module. Show that the following construction describes a sheaf $\tilde{M}$ on the distinguished base. To $D(f)$ we associate $M_f = M \otimes_R R_f$; the restriction map is the "obvious" one. This is a sheaf of $\mathcal{O}_{\operatorname{Spec} R}$-modules! We get a natural bijection: rings, modules $\leftrightarrow$ Affine schemes, $\tilde{M}$.

Useful fact for later: As a consequence, note that if $(f_1, \dots, f_r) = R$, we have identified M with a specific submodule of $M_{f_1} \times \cdots \times M_{f_r}$. Even though $M \to M_{f_i}$ may not be an inclusion for any f_i, $M \to M_{f_1} \times \cdots \times M_{f_r}$ *is* an inclusion. We don't care yet, but we'll care about this later, and I'll invoke this fact. (Reason: we'll want to show that if M has some nice property, then M_f does too, which will be easy. We'll also want to show that if $(f_1, \dots, f_n) = R$, then if M_{f_i} have this property, then M does too.)

1.3. Brief fun: The Chinese Remainder Theorem is a *geometric* fact. I want to show you that the Chinese Remainder theorem is embedded in what we've done, which shouldn't be obvious to you. I'll show this by example. The Chinese Remainder Theorem says that knowing an integer modulo 60 is the same as knowing an integer modulo 3, 4, and 5. Here's how to see this in the language of schemes. What is $\operatorname{Spec} \mathbb{Z}/(60)$? What are the primes of this ring? Answer: those prime ideals containing (60), i.e. those primes dividing 60, i.e. (2), (3), and (5). So here is my picture of the scheme [3 dots]. They are all closed points, as these are all maximal ideals, so the topology is the discrete topology. What are the stalks? You can check that they are $\mathbb{Z}/4$, $\mathbb{Z}/3$, and $\mathbb{Z}/5$. My picture is actually like this (draw a small arrow on (2)): the scheme has nilpotents here ($2^2 \equiv 0 \pmod 4$). So what are global sections on this scheme? They are sections on this open set (2), this other open set (3), and this third open set (5). In other words, we have a natural isomorphism of rings

$$\mathbb{Z}/60 \to \mathbb{Z}/4 \times \mathbb{Z}/3 \times \mathbb{Z}/5.$$

On a related note:

1.4. *Exercise.* Show that the disjoint union of a *finite* number of affine schemes is also an affine scheme. (Hint: say what the ring is.)

This is *always* false for an infinite number of affine schemes:

1.5. *Unimportant exercise.* An infinite disjoint union of (non-empty) affine schemes is not an affine scheme. (One-word hint: quasicompactness.)

1.6. Stalks of this sheaf: germs, and values at a point. Like every sheaf, the structure sheaf has stalks, and we shouldn't be surprised if they are interesting from an algebraic point of view. In fact, we have seen them before.

1.7. *Exercise.* Show that the stalk of $\mathcal{O}_{\operatorname{Spec} R}$ at the point $[\mathfrak{p}]$ is the ring $R_{\mathfrak{p}}$. (Hint: use distinguished open sets in the direct limit you use to define the stalk. In the course of doing this, you'll discover a useful principle. In the concrete definition of stalk, the elements were sections of the sheaf over *some* open set containing our point, and two sections over different open sets were considered the same if they agreed on some smaller open set. In fact, you can just consider elements of your base when doing this. This is called a *cofinal system* in the *directed set*.) This is yet another reason to like the notion of a sheaf on a base.

The *residue field of a scheme at a point* is the local ring modulo its maximal ideal.

Essentially the same argument will show that the stalk of the sheaf $\tilde{M}$ at $[\mathfrak{p}]$ is $M_{\mathfrak{p}}$.

So now we can make precise some of our intuition. Suppose $[\mathfrak{p}]$ is a point in some open set U of $\operatorname{Spec} R$. For example, say $R = k[x, y]$, $\mathfrak{p} = (x)$ (draw picture), and $U = \mathbb{A}^2 - (0, 0)$. (First, make sure you see that this *is* an open set! $(0, 0) = V((x, y))$ is indeed closed. Make sure you see that $[\mathfrak{p}]$ indeed is in U.)

- Then a function on U, i.e. a section of $\mathcal{O}_{\operatorname{Spec} R}$ over U, has a *germ near* $[\mathfrak{p}]$, which is an element of $R_{\mathfrak{p}}$. Note that this is a local ring, with maximal ideal $\mathfrak{p}R_{\mathfrak{p}}$. In our example, consider the function $(3x^4 + x^2 + xy + y^2)/(3x^2 + xy + y^2 + 1)$, which is defined on the open set $D(3x^2 + xy + y^2 + 1)$. Because there are no factors of x in the denominator, it is indeed in $R_{\mathfrak{p}}$.
- A germ has a *value* at $[\mathfrak{p}]$, which is an element of $R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}$. (This is isomorphic to $\operatorname{Frac}(R/\mathfrak{p})$, the fraction field of the quotient domain.) So the value of a function at a point always takes values in a field. In our example, to see the value of our germ at $x = 0$, we simply set $x = 0$! So we get the value $y^2/(y^2 + 1)$, which is certainly in $\operatorname{Frac} k[y]$.
- We say that the germ *vanishes* at $\mathfrak{p}$ if the value is zero. In our example, the germ doesn't vanish at $\mathfrak{p}$.

If anything makes you nervous, you should make up an example to assuage your nervousness. (Example: $27/4$ is a regular function on $\operatorname{Spec} \mathbb{Z} - \{(2), (7)\}$. What is its value at (5)? Answer: $2/(-1) \equiv -2 \pmod 5$. What is its value at (0)? Answer: $27/4$. Where does it vanish? At (3).)

1.8. An extended example. I now want to work through an example with you, to show that this distinguished base is indeed something that you can work with. Let $R = k[x, y]$, so $\operatorname{Spec} R = \mathbb{A}_k^2$. If you want, you can let k be $\mathbb{C}$, but that won't be relevant. Let's work out the space of functions on the open set $U = \mathbb{A}^2 - (0, 0)$.

It is a non-obvious fact that you can't cut out this set with a single equation, so this isn't a distinguished open set. We'll see why fairly soon. But in any case, even if we're not sure if this is a distinguished open set, we can describe it as the union of two things which *are* distinguished open sets. If I throw out the x axis, i.e. the set $y = 0$, I get the distinguished open set $D(y)$. If I throw out the y axis, i.e. $x = 0$, I get the distinguished open set $D(x)$. So $U = D(x) \cup D(y)$. (Remark: $U = \mathbb{A}^2 - V(x, y)$ and $U = D(x) \cup D(y)$. Coincidence? I think not!) We will find the functions on U by gluing together functions on $D(x)$ and $D(y)$.

What are the functions on $D(x)$? They are, by definition, $R_x = k[x, y, 1/x]$. In other words, they are things like this: $3x^2 + xy + 3y/x + 14/x^4$. What are the functions on $D(y)$? They are, by definition, $R_y = k[x, y, 1/y]$. Note that $R \hookrightarrow R_x, R_y$. This is because x and y are not zero-divisors. (In fact, R is an integral domain — it has no zero-divisors, besides 0 — so localization is always an inclusion.) So we are looking for functions on $D(x)$ and $D(y)$ that agree on $D(x) \cap D(y) = D(xy)$, i.e. they are just the same function. Well, which things of this first form are also of the second form? Just old-fashioned polynomials —

$$\Gamma(U, \mathcal{O}_{\mathbb{A}^2}) \equiv k[x, y]. \tag{1}$$

In other words, we get no extra functions by throwing out this point. Notice how easy that was to calculate!

This is interesting: any function on $\mathbb{A}^2 - (0, 0)$ extends over all of $\mathbb{A}^2$. (Aside: This is an analog of Hartogs' theorem in complex geometry: you can extend a holomorphic function defined on the complement of a set of codimension at least two on a complex manifold over the missing set. This will work more generally in the algebraic setting: you can extend over points in codimension at least 2 not only if they are smooth, but also if they are mildly singular — what we will call *normal*.)

We can now see that this is not an affine scheme. Here's why: otherwise, if $(U, \mathcal{O}_{\mathbb{A}^2}|_U) = (\operatorname{Spec} S, \mathcal{O}_{\operatorname{Spec} S})$, then we can recover S by taking global sections:

$$S = \Gamma(U, \mathcal{O}_{\mathbb{A}^2}|_U),$$

which we have already identified in (1) as $k[x, y]$. So if U is affine, then $U = \mathbb{A}^2_k$. But we get more: we can recover the points of $\operatorname{Spec} S$ by taking the primes of S. In particular, the prime ideal (x, y) of S should cut out a point of $\operatorname{Spec} S$. But on U, $V(x) \cap V(y) = \emptyset$. Conclusion: U is *not* an affine scheme. (If you are ever looking for a counterexample to something, and you are expecting one involving a non-affine scheme, keep this example in mind!)

It is however a scheme.

Again, let me repeat the *definition of a* **scheme**. It is a topological space X, along with a sheaf of rings $\mathcal{O}_X$, such that any point $x \in X$ has a neighborhood U such that $(U, \mathcal{O}_X|_U)$ is an affine scheme (i.e. we have a homeomorphism of U with some $\operatorname{Spec} R$, say $f : U \to \operatorname{Spec} R$, and an isomorphism $\mathcal{O}_X|_U \cong \mathcal{O}_R$, where the two spaces are identified). The scheme can be denoted $(X, \mathcal{O}_X)$, although it is often denoted X, with the structure sheaf implicit.

I stated earlier in the notes, Exercise 1.1 (and at roughly at this point in the class): If we take the underlying subset of $D(f)$ with the restriction of the sheaf $\mathcal{O}_{\operatorname{Spec} R}$, we obtain the scheme $\operatorname{Spec} R_f$.

If X is a scheme, and U is *any* open subset, then $(U, \mathcal{O}_X|_U)$ is also a scheme. **Exercise.** Prove this. $(U, \mathcal{O}_X|_U)$ is called an *open subscheme of* U. If U is also an affine scheme, we often say U is an *affine open subset*, or an *affine open subscheme*, or sometimes informally just an *affine open*. For an example, $D(f)$ is an affine open subscheme of $\operatorname{Spec} R$.

1.9. *Exercise.* Show that if X is a scheme, then the affine open sets form a base for the Zariski topology. (Warning: they don't form a nice base, as we'll see in Exercise 1.11 below. However, in "most nice situations" this will be true, as we will later see, when we define the analogue of "Hausdorffness", called separatedness.)

You've already seen two examples of non-affine schemes: an infinite disjoint union of non-empty schemes, and $\mathbb{A}^2 - (0,0)$. I want to give you two more important examples. They are important because they are the first examples of fundamental behavior, the first pathological, and the second central.

First, I need to tell you how to glue two schemes together. Suppose you have two schemes $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$, and open subsets $U \subset X$ and $V \subset Y$, along with a homeomorphism $U \cong V$, and an isomorphism of structure sheaves $(U, \mathcal{O}_X|_U) \cong (V, \mathcal{O}_Y|_V)$. Then we can glue these together to get a single scheme. Reason: let W be X and Y glued together along the isomorphism $U \cong V$. Then problem 9 on the second problem set shows that the structure sheaves can be glued together to get a sheaf of rings. Note that this is indeed a scheme: any point has a neighborhood that is an affine scheme. (Do you see why?)

So I'll give you two non-affine schemes. In both cases, I will glue together two copies of the affine line $\mathbb{A}^1_k$. Again, if it makes you feel better, let $k = \mathbb{C}$, but it really doesn't matter; this is the last time I'll say this.

Let $X = \operatorname{Spec} k[t]$, and $Y = \operatorname{Spec} k[u]$. Let $U = D(t) = \operatorname{Spec} k[t, 1/t] \subset X$ and $V = D(u) = \operatorname{Spec} k[u, 1/u] \subset Y$.

We will get example 1 by gluing X and Y together along U and V. We will get example 2 by gluing X and Y together along U and V.

Example 1: the affine line with the doubled origin. Consider the isomorphism $U \cong V$ via the isomorphism $k[t, 1/t] \cong k[u, 1/u]$ given by $t \leftrightarrow u$. Let the resulting scheme be X. The picture looks like this [line with doubled origin]. This is called the *affine line with doubled origin*.

As the picture suggests, intuitively this is an analogue of a failure of Hausdorffness. $\mathbb{A}^1$ itself is not Hausdorff, so we can't say that it is a failure of Hausdorffness. We see this as weird and bad, so we'll want to make up some definition that will prevent this from happening. This will be the notion of *separatedness*. This will answer other of our prayers as well. For example, on a separated scheme, the "affine base of the Zariski topology" is nice — the intersection of two affine open sets will be affine.

1.10. *Exercise.* Show that X is not affine. Hint: calculate the ring of global sections, and look back at the argument for $\mathbb{A}^2 - (0,0)$.

1.11. *Exercise.* Do the same construction with $\mathbb{A}^1$ replaced by $\mathbb{A}^2$. You'll have defined the *affine plane with doubled origin*. Use this example to show that the affine base of the topology isn't a nice base, by describing two affine open sets whose intersection is not affine.

Example 2: the projective line. Consider the isomorphism $U \cong V$ via the isomorphism $k[t, 1/t] \cong k[u, 1/u]$ given by $t \leftrightarrow 1/u$. The picture looks like this [draw it]. Call the resulting scheme the *projective line over the field* k, $\mathbb{P}^1_k$.

Notice how the points glue. Let me assume that k is algebraically closed for convenience. (You can think about how this changes otherwise.) On the first affine line, we have the closed (= "old-fashioned") points $(t - a)$, which we think of as "a on the t-line", and we have the generic point. On the second affine line, we have closed points that are "b on the u-line", and the generic point. Then a on the t-line is glued to $1/a$ on the u-line (if $a \neq 0$ of course), and the generic point is glued to the generic point (the ideal (0) of $k[t]$ becomes the ideal (0) of $k[t, 1/t]$ upon localization, and the ideal (0) of $k[u]$ becomes the ideal (0) of $k[u, 1/u]$. And (0) in $k[t, 1/t]$ is (0) in $k[u, 1/u]$ under the isomorphism $t \leftrightarrow 1/u$).

We can interpret the closed ("old-fashioned") points of $\mathbb{P}^1$ in the following way, which may make this sound closer to the way you have seen projective space defined earlier. The points are of the form $[a; b]$, where a and b are not both zero, and $[a; b]$ is identified with $[ac; bc]$ where $c \in k^*$. Then if $b \neq 0$, this is identified with a/b on the t-line, and if $a \neq 0$, this is identified with b/a on the u-line.

1.12. *Exercise.* Show that $\mathbb{P}^1_k$ is not affine. Hint: calculate the ring of global sections.

This one I will do for you.

The global sections correspond to sections over X and sections over Y that agree on the overlap. A section on X is a polynomial $f(t)$. A section on Y is a polynomial $g(u)$. If I restrict $f(t)$ to the overlap, I get something I can still call $f(t)$; and ditto for $g(u)$. Now we want them to be equal: $f(t) = g(1/t)$. How many polynomials in t are at the same time polynomials in $1/t$? Not very many! Answer: only the constants k. Thus $\Gamma(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}) = k$. If $\mathbb{P}^1$ were affine, then it would be $\operatorname{Spec} \Gamma(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}) = \operatorname{Spec} k$, i.e. one point. But it isn't — it has lots of points.

Note that we have proved an analog of a theorem: the only holomorphic functions on $\mathbb{CP}^1$ are the constants!

1.13. *Important exercise.* Figure out how to define projective n-space $\mathbb{P}^n_k$. Glue together $n+1$ opens each isomorphic to $\mathbb{A}^n_k$. Show that the only global sections of the structure sheaf are the constants, and hence that $\mathbb{P}^n_k$ is not affine if $n > 0$. (Hint: you might fear that you will need some delicate interplay among all of your affine opens, but you will only need two of your opens to see this. There is even some geometric intuition behind this: the complement of the union of two opens has codimension 2. But "Hartogs' Theorem" says that any function defined on this union extends to be a function on all of projective space. Because we're expecting to see only constants as functions on all of projective space, we should already see this for this union of our two affine open sets.)

Exercise. The closed points of $\mathbb{P}^n_k$ (if k is algebraically closed) may be interpreted in the same way as we interpreted the points of $\mathbb{P}^1_k$. The points are of the form $[a_0; \ldots; a_n]$, where the a_i are not all zero, and $[a_0; \ldots; a_n]$ is identified with $[ca_0; \ldots; ca_n]$ where $c \in k^*$.

We will later give another definition of projective space. Your definition will be handy for computing things. But there is something unnatural about it — projective space is highly symmetric, and that isn't clear from your point of view.

Note that your definition will give a definition of $\mathbb{P}^n_R$ for any *ring* R. This will be useful later.

1.14. *Example.* Here is an example of a function on an open subset of a scheme that is a bit surprising. On $X = \operatorname{Spec} k[w, x, y, z]/(wx - yz)$, consider the open subset $D(y) \cup D(w)$. Show that the function x/y on $D(y)$ agrees with z/w on $D(w)$ on their overlap $D(y) \cap D(w)$. Hence they glue together to give a section. Justin points out that you may have seen this before when thinking about analytic continuation in complex geometry — we have a "holomorphic" function the description x/y on an open set, and this description breaks down elsewhere, but you can still "analytically continue" it by giving the function a different definition.

Follow-up for curious experts: This function has no "single description" as a well-defined expression in terms of w, x, y, z! there is lots of interesting geometry here. Here is a glimpse, in terms of words we have not yet defined. $\operatorname{Spec} k[w, x, y, z]$ is $\mathbb{A}^4$, and is, not surprisingly, 4-dimensional. We are looking at the set X, which is a hypersurface, and is 3-dimensional. It is a cone over a smooth quadric surface in $\mathbb{P}^3$ [show them hyperboloid of one sheet, and point out the two rulings]. $D(y)$ is X minus some hypersurface, so we are throwing away a codimension 1 locus. $D(z)$ involves throwing another codimension 1 locus. You might think that their intersection is then codimension 2, and that maybe failure of extending this weird function to a global polynomial comes because of a failure of our Hartogs'-type theorem, which will be a failure of normality. But that's not true — $V(y) \cap V(z)$ is in fact codimension 1 — so no Hartogs-type theorem holds. Here is what is actually going on. $V(y)$ involves throwing away the (cone over the) union of two lines l and m_1, one in each "ruling" of the surface, and $V(z)$ also involves throwing away the (cone over the) union of two lines l and m_2. The intersection is the (cone over the) line l, which is a codimension 1 set. Neat fact: despite being "pure codimension 1", it is not cut out even set-theoretically by a single equation. (It is hard to get an example of this behavior. This example is the simplest example I know.) This means that any expression $f(w, x, y, z)/g(w, x, y, z)$ for our function cannot correctly describe our function on $D(y) \cup D(z)$ — at some point of $D(y) \cup D(z)$ it must be $0/0$. Here's why. Our function can't be defined on $V(y) \cap V(z)$, so g must vanish here. But then g can't vanish just on the cone over l — it must vanish elsewhere too. (For the experts among the experts: here is why the cone over l is not cut out set-theoretically by a single equation. If $l = V(f)$, then $D(f)$ is affine. Let l' be another line in the same ruling as l, and let $C(l)$ (resp. l' be the cone over l (resp. l'). Then $C(l')$ can be given the structure of a closed subscheme of $\operatorname{Spec} k[w, x, y, z]$, and can be given the structure of $\mathbb{A}^2$. Then $C(l') \cap V(f)$ is a closed subscheme of $D(f)$. Any closed subscheme of an affine scheme is affine. But $l \cap l' = \emptyset$, so the cone over l intersects the cone over l' is a point, so $C(l') \cap V(f)$ is $\mathbb{A}^2$ minus a point, which we've seen is not affine, contradiction.)

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FOUNDATIONS OF ALGEBRAIC GEOMETRY CLASS 8

CONTENTS

Last day: $\tilde{M}$, sheaf associated to R-module M; Chinese remainder theorem; germs and value at a point of the structure sheaf; non-affine schemes $\mathbb{A}^2 - (0,0)$, line with doubled origin, $\mathbb{P}^n$.

Today: irreducible, connected, quasicompact, reduced, dimension.

1. PROPERTIES OF SCHEMES

We're now going to define properties of schemes. We'll start with some topological properties.

I've already defined what it means for a topological space to be *irreducible*: if X is the union of two closed subsets $U \cup V$, then either $X = U$ or $X = V$.

Problem A4 on problem set 3 implies that $\mathbb{A}^n_k$ is irreducible. There is a one-line answer. This argument "behaves well under gluing", yielding:

1.1. Exercise. Show that $\mathbb{P}^n_k$ is irreducible.

1.2. Exercise. You showed earlier that for affine schemes, there is a bijection between irreducible closed subsets and points. Show that this is true of schemes as well.

In the examples we have considered, the spaces have naturally broken up into some obvious pieces. Let's make that a bit more precise.

A topological space X is called *Noetherian* if it satisfies the descending chain condition for closed subsets: any sequence $Z_1 \supseteq Z_2 \supset \cdots$ of closed subsets eventually stabilizes: there is an r such that $Z_r = Z_{r+1} = \cdots$.

I showed some examples on $\mathbb{A}^2$, to show that it can take arbitrarily long to stabilize.

Date: Monday, October 24, 2005. Small correction January 31, 2007.

All of the cases we have considered have this property, but that isn't true of all rings. The key characteristic all of our examples have had in common is that the rings were *Noetherian*. Recall that a ring is *Noetherian* if *ascending* sequence $I_1 \subset I_2 \subset \cdots$ of closed ideals eventually stabilizes: there is an r such that $I_r = I_{r+1} = \cdots$.

Here are some quick facts about Noetherian rings. You should be able to prove them all.

- Fields are Noetherian. $\mathbb{Z}$ is Noetherian.
- If R is Noetherian, and I is any ideal, then R/I is Noetherian.
- If R is Noetherian, and S is any multiplicative set, then $S^{-1}R$ is Noetherian.
- In a Noetherian ring, any ideal is finitely generated. Any submodule of a finitely generated module over a Noetherian ring is finitely generated.

The next fact is non-trivial.

1.3. *The Hilbert basis theorem. — If R is Noetherian, then so is $R[x]$.*

Proof omitted. (This was done in Math 210.)

(I then discussed the game of Chomp. The fact that the game of infinite chomp is guaranteed to end is an analog of the Hilbert basis theorem. In fact, this is a consequence of the Hilbert basis theorem — the fact that infinite chomp is guaranteed to end corresponds to the fact that any ascending chain of monomial ideals in $k[x, y]$ must eventually stabilize. I learned of this cute fact from Rahul Pandharipande. If you prove the Chomp problem, you'll understand how to prove the Hilbert basis theorem.)

Using these results, then any polynomial ring over any field, or over the integers, is Noetherian — and also any quotient or localization thereof. Hence for example any finitely-generated algebra over k or $\mathbb{Z}$, or any localization thereof is Noetherian.

1.4. Exercise. Prove the following. If R is Noetherian, then $\operatorname{Spec} R$ is a Noetherian topological space. If X is a scheme that has a finite cover $X = \cup_{i=1}^{n} \operatorname{Spec} R_i$ where R_i is Noetherian, then X is a Noetherian topological space.

Thus $\mathbb{P}_k^n$ and $\mathbb{P}_{\mathbb{Z}}^n$ are Noetherian topological spaces: we built them by gluing together a finite number of Spec's of Noetherian rings.

If X is a topological space, and Z is an irreducible closed subset not contained in any larger irreducible closed subset, Z is said to be an *irreducible component* of X. (I drew a picture.)

1.5. Exercise. If R is any ring, show that the irreducible components of $\operatorname{Spec} R$ are in bijection with the minimal primes of R. (Here minimality is with respect to inclusion.)

For example, the only minimal prime of $k[x, y]$ is (0). What are the minimal primes of $k[x, y]/(xy)$?

1.6. *Proposition. — Suppose* X *is a Noetherian topological space. Then every non-empty closed subset* Z *can be expressed uniquely as a finite union* $Z = Z_1 \cup \cdots \cup Z_n$ *of irreducible closed subsets, none contained in any other.*

As a corollary, this implies that a Noetherian ring R has only finitely many minimal primes.

Proof. The following technique is often called *Noetherian induction*, for reasons that will become clear.

Consider the collection of closed subsets of X that *cannot* be expressed as a finite union of irreducible closed subsets. We will show that it is empty. Otherwise, let Y_1 be one such. If it properly contains another such, then choose one, and call it Y_2. If this one contains another such, then choose one, and call it Y_3, and so on. By the descending chain condition, this must eventually stop, and we must have some Y_r that cannot be written as a finite union of irreducible closed subsets, but every closed subset contained in it can be so written. But then Y_r is not itself irreducible, so we can write $Y_r = Y' \cup Y''$ where Y' and Y'' are both proper closed subsets. Both of these by hypothesis can be written as the union of a finite number of irreducible subsets, and hence so can Y_r, yielding a contradiction. Thus each closed subset can be written as a finite union of irreducible closed subsets. We can assume that none of these irreducible closed subsets contain any others, by discarding some of them.

We now show uniqueness. Suppose

$$Z = Z_1 \cup Z_2 \cup \cdots \cup Z_r = Z'_1 \cup Z'_2 \cup \cdots \cup Z'_s$$

are two such representations. Then $Z'_1 \subset Z_1 \cup Z_2 \cup \cdots \cup Z_r$, so $Z'_1 = (Z_1 \cap Z'_1) \cup \cdots \cup (Z_r \cap Z'_1)$. Now Z'_1 is irreducible, so one of these is Z'_1 itself, say (without loss of generality) $Z_1 \cap Z'_1$. Thus $Z'_1 \subset Z_1$. Similarly, $Z_1 \subset Z'_a$ for some a; but because $Z'_1 \subset Z_1 \subset Z'_a$, and Z'_1 is contained in no other Z'_i, we must have $a = 1$, and $Z'_1 = Z_1$. Thus each element of the list of Z's is in the list of Z''s, and vice versa, so they must be the same list. □

1.7. Connectedness and quasicompactness.

Definition. A topological space X is connected if it cannot be written as the disjoint union of two non-empty open sets.

We say that a subset Y of X is a *connected component* if it is connected, and both open and closed. **Remark added later: Thanks to Anssi for pointing out that this is not the usual definition of connected component. The usual definition, which deals with more pathological situations, implies this one. At some point I might update these notes and say more.**

1.8. Exercise. Show that an irreducible topological space is connected.

1.9. Exercise. Give (with proof!) an example of a scheme that is connected but reducible.

We have already defined **quasicompact**.

1.10. Exercise. Show that a finite union of affine schemes is quasicompact. (Hence $\mathbb{P}_k^n$ is quasicompact.) Show that every closed subset of an affine scheme is quasicompact. Show that every closed subset of a quasicompact scheme is quasicompact.

The last topological property I should discuss is *dimension*. But that will take me some time, and it will involve some non-topological issues, so I'll first talk about an important non-topological property. Remember that one of the alarming things about schemes is that functions are not determined by their values at points, and that was because of the presence of *nilpotents*.

1.11. Definition. We will say that a ring is *reduced* if it has no nilpotents. A scheme is *reduced* if $\mathcal{O}_X(U)$ has no nonzero nilpotents for any open set U of X.

An example of a nonreduced affine scheme is $k[x,y]/(xy,x^2)$. Picture: y-axis with some fuzz at the origin (I drew this). The fuzz indicates that there is some nonreducedness going on at the origin. Here are two different functions: y and $x+y$. Their values agree at all points. They are actually the same function on the open set $D(y)$, which is not surprising, as $D(y)$ is reduced, as the next exercise shows.

1.12. Exercise. Show that $\left(k[x,y]/(xy,x^2)\right)_y$ has no nilpotents. (Hint: show that it is isomorphic to another ring, by considering the geometric picture.)

1.13. Exercise. Show that a scheme is reduced if and only if none of the stalks have nilpotents. Hence show that if f and g are two functions on a reduced scheme that agree at all points, then $f = g$.

Definition. A scheme is *integral* if $\mathcal{O}_X(U)$ is an integral domain for each open set U of X.

1.14. Exercise. Show that an affine scheme $\operatorname{Spec} R$ is integral if and only if R is an integral domain.

1.15. Exercise. Show that a scheme X is integral if and only if it is irreducible and reduced.

1.16. Exercise. Suppose X is an integral scheme. Then X (being irreducible) has a generic point η. Suppose $\operatorname{Spec} R$ is any non-empty affine open subset of X. Show that the stalk at η, $\mathcal{O}_{X,\eta}$, is naturally $\operatorname{Frac} R$. This is called the *function field* of X. It can be computed on any non-empty open set of X (as any such open set contains the generic point).

1.17. Exercise. Suppose X is an integral scheme. Show that the restriction maps $\operatorname{res}_{U,V} : \mathcal{O}_X(U) \to \mathcal{O}_X(V)$ are inclusions so long as $V \neq \emptyset$. Suppose $\operatorname{Spec} R$ is any non-empty affine open subset of X (so R is an integral domain). Show that the natural map $\mathcal{O}_X(U) \to \mathcal{O}_{X,\eta} = \operatorname{Frac} R$ (where U is any non-empty open set) is an inclusion.

2. DIMENSION

Our goal is to define the dimension of schemes. This should agree with, and generalize, our geometric intuition. (Careful: if you are thinking over the complex numbers, our dimensions will be complex dimensions, and hence half that of the "real" picture.) We will also use it to prove things; as a preliminary example, we will classify the prime ideals of $k[x, y]$.

It turns out that the right definition is purely topological — it just depends on the topological space, and not at all on the structure sheaf. Define **dimension** by Krull dimension: the supremum of lengths of chains of closed irreducible sets, starting indexing with 0. This dimension is allowed to be ∞. Define the Krull dimension of a ring to be the Krull dimension of its topological space. It is one less than the length of the longest chain of nested prime ideals you can find. (You might think a Noetherian ring has finite dimension, but this isn't necessarily true. For a counterexample by Nagata, who is the master of all counterexamples, see Eisenbud's *Commutative Algebra with a View to Algebraic Geometry*, p. 231.)

(Scholars of the empty set can take the dimension of the empty set to be $-\infty$.)

Obviously the Krull dimension of a ring R is the same as the Krull dimension of $R/\mathfrak{N}$: dimension doesn't care about nilpotents.

For example: We have identified the prime ideals of $k[t]$, so we can check that $\dim \mathbb{A}^1 = 1$. Similarly, $\dim \operatorname{Spec} \mathbb{Z} = 1$. Also, $\dim \operatorname{Spec} k = 0$, and $\dim \operatorname{Spec} k[x]/x^2 = 0$.

Caution: if Z is the union of two closed subsets X and Y, then $\dim_Z = \max(\dim X, \dim Y)$. In particular, if Z is the disjoint union of something of dimension 2 and something of dimension 1, then it has dimension 2. Thus dimension is not a "local" characteristic of a space. This sometimes bothers us, so we will often talk about dimensions of irreducible topological spaces. If a topological space can be expressed as a finite union of irreducible subsets, then say that it is *equidimensional* or *pure dimensional* (resp. equidimensional of dimension n or pure dimension n) if each of its components has the same dimension (resp. they are all of dimension n).

The notion of *codimension* of something equidimensional in something equidimensional makes good sense (as the difference of the two dimensions). Caution (added Nov. 6): there is another possible definition of codimension, in terms of *height*, defined later. Hartshorne uses this second definition. These two definitions can disagree — see e.g. the example of "height behaving badly" in the Class 9 notes. So we will be very cautious in using then word "codimension".

An equidimensional dimension 1 (resp. 2, n) topological space is said to be a *curve* (resp. surface, n-fold).

2.1. Reality check. Show that $\dim R/\mathfrak{p} \leq \dim R$, where $\mathfrak{p}$ is prime. Hope: equality holds if and only if $\mathfrak{p} = 0$ or $\dim R/\mathfrak{p} = \infty$. It is immediate that if R is a finite-dimensional domain, and $\mathfrak{p} \neq 0$, then we have inequality.

Warning: in all of the examples we have looked at, they behave well, but dimension can behave quite pathologically. But in good situations, including ones that come up more naturally in nature, it doesn't. For example, in cases involving a finite number variables over a field, dimension follows our intuition. More precisely:

2.2. *Big Theorem of today.* — Suppose R is a finitely-generated domain over a field k. Then $\dim \operatorname{Spec} R$ is the transcendence degree of the fraction field $\operatorname{Frac}(R)$ over k.

(By "finitely generated domain over k", I mean "a finitely generated k-algebra that is an integral domain". I'm just trying to save ink.)

Note that these finitely generated domains over k can each be described as the ring of functions on an irreducible subset of some $\mathbb{A}^n$: given such a domain, choose generators $x_1, \dots, x_n$. Conversely, if $\mathfrak{p} \subset k[x_1, \dots, x_n]$ is any prime ideal, then $\dim \operatorname{Spec} k[x_1, \dots, x_n]/\mathfrak{p}$ is the transcendence degree of $k[x_1, \dots, x_n]/\mathfrak{p}$ over k.

Before getting to the proof, I want to discuss some consequences.

2.3. *Corollary.* — $\dim \mathbb{A}^n_k = n$.

We can now confirm that we have named all the primes of $k[x, y]$ where k is algebraically closed. Recall that we have discovered the primes (0), $f(x, y)$ where f is irreducible, and $(x - a, y - b)$ where $a, b \in k$. By the Nullstellensatz, we have found all the closed points, so we have found all the irreducible subsets of dimension 0. As $\mathbb{A}^2_k$ is irreducible, there is only one irreducible subset of dimension 2. So it remains to show that all the irreducible subsets of dimension 1 are of the form $V(f(x, y))$, where f is an irreducible polynomial. Suppose $\mathfrak{p}$ is a prime ideal corresponding to an irreducible subset of dimension 1. Suppose $g \in \mathfrak{p}$ is non-zero. Factor g into irreducibles: $g_1 \cdots g_n \in \mathfrak{p}$. Then as $\mathfrak{p}$ is prime, one of the g_i's, say g_1, lies in $\mathfrak{p}$. Thus $(g_1) \subset \mathfrak{p}$. Now (g_1) is a prime ideal, and hence cuts out an irreducible subset, which contains $V(\mathfrak{p})$. It can't strictly contain $V(\mathfrak{p})$, as its dimension is no bigger than 1, and the dimension of $V(\mathfrak{p})$ is also 1. Thus $V((g_1)) = V(\mathfrak{p})$. But they are both prime ideals, and by the bijection between irreducible closed subsets and prime ideals, we have $\mathfrak{p} = (g_1)$.

Here are two more exercises added to the notes on November 5.

2.4. Exercise: Nullstellensatz from dimension theory. (a) Prove a microscopically stronger version of the weak Nullstellensatz: Suppose $R = k[x_1, \dots, x_n]/I$, where k is an algebraically closed field and I is some ideal. Then the maximal ideals are precisely those of the form $(x_1 - a_1, \dots, x_n - a_n)$, where $a_i \in k$.
(b) Suppose $R = k[x_1, \dots, x_n]/I$ where k is not necessarily algebraically closed. Show that every maximal ideal of R has a residue field that is a finite extension of k. [Hint for both: the maximal ideals correspond to dimension 0 points, which correspond to transcendence

degree 0 extensions of k, i.e. finite extensions of k. If k is algebraically closed, the maximal ideals correspond to surjections $f : k[x_1, \dots, x_n] \to k$. Fix one such surjection. Let $a_i = f(x_i)$, and show that the corresponding maximal ideal is $(x_1 - a_1, \dots, x_n - a_n)$.]

2.5. Important Exercise. Suppose X is an integral scheme, that can be covered by open subsets of the form $\operatorname{Spec} R$ where R is a finitely generated domain over k. Then $\dim X$ is the transcendence degree of the function field (the stalk at the generic point) $\mathcal{O}_{X,\eta}$ over k. Thus (as the generic point lies in all non-empty open sets) the dimension can be computed in any open set of X.

Here is an application that you might reasonably have wondered about before thinking about algebraic geometry. I don't think there is a simple proof, but maybe I'm wrong.

2.6. *Exercise.* Suppose $f(x,y)$ and $g(x,y)$ are two complex polynomials ($f, g \in \mathbb{C}[x,y]$). Suppose f and g have no common factors. Show that the system of equations $f(x,y) = g(x,y) = 0$ has a finite number of solutions.

Let's start to prove the big theorem! If R is a finitely generated domain over k, temporarily define $\dim_{tr} R = \dim_{tr} \operatorname{Spec} R$ to be the transcendence degree of $\operatorname{Frac}(R)$ over k. We wish to show that $\dim_{tr} R = \dim R$. After proving the big theorem, we will discard the temporary notation $\dim_{tr}$.

2.7. *Lemma. — Suppose R is an integral domain over k (not necessarily finitely generated, although that is the case we will care most about), and $\mathfrak{p} \subset R$ a prime. Then $\dim_{tr} R \geq \dim_{tr} R/\mathfrak{p}$, with equality if and only $\mathfrak{p} = (0)$, or $\dim_{tr} R/\mathfrak{p} = \infty$.*

You should have a picture in your mind when you hear this: if you have an irreducible space of finite dimension, then any proper subspace has strictly smaller dimension — certainly believable!

This implies that $\dim R \leq \dim_{tr} R$.

Proof. You can quickly check that if $\mathfrak{p} = (0)$ or $\dim_{tr} R/\mathfrak{p} = \infty$ then we have equality, so we'll assume that $\mathfrak{p} \neq (0)$, and $\dim_{tr} R/\mathfrak{p} = n < \infty$. Choose $x_1, \dots, x_n$ in R such that their residues $\overline{x}_1, \dots, \overline{x}_n$ are algebraically independent. Choose any $y \neq 0$ in $\mathfrak{p}$. Assume for the sake of contradiction that $\dim_{tr} R = n$. Then $y, x_1, \dots, x_n$ cannot be algebraically independent over k, so there is some irreducible polynomial $f(Y, X_1, \dots, X_n) \in k[Y, X_1, \dots, X_n]$ such that $f(y, x_1, \dots, x_n) = 0$ (in R). This irreducible f is not (a multiple of) Y, as otherwise $f(y, x_1, \dots, x_n) = y \neq 0$ in R. Hence f contains monomials that are not multiples of Y, so $F(X_1, \dots, X_n) := f(0, X_1, \dots, X_n) \in k[X_1, \dots, X_n]$ is non-zero. Reducing $f(y, x_1, \dots, x_n) = 0$ modulo $\mathfrak{p}$ gives us

$$F(\overline{x}_1, \dots, \overline{x}_n) = f(0, \overline{x}_1, \dots, \overline{x}_n) = 0 \qquad \text{in } R/\mathfrak{p}$$

contradicting the algebraic independence of $\overline{x}_1, \dots, \overline{x}_n$. □

At the end of the class, I stated the following, which will play off of our lemma to prove the big theorem.

2.8. *Krull's principal ideal theorem (transcendence degree version). — Suppose* R *is a finitely generated domain over* k, $f \in R$, $\mathfrak{p}$ *a minimal prime of* R/f. *Then if* $f \neq 0$, $\dim_{tr} R/\mathfrak{p} = \dim_{tr} R - 1$.

This is best understood geometrically:

2.9. *Theorem (geometric interpretation of Krull). — Suppose* $X = \operatorname{Spec} R$ *where* R *is a finitely generated domain over* k, $g \in R$, Z *an irreducible component of* $V(g)$. *Then if* $g \neq 0$, $\dim_{tr} Z = \dim_{tr} X - 1$.

In other words, if you have some irreducible space of finite dimension, then any non-zero function on it cuts out a set of *pure* codimension 1.

We'll see how these two geometric statements will quickly combine to prove our big theorem.

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FOUNDATIONS OF ALGEBRAIC GEOMETRY CLASS 9

CONTENTS

Last day: irreducible, connected, quasicompact, reduced, dimension.

Today: Krull's Principal Ideal Theorem, height, affine communication lemma, properties of schemes: locally Noetherian, Noetherian, finite type S-scheme, locally of finite type S-scheme, normal.

I realize now that you may not have seen the notion of transcendence degree. I'll tell you the main thing you need to know about it, which I hope you will find believable. Suppose K/k is a finitely generated field extension. Then any two maximal sets of algebraically independent elements of K over k (i.e. any set with no algebraic relation) have the same size (a non-negative integer or ∞). If this size is finite, say n, and $x_1, \ldots, x_n$ is such a set, then $K/k(x_1, \ldots, x_n)$ is necessarily a finitely generated algebraic extension, i.e. a finite extension. (Such a set $x_1, \ldots, x_n$ is called a transcendence basis, and n is called the *transcendence degree*.) A short and well-written proof of this fact is in *Mumford's Red Book of Varieties and Schemes*.

1. DIMENSION, CONTINUED

Last day, I defined the dimension of a scheme. I defined the dimension (or Krull dimension) as the supremum of lengths of chains of closed irreducible sets, starting indexing with 0. This dimension is allowed to be ∞. For example: a Noetherian topological space has a finite dimension. The Krull dimension of a ring is the Krull dimension of its topological space. It is one less than the length of the longest chain of nested prime ideals you can find.

We are in the midst of proving the following result, which lets us understand dimension when working in good situations.

1.1. *Big Theorem of last day. — Suppose* R *is a finitely-generated domain over a field* k*. Then* $\dim \operatorname{Spec} R$ *is the transcendence degree of the fraction field* $\operatorname{Frac}(R)$ *over* k*.*

Date: Wednesday, October 26, 2005. Small update January 31, 2007.

1.2. Exercise. Suppose X is an integral scheme, that can be covered by open subsets of the form $\operatorname{Spec} R$ where R is a finitely generated domain over k. Then $\dim X$ is the transcendence degree of the function field (the stalk at the generic point) $\mathcal{O}_{X,\eta}$ over k. Thus (as the generic point lies in all non-empty open sets) the dimension can be computed in any open set of X.

The proof of the big theorem will rely on two different facts pulling in opposite directions. The first is the following lemma, which we proved.

1.3. *Lemma. — Suppose* R *is an integral domain over* k *(not necessarily finitely generated, although that is the case we will care most about), and* $\mathfrak{p} \subset R$ *a prime. Then* $\dim_{tr} R \geq \dim_{tr} R/P$, *with equality if and only* $\mathfrak{p} = (0)$, *or* $\dim_{tr} R/P = \infty$.

You should have a picture in your mind when you hear this: if you have an irreducible space of finite dimension, then any proper subspace has strictly smaller dimension — certainly believable!

This implies that $\boxed{\dim R \leq \dim_{tr} R}$. (Think this through!)

The other fact we'll use is Krull's Principal Ideal Theorem. This result is one of the few hard facts I'll not prove. We may prove it later in the class (possibly in a problem set), and you can also read a proof in Mumford's *Red Book*, in §I.7, where you'll find much of this exposition.

1.4. *Krull's Principal Ideal Theorem (transcendence degree version). — Suppose* R *is a finitely generated domain over* k, $f \in R$, $\mathfrak{p}$ *a minimal prime of* R/f. *Then if* $f \neq 0$, $\dim_{tr} R/\mathfrak{p} = \dim_{tr} R - 1$.

This is best understood geometrically: if you have some irreducible space of finite dimension, then any non-zero function on it cuts out a set of *pure* codimension 1. Somewhat more precisely:

1.5. *Theorem (geometric interpretation of Krull). — Suppose* $X = \operatorname{Spec} R$ *where* R *is a finitely generated domain over* k, $g \in R$, Z *an irreducible component of* $V(g)$. *Then if* $g \neq 0$, $\dim_{tr} Z = \dim_{tr} X - 1$.

Before I get to the proof of the theorem, I want to point out that this is useful on its own. Consider the scheme $\operatorname{Spec} k[w, x, y, z]/(wx - yz)$. What is its dimension? It is cut out by one non-zero equation $wx - yz$ in $\mathbb{A}^4$, so it is a threefold.

1.6. Exercise. What is the dimension of $\operatorname{Spec} k[w, x, y, z]/(wx - yz, x^{17} + y^{17})$? (Be careful to check they hypotheses before invoking Krull!)

1.7. Exercise. Show that $\operatorname{Spec} k[w, x, y, z]/(wz - xy, wy - x^2, xz - y^2)$ is an integral *surface*. You might expect it to be a curve, because it is cut out by three equations in 4-space. (Remark for experts: this is not a random ideal. In language we will later make precise: it is the affine cone over a curve in $\mathbb{P}^3$. This curve is called the *twisted cubic*. It is in some sense the simplest curve in $\mathbb{P}^3$ not contained in a hyperplane. You can think of it as the points of the form (t, t^2, t^3) in $\mathbb{A}^3$. Indeed, you'll notice that $(w, x, y, z) = (a, at, at^2, at^3)$ satisfies the equations above. It turns out that you actually need three equations to cut out this surface. The first equation cuts out a threefold in four-space (by Krull's theorem, see later). The second equation cuts out a surface: our surface, and another surface. The third equation cuts out our surface. One last aside: notice once again that the cone over the quadric surface $k[w, x, y, z]/(wz - xy)$ makes an appearance.)

We'll now put together our lemma, and this geometric interpretation of Krull. Notice the interplay between the two: the first says that the dimension definitely drops when you take a proper irreducible closed subset. The second says that you can arrange for it to drop by precisely 1.

I proved the following result, which I didn't end up using.

1.8. *Proposition. — Suppose* X *is the* Spec *of a finitely generated domain over* k*, and* Z *is an irreducible closed subset, maximal among all proper irreducible closed subsets of* X*. (I gave a picture here.) Then* $\dim_{tr} Z = \dim_{tr} X - 1$.

(We certainly have $\dim_{tr} Z \leq \dim_{tr} X - 1$ by our lemma.)

Proof. Suppose $Z = V(\mathfrak{p})$ where $\mathfrak{p}$ is prime. Choose any non-zero $g \in \mathfrak{p}$. By Krull's theorem, the components of $V(g)$ are have $\dim_{tr} = \dim_{tr} X - 1$. Z is contained in one of the components. By the maximality of Z, Z *is* one of the components. □

1.9. *Proof of big theorem.* We prove it by induction on $\dim_{tr} X$. The base case $\dim_{tr} X = 0$ is easy: by our lemma, $\dim X \leq \dim_{tr} X$, so $\dim X = 0$.

Now assume that $\dim_{tr} X = n$. As $\dim X \leq \dim_{tr} X$, our goal will be to produce a chain of $n + 1$ irreducible closed subsets. Say $X = \operatorname{Spec} R$. Choose any $g \neq 0$ in R. Choose any component Z of $V(g)$. Then $\dim_{tr} Z = n - 1$ by Krull's theorem, and the inductive hypothesis, so we can find a chain of n irreducible closed subsets descending from Z. We're done. □

I gave a geometric picture of both. Note that equality needn't hold in the first case.

The big theorem is about the dimension of finitely generated domains over k. For such rings, dimension is well-behaved. This set of rings behaves well under quotients; I want to show you that it behaves well under localization as well.

1.10. *Proposition. — Suppose* R *is a finitely generated domain over* k*, and* $\mathfrak{p}$ *is a prime ideal. Then* $\dim R_{\mathfrak{p}} = \dim R - \dim R/\mathfrak{p}$.

The scheme-theoretic version of this statement about rings is: $\dim_{\mathcal{O}_{Z,X}} = \dim X - \dim Z$.

1.11. Exercise. Prove this. (I gave a geometric explanation of why this is true, which you can take as a "hint" for this exercise.) In the course of this exercise, you will show the important fact that if $n = \dim R$, then any chain of prime ideals can be extended to a chain of prime ideals of length n. Further, given a prime ideal, you can tell where it is in any chain by looking at the transcendence degree of its quotient field. This is a particularly nice feature of polynomial rings, that will not hold even for Noetherian rings in general (see the next section).

2. Height, and Krull's Principal Ideal Theorem

This is a good excuse to tell you a definition in algebra. Definition: the *height* of the prime ideal $\mathfrak{p}$ in R is $\dim R_{\mathfrak{p}}$. Algebraic translation: it is the supremum of lengths of chains of primes contained in $\mathfrak{p}$.

This is a good but imperfect version of codimension. For finitely generated domains over k, the two notions agree, by Proposition 1.10. An example of a pathology is given below.

With this definition of height, I can state a more general version of Krull's Principal Ideal Theorem.

2.1. *Krull's Principal Ideal Theorem. — Suppose R is a Noetherian ring, and $f \in A$ an element which is not a zero divisor. Then every minimal prime $\mathfrak{p}$ containing f has height 1.* (Atiyah-Macdonald p. 122)

(We could have $V(f) = \emptyset$, if f is a unit — but that doesn't violate the statement.)

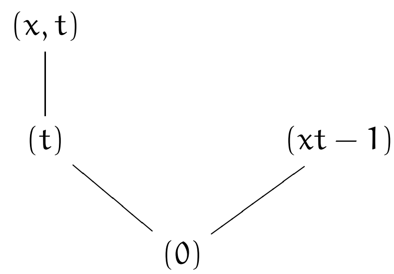
The geometric picture is the same as before: "If f is not a zero-divisor, the codimension is 1."

It is possible that I will give a proof later in the course. Either I'll give an algebraic proof in the notes, or I will give a geometric proof in class, using concepts we have not yet developed. (I'll be careful to make sure the argument is not circular!)

2.2. Important Exercise. (This will be useful soon.) (a) Suppose $X = \operatorname{Spec} R$ where R is a Noetherian domain, and Z is an irreducible component of $V(r_1, \ldots, r_n)$, where $r_1, \ldots, r_n \in R$. Show that the height of (the prime associated to) Z is at most n. Conversely, suppose $X = \operatorname{Spec} R$ where R is a Noetherian domain, and Z is an irreducible subset of height n. Show that there are $f_1, \ldots, f_n \in R$ such that Z is an irreducible component of $V(f_1, \ldots, f_n)$.
(b) (application to finitely generated k-algebras) Suppose $X = \operatorname{Spec} R$ where R is a finitely generated domain over k, and Z is an irreducible component of $V(r_1, \ldots, r_n)$, where $r_1, \ldots, r_n \in R$. Show that $\dim Z \geq \dim X - n$. Conversely, suppose $X = \operatorname{Spec} R$ where R is a Noetherian domain, and Z is an irreducible subset of codimension n. Show that there are $f_1, \ldots, f_n \in R$ such that Z is an irreducible component of $V(f_1, \ldots, f_r)$.

2.3. Important but straightforward exercise. If R is a finitely generated domain over k, show that $\dim R[x] = \dim R + 1$. If R is a Noetherian ring, show that $\dim R[x] \geq \dim R + 1$. (Fact, proved later: if R is a Noetherian ring, then $\dim R[x] = \dim R + 1$. We'll prove this later. You may use this fact in exercises in later weeks.)

We now show how the height can behave badly. Let $R = k[x]_{(x)}[t]$. In other words, elements of R are polynomials in t, whose coefficients are quotients of polynomials in x, where no factors of x appear in the denominator. R is a domain. $(xt - 1)$ is not a zero divisor. You can verify that $R/(xt-1) \cong k[x]_{(x)}[1/x] \cong k(x)$ — "in $k[x]_{(x)}$, we may divide by everything but x, and now we are allowed to divide by x as well" — so $R/(xt-1)$ is a field. Thus $(xt-1)$ is not just prime but also maximal. By Krull's theorem, $(xt-1)$ is height 1. Thus $(0) \subset (xt-1)$ is a maximal chain. However, R has dimension at least 2: $(0) \subset (t) \subset (x,t)$ is a chain of primes of length 3. (In fact, R has dimension precisely 2: $k[x]_{(x)}$ has dimension 1, and the fact mentioned in the previous exercise 2.3 implies $\dim k[x]_{(x)}[t] = \dim k[x]_{(x)} + 1 = 2$.) Thus we have a height 1 prime in a dimension 2 ring that is "codimension 2". A picture of this lattice of ideals is below.



(This example comes from geometry; it is enlightening to draw a picture. $k[x]_{(x)}$ corresponds to a germ of $\mathbb{A}^1_k$ near the origin, and $k[x]_{(x)}[t]$ corresponds to "this $\times$ the affine line".) For this reason, codimension is a badly behaved notion in Noetherian rings in general.

I find it disturbing that this misbehavior turns up even in a relative benign-looking ring.

3. Properties of schemes that can be checked "affine-locally"

Now I want to describe a host of important properties of schemes. All of these are "affine-local" in that they can be checked on any affine cover, by which I mean a covering by open affine sets.

Before I get going, I want to point out something annoying in the definition of schemes. I've said that a scheme is a topological space with a sheaf of rings, that can be covered by affine schemes. There is something annoying about this description that I find hard to express. We have all these affine opens in the cover, but we don't know how to communicate between any two of them. Put a different way, if I have an affine cover, and you have an affine cover, and we want to compare them, and I calculate something on my cover, there should be some way of us getting together, and figuring out how to translate my calculation over onto your cover. (I'm not sure if you buy what I'm trying to sell here.) The affine communication lemma I'll soon describe will do this for us.

3.1. *Remark.* In our limited examples so far, any time we've had an affine open subset of an affine scheme $\operatorname{Spec} S \subset \operatorname{Spec} R$, in fact $\operatorname{Spec} S = D(f)$ for some f. But this is not always true, and we will eventually have an example. (We'll first need to define elliptic curves!)

3.2. *Proposition. — Suppose* $\operatorname{Spec} A$ *and* $\operatorname{Spec} B$ *are affine open subschemes of a scheme* X. *Then* $\operatorname{Spec} A \cap \operatorname{Spec} B$ *is the union of open sets that are simultaneously distinguished open subschemes of* $\operatorname{Spec} A$ *and* $\operatorname{Spec} B$.

Proof. (This is best seen with a picture, which unfortunately won't be in the notes.) Given any $\mathfrak{p} \in \operatorname{Spec} A \cap \operatorname{Spec} B$, we produce an open neighborhood of $\mathfrak{p}$ in $\operatorname{Spec} A \cap \operatorname{Spec} B$ that is simultaneously distinguished in both $\operatorname{Spec} A$ and $\operatorname{Spec} B$. Let $\operatorname{Spec} A_f$ be a distinguished open subset of $\operatorname{Spec} A$ contained in $\operatorname{Spec} A \cap \operatorname{Spec} B$. Let $\operatorname{Spec} B_g$ be a distinguished open subset of $\operatorname{Spec} B$ contained in $\operatorname{Spec} A_f$. Then $g \in \Gamma(\operatorname{Spec} B, \mathcal{O}_X)$ restricts to an element $g' \in \Gamma(\operatorname{Spec} A_f, \mathcal{O}_X) = A_f$. The points of $\operatorname{Spec} A_f$ where g vanishes are precisely the points of $\operatorname{Spec} A_f$ where g' vanishes (cf. earlier exercise), so

$$\begin{aligned} \operatorname{Spec} B_g &= \operatorname{Spec} A_f \setminus \{\mathfrak{p} : g' \in \mathfrak{p}\} \\ &= \operatorname{Spec} (A_f)_{g'}. \end{aligned}$$

If $g' = g''/f^n$ ($g'' \in A$) then $\operatorname{Spec}(A_f)_{g'} = \operatorname{Spec} A_{fg''}$, and we are done. □

3.3. *Affine communication lemma. — Let* P *be some property enjoyed by some affine open sets of a scheme* X, *such that*

(i) *if* $\operatorname{Spec} R \hookrightarrow X$ *has* P *then for any* $f \in R$, $\operatorname{Spec} R_f \hookrightarrow X$ *does too.*
(ii) *if* $(f_1, \dots, f_n) = R$, *and* $\operatorname{Spec} R_{f_i} \hookrightarrow X$ *has* P *for all* i, *then so does* $\operatorname{Spec} R \hookrightarrow X$.

Suppose that $X = \cup_{i \in I} \operatorname{Spec} R_i$ *where* $\operatorname{Spec} R_i$ *is an affine, and* R_i *has property* P. *Then every other open affine subscheme of* X *has property* P *too.*

Proof. (This is best done with a picture.) Cover $\operatorname{Spec} R$ with a finite number of distinguished opens $\operatorname{Spec} R_{g_j}$, each of which is distinguished in some R_{f_i}. This is possible by Proposition 3.2 and the quasicompactness of $\operatorname{Spec} R$. By (i), each $\operatorname{Spec} R_{g_j}$ has P. By (ii), $\operatorname{Spec} R$ has P. □

By choosing P appropriately, we define some important properties of schemes.

3.4. *Proposition. — Suppose* R *is a ring, and* $(f_1, \dots, f_n) = R$.

(a) *If* R *is a Noetherian ring, then so is* R_{f_i}. *If each* R_{f_i} *is Noetherian, then so is* R.
(b) *If* R *has no nonzero nilpotents (i.e.* 0 *is a radical ideal), then* R_{f_i} *also has no nonzero nilpotents. If no* R_{f_i} *has a nonzero nilpotent, then neither does* R. **Do we say "a ring is reduced? radical?"**

(c) *Suppose* A *is a ring, and* R *is an* A*-algebra. If* R *is a finitely generated* A*-algebra, then so is* R_{f_i}*. If each* R_{f_i} *is a finitely-generated* A*-algebra, then so is* R*. (I didn't say this in class, so I'll say it on Monday.)*
(d) *Suppose* R *is an integral domain. If* R *is integrally closed, then so is* R_{f_i}*. If each* R_{f_i} *is integrally closed, then so is* R*.*

We'll prove these shortly. But given this, I want to make some definitions.

3.5. Important Definitions. Suppose X is a scheme.

- If X can be covered by affine opens $\operatorname{Spec} R$ where R is Noetherian, we say that X is a *locally Noetherian scheme*. If in addition X is quasicompact, or equivalently can be covered by finitely many such affine opens, we say that X is a *Noetherian scheme* **Exercise.** Show that the underlying topological space of a Noetherian scheme is Noetherian. **Exercise.** Show that all open subsets of a Noetherian scheme are quasicompact.
- If X can be covered by affine opens $\operatorname{Spec} R$ where R is reduced (nilpotent-free), we say that X is *reduced*. **Exercise:** Check that this agrees with our earlier definition. This definition is advantageous: our earlier definition required us to check that the ring of functions over *any* open set is nilpotent free. This lets us check in an affine cover. Hence for example $\mathbb{A}^n_k$ and $\mathbb{P}^n_k$ are reduced.
- Suppose A is a ring (e.g. A is a field k), and $\Gamma(X, \mathcal{O}_X)$ is an A-algebra. Then we say that X is an A*-scheme*, or a *scheme over* A. Suppose X is an A-scheme. (Then for any non-empty U, $\Gamma(U, \mathcal{O}_X)$ is naturally an A-algebra.) If X can be covered by affine opens $\operatorname{Spec} R$ where R is a *finitely generated* A-algebra, we say that X is *locally of finite type over* A, or that it is a *locally of finite type* A*-scheme*. (My apologies for this cumbersome terminology; it will make more sense later.) If furthermore X is quasicompact, X is *finite type over* A, or a *finite type* A*-scheme*.
- If X is integral, and can be covered by affine opens $\operatorname{Spec} R$ where R is a integrally closed, we say that X is *normal*. (Thus in my definition, normality can only apply to integral schemes. I may want to patch this later.) **Exercise.** If R is a unique factorization domain, show that $\operatorname{Spec} R$ is integrally closed. Hence $\mathbb{A}^n_k$ and $\mathbb{P}^n_k$ are both normal.

Proof. (a) (i) If $I_1 \subsetneq I_2 \subsetneq I_3 \subsetneq \cdots$ is a strictly increasing chain of ideals of R_f, then we can verify that $J_1 \subsetneq J_2 \subsetneq J_3 \subsetneq \cdots$ is a strictly increasing chain of ideals of R, where

$$J_j = \{r \in R : r \in I_j\}$$

where $r \in I_j$ means "the image in R_f lies in I_j". (We think of this as $I_j \cap R$, except in general R needn't inject into R_{f_i}.) Clearly J_j is an ideal of R. If $x/f^n \in I_{j+1} \setminus I_j$ where $x \in R$, then $x \in J_{j+1}$, and $x \notin J_j$ (or else $x(1/f)^n \in J_j$ as well). (ii) Suppose $I_1 \subsetneq I_2 \subsetneq I_3 \subset \cdots$ is a strictly increasing chain of ideals of R. Then for each $1 \leq i \leq n$,

$$I_{i,1} \subset I_{i,2} \subset I_{i,3} \subset \cdots$$

is an increasing chain of ideals in R_{f_i}, where $I_{i,j} = I_j \otimes_R R_{f_i}$. We will show that for each j, $I_{i,j} \subsetneq I_{i,j+1}$ for some i; the result will then follow.

(b) **Exercise.**

(c) (I'll present this on Monday.) (i) is clear: if R is generated over S by $r_1, \dots, r_n$, then R_f is generated over S by $r_1, \dots, r_n, 1/f$.

(ii) Here is the idea; I'll leave this as an **exercise** for you to make this work. We have generators of R_i: r_{ij}/f_i^j, where $r_{ij} \in R$. I claim that $\{r_{ij}\}_{ij} \cup \{f_i\}_j$ generate R as a S-algebra. Here's why. Suppose you have any $r \in R$. Then in R_{f_i}, we can write r as some polynomial in the r_{ij}'s and f_i, divided by some huge power of f_i. So "in each R_{f_i}, we have described r in the desired way", except for this annoying denominator. Now use a partition of unity type argument to combine all of these into a single expression, killing the denominator. Show that the resulting expression you build still agrees with r in each of the R_{f_i}. Thus it is indeed r.

(d) (i) is easy. If $x^n + a_{n-1}x^{n-1} + \cdots + a_0 = 0$ where $a_i \in R_f$ has a root in the fraction field. Then we can easily show that the root lies in R_f, by multiplying by enough f's to kill the denominator, then replacing $f^a x$ by y. That is likely incomprehensible, so I'll leave this as an **exercise**.

(ii) (This one involves a neat construction.) Suppose R is not integrally closed. We show that there is some f_i such that R_{f_i} is also not integrally closed. Suppose

$$x^n + a_{n-1}x^{n-1} + \cdots + a_0 = 0 \tag{1}$$

(with $a_i \in R$) has a solution s in $\mathrm{Frac}(R)$. Let I be the "ideal of denominators" of s:

$$I := \{r \in R : rs \in R\}.$$

(Note that I is clearly an ideal of R.) Now $I \neq R$, as $1 \notin I$. As $(f_1, \dots, f_n) = R$, there must be some $f_i \notin I$. Then $s \notin R_{f_i}$, so equation (1) in $R_{f_i}[x]$ shows that R_{f_i} is not integrally closed as well, as desired. ☐

3.6. *Unimportant Exercise relating to the proof of (d).* One might naively hope from experience with unique factorization domains that the ideal of denominators is principal. This is not true. As a counterexample, consider our new friend $R = k[a, b, c, d]/(ad - bc)$, and $a/c = b/d \in \mathrm{Frac}(R)$. Then it turns out that $I = (c, d)$, which is not principal. We'll likely show that it is not principal at the start of the second quarter. (I could give a one-line explanation right now, but this topic makes the most sense when we talk about Zariski tangent spaces.)

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FOUNDATIONS OF ALGEBRAIC GEOMETRY CLASS 10

CONTENTS

Last day: Krull's Principal Ideal Theorem, height, affine communication lemma, properties of schemes: locally Noetherian, Noetherian, finite type S-scheme, locally of finite type S-scheme, normal

Today: finite type A-scheme, locally of finite type A-scheme, projective schemes over A or k.

Problem set 4 is out today (on the web), and problem set 3 is due today. As always, feedback is most welcome. How are the problem sets pitched? I don't want to make them too grueling, but I'd like to give you enough so that you can get a grip on the concepts. I've noticed that some of you are going after the hardest questions, and others are trying easier questions, and that's fine with me.

There is a notion that I have been using implicitly, and I should have made it explicit by now. It's the notion of what I mean by when two schemes are the *isomorphic*. An *isomorphism* of two schemes $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ is the following data: (i) it is a homeomorphism between X and Y (the identification of the *sets* and *topologies*). Then we can think of $\mathcal{O}_X$ and $\mathcal{O}_Y$ are sheaves (of rings) on the same space, via this homeomorphism. (ii) It is the data of an isomorphism of sheaves $\mathcal{O}_X \leftrightarrow \mathcal{O}_Y$.

Last day, I introduced the affine communication lemma. this lemma will come up repeatedly in the future.

0.1. *Affine communication theorem. — Let P be some property enjoyed by some affine open sets of a scheme X, such that*

(i) *if* $\operatorname{Spec} R \hookrightarrow X$ *has* P *then for any* $f \in R$, $\operatorname{Spec} R_f \hookrightarrow X$ *does too.*
(ii) *if* $(f_1, \dots, f_n) = R$, *and* $\operatorname{Spec} R_{f_i} \hookrightarrow X$ *has* P *for all* i, *then so does* $\operatorname{Spec} R \hookrightarrow X$.

Suppose that $X = \cup_{i \in I} \operatorname{Spec} R_i$ *where* $\operatorname{Spec} R_i$ *is an affine, and* R_i *has property* P. *Then every other open affine subscheme of* X *has property* P *too.*

Date: Monday, October 31 (Hallowe'en), 2005. Small update January 31, 2007.

By choosing P appropriately, we define some important properties of schemes. I gave several examples. Here is one last example.

0.2. *Proposition. — Suppose* R *is a ring, and* $(f_1, \ldots, f_n) = R$. *Suppose* A *is a ring, and* R *is an* A*-algebra. (i) If* R *is a finitely generated* A*-algebra, then so is* R_{f_i}. *(ii) If each* R_{f_i} *is a finitely-generated* A*-algebra, then so is* R.

This of course leads to a corresponding definition.

0.3. Important Definition. Suppose X is a scheme, and A is a ring (e.g. A is a field k), and $\Gamma(X, \mathcal{O}_X)$ is an A-algebra. Note that $\Gamma(U, \mathcal{O}_X)$ is an A-algebra for all non-empty U. Then we say that X is an *A-scheme*, or a *scheme over A*. Suppose X is an A-scheme. If X can be covered by affine opens $\operatorname{Spec} R$ where R is a *finitely generated* A-algebra, we say that X is *locally of finite type over* A, or that it is a *locally of finite type A-scheme*. (My apologies for this cumbersome terminology; it will make more sense later.) If furthermore X is quasicompact, X is *finite type over* A, or a *finite type A-scheme*.

Proof of Proposition 0.2. (i) is clear: if R is generated over A by $r_1, \ldots, r_n$, then R_f is generated over A by $r_1, \ldots, r_n, 1/f$.

(ii) Here is the idea; I'll leave this as an **exercise** for you to make this work. We have generators of R_{f_i}: r_{ij}/f_i^j, where $r_{ij} \in R$. I claim that $\{r_{ij}\}_{ij} \cup \{f_i\}_i$ generate R as a A-algebra. Here's why. Suppose you have any $r \in R$. Then in R_{f_i}, we can write r as some polynomial in the r_{ij}'s and f_i, divided by some huge power of f_i. So "in each R_{f_i}, we have described r in the desired way", except for this annoying denominator. Now use a partition of unity type argument to combine all of these into a single expression, killing the denominator. Show that the resulting expression you build still agrees with r in each of the R_{f_i}. Thus it is indeed r. $\square$

1. Projective k-schemes and A-schemes: a concrete example

I now want to tell you about an important class of schemes.

Our building blocks of schemes are affine schemes. For example, affine finite type k-schemes correspond to finitely generated k-algebras. Once you pick generators of the algebra, say $x_1, \ldots, x_n$, then you can think of the scheme as sitting in n-space. More precisely, suppose R is a finitely-generated k-algebra, say

$$R = k[x_1, \ldots, x_n]/I.$$

Then at least as a topological space, it is a closed subset of $\mathbb{A}^n$, with set $V(I)$. (We will later be able to say that it is a *closed subscheme*, but we haven't yet defined this phrase.)

Different choices of generators give us different ways of seeing $\operatorname{Spec} R$ as sitting in some affine space. These affine schemes already are very interesting. But when you glue them together, you can get even more interesting things. I'll now tell you about projective schemes.

As a warm-up, let me discuss $\mathbb{P}^n_k$ again.

Intuitive idea: We think of closed points of $\mathbb{P}^n$ as $[x_0; x_1; \cdots; x_n]$, not all zero, with an equivalence relation $[x_0; \cdots; x_n] = [\lambda x_0; \cdots; \lambda x_n]$. $x_0^2 + x_2^2$ isn't a function on $\mathbb{P}^n$. But $x_0^2 + x_2^2 = 0$ makes sense. And $(x_0^2 + x_2^2)/(x_1^2 + x_2 x_3)$ *is* a function on $\mathbb{P}^2 - \{x_1^2 + x_2 x_3 = 0\}$. We have $n+1$ patches, corresponding to $x_i = 0$ ($0 \leq i \leq n$). Where $x_0 \neq 0$, we have a patch $[x_0; x_1; x_2] = [1; u_1; u_2]$, and similarly for $x_1 \neq 0$ and $x_2 \neq 0$.

More precisely: We defined $\mathbb{P}^n$ by gluing together $n+1$ copies of $\mathbb{A}^n$. Let me show you this in the case of $\mathbb{P}^2_k =$ "$\{[x_0; x_1; x_2]\}$". Let's pick co-ordinates wisely. The first patch is $U_0 = \{x_0 \neq 0\}$. We imagine $[x_0; x_1; x_2] = [1; x_{1/0}; x_{2/0}]$. The patch will have coordinates $x_{1/0}$ and $x_{2/0}$, i.e. it is $\operatorname{Spec} k[x_{1/0}, x_{2/0}]$.

Similarly, the second patch is $U_1 = \{x_1 \neq 0\} = \operatorname{Spec} k[x_{0/1}, x_{2/1}]$. We imagine $[x_0; x_1; x_2] = [x_{0/1}; 1; x_{2/1}]$.

Finally, the third patch is $U_2 = \{x_2 \neq 0\} = \operatorname{Spec} k[x_{0/2}, x_{1/2}]$, with "$[x_0; x_1; x_2] = [x_{0/2}; x_{1/2}; 1]$".

We glue U_0 along $x_{1/0} \neq 0$ to U_1 along $x_{0/1} \neq 0$. Our identification (from $[1; x_{1/0}; x_{2/0}] = [x_{0/1}; 1; x_{2/1}]$) is given by $x_{1/0} = 1/x_{0/1}$ and $x_{2/0} = x_{2/1} x_{1/0}$. $U_{01} := U_0 \cap U_1 = \operatorname{Spec} k[x_{1/0}, x_{2/0}, 1/x_{1/0}] \cong \operatorname{Spec} k[x_{0/1}, x_{2/1}, 1/x_{0/1}]$, where the isomorphism was as just described.

We similarly glue together U_0 and U_2, and U_1 and U_2. You could show that all this is compatible, and you could imagine that this is annoying to show. I'm not going to show you the details, because I'll give you a slick way around this naive approach fairly soon.

Suppose you had a homogeneous polynomial, such as $x_0^2 + x_1^2 = x_2^2$. (Intuition: I want a homogeneous polynomial, because in my intuitive notion of projective space as $[x_0; \cdots; x_n]$, I can make sense of where a homogeneous polynomial vanishes, but I can't make as good sense of where an inhomogeneous polynomial vanishes.)

Then I claim that this defines a scheme "in" projective space (in the same way that $\operatorname{Spec} k[x_1, \ldots, x_n]/I$ was a scheme "in" $\mathbb{A}^n$). Here's how. In the patch U_0, I interpret this as $1 + x_{1/0}^2 = x_{2/0}^2$. In patch U_1, I interpret it as $x_{0/1}^2 + 1 = x_{2/1}^2$. On the overlap U_{01}, these two equations are the same: the first equation in $\operatorname{Spec} k[x_{1/0}, x_{2/0}, 1/x_{1/0}]$ is the second equation in $\operatorname{Spec} k[x_{0/1}, x_{2/1}, 1/x_{0/1}]$ [do algebra], unsurprisingly. So piggybacking on that annoying calculation that $\mathbb{P}^2$ consists of 3 pieces glued together nicely is the fact that this scheme consists of three schemes glued together nicely. Similarly, any homogeneous polynomials $x_0, \ldots, x_n$ describes some nice scheme "in" $\mathbb{P}^n$.

1.1. Exercise. Show that an irreducible homogeneous polynomial in $n+1$ variables (over a field k) describes an integral scheme of dimension $n-1$. We think of this as a "hypersurface in $\mathbb{P}^n_k$". Definition: The degree of the hypersurface is the degree of the polynomial. (Other definitions: degree 1 = hyperplane, degree 2, 3, … = quadric hypersurface, cubic, quartic, quintic, sextic, septic, octic, …; a quadric curve is usually called a conic curve, or a conic for short.) Remark: $x_0^2 = 0$ is degree 2.

I could similarly do this with a bunch of homogeneous polynomials. For example:

1.2. Exercise. Show that $wx = yz, x^2 = wy, y^2 = xz$ describes an irreducible curve in $\mathbb{P}^3_k$ (the twisted cubic!).

1.3. Tentative definitions. Any scheme described in this way ("in $\mathbb{P}^n_k$") is called a *projective* k*-scheme*. We're not using anything about k being a ring, so similarly if A is a ring, we can define a projective A-scheme. (I did the case $A = k$ first because that's the more classical case.) If I is the ideal in $A[x_0, \dots, x_n]$ generated by these homogeneous polynomials, we say that the scheme we have constructed is $\operatorname{Proj} A[x_0, \dots, x_n]/I$.

1.4. Examples of projective k-schemes "in" $\mathbb{P}^2_k$: $x = 0$ ("line"), $x^2 + y^2 = z^2$ ("conic"). $wx = yz$ ("smooth quadric surface"). $y^2 z = x^3 - xz$ ("smooth cubic curve"). ($\mathbb{P}^2_k$)

You imagine that we will have a map $\operatorname{Proj} A[x_0, \dots, x_n]/I$ to $\operatorname{Spec} A$. And indeed we will once we have a definition of morphisms of schemes.

The *affine cone* of $\operatorname{Proj} R$ is $\operatorname{Spec} R$. The picture to have in mind is an actual cone. (I described it in the cases above, §1.4.) Intuitively, you could imagine that if you discarded the origin, you would get something that would project onto $\operatorname{Proj} R$. That will be right, but right now we don't know what maps of schemes are.

The *projective cone* of $\operatorname{Proj} R$ is $\operatorname{Proj} R[T]$, where T is one more variable. For example, the cone corresponding to the conic $\operatorname{Proj} k[x,y,z]/(x^2 + y^2 = z^2)$ is $\operatorname{Proj} k[x,y,z,T]/(x^2 + y^2 + z^2)$. I then discussed this in the cases above, in §1.4.

1.5. Exercise. Show that the projective cone of $\operatorname{Proj} R$ has an open subscheme that is the affine cone, and that its complement $V(T)$ can be associated with $\operatorname{Proj} R$ (as a topological space). (More precisely, setting $T = 0$ cuts out a scheme isomorphic to $\operatorname{Proj} R$.)

2. A more general notion of Proj

Let's abstract these notions. In the examples we've been doing, we have a graded ring $S = k[x_0, \dots, x_n]/I$ where I is a *homogeneous ideal* (i.e. I is generated by homogeneous elements of $k[x_0, \dots, x_n]$). Here we are taking the usual grading on $k[x_0, \dots, x_n]$, where each x_i has weight 1. Then S is also a graded ring, and we'll call its graded pieces S_0, S_1, etc. (In a graded ring: $S_m \times S_n \to S_{m+n}$. Note that S_0 is a subring, and S is a S_0-algebra.)

Notice in our example that $S_0 = k$, and S is generated over S_0 by S_1.

2.1. Definition. Assume for the rest of the day that S_* is a graded ring (with grading $\mathbb{Z}^{\geq 0}$). It is automatically a module over S_0. Suppose S_0 is a module over some ring A. (Imagine that $A = S_0 = k$.) Now $S_+ := \oplus_{i>0} S_i$ is an ideal, which we will call the *irrelevant ideal*; suppose that it is a finitely generated ideal.

2.2. Exercise. Show that S_* is a finitely-generated S_0-algebra.

Here is an example to keep in mind: $S_* = k[x_0, x_1, x_2]$ (with the usual grading). In this case we will build $\mathbb{P}^2_k$.

I will now define the scheme, that I will denote $\operatorname{Proj} S_*$. I will define it as a *set*, with a *topology*, and a *structure sheaf*. It will be enlightening to picture this in terms of the *affine cone* $\operatorname{Spec} S_*$. We will think of $\operatorname{Proj} S_*$ as the affine cone, minus the origin, modded out by multiplication by scalars.

The points of $\operatorname{Proj} S_*$ are defined to be the homogeneous prime ideals, except for any ideal containing the irrelevant ideal. (I waved my hands in the air linking this to $\operatorname{Spec} S_*$.)

We'll define the topology by defining the closed subsets. The closed subsets are of the form $V(I)$, where I is a homogeneous ideal. Particularly important open sets will the *distinguished open sets* $D(f) = \operatorname{Proj} S_* - V(f)$, where $f \in S_+$ is homogeneous. They form a base for the same reason as the analogous distinguished open sets did in the affine case.

Note: If $D(f) \subset D(g)$, then $f^n \in (g)$ for some n, and vice versa. We've done this before in the affine case .

Clearly $D(f) \cap D(g) = D(fg)$, by the same immediate argument as in the affine case.

We define $\mathcal{O}_{\operatorname{Proj} S_*}(D(f)) = (S_f)_0$, where $(S_f)_0$ means the 0-graded piece of the graded ring (S_f). As before, we check that this is well-defined (i.e. if $D(f) = D(f')$, then we are defining the same ring). In our example of $S_* = k[x_0, x_1, x_2]$, if we take $f = x_0$, we get $(k[x_0, x_1, x_2]_{x_0})_0 := k[x_{1/0}, x_{2/0}]$.

We now check that this is a sheaf. I could show that this is a sheaf on the base, and the argument would be the same. But instead, here is a trickier argument: I claim that

$$(D(f), \mathcal{O}_{\operatorname{Proj} S_*}) \cong \operatorname{Spec}(S_f)_0.$$

You can do this by showing that the distinguished base elements of $\operatorname{Proj} R$ contained in $D(f)$ are precisely the distinguished base elements of $\operatorname{Spec}(S_f)_0$, and the two sheaves have identifiable sections, and the restriction maps are the same.

2.3. Important Exercise. Do this. (Caution: don't assume $\deg f = 1$.)

2.4. *Example:* $\mathbb{P}^n_A$. $\mathbb{P}^n_A = \operatorname{Proj} A[x_0, \dots, x_n]$. This is great, because we didn't have to do any messy gluing.

2.5. Exercise. Check that this agrees with our earlier version of projective space.

2.6. *Exercise.* Show that $Y = \mathbb{P}^2 - (x^2 + y^2 + z^2 = 0)$ is affine, and find its corresponding ring (= find its ring of global sections).

We like this definition for a more abstract reason. Let V be an $n+1$-dimensional vector space over k. (Here k can be replaced by A as well.) Let $\operatorname{Sym}^{\cdot} V^* = k \oplus V^* \oplus \operatorname{Sym}^2 V^* \oplus \cdots$. The dual here may be confusing; it's here for reasons that will become apparent far later.)

If for example V is the dual of the vector space with basis associated to $x_0, \dots, x_n$, we would have $\operatorname{Sym}^{\cdot} V^* = k[x_0, \dots, x_n]$. Then we can define $\operatorname{Proj} \operatorname{Sym}^{\cdot} V^*$. (This is often called $\mathbb{P}V$.) I like this definition because it doesn't involved choosing a basis of V. [Picture of vector bundle, and its projectivization.]

If S_* is generated by S_1 (as a S_0-algebra), then $\operatorname{Proj} S_*$ "sits in $\mathbb{P}^n_A$". (Terminology: *generated in degree* 1.) $k[\operatorname{Sym}^* S_1] = k[x, y, z] \rightarrow\rightarrow S_*$ implies $S = k[x, y, z]/I$, where I is a homogeneous ideal. Example: $S_* = k[x, y, z]/(x^2 + y^2 - z^2)$. It sits naturally in $\mathbb{P}^2$.

Next day: I'll describe some nice properties of projective S_0-schemes.

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FOUNDATIONS OF ALGEBRAIC GEOMETRY CLASS 11

CONTENTS

Last day: finite type A-scheme, locally of finite type A-scheme, projective schemes over A or k.

Today: Smoothness=regularity=nonsingularity, Zariski tangent space and related notions, Nakayama's Lemma.

Warning: I've changed problem B6 to make it more general (reposted on web). The proof is the same as the original problem, but I'll use it in this generality.

1. Projective k-schemes and projective A-schemes

Last day, I defined $\operatorname{Proj} S_*$ where: S_* is a graded ring (with grading $\mathbb{Z}^{\geq 0}$). Last day I said: Suppose S_0 is an A-algebra. I've changed my mind: I'd like to take $S_0 = A$. $S_+ := \oplus_{i>0} S_i$ is the *irrelevant ideal*; suppose that it is finitely generated over S.

Set: The points of $\operatorname{Proj} S_*$ are defined to be the homogeneous prime ideals, except for any ideal containing the irrelevant ideal.

Topology: The closed subsets are of the form $V(I)$, where I is a homogeneous ideal. Particularly important open sets will the *distinguished open sets* $D(f) = \operatorname{Proj} S_* - V(f)$, where $f \in S_+$ is homogeneous. They form a base.

Structure sheaf: $\mathcal{O}_{\operatorname{Proj} S_*}(D(f)) := (S_f)_0$, where $(S_f)_0$ means the 0-graded piece of the graded ring (S_f). This is a sheaf. One method:

$$(D(f), \mathcal{O}_{\operatorname{Proj} S_*}|_{D(f)}) \cong \operatorname{Spec}(S_f)_0.$$

1.1. If S_* is generated by S_1 (as an S_0-algebra — we say S_* is *generated in degree* 1), say by $n+1$ elements $x_0, \dots, x_n$, then $\operatorname{Proj} S_*$ “sits in $\mathbb{P}^n_A$” as follows. (X “in” Y currently means that the topological space of X is a subspace of the topological space of Y.) Consider A^{n+1} as a free module with generators $t_0, \dots, t_n$ associated to $x_0, \dots, x_n$.

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$k[\mathrm{Sym}^* A^{n+1}] = k[t_0, t_1, \dots, t_n] \twoheadrightarrow S_*$ implies $S = k[t_0, t_1, \dots t_n]/I$, where I is a homogeneous ideal. Example: $S_* = k[x, y, z]/(x^2 + y^2 - z^2)$ sits naturally in $\mathbb{P}^2$.

1.2. Easy exercise (silly example). $\mathbb{P}^0_A = \mathrm{Proj}\, A[T] \cong \mathrm{Spec}\, A$. Thus "$\mathrm{Spec}\, A$ is a projective A-scheme".

Here are some useful facts.

A **quasiprojective A-scheme** is an open subscheme of a projective A-scheme. The "A" is omitted if it is clear from the context; often A is some field.)

1.3. Exercise. Show that all projective A-schemes are quasicompact. (Translation: show that any projective A-scheme is covered by a finite number of affine open sets.) Show that $\mathrm{Proj}\, S_*$ is finite type over $A = S_0$. If S_0 is a Noetherian ring, show that $\mathrm{Proj}\, S_*$ is a Noetherian scheme, and hence that $\mathrm{Proj}\, S_*$ has a finite number of irreducible components. Show that any quasiprojective scheme is locally of finite type over A. If A is Noetherian, show that any quasiprojective A-scheme is quasicompact, and hence of finite type over A. Show this need not be true if A is not Noetherian.

I'm now going to ask a somewhat rhetorical question. It's going to sound complicated because of all the complicated words in it. But all the complicated words just mean simple concepts.

Question (open for now): are there any quasicompact finite type k-schemes that are not quasiprojective? (Translation: if we're gluing together a finite number of schemes each sitting in some $\mathbb{A}^n$, can we ever get something not quasiprojective?) The difficulty of answering this question shows that this is a good notion! We will see before long that the affine line with the doubled origin is not projective, but we'll call that kind of bad behavior "non-separated", and then the question will still stand: is every separated quasicompact finite type k-scheme quasiprojective?

1.4. Exercise. Show that $\mathbb{P}^n_k$ is normal. More generally, show that $\mathbb{P}^n_R$ is normal if R is a Unique Factorization Domain.

I said earlier that the *affine cone* is $\mathrm{Spec}\, S_*$. (We'll soon see that we'll have a map from cone minus origin to Proj.) The *projective cone* of $\mathrm{Proj}\, S_*$ is $\mathrm{Proj}\, S_*[T]$. We have an intuitive picture of both.

1.5. Exercise (better version of exercise from last day). Show that the projective cone of $\mathrm{Proj}\, S_*$ has an open subscheme $D(T)$ that is the affine cone, and that its complement $V(T)$ can be identified with $\mathrm{Proj}\, S_*$ (as a topological space). More precisely, setting $T = 0$ "cuts out" a scheme isomorphic to $\mathrm{Proj}\, S_*$ — see if you can make that precise.

A lot of what we did for affine schemes generalizes quite easily, as you'll see in these exercises.

1.6. Exercise. Show that the irreducible subsets of dimension $n-1$ of $\mathbb{P}^n_k$ correspond to homogeneous irreducible polynomials modulo multiplication by non-zero scalars.

1.7. Exercise.

(a) Suppose I is any homogeneous ideal, and f is a homogeneous element. Suppose f vanishes on $V(I)$. Show that $f^n \in I$ for some n. (Hint: mimic the proof in the affine case.)
(b) If $Z \subset \operatorname{Proj} S_*$, define $I(\cdot)$. Show that it is a homogeneous ideal. For any two subsets, show that $I(Z_1 \cup Z_2) = I(Z_1) \cap I(Z_2)$.
(c) For any homogeneous ideal I with $V(I) \neq \emptyset$, show that $I(V(I)) = \sqrt{I}$. [They may need the next exercise for this.]
(d) For any subset $Z \subset \operatorname{Proj} S_*$, show that $V(I(Z)) = \overline{Z}$.

1.8. Exercise. Show that the following are equivalent. (a) $V(I) = \emptyset$ (b) for any f_i (i in some index set) generating I, $\cup D(f_i) = \operatorname{Proj} S_*$ (c) $\sqrt{I} \supset S_+$.

Now let's go back to some interesting geometry. Here is a useful construction. Define $S_{n*} := \oplus_i S_{ni}$. (We could rescale our degree, so "old degree" n is "new degree" 1.)

1.9. Exercise. Show that $\operatorname{Proj} S_{n*}$ is isomorphic to $\operatorname{Proj} S_*$.

1.10. Exercise. Suppose S_* is generated over S_0 by $f_1, \ldots, f_n$. Suppose $d = \operatorname{lcm}(\deg f_1, \ldots, \deg f_n)$. Show that S_{d*} is generated in "new" degree 1 (= "old" degree d). (Hint: I like to show this by induction on the size of the set $\{\deg f_1, \ldots, \deg f_n\}$.) This is handy, because we can stick every Proj in some projective space via the construction of 1.1.

1.11. Exercise. If S_* is a Noetherian domain over k, and $\operatorname{Proj} S_*$ is non-empty show that $\dim \operatorname{Spec} S_* = \dim \operatorname{Proj} S_* + 1$. (Hint: throw out the origin. Look at a distinguished $D(f)$ where $\deg f = 1$. Use the fact mentioned in Exercise 2.3 of Class 9. By the previous exercise, you can assume that S_* is generated in degree 1 over $S_0 = A$.)

Example: Suppose $S_* = k[x, y]$, so $\operatorname{Proj} S_* = \mathbb{P}^1_k$. Then $S_{2*} = k[x^2, xy, y^2] \subset k[x, y]$. What is this subring? Answer: let $u = x^2, v = xy, w = y^2$. I claim that $S_{2*} = k[u, v, w]/(uw - v^2)$.

1.12. Exercise. Prove this.

We have a graded ring with three generators; thus we think of it as sitting "in" $\mathbb{P}^2$. This is $\mathbb{P}^1$ as a conic in $\mathbb{P}^2$.

1.13. *Side remark: diagonalizing quadrics.* Suppose k is an algebraically closed field of characteristic not 2. Then any quadratic form in n variables can be "diagonalized" by changing coordinates to be a sum of squares (e.g. $uw - v^2 = ((u+v)/2)^2 + (i(u-v)/2)^2 + (iv)^2$), and the number of such squares is invariant of the change of coordinates. (Reason:

write the quadratic form on $x_1, \dots, x_n$ as

$$\begin{pmatrix} x_1 & \cdots & x_n \end{pmatrix} M \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

where M is a symmetric matrix — here you are using characteristic $\neq 2$. Then diagonalize M — here you are using algebraic closure.) Thus the conics in $\mathbb{P}^2$, up to change of coordinates, come in only a few flavors: sums of 3 squares (e.g. our conic of the previous exercise), sums of 2 squares (e.g. $y^2 - x^2 = 0$, the union of 2 lines), a single square (e.g. $x^2 = 0$, which looks set-theoretically like a line), and 0 (not really a conic at all). Thus we have proved: any plane conic (over an algebraically closed field of characteristic not 2) that can be written as the sum of three squares is isomorphic to $\mathbb{P}^1$.

We now soup up this example.

1.14. Exercise. Show that $\operatorname{Proj} S_{3*}$ is the *twisted cubic* "in" $\mathbb{P}^3$.

1.15. Exercise. Show that $\operatorname{Proj} S_{d*}$ is given by the equations that

$$\begin{pmatrix} y_0 & y_1 & \cdots & y_{d-1} \\ y_1 & y_2 & \cdots & y_d \end{pmatrix}$$

is rank 1 (i.e. that all the 2×2 minors vanish).

This is called the *degree* d *rational normal curve* "in" $\mathbb{P}^d$.

More generally, if $S_* = k[x_0, \dots, x_n]$, then $\operatorname{Proj} S_{d*} \subset \mathbb{P}^{N-1}$ (where N is the number of degree d polynomials in $x_0, \dots, x_n$) is called the d-*uple embedding* or d-*uple Veronese embedding*. **Exercise.** Show that $N = \binom{n+d}{d}$.

1.16. Exercise. Find the equations cutting out the *Veronese surface* $\operatorname{Proj} S_{2*}$ where $S_* = k[x_0, x_1, x_2]$, which sits naturally in $\mathbb{P}^5$.

1.17. *Example.* If we put a non-standard weighting on the variables of $k[x_1, \dots, x_n]$ — say we give x_i degree d_i — then $\operatorname{Proj} k[x_1, \dots, x_n]$ is called *weighted projective space* $\mathbb{P}(d_1, d_2, \dots, d_n)$.

1.18. Exercise. Show that $\mathbb{P}(m, n)$ is isomorphic to $\mathbb{P}^1$. Show that $\mathbb{P}(1, 1, 2) \cong \operatorname{Proj} k[u, v, w, z]/(uw - v^2)$. Hint: do this by looking at the even-graded parts of $k[x_0, x_1, x_2]$, cf. Exercise 1.9. (Picture: this is a projective cone over a conic curve.)

1.19. *Important exercise for later.* (a) *(Hypersurfaces meet everything of dimension at least* 1 *in projective space — unlike in affine space.)* Suppose X is a closed subset of $\mathbb{P}^n_k$ of dimension at least 1, and H a nonempty hypersurface in $\mathbb{P}^n_k$. Show that H meets X. (Hint: consider the affine cone, and note that the cone over H contains the origin. Use Krull's Principal Ideal Theorem.)

(b) (Definition: Subsets in $\mathbb{P}^n$ cut out by linear equations are called *linear subspaces*. Dimension $1, 2$ linear subspaces are called *lines* and *planes* respectively.) Suppose $X \hookrightarrow \mathbb{P}^n_k$ is a closed subset of dimension r. Show that any codimension r linear space meets X. (Hint: Refine your argument in (a).)
(c) Show that there is a codimension $r+1$ complete intersection (codimension $r+1$ set that is the intersection of $r+1$ hypersurfaces) missing X. (The key step: show that there is a hypersurface of sufficiently high degree that doesn't contain every generic point of X.) If k is infinite, show that there is a codimension $r+1$ linear subspace missing X. (The key step: show that there is a hyperplane not containing any generic point of a component of X.)

1.20. Exercise. Describe all the lines on the quadric surface $wx - yz = 0$ in $\mathbb{P}^3_k$. (Hint: they come in two "families", called the *rulings* of the quadric surface.)

Hence by Remark 1.13, if we are working over an algebraically closed field of characteristic not 2, we have shown that all rank 4 quadric surfaces have two rulings of lines.

2. "SMOOTHNESS" = REGULARITY = NONSINGULARITY

The last property of schemes I want to discuss is something very important: when they are "smooth". For unfortunate historic reasons, *smooth* is a name given to certain morphisms of schemes, but I'll feel free to use this to use it also for schemes themselves. The more correct terms are *regular* and *nonsingular*. A point of a scheme that is not smooth=regular=nonsingular is, not surprisingly, *singular*.

The best way to describe this is by first defining the tangent space to a scheme at a point, what we'll call the *Zariski tangent space*. This will behave like the tangent space you know and love at smooth points, but will also make sense at other points. In other words, geometric intuition at the smooth points guides the definition, and then the definition guides the algebra at all points, which in turn lets us refine our geometric intuition.

This definition is short but surprising. I'll have to convince you that it deserves to be called the tangent space. I've always found this tricky to explain, and that is because we want to show that it agrees with our intuition; but unfortunately, our intuition is crappier than we realize. So I'm just going to define it for you, and later try to convince you that it is reasonable.

Suppose A is a ring, and $\mathfrak{m}$ is a point. Translation: we have a point $[\mathfrak{m}]$ on a scheme $\operatorname{Spec} A$. Let $k = A/\mathfrak{m}$ be the residue field. Then $\mathfrak{m}/\mathfrak{m}^2$ is a vector space over the residue field $A/\mathfrak{m}$: it is an A-module, and $\mathfrak{m}$ acts like 0. This is defined to be the **Zariski cotangent space**. The dual is the **Zariski tangent space**. Elements of the Zariski cotangent space are called **cotangent vectors** or **differentials**; elements of the tangent space are called **tangent vectors**.

Note: This is intrinsic; it doesn't depend on any specific description of the ring itself (e.g. the choice of generators over a field k = choice of embedding in affine space). An interesting feature: in some sense, the cotangent space is more algebraically natural than the tangent space. There is a moral reason for this: the cotangent space is more naturally determined in terms of functions on a space, and we are very much thinking about schemes in terms of "functions on them". This will come up later.

I'm now going to give you a bunch of plausibility arguments that this is a reasonable definition.

First, I'll make a moral argument that this definition is plausible for the cotangent space of the origin of $\mathbb{A}^n$. Functions on $\mathbb{A}^n$ should restrict to a linear function on the tangent space. What function does $x^2 + xy + x + y$ restrict to "near the origin"? You will naturally answer: $x + y$. Thus we "pick off the linear terms". Hence $\mathfrak{m}/\mathfrak{m}^2$ are the linear functionals on the tangent space, so $\mathfrak{m}/\mathfrak{m}^2$ is the cotangent space.

Here is a second argument, for those of you who think of the tangent space as the space of derivations. (I didn't say this in class, because I didn't realize that many of you thought in this way until later.) More precisely, in differential geometry, the tangent space at a point is sometimes defined as the vector space of derivations at that point. A derivation is a function that takes in functions near the point that vanish at the point, and gives elements of the field k, and satisfies the Leibniz rule $(fg)' = f'g + g'f$. Translation: a derivation is a map $\mathfrak{m} \to k$. But $\mathfrak{m}^2 \to 0$, as if $f(p) = g(p) = 0$, then $(fg)'(p) = f'(p)g(p) + g'(p)f(p) = 0$. Thus we have a map $\mathfrak{m}/\mathfrak{m}^2 \to k$, i.e. an element of $(\mathfrak{m}/\mathfrak{m}^2)^*$. **Exercise (for those who think in this way).** Check that this is reversible, i.e. that any map $\mathfrak{m}/\mathfrak{m}^2 \to k$ gives a derivation — i.e., check the Leibniz rule.

2.1. Here is an old-fashioned example to help tie this down to earth. This is not currently intended to be precise. In $\mathbb{A}^3$, we have a curve cut out by $x + y + z^2 + xyz = 0$ and $x - 2y + z + x^2y^2z^3 = 0$. What is the tangent line near the origin? (Is it even smooth there?) Answer: the first surface looks like $x + y = 0$ and the second surface looks like $x - 2y + z = 0$. The curve has tangent line cut out by $x + y = 0$ and $x - 2y + z = 0$. It is smooth (in the analytic sense). I give questions like this in multivariable calculus. The students do a page of calculus to get the answer, because I can't tell them to just pick out the linear terms.

Another example: $x + y + z^2 = 0$ and $x + y + x^2 + y^4 + z^5 = 0$ cuts out a curve, which obviously passes through the origin. If I asked my multivariable calculus students to calculate the tangent line to the curve at the origin, they would do a page of calculus which would boil down to picking off the linear terms. They would end up with the equations $x + y = 0$ and $x + y = 0$, which cuts out a plane, not a line. They would be disturbed, and I would explain that this is because the curve isn't smooth at a point, and their techniques don't work. We on the other hand bravely declare that the cotangent space is cut out by $x + y = 0$, and *define* this as a singular point. (Intuitively, the curve near the origin is very close to lying in the plane $x + y = 0$.) Notice: the cotangent space jumped up in dimension from what it was "supposed to be", not down.

2.2. *Proposition. — Suppose* $(A, \mathfrak{m})$ *is a Noetherian local ring. Then* $\dim A \leq \dim_k \mathfrak{m}/\mathfrak{m}^2$.

We'll prove this on Friday.

If equality holds, we say that A is **regular** at $\mathfrak{p}$. If A is a local ring, then we say that A is a **regular local ring**. If A is regular at all of its primes, we say that A is a **regular ring**.

A scheme X is **regular** or **nonsingular** or **smooth** at a point p if the local ring $\mathcal{O}_{X,p}$ is regular. It is **singular** at the point otherwise. A scheme is **regular** or **nonsingular** or **smooth** if it is regular at all points. It is **singular** otherwise (i.e. if it is singular at *at least one* point.

In order to prove Proposition 2.2, we're going to pull out another algebraic weapon: Nakayama's lemma. This was done in Math 210, so I didn't discuss it in class. You should read this short exposition. If you have never seen Nakayama before, you'll see a complete proof. If you want a refresher, here it is. And even if you are a Nakayama expert, please take a look, because there are several related facts that go by the name of Nakayama's Lemma, and we should make sure we're talking about the same one(s). Also, this will remind you that the proof wasn't frightening and didn't require months of previous results.

2.3. *Nakayama's Lemma version 1.* — *Suppose* R *is a ring,* I *an ideal of* R*, and* M *is a finitely-generated* R*-module. Suppose* $M = IM$*. Then there exists an* $a \in R$ *with* $a \equiv 1 \pmod{I}$ *with* $aM = 0$*.*

Proof. Say M is generated by $m_1, \ldots, m_n$. Then as $M = IM$, we have $m_i = \sum_j a_{ij} m_j$ for some $a_{ij} \in I$. Thus

$$(\mathrm{Id}_n - A)\begin{pmatrix} m_1 \\ \vdots \\ m_n \end{pmatrix} = 0 \tag{1}$$

where Id_n is the $n \times n$ identity matrix in R, and $A = (a_{ij})$. We can't quite invert this matrix, but we almost can. Recall that any $n \times n$ matrix M has an adjoint $\mathrm{adj}(M)$ such that $\mathrm{adj}(M)M = \det(M)\mathrm{Id}_n$. The coefficients of $\mathrm{adj}(M)$ are polynomials in the coefficients of M. (You've likely seen this in the form a formula for M^{-1} when there is an inverse.) Multiplying both sides of (1) on the left by $\mathrm{adj}(\mathrm{Id}_n - A)$, we obtain

$$\det(\mathrm{Id}_n - A)\begin{pmatrix} m_1 \\ \vdots \\ m_n \end{pmatrix} = 0.$$

But when you expand out $\det(\mathrm{Id}_n - A)$, you get something that is $1 \pmod{I}$. □

Here is why you care: Suppose I is contained in all maximal ideals of R. (The intersection of all the maximal ideals is called the *Jacobson radical*, but I won't use this phrase. Recall that the nilradical was the intersection of the *prime ideals* of R.) Then I claim that any $a \equiv 1 \pmod{I}$ is invertible. For otherwise $(a) \neq R$, so the ideal (a) is contained in some maximal ideal $\mathfrak{m}$ — but $a \equiv 1 \pmod{\mathfrak{m}}$, contradiction. Then as a is invertible, we have the following.

2.4. *Nakayama's Lemma version 2. — Suppose* R *is a ring,* I *an ideal of* R *contained in all maximal ideals, and* M *is a finitely-generated* R*-module. (Most interesting case:* R *is a local ring, and* I *is the maximal ideal.) Suppose* $M = IM$*. Then* $M = 0$*.*

2.5. Important exercise (Nakayama's lemma version 3). Suppose R is a ring, and I is an ideal of R contained in all maximal ideals. Suppose M is a *finitely generated* R-module, and $N \subset M$ is a submodule. If $N/IN \hookrightarrow M/IM$ an isomorphism, then $M = N$.

2.6. Important exercise (Nakayama's lemma version 4). Suppose $(R, \mathfrak{m})$ is a local ring. Suppose M is a finitely-generated R-module, and $f_1, \dots, f_n \in M$, with (the images of) $f_1, \dots, f_n$ generating $M/\mathfrak{m}M$. Then $f_1, \dots, f_n$ generate M. (In particular, taking $M = \mathfrak{m}$, if we have generators of $\mathfrak{m}/\mathfrak{m}^2$, they also generate $\mathfrak{m}$.)

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FOUNDATIONS OF ALGEBRAIC GEOMETRY CLASS 12

CONTENTS

Last day: smoothness=regularity=nonsingularity, Zariski tangent space and related notions, Nakayama's Lemma.

Today: Jacobian criterion, Euler test, characterizations of discrete valuation rings = dimension 1 Noetherian regular local rings

1. "SMOOTHNESS" = REGULARITY = NONSINGULARITY, CONTINUED

Last day, I defined the Zariski tangent space. Suppose A is a ring, and $\mathfrak{m}$ is a maximal ideal, with residue field $k = A/\mathfrak{m}$. Then $\mathfrak{m}/\mathfrak{m}^2$, a vector space over k, is the **Zariski cotangent space**. The dual is the **Zariski tangent space**. Elements of the Zariski cotangent space are called **cotangent vectors** or **differentials**; elements of the tangent space are called **tangent vectors**.

I tried to convince you that this was a reasonable definition. I also asked you what your private definition of tangent space or cotangent space was, so I could convince you that this is the right algebraic notion. A couple of you think of tangent vectors as *derivations*, and in this case, the connection is very fast. I've put it in the Class 11 notes, so please check it out if you know what derivations are.

Last day, I stated the following proposition.

1.1. *Proposition. — Suppose* $(A, \mathfrak{m})$ *is a Noetherian local ring. Then* $\dim A \leq \dim_k \mathfrak{m}/\mathfrak{m}^2$.

We'll prove this in a moment.

If equality holds, we say that A is **regular** at $\mathfrak{m}$. If A is a local ring, then we say that A is a **regular local ring**. If A is regular at all of its primes, we say that A is a **regular ring**.

A scheme X is **regular** or **nonsingular** or **smooth** at a point p if the local ring $\mathcal{O}_{X,p}$ is regular. It is **singular** at the point otherwise. A scheme is **regular** or **nonsingular** or

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smooth if it is regular at all points. It is **singular** otherwise (i.e. if it is singular at *at least one* point).

1.2. Exercise. Show that if A is a Noetherian local ring, then A has finite dimension. (*Warning:* Noetherian rings in general could have infinite dimension.)

In order to prove Proposition 1.1, we're going to use Nakayama's Lemma, which hopefully you've looked at.

The version we'll use is:

1.3. Important exercise (Nakayama's lemma version 4). Suppose $(R, \mathfrak{m})$ is a local ring. Suppose M is a finitely-generated R-module, and $f_1, \dots, f_n \in M$, with (the images of) $f_1, \dots, f_n$ generating $M/\mathfrak{m}M$. Then $f_1, \dots, f_n$ generate M. (In particular, taking $M = \mathfrak{m}$, if we have generators of $\mathfrak{m}/\mathfrak{m}^2$, they also generate $\mathfrak{m}$.) Translation: if we have a set of generators of a *finitely generated* module modulo a finite ideal, then they generate the entire module.

Proof of Proposition 1.1: Note that $\mathfrak{m}$ is finitely generated (as R is Noetherian), so $\mathfrak{m}/\mathfrak{m}^2$ is a finitely generated $R/\mathfrak{m} = k$-module, hence finite-dimensional. Say $\dim_k \mathfrak{m}/\mathfrak{m}^2 = n$. Choose n elements of $\mathfrak{m}/\mathfrak{m}^2$, and lift them to elements $f_1, \dots, f_n$ of $\mathfrak{m}$. Then by Nakayama's lemma, $(f_1, \dots, f_n) = \mathfrak{m}$.

Problem B6 on problem set 4 (newest version!) includes the following: Suppose $X = \operatorname{Spec} R$ where R is a Noetherian domain, and Z is an irreducible component of $V(f_1, \dots, f_n)$, where $f_1, \dots, f_n \in R$. Show that the height of Z (as a prime ideal) is at most n.

In this case, $V((f_1, \dots, f_n)) = V(\mathfrak{m})$ is just the point $[\mathfrak{m}]$, so the height of $\mathfrak{m}$ is at most n. Thus the longest chain of prime ideals containing $\mathfrak{m}$ is at most $n + 1$. But this is also the longest chain of prime ideals in X (as $\mathfrak{m}$ is the unique maximal ideal), so $n \geq \dim X$. $\square$

Computing the Zariski-tangent space is actually quite hands-on, because you can compute it in a multivariable calculus way.

For example, last day I gave some motivation, by saying that $x + y + 3z + y^3 = 0$ and $2x + z^3 + y^2 = 0$ cut out a curve in $\mathbb{A}^3$, which is nonsingular at the origin, and that the tangent space at the origin is cut out by $x + y + 3z = 2x = 0$. This can be made precise through the following exercise.

1.4. *Important exercise.* Suppose A is a ring, and $\mathfrak{m}$ a maximal ideal. If $f \in \mathfrak{m}$, show that the dimension of the Zariski tangent space of $\operatorname{Spec} A$ at $[\mathfrak{m}]$ is the dimension of the Zariski tangent space of $\operatorname{Spec} A/(f)$ at $[\mathfrak{m}]$, or one less. (Hint: show that the Zariski tangent space of $\operatorname{Spec} A/(f)$ is "cut out" in the Zariski tangent space of $\operatorname{Spec} A$ by the linear equation $f \pmod{\mathfrak{m}^2}$.)

1.5. *Exercise.* Find the dimension of the Zariski tangent space at the point $[(2,x)]$ of $\mathbb{Z}[2i] \cong \mathbb{Z}[x]/(x^2+4)$. Find the dimension of the Zariski tangent space at the point $[(2,x)]$ of $\mathbb{Z}[\sqrt{2}i] \cong \mathbb{Z}[x]/(x^2+2)$.

1.6. *Exercise (the Jacobian criterion for checking nonsingularity).* Suppose k is an algebraically closed field, and X is a finite type k-scheme. Then locally it is of the form $\operatorname{Spec} k[x_1,\dots,x_n]/(f_1,\dots,f_r)$. Show that the Zariski tangent space at the closed point p (with residue field k, by the Nullstellensatz) is given by the cokernel of the Jacobian map $k^r \to k^n$ given by the Jacobian matrix

$$J = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(p) & \cdots & \frac{\partial f_r}{\partial x_1}(p) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_1}{\partial x_n}(p) & \cdots & \frac{\partial f_r}{\partial x_n}(p) \end{pmatrix}. \tag{1}$$

(This is just making precise our example of a curve in $\mathbb{A}^3$ cut out by a couple of equations, where we picked off the linear terms.) Possible hint: "translate p to the origin," and consider linear terms. See also the exercise two previous to this one.

You might be alarmed: what does $\frac{\partial f}{\partial x_1}$ mean?! Do you need deltas and epsilons? No! Just define derivatives formally, e.g.

$$\frac{\partial}{\partial x_1}(x_1^2 + x_1x_2 + x_2^2) = 2x_1 + x_2.$$

1.7. Exercise: Dimension theory implies the Nullstellensatz. In the previous exercise, l is necessarily only a finite extension of k, as this exercise shows. (a) Prove a microscopically stronger version of the weak Nullstellensatz: Suppose $R = k[x_1,\dots,x_n]/I$, where k is an algebraically closed field and I is some ideal. Then the maximal ideals are precisely those of the form $(x_1 - a_1,\dots,x_n - a_n)$, where $a_i \in k$.
(b) Suppose $R = k[x_1,\dots,x_n]/I$ where k is not necessarily algebraically closed. Show every maximal ideal of R has residue field that are finite extensions of k. (Hint for both: the maximal ideals correspond to dimension 0 points, which correspond to transcendence degree 0 extensions of k, i.e. finite extensions of k. If $k = \overline{k}$, the maximal ideals correspond to surjections $f : k[x_1,\dots,x_n] \to k$. Fix one such surjection. Let $a_i = f(x_i)$, and show that the corresponding maximal ideal is $(x_1 - a_1,\dots,x_n - a_n)$.) This exercise is a bit of an aside — it belongs in class 8, and I've also put it in those notes.

1.8. Exercise. Show that the singular *closed* points of the hypersurface $f(x_1,\dots,x_n) = 0$ in $\mathbb{A}^n_k$ are given by the equations $f = \frac{\partial f}{\partial x_1} = \cdots = \frac{\partial f}{\partial x_n} = 0$.

1.9. Exercise. Show that $\mathbb{A}^1$ and $\mathbb{A}^2$ are nonsingular. (Make sure to check nonsingularity at the non-closed points! Fortunately you know what all the points of $\mathbb{A}^2$ are; this is trickier for $\mathbb{A}^3$.)

In the previous exercise, you'll use the fact that the local ring at the generic point of $\mathbb{A}^2$ is dimension 0, and the local ring at generic point at a curve in $\mathbb{A}^2$ is 1.

Let's apply this technology to an arithmetic situation.

1.10. Exercise. Show that $\operatorname{Spec}\mathbb{Z}$ is a nonsingular curve.

Here are some fun comments: What is the derivative of 35 at the prime 5? Answer: $35 \pmod{25}$, so 35 has the same "slope" as 10. What is the derivative of 9, which doesn't vanish at 5? Answer: the notion of derivative doesn't apply there. You'd think that you'd want to subtract its value at 5, but you can't subtract "$4 \pmod 5$" from the integer 9. Also, $35 \pmod 2)5$ you might *think* you want to restate as $7 \pmod 5$, by dividing by 5, but that's morally wrong — you're dividing by a particular choice of generator 5 of the maximal ideal of the 5-adics $\mathbb{Z}_5$; in this case, one appears to be staring you in the face, but in general that won't be true. Follow-up fun: you can talk about the derivative of a function only for functions vanishing at a point. And you can talk about the second derivative of a function only for functions that vanish, and whose first derivative vanishes. For example, 75 has second derivative $75 \pmod 1)25$ at 5. It's pretty flat.

1.11. Exercise. Note that $\mathbb{Z}[i]$ is dimension 1, as $\mathbb{Z}[x]$ has dimension 2 (problem set exercise), and is a domain, and x^2+1 is not 0, so $\mathbb{Z}[x]/(x^2+1)$ has dimension 1 by Krull. Show that $\operatorname{Spec}\mathbb{Z}[i]$ is a nonsingular curve. (This is intended for people who know about the primes of the Gaussian integers $\mathbb{Z}[i]$.)

1.12. Exercise. Show that there is one singular point of $\operatorname{Spec}\mathbb{Z}[2i]$, and describe it.

1.13. Handy Exercise (the Euler test for projective hypersurfaces). There is an analogous Jacobian criterion for hypersurfaces $f = 0$ in $\mathbb{P}^n_k$. Show that the singular *closed* points correspond to the locus $f = \frac{\partial f}{\partial x_1} = \cdots = \frac{\partial f}{\partial x_n} = 0$. If the degree of the hypersurface is not divisible by the characteristic of any of the residue fields (e.g. if we are working over a field of characteristic 0), show that it suffices to check $\frac{\partial f}{\partial x_1} = \cdots = \frac{\partial f}{\partial x_n} = 0$. (Hint: show that f lies in the ideal $(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n})$). (Fact: this will give the singular points in general. I don't want to prove this, and I won't use it.)

1.14. Exercise. Suppose k is algebraically closed. Show that $y^2 z = x^3 - xz^2$ in $\mathbb{P}^2_k$ is an irreducible nonsingular curve. (This is for practice.) Warning: I didn't say $\operatorname{char} k = 0$.

1.15. Exercises. Find all the singular closed points of the following plane curves. Here we work over a field of characteristic 0 for convenience.

(a) $y^2 = x^2 + x^3$. This is called a *node*.
(b) $y^2 = x^3$. This is called a *cusp*.
(c) $y^2 = x^4$. This is called a *tacnode*.

(I haven't given precise definitions for node, cusp, or tacnode. You may want to think about what they might be.)

1.16. Exercise. Show that the twisted cubic $\operatorname{Proj} k[w, x, y, z]/(wz - xy, wy - x^2, xz - y^2)$ is nonsingular. (You can do this by using the fact that it is isomorphic to $\mathbb{P}^1$. I'd prefer you to do this with the explicit equations, for the sake of practice.)

1.17. Exercise. Show that the only dimension 0 Noetherian regular local rings are fields. (Hint: Nakayama.)

2. Dimension 1 Noetherian regular local rings = discrete valuation rings

The case of dimension 1 is also very important, because if you understand how primes behave that are separated by dimension 1, then you can use induction to prove facts in arbitrary dimension. This is one reason why Krull is so useful.

A dimension 1 Noetherian regular local ring can be thought of as a "germ of a curve". Two examples to keep in mind are $k[x]_{(x)} = \{f(x)/g(x) : x \not| g(x)\}$ and $\mathbb{Z}_{(5)} = \{a/b : 5 \not| b\}$.

The purpose of this section is to give a long series of equivalent definitions of these rings.

Theorem. *Suppose* $(R, \mathfrak{m})$ *is a Noetherian dimension* 1 *local ring. The following are equivalent.*
(a) R *is regular.*

Informal translation: R is a germ of a smooth curve.

(b) $\mathfrak{m}$ *is principal.* If R is regular, then $\mathfrak{m}/\mathfrak{m}^2$ is one-dimensional. Choose any element $t \in \mathfrak{m} - \mathfrak{m}^2$. Then t generates $\mathfrak{m}/\mathfrak{m}^2$, so generates $\mathfrak{m}$ by Nakayama's lemma. Such an element is called a *uniformizer*. (Warning: we needed to know that $\mathfrak{m}$ was finitely generated to invoke Nakayama — but fortunately we do, thanks to the Noetherian hypothesis!)

Conversely, if $\mathfrak{m}$ is generated by one element t over R, then $\mathfrak{m}/\mathfrak{m}^2$ is generated by one element t over $R/\mathfrak{m} = k$.

(c) All ideals are of the form $\mathfrak{m}^n$ *or* 0. Suppose $(R, \mathfrak{m}, k)$ is a Noetherian regular local ring of dimension 1. Then I claim that $\mathfrak{m}^n \neq \mathfrak{m}^{n+1}$ for any n. Proof: Otherwise, $\mathfrak{m}^n = \mathfrak{m}^{n+1} = \mathfrak{m}^{n+2} = \cdots$. Then $\cap_i \mathfrak{m}^i = \mathfrak{m}^n$. But $\cap_i \mathfrak{m}^i = (0)$. **(I'd given a faulty reason for this. I owe you this algebraic fact.)** Then as $t^n \in \mathfrak{m}^n$, we must have $t^n = 0$. But R is a domain, so $t = 0$ — but $t \in \mathfrak{m} - \mathfrak{m}^2$.

I next claim that $\mathfrak{m}^n/\mathfrak{m}^{n+1}$ is dimension 1. Reason: $\mathfrak{m}^n = (t^n)$. So $\mathfrak{m}^n$ is generated as as a R-module by one element, and $\mathfrak{m}^n/(\mathfrak{m}\mathfrak{m}^n)$ is generated as a $(R/\mathfrak{m} = k)$-module by 1 element, so it is a one-dimensional vector space.

So we have a chain of ideals $R \supset \mathfrak{m} \supset \mathfrak{m}^2 \supset \mathfrak{m}^3 \supset \cdots$ with $\cap \mathfrak{m}^i = (0)$. We want to say that there is no room for any ideal besides these, because "each pair is "separated by dimension 1", and there is "no room at the end". Proof: suppose $I \subset R$ is an ideal. If $I \neq (0)$, then there is some n such that $I \subset \mathfrak{m}^n$ but $I \not\subset \mathfrak{m}^{n+1}$. Choose some $u \in I - \mathfrak{m}^{n+1}$. Then $(u) \subset I$. But u generates $\mathfrak{m}^n/\mathfrak{m}^{n+1}$, hence by Nakayama it generates $\mathfrak{m}^n$, so we have

$\mathfrak{m}^n \subset I \subset \mathfrak{m}^n$, so we are done. Conclusion: in a Noetherian local ring of dimension 1, regularity implies all ideals are of the form $\mathfrak{m}^n$ or (0).

Conversely, suppose we have a dimension 1 Noetherian local domain that is not regular, so $\mathfrak{m}/\mathfrak{m}^2$ has dimension at least 2. Choose any $u \in \mathfrak{m} - \mathfrak{m}^2$. Then $(u, \mathfrak{m}^2)$ is an ideal, but $\mathfrak{m} \subsetneq (u, \mathfrak{m}^2) \subsetneq \mathfrak{m}^2$. We've thus shown that (c) is equivalent to the previous cases.

(d) R is a principal ideal domain. I didn't do this in class. **Exercise.** Show that (d) is equivalent to (a)–(c).

(e) R is a discrete valuation ring. I will now define something for you that will be a very nice way of describing such rings, that will make precise some of our earlier vague ramblings. We'll have to show that this definition accords with (a)–(d) of course.

Suppose K is a field. A *discrete valuation* on K is a *surjective homomorphism* $\nu : K^* \to \mathbb{Z}$ (homomorphism: $\nu(xy) = \nu(x) + \nu(y)$) satisfying

$$\nu(x + y) \geq \min(\nu(x), \nu(y)).$$

Suggestive examples: (i) *(the 5-adic valuation)* $K = \mathbb{Q}$, $\nu(r)$ is the "power of 5 appearing in r", e.g. $\nu(35/2) = 1$, $\nu(27/125) = -3$.

(ii) $K = k(x)$, $\nu(f)$ is the "power of x appearing in f".

Then $0 \cup \{x \in K^* : \nu(x) \geq 0\}$ is a ring. It is called the *valuation ring* of ν.

2.1. Exercise. Describe the valuation rings in those two examples. Hmm — they are familiar-looking dimension 1 Noetherian local rings. What a coincidence!

2.2. Exercise. Show that $0 \cup \{x \in K^* : \nu(x) \geq 1\}$ is the unique maximal ideal of the valuation ring. (Hint: show that everything in the complement is invertible.) Thus the valuation ring is a local ring.

An integral domain A is called a *discrete valuation ring* if there exists a discrete valuation ν on its fraction field $K = \mathrm{Frac}(A)$.

Now if R is a Noetherian regular local ring of dimension 1, and t is a uniformizer (generator of $\mathfrak{m}$ as an ideal = dimension of $\mathfrak{m}/\mathfrak{m}^2$ as a k-vector space) then any non-zero element r of R lies in some $\mathfrak{m}^n - \mathfrak{m}^{n+1}$, so $r = t^n u$ where u is a unit (as t^n generates $\mathfrak{m}^n$ by Nakayama, and so does r), so $\mathrm{Frac}\, R = R_t = R[1/t]$. So any element of $\mathrm{Frac}\, R$ can be written uniquely as ut^n where u is a unit and $n \in \mathbb{Z}$. Thus we can define a valuation $\nu(ut^n) = n$, and we'll quickly see that it is a discrete valuation (**exercise**). Thus (a)-(c) implies (d).

Conversely, suppose $(R, \mathfrak{m})$ is a discrete valuation ring. Then I claim it is a Noetherian regular local ring of dimension 1. **Exercise.** Check this. (Hint: Show that the ideals are all of the form (0) or $I_n = \{r \in R : \nu(r) \geq n\}$, and I_1 is the only prime of the second sort. Then we get Noetherianness, and dimension 1. Show that I_1/I_2 is generated by any element of $I_1 - I_2$.)

Exercise/Corollary. There is only one discrete valuation on a discrete valuation ring.

Thus whenever you see a regular local ring of dimension 1, we have a valuation on the fraction field. If the valuation of an element is $n > 0$, we say that the element has a *zero of order* n. If the valuation is $-n < 0$, we say that the element has a *pole of order* n.

So we can finally make precise the fact that 75/34 has a double zero at 5, and a single pole at 2! Also, you can easily figure out the zeros and poles of $x^3(x+y)/(x^2+xy)^3$ on $\mathbb{A}^2$. Note that we can only make sense of zeros and poles at *nonsingular points of codimension* 1.

Definition. More generally: suppose X is a locally Noetherian scheme. Then for any regular height(=codimension) 1 points (i.e. any point p where $\mathcal{O}_{X,p}$ is a regular local ring of dimension 1), we have a valuation ν. If f is any non-zero element of the fraction field of $\mathcal{O}_{X,p}$ (e.g. if X is integral, and f is a non-zero element of the function field of X), then if $\nu(f) > 0$, we say that the element has a *zero of order* $\nu(f)$, and if $\nu(f) < 0$, we say that the element has a *pole of order* $-\nu(f)$.

Exercise. Suppose X is a regular integral Noetherian scheme, and $f \in \mathrm{Frac}(\Gamma(X, \mathcal{O}_X))^*$ is a non-zero element of its function field. Show that f has a finite number of zeros and poles.

Finally:

(f) $(R, \mathfrak{m})$ *is a unique factorization domain,*

(g) R *is integrally closed in its fraction field* $K = \mathrm{Frac}(R)$.

(a)-(e) clearly imply (f), because we have the following stupid unique factorization: each non-zero element of r can be written uniquely as ut^n where $n \in \mathbb{Z}^{\geq 0}$ and u is a unit.

(f) implies (g), because checked earlier that unique factorization domains are always integrally closed in its fraction field.

So it remains to check that (g) implies (a)-(e). This is straightforward, but for the sake of time, I'm not going to give the proof in class. But in the interests of scrupulousness, I'm going to give you a full proof in the notes. It will take us less than half a page. This is the only tricky part of this entire theorem.

2.3. Fact. Suppose $(S, \mathfrak{n})$ is a Noetherian local domain of dimension 0. Then $\mathfrak{n}^n = 0$ for some n. (I had earlier given this as an exercise, with an erroneous hint. I may later add a proof to the notes.)

2.4. Exercise. Suppose A is a subring of a ring B, and $x \in B$. Suppose there is a faithful $A[x]$-module M that is finitely generated as an A-module. Show that x is integral over A.

(Hint: look carefully at the proof of Nakayama's Lemma version 1 in the Class 11 notes, and change a few words.)

Proof that (f) implies (b). Suppose $(R, \mathfrak{m})$ is a Noetherian local domain of dimension 1, that is integrally closed in its fraction field $K = \operatorname{Frac}(R)$. Choose any $r \in R \neq 0$. Then $S = R/(r)$ is dimension 0, and is Noetherian and local, so if $\mathfrak{n}$ is its maximal ideal, then there is some n such that $\mathfrak{n}^n = 0$ but $\mathfrak{n}^{n-1} \neq 0$ by Exercise 2.3. Thus $\mathfrak{m}^n \subseteq (r)$ but $\mathfrak{m}^{n-1} \not\subset (r)$. Choose $s \in \mathfrak{m}^{n-1} - (r)$. Consider $x = r/s$. Then $x^{-1} \notin R$, so as R is integrally closed, x^{-1} is not integral over R.

Now $x^{-1}\mathfrak{m} \not\subset \mathfrak{m}$ (or else $x^{-1}\mathfrak{m} \subset \mathfrak{m}$ would imply that $\mathfrak{m}$ is a faithful $R[x^{-1}]$-module, contradicting Exercise 2.4). But $x^{-1}\mathfrak{m} \subset R$. Thus $x^{-1}\mathfrak{m} = R$, from which $\mathfrak{m} = xR$, so $\mathfrak{m}$ is principal. □

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FOUNDATIONS OF ALGEBRAIC GEOMETRY CLASS 13

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CONTENTS

Last day: Jacobian criterion, Euler test, characterizations of discrete valuation rings = Noetherian regular local rings

Today: discrete valuation rings (conclusion), cultural facts to know about regular local rings, the distinguished affine base of the topology, 2 definitions of quasicoherent sheaf.

Problem set 5 is out today.

I'd like to start with some words on height versus codimension. Suppose R is an integral domain, and $\mathfrak{p}$ is a prime ideal. Thus geometrically we are thinking of an irreducible topological space $\operatorname{Spec} R$, and an irreducible closed subset $\overline{[\mathfrak{p}]}$. Then we have:

$$\dim R/\mathfrak{p} + \operatorname{height} \mathfrak{p} := \dim R/\mathfrak{p} + \dim R_{\mathfrak{p}} \leq \dim R.$$

The reason is as follows: $\dim R$ is one less than the length of the longest chain of prime ideals of R. $\dim R/\mathfrak{p}$ is one less than the length of the longest chain of prime ideals containing $\mathfrak{p}$. $\dim R_{\mathfrak{p}}$ is the length of the longest chain of prime ideals contained in $\mathfrak{p}$. In the homework, you've shown that if R is a finitely generated domain over k, then we have equality, because we can compute dimension using transcendence degree. Hence through any $\mathfrak{p} \subset R$, we can string a "longest chain". Thus we even know that we have equality if R is a *localization* of a finitely generated domain over k.

However, this is false in general. In the class 9 notes, I've added an elementary example to show that you can have the following strange situation: $R = k[x]_{(x)}[t]$ has dimension 2, it is easy to find a chain of prime ideals of length 3:

$$(0) \subset (t) \subset (x, t).$$

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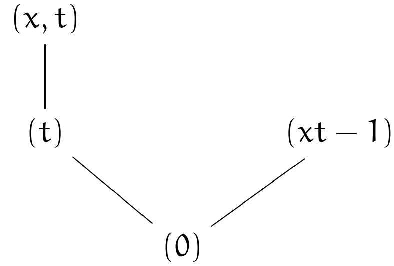
However, the ideal $(xt-1)$ is prime, and height 1 (there is no prime between it and (0)), and maximal.



The details are easy. Thus we have a dimension 0 subset of a dimension 2 set, but it is height 1. Thus it is dangerous to define codimension as height, because you might say something incorrect accidentally.

This example comes from a geometric picture, and if you're curious as to what it is, ask me after class.

There is one more idea I wanted to mention to you, to advertise a nice consequence of the idea of Zariski tangent space.

Problem. Consider the ring $R = k[x,y,z]/(xy-z^2)$. Show that (x,z) is not a principal ideal.

As $\dim R = 2$, and $R/(x,z) \cong k[y]$ has dimension 1, we see that this ideal is height 1 (as height behaves as codimension should for finitely generated k-domains!). Our geometric picture is that $\operatorname{Spec} R$ is a cone (we can diagonalize the quadric as $xy - z^2 = ((x+y)/2)^2 - ((x-y)/2)^2 - z^2$, at least if $\operatorname{char} k \neq 2$), and that (x,z) is a ruling of the cone. This suggests that we look at the cone point.

Solution. Let $\mathfrak{m} = (x,y,z)$ be the maximal ideal corresponding to the origin. Then $\operatorname{Spec} R$ has Zariski tangent space of dimension 3 at the origin, and $\operatorname{Spec} R/(x,z)$ has Zariski tangent space of dimension 1 at the origin. But $\operatorname{Spec} R/(f)$ must have Zariski tangent space of dimension at least 2 at the origin.

Exercise. Show that $(x,z) \subset k[w,x,y,z]/(wz-xy)$ is a height 1 ideal that is not principal. (What is the picture?)

1. Dimension 1 Noetherian regular local rings = discrete valuation rings

Last day we mostly proved the following.

Theorem. *Suppose* $(R,\mathfrak{m})$ *is a Noetherian dimension* 1 *local ring. The following are equivalent.*

(a) R *is regular.*
(b) $\mathfrak{m}$ *is principal.*
(c) *All ideals are of the form* $\mathfrak{m}^n$ *or* 0.
(d) R *is a principal ideal domain.*
(e) R *is a discrete valuation ring.*

(f) $(R, \mathfrak{m})$ *is a unique factorization domain,*
(g) R *is integrally closed in its fraction field* $K = \mathrm{Frac}(R)$.

I didn't state (d) in class, but I included it as an exercise, as it is easy to connect to the others. Other than that, I connected (a) to (e), and showed that they implies (f), which in turn implies (g). All of the arguments were quite short. I didn't show that (g) implies (a)-(e), but I included it in the notes for Friday's class. I find this the trickiest part of the argument, but it is still quite short, less than half a page.

I'd like to repeat what I said on Friday about the consequences of this characterization of discrete valuation rings (DVR's).

Whenever you see a Noetherian regular local ring of dimension 1, we have a valuation on the fraction field. If the valuation of an element is $n > 0$, we say that the element has a *zero of order* n. If the valuation is $-n < 0$, we say that the element has a *pole of order* n.

Definition. More generally: suppose X is a locally Noetherian scheme. Then for any regular height(=codimension) 1 points (i.e. any point p where $\mathcal{O}_{X,p}$ is a regular local ring), we have a valuation ν. If f is any non-zero element of the fraction field of $\mathcal{O}_{X,p}$ (e.g. if X is integral, and f is a non-zero element of the function field of X), then if $\nu(f) > 0$, we say that the element has a *zero of order* $\nu(f)$, and if $\nu(f) < 0$, we say that the element has a *pole of order* $-\nu(f)$.

We aren't yet allowed to discuss order of vanishing at a point that is not regular codimension 1. One can make a definition, but it doesn't behave as well as it does when have you have a discrete valuation.

Exercise. Suppose X is an integral Noetherian scheme, and $f \in \mathrm{Frac}(\Gamma(X, \mathcal{O}_X))^*$ is a non-zero element of its function field. Show that f has a finite number of zeros and poles. (Hint: reduce to $X = \operatorname{Spec} R$. If $f = f_1/f_2$, where $f_i \in R$, prove the result for f_i.)

Now I'd like to discuss the geometry of normal Noetherian schemes. Suppose R is an Noetherian integrally closed domain. Then it is *regular in codimension* 1 (translation: all its codimension at most 1 points are regular). If R is dimension 1, then obviously R is nonsingular=regular=smooth.

Example: $\operatorname{Spec} \mathbb{Z}[i]$ is smooth. Reason: it is dimension 1, and $\mathbb{Z}[i]$ is a unique factorization domain, hence its Spec is normal.

Remark: A (Noetherian) scheme can be singular in codimension 2 and still normal. Example: you have shown that the cone $x^2+y^2 = z^2$ in $\mathbb{A}^3$ is normal (PS4, problem B4), but it is clearly singular at the origin (the Zariski tangent space is visibly three-dimensional).

So integral (locally Noetherian) schemes can be singular in codimension 2. But their singularities turn out to be not so bad. I mentioned earlier, before we even knew what normal schemes were, that they satisfied "Hartogs Theorem", that you could extend functions over codimension 2 sets.

Remark: We know that for Noetherian rings we have inclusions:

$$\{\text{regular in codimension } 1\} \supset \{\text{integrally closed}\} \supset \{\text{unique factorization domain}\}.$$

Here are two examples to show you that these inclusions are strict.

Exercise. Let R be the subring $k[x^3, x^2, xy, y] \subset k[x, y]$. (The idea behind this example: I'm allowing all monomials in $k[x, y]$ except for x.) Show that it is not integrally closed (easy — consider the "missing x"). Show that it is regular in codimension 1 (hint: show it is dimension 2, and when you throw out the origin you get something nonsingular, by inverting x^2 and y respectively, and considering R_{x^2} and R_y).

Exercise. You have checked that if $k = \mathbb{C}$, then $k[w, x, y, z]/(wx - yz)$ is integrally closed (PS4, problem B5). Show that it is not a unique factorization domain. (The most obvious possibility is to do this "directly", but this might be hard. Another possibility, faster but less intuitive, is to prove the intermediate result that *in a unique factorization domain, any height* 1 *prime is principal*, and considering an exercise from earlier today that $w = z = 0$ is not principal.)

2. Good facts to know about regular local rings

There are some other important facts to know about regular local rings. In this class, I'm trying to avoid pulling any algebraic facts out of nowhere. As a rule of thumb, anything that you wouldn't see in Math 210, I consider "pulled out of nowhere". Even the harder facts from 210, I'm happy to give you a proof of, if you ask — none of those facts require more than a page of proof. To my knowledge, the only facts I've pulled out of nowhere to date are Krull's Principal Ideal Theorem, and its transcendence degree form. I might even type up a short proof of Krull's Theorem, and put it in the notes, so even that won't come out of nowhere.

Now, smoothness is an easy intuitive concept, but it is algebraically hard — harder than dimension. A sign of this is that I'm going to have to pull three facts out of nowhere. I think it's good for you to see these facts, but I'm going to try to avoid using these facts in the future. So consider them as mainly for culture.

Suppose $(R, \mathfrak{m})$ is a Noetherian regular local ring.

Fact 1. Any localization of R at a prime is also a regular local ring (Eisenbud's *Commutative Algebra*, Cor. 19.14, p. 479).

Hence to check if $\operatorname{Spec} R$ is nonsingular, then it suffices to check at closed points (at maximal ideals). For example, to check if $\mathbb{A}^3$ is nonsingular, you can check at all closed points, because all other points are obtained by localizing further. (You should think about this — it is confusing because of the order reversal between primes and closed subsets.)

Exercise. Show that on a Noetherian scheme, you can check nonsingularity by checking at closed points. (Caution: a scheme in general needn't have any closed points!) You are allowed to use the unproved fact from the notes, that any localization of a regular local ring is regular.

2.1. Less important exercise. Show that there is a nonsingular hypersurface of degree d. Show that there is a Zariski-open subset of the space of hypersurfaces of degree d. The two previous sentences combine to show that the nonsingular hypersurfaces form a Zariski-open set. Translation: almost all hypersurfaces are smooth.

Fact 2. ("leading terms", proved in an important case) The natural map $\mathrm{Sym}^n(\mathfrak{m}/\mathfrak{m}^2) \to \mathfrak{m}^n/\mathfrak{m}^{n+1}$ is an isomorphism. Even better, the following diagram commutes:

$$\begin{array}{ccc} (\mathfrak{m}^i/\mathfrak{m}^{i+1}) \times (\mathfrak{m}^j/\mathfrak{m}^{j+1}) & \xrightarrow{\times} & \mathfrak{m}^{i+j}/\mathfrak{m}^{i+j+1} \\ \downarrow \sim & & \downarrow \sim \\ \mathrm{Sym}^i(\mathfrak{m}/\mathfrak{m}^2) \times \mathrm{Sym}^j(\mathfrak{m}/\mathfrak{m}^2) & \xrightarrow{\times} & \mathrm{Sym}^{i+j}(\mathfrak{m}/\mathfrak{m}^2) \end{array}$$

Easy Exercise. Suppose $(R, \mathfrak{m}, k)$ is a regular Noetherian local ring of dimension n. Show that $\dim_k(\mathfrak{m}^i/\mathfrak{m}^{i+1}) = \binom{n+i-1}{i}$.

Exercise. Show that Fact 2 also implies that $(R, \mathfrak{m})$ is a domain. (Hint: show that if $f, g \neq 0$, then $fg \neq 0$, by considering the leading terms.)

I don't like facts pulled out of nowhere, so I want to prove it in an important case. Suppose $(R, \mathfrak{m})$ is a Noetherian local ring containing its residue field k: $k \hookrightarrow R \twoheadrightarrow R/\mathfrak{m} = k$. (For example, if k is algebraically closed, this is true for all local rings of finite type k-schemes at maximal ideals, by the Nullstellensatz. But it is not true if $(R, \mathfrak{m}) = (\mathbb{Z}_p, p\mathbb{Z}_p)$, as the residue field $\mathbb{F}_p$ is not a subring of $\mathbb{Z}_p$.)

Suppose R is a regular of dimension n, with $x_1, \ldots, x_n \in R$ generating $\mathfrak{m}/\mathfrak{m}^2$ as a vector space (and hence $\mathfrak{m}$ as an ideal, by Nakayama's lemma). Then we get a natural map $k[t_1, \ldots, t_n] \to R$, taking t_i to x_i.

2.2. *Theorem. — Suppose* $(R, \mathfrak{m})$ *is a Noetherian regular local ring containing its residue field* k*:* $k \hookrightarrow R \twoheadrightarrow R/\mathfrak{m} = k$. *Then* $k[t_1, \ldots, t_n]/(t_1, \ldots, t_n)^m \to R/\mathfrak{m}^m$ *is an isomorphism for all* m.

Proof: See Section 3.

To interpret this better, and to use it: define the inverse limit $\hat{R} := \lim_{\leftarrow} R/\mathfrak{m}^n$. This is the *completion* of R at $\mathfrak{m}$. (We can complete any ring at any ideal of course.) For example, if $S = k[x_1, \ldots, x_n]$, and $\mathfrak{n} = (x_1, \ldots, x_n)$, then $\hat{S} = k[[x_1, \ldots, x_n]]$, power series in n variables. We have a good intuition for for power series, so we will be very happy with the next result.

2.3. *Theorem. — Suppose* R *contains its residue field* k*:* $k \hookrightarrow R \twoheadrightarrow R/\mathfrak{m} = k$ *. Then the natural map* $k[[t_1, \dots, t_n]] \to \hat{R}$ *taking* t_i *to* x_i *is an isomorphism.*

This follows immediately from the previous theorem, as both sides are inverse limits of the same things. I'll now give some consequences.

Note that $R \hookrightarrow \hat{R}$. Here's why. (Recall the interpretation of inverse limit: you can interpret $\hat{R}$ as a subring of $R/\mathfrak{m} \times R/\mathfrak{m}^2 \times R/\mathfrak{m}^3 \times \cdots$ such that if $j > i$, the jth element maps to the ith factor under the natural quotient map.) What can go to 0 in $\hat{R}$? It is something that lies in $\mathfrak{m}^n$ for all n. But $\cap_i \mathfrak{m}^i = 0$ (a fact I stated in class when discussing Nakayama — *I owe you a proof of this*), so the map is injective. (Important note: We aren't assuming regularity of R in this argument!!)

Thus we can think of the map $R \to \hat{R}$ as a power series expansion.

This implies the "leading term" fact in this case (where the local ring contains the residue field). (**Exercise:** Prove this. This isn't hard; it's a matter of making sure you see what the definitions are.) Hence in this case we have proved that R is a domain.

We go back to stating important **facts** that we will try not to use.

Fact 3. Not only is $(R, \mathfrak{m})$ a domain, it is a unique factorization domain, which we have shown implies integrally closed in its fraction field. Reference: Eisenbud Theorem 19.19, p. 483. This implies that regular schemes are normal. Reason: integrally closed iff all local rings are integrally closed domains. I'll explain why later.

3. Promised proof of Theorem 2.2

Let's now set up the proof of Theorem 2.2, with a series of exercises.

3.1. Exercise. If S is a Noetherian ring, show that $S[[t]]$ is Noetherian. (Hint: Suppose $I \subset S[[t]]$ is an ideal. Let $I_n \subset S$ be the coefficients of t^n that appear in the elements of I form an ideal. Show that $I_n \subset I_{n+1}$, and that I is determined by $(I_0, I_1, I_2, \dots)$.)

3.2. Corollary. $k[[t_1, \dots, t_n]]$ is a Noetherian local ring.

3.3. Exercise. Show that $\dim k[[t_1, \dots, t_n]]$ is dimension n. (Hint: find a chain of $n+1$ prime ideals to show that the dimension is at least n. For the other inequality, use Krull.)

3.4. Exercise. If R is a Noetherian local ring, show that $\hat{R} := \lim_{\leftarrow} R/\mathfrak{m}^n$ is a Noetherian local ring. (Hint: Suppose $\mathfrak{m}/\mathfrak{m}^2$ is finite-dimensional over k, say generated by $x_1, \dots, x_n$. Describe a surjective map $k[[t_1, \dots, t_n]] \to \hat{R}$.)

We now outline the proof of the Theorem, as an extended exercise. (This is hastily and informally written.)

Suppose $\mathfrak{p} \subset R$ is a prime ideal. Define $\hat{\mathfrak{p}} \subset \hat{R}$ by $\mathfrak{p}/\mathfrak{m}^m \subset R/\mathfrak{m}^m$. Show that $\hat{\mathfrak{p}}$ is a prime ideal of $\hat{R}$. (Hint: if $f, g \notin \mathfrak{p}$, then let m_f, m_g be the first "level" where they are not in $\mathfrak{p}$ (i.e. the smallest m such that $f \notin \mathfrak{p}/\mathfrak{m}^{m+1}$). Show that $fg \notin \mathfrak{p}$ by showing that $fg \notin \mathfrak{p}/\mathfrak{m}^{m_f+m_g+1}$.)

Show that if $\mathfrak{p} \subset \mathfrak{q}$, then $\hat{\mathfrak{p}} \subset \hat{\mathfrak{q}}$. Hence show that $\dim \hat{R} \geq \dim R$. But also $\dim \hat{R} \leq \dim \mathfrak{m}/\mathfrak{m}^2 = \dim R$. Thus $\dim \hat{R} = \dim R$.

We're now ready to prove the Theorem. We wish to show that $k[[t_1, \dots, t_n]] \to \hat{R}$ is injective; we already know it is surjective. Suppose $f \in k[[t_1, \dots, t_n]] \mapsto 0$, so we get a map $k[[t_1, \dots, t_n]]/f$ surjects onto $\hat{R}$. Now f is not a zero-divisor, so by Krull, the left side has dimension $n-1$. But then any quotient of it has dimension at most $n-1$, contradiction. $\square$

4. Toward quasicoherent sheaves: the distinguished affine base

Schemes generalize and geometrize the notion of "ring". It is now time to define the corresponding analogue of "module", which is a quasicoherent sheaf.

One version of this notion is that of a sheaf of $\mathcal{O}_X$-modules. They form an abelian category, with tensor products. (That might be called a tensor category — I should check.)

We want a better one — a subcategory of $\mathcal{O}_X$-modules. Because these are the analogues of modules, we're going to define them in terms of affine open sets of the scheme. So let's think a bit harder about the structure of affine open sets on a general scheme X. I'm going to define what I'll call the *distinguished affine base* of the Zariski topology. This won't be a base in the sense that you're used to. (For experts: it is a first example of a *Grothendieck topology*.) It is more akin to a base.

The open sets are the affine open subsets of X. We've already observed that this forms a base. But forget about that.

We like distinguished opens $\operatorname{Spec} R_f \hookrightarrow \operatorname{Spec} R$, and we don't really understand open immersions of one random affine in another. So we just remember the "nice" inclusions.

Definition. The *distinguished affine base* of a scheme X is the data of the affine open sets and the distinguished inclusions.

(Remark we won't need, but is rather fundamental: what we are using here is that we have a collection of open subsets, and *some* subsets, such that if we have any $x \in U, V$ where U and V are in our collection of open sets, there is some W containing x, and contained in U and V such that that $W \hookrightarrow U$ and $W \hookrightarrow V$ are both in our collection of inclusions. In the case we are considering here, this is the key Proposition in Class 9 that given any two affine opens $\mathrm{Spec}\, A$, $\mathrm{Spec}\, B$ in X, $\mathrm{Spec}\, A \cap \mathrm{Spec}\, B$ could be covered by affine opens that were simultaneously distinguished in $\mathrm{Spec}\, A$ and $\mathrm{Spec}\, B$. This is a *cofinal* condition.)

We can define a sheaf on the distinguished affine base in the obvious way: we have a set (or abelian group, or ring) for each affine open set, and we know how to restrict to distinguished open sets.

Given a sheaf $\mathcal{F}$ on X, we get a sheaf on the distinguished affine base. You can guess where we're going: we'll show that all the information of the sheaf is contained in the information of the sheaf on the distinguished affine base.

As a warm-up: We can recover stalks. Here's why. $\mathcal{F}_x$ is the direct limit $\lim_{\rightarrow}(f \in \mathcal{F}(U))$ where the limit is over all open sets contained in U. We compare this to $\lim_{\rightarrow}(f \in \mathcal{F}(U))$ where the limit is over all affine open sets, and all distinguished inclusions. You can check that the elements of one correspond to elements of the other. (Think carefully about this! It corresponds to the fact that the basic elements are cofinal in this directed system.)

4.1. Exercise. Show that a section of a sheaf on the distinguished affine base is determined by the section's germs.

4.2. *Theorem. —*

(a) *A sheaf on the distinguished affine base $\mathcal{F}^b$ determines a unique sheaf $\mathcal{F}$, which when restricted to the affine base is $\mathcal{F}^b$. (Hence if you start with a sheaf, and take the sheaf on the distinguished affine base, and then take the induced sheaf, you get the sheaf you started with.)*

(b) *A morphism of sheaves on an affine base determines a morphism of sheaves.*

(c) *A sheaf of $\mathcal{O}_X$-modules "on the distinguished affine base" yields an $\mathcal{O}_X$-module.*

Proof of (a). (Two comments: this is very reminiscent of our sheafification argument. It also trumps our earlier theorem on sheaves on a nice base.)

Suppose $\mathcal{F}^b$ is a sheaf on the distinguished affine base. Then we can define stalks.

For *any* open set U of X, define

$$\mathcal{F}(U) := \{(f_x \in \mathcal{F}^b_x)_{x \in U} : \forall x \in U, \exists U_X \text{ with } x \subset U_x \subset U, F^x \in \mathcal{F}^b(U_x) : F^x_y = f_y \forall y \in U_x\}$$

where each U_x is in our base, and F^x_y means "the germ of F^x at y". (As usual, those who want to worry about the empty set are welcome to.)

This is a sheaf: convince yourself that we have restriction maps, identity, and gluability, really quite easily.

I next claim that if U is in our base, that $\mathcal{F}(U) = \mathcal{F}^b(U)$. We clearly have a map $\mathcal{F}^b(U) \rightarrow \mathcal{F}(U)$. For the map $\mathcal{F}(U) \rightarrow \mathcal{F}^b(U)$: **gluability exercise** (a bit subtle).

These are isomorphisms, because elements of $\mathcal{F}(U)$ are determined by stalks, as are elements of $\mathcal{F}^b(U)$.

(b) Follows as before.

(c) **Exercise.** □

5. Quasicoherent sheaves

We now define a **quasicoherent sheaf**. In the same way that a scheme is defined by "gluing together rings", a quasicoherent sheaf over that scheme is obtained by "gluing together modules over those rings". We will give two equivalent definitions; each definition is useful in different circumstances. The first just involves the distinguished topology.

Definition 1. An $\mathcal{O}_X$-module $\mathcal{F}$ is a **quasicoherent sheaf** if for every affine open $\operatorname{Spec} R$ and distinguished affine open $\operatorname{Spec} R_f$ thereof, the restriction map $\phi : \Gamma(\operatorname{Spec} R, \mathcal{F}) \to \Gamma(\operatorname{Spec} R_f, \mathcal{F})$ factors as:

$$\phi : \Gamma(\operatorname{Spec} R, \mathcal{F}) \to \Gamma(\operatorname{Spec} R, \mathcal{F})_f \cong \Gamma(\operatorname{Spec} R_f, \mathcal{F}).$$

The second definition is more directly "sheafy". Given a ring R and a module M, we defined a sheaf $\tilde{M}$ on $\operatorname{Spec} R$ long ago — the sections over $D(f)$ were M_f.

Definition 2. An $\mathcal{O}_X$-module $\mathcal{F}$ is a *quasicoherent sheaf* if for every affine open $\operatorname{Spec} R$,

$$\mathcal{F}|_{\operatorname{Spec} R} \cong \Gamma(\widetilde{\operatorname{Spec} R}, \mathcal{F}).$$

(The "wide tilde" is suposed to cover the entire right side $\Gamma(\operatorname{Spec} R, \mathcal{F})$.) This isomorphism is as sheaves of $\mathcal{O}_X$-modules.

Hence by this definition, the sheaves on $\operatorname{Spec} R$ correspond to R-modules. Given an R-module M, we get a sheaf $\tilde{M}$. Given a sheaf $\mathcal{F}$ on $\operatorname{Spec} R$, we get an R-module $\Gamma(X, \mathcal{F})$. These operations are inverse to each other. So in the same way as schemes are obtained by gluing together rings, quasicoherent sheaves are obtained by gluing together modules over those rings.

By Theorem 4.2, we have:

Definition 2'. An $\mathcal{O}_X$-module on the distinguished affine base yields an $\mathcal{O}_X$-module.

5.1. *Proposition. — Definitions 1 and 2 are the same.*

Proof. Clearly Definition 2 implies Definition 1. (Recall that the definition of $\tilde{M}$ was in terms of the distinguished topology on $\operatorname{Spec} R$.) We now show that Definition 1 implies Definition 2. We use Theorem 4.2. By Definition 1, the sections over any distinguished open $\operatorname{Spec} R_f$ of $\mathcal{M}$ on $\operatorname{Spec} R$ is precisely $\Gamma(\operatorname{Spec} R, \mathcal{M})_f$, i.e. the sections of $\Gamma(\widetilde{\operatorname{Spec} R}, \mathcal{M})$ over $\operatorname{Spec} R_f$, and the restriction maps agree. Thus the two sheaves agree. □

We like Definition 1 because it says that to define a quasicoherent sheaf of $\mathcal{O}_X$-modules is that we just need to know what it is on all affine open sets, and that it behaves well under inverting single elements.

One reason we like Definition 2 is that it glues well.

5.2. *Proposition (quasicoherence is affine-local). — Let* X *be a scheme, and* $\mathcal{M}$ *a sheaf of* $\mathcal{O}_X$*-modules. Then let* P *be the property of affine open sets that* $\mathcal{M}|_{\operatorname{Spec} R} \cong \Gamma(\widetilde{\operatorname{Spec} R, \mathcal{M}})$*. Then* P *is an affine-local property.*

We will prove this next day.

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FOUNDATIONS OF ALGEBRAIC GEOMETRY CLASS 14

CONTENTS

Last day: discrete valuation rings (conclusion), cultural facts to know about regular local rings, the distinguished affine base of the topology, 2 definitions of quasicoherent sheaf.

Today: quasicoherence is affine-local, (locally) free sheaves and vector bundles, invertible sheaves and line bundles, torsion-free sheaves, quasicoherent sheaves of ideals and closed subschemes.

Last day, we defined the distinguished affine base of the Zariski topology of a scheme.

We showed that the information contained in a sheaf was precisely the information contained in a sheaf on the distinguished affine base.

0.1. *Theorem. —*

(a) *A sheaf on the distinguished affine base $\mathcal{F}^b$ determines a unique sheaf $\mathcal{F}$, which when restricted to the affine base is $\mathcal{F}^b$. (Hence if you start with a sheaf, and take the sheaf on the distinguished affine base, and then take the induced sheaf, you get the sheaf you started with.)*
(b) *A morphism of sheaves on an affine base determines a morphism of sheaves.*
(c) *A sheaf of $\mathcal{O}_X$-modules "on the distinguished affine base" yields an $\mathcal{O}_X$-module.*

We then gave two definitions of quasicoherent sheaves.

Definition 1. An $\mathcal{O}_X$-module $\mathcal{F}$ is a **quasicoherent sheaf** if for every affine open $\operatorname{Spec} R$ and distinguished affine open $\operatorname{Spec} R_f$ thereof, the restriction map $\phi : \Gamma(\operatorname{Spec} R, \mathcal{F}) \to$

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$\Gamma(\operatorname{Spec} R_f, \mathcal{F})$ factors as:

$$\phi : \Gamma(\operatorname{Spec} R, \mathcal{F}) \to \Gamma(\operatorname{Spec} R, \mathcal{F})_f \cong \Gamma(\operatorname{Spec} R_f, \mathcal{F}).$$

Definition 2. An $\mathcal{O}_X$-module $\mathcal{F}$ is a *quasicoherent sheaf* if for every affine open $\operatorname{Spec} R$,

$$\mathcal{F}|_{\operatorname{Spec} R} \cong \Gamma(\widetilde{\operatorname{Spec} R, \mathcal{F}}).$$

This isomorphism is as sheaves of $\mathcal{O}_X$-modules.

By part (c) of the above Theorem, an $\mathcal{O}_X$-module on the distinguished affine base yields an $\mathcal{O}_X$-module, so these two notions are equivalent. Thus to give a quasicoherent sheaf, I just need to give you a module for each affine open, and have them behave well with respect to restriction. (That's a priori a little weaker than definition 2, where we actually need an $\mathcal{O}_X$-module.)

Last time I proved:

0.2. *Proposition. — Definitions 1 and 2 are the same.*

1. ONWARDS!

1.1. *Proposition (quasicoherence is affine-local). — Let X be a scheme, and $\mathcal{F}$ a sheaf of $\mathcal{O}_X$-modules. Then let P be the property of affine open sets that $\mathcal{F}|_{\operatorname{Spec} R} \cong \Gamma(\widetilde{\operatorname{Spec} R, \mathcal{F}})$. Then P is an affine-local property.*

Proof. By the Affine Communication Lemma , we must check two things. Clearly if $\operatorname{Spec} R$ has property P, then so does the distinguished open $\operatorname{Spec} R_f$: if M is an R-module, then $\tilde{M}|_{\operatorname{Spec} R_f} \cong \tilde{M}_f$ as sheaves of $\mathcal{O}_{\operatorname{Spec} R_f}$-modules (both sides agree on the level of distinguished opens and their restriction maps).

We next show the second hypothesis of the Affine Communication Lemma. Suppose we have modules $M_1, \ldots, M_n$, where M_i is an R_{f_i}-module, along with isomorphisms $\phi_{ij} : (M_i)_{f_j} \to (M_j)_{f_i}$ of $R_{f_i f_j}$-modules ($i \neq j$; where $\phi_{ij} = \phi_{ji}^{-1}$). We want to construct an M such that $\tilde{M}$ gives us $\tilde{M}_i$ on $D(f_i) = \operatorname{Spec} R_{f_i}$, or equivalently, isomorphisms $\Gamma(D(f_i), \tilde{M}) \cong M_i$, with restriction maps

$$\begin{array}{ccc} \Gamma(D(f_i), \tilde{M}) & & \Gamma(D(f_j), \tilde{M}) \\ \downarrow & & \downarrow \\ \Gamma(D(f_i), \tilde{M})_{f_j} & \xleftrightarrow{\cong} & \Gamma(D(f_j), \tilde{M})_{f_i} \end{array}$$

that agree with ϕ_{ij}.

We already know what M should be. Consider elements of $M_1 \times \cdots \times M_n$ that "agree on overlaps"; let this set be M. Then

$$0 \to M \to M_1 \times \cdots \times M_n \to M_{12} \times M_{13} \times \cdots \times M_{(n-1)n}$$

is an exact sequence (where $M_{ij} = (M_i)_{f_j} \cong (M_j)_{f_i}$, and the latter morphism is the "difference" morphism). So M is a kernel of a morphism of R-modules, hence an R-module. We show that $M_i \cong M_{f_i}$; for convenience we assume $i = 1$. Localization is exact, so

$$(1) \quad 0 \to M_{f_1} \to M_1 \times (M_2)_{f_1} \times \cdots \times (M_n)_{f_1} \to M_{12} \times \cdots \times (M_{23})_{f_1} \times \cdots \times (M_{(n-1)n})_{f_1}$$

Then by interpreting this exact sequence, you can verify that the kernel is M_1. I gave one proof in class, and I'd like to give two proofs here. We know that $\cup_{i=2}^n D(f_1 f_i)$ is a distinguished cover of $D(f_1) = \operatorname{Spec} R_1$. So we have an exact sequence

$$0 \to M_1 \to (M_1)_{f_2} \times \cdots \times (M_1)_{f_n} \to (M_1)_{f_2 f_3} \times \cdots \times (M_1)_{f_{n-1} f_n}.$$

Put two copies on top of each other, and add vertical isomorphisms, alternating between identity and the negative of the identity:

$$\begin{array}{ccccccc} 0 & \longrightarrow & M_1 & \longrightarrow & (M_1)_{f_2} \times \cdots \times (M_1)_{f_n} & \longrightarrow & (M_1)_{f_2 f_3} \times \cdots \times (M_1)_{f_{n-1} f_n} \\ & & \downarrow \mathrm{id} & & \downarrow -\mathrm{id} & & \\ 0 & \longrightarrow & M_1 & \longrightarrow & (M_1)_{f_2} \times \cdots \times (M_1)_{f_n} & \longrightarrow & (M_1)_{f_2 f_3} \times \cdots \times (M_1)_{f_{n-1} f_n} \end{array}$$

Then the *total complex* of this *double complex* is exact as well (**exercise**). (The total complex is obtained as follows. The terms are obtained by taking the direct sum in each southwest-to-northeast diagonal. This is a baby case of something essential so check it, if you've never seen it before!). But this is the same sequence as (1), except M_{f_1} replaces M_1, so we have our desired isomorphism.

Here is a second proof that the sequence

$$(2) \quad 0 \to M_1 \to M_1 \times (M_2)_{f_1} \times \cdots \times (M_n)_{f_1} \to M_{12} \times \cdots \times (M_{23})_{f_1} \times \cdots \times (M_{(n-1)n})_{f_1}$$

is exact. To check exactness of a complex of R-modules, it suffices to check exactness "at each prime $\mathfrak{p}$". In other words, if a complex is exact once tensored with $R_\mathfrak{p}$ for all $\mathfrak{p}$, then it was exact to begin with. Now note that if N is an R-module, then $(N_{f_i})_\mathfrak{p}$ is 0 if $f_i \in \mathfrak{p}$, and $N_\mathfrak{p}$ otherwise. Hence after tensoring with $R_\mathfrak{p}$, each term in (2) is either 0 or $N_\mathfrak{p}$, and the reader will quickly verify that the resulting complex is exact. (If any reader thinks I should say a few words as to why this is true, they should let me know, and I'll add a bit to these notes. I'm beginning to think that I should re-work some of my earlier arguments, including for example base gluability and base identity of the structure sheaf, in this way.) □

At this point, you probably want an example. I'll give you a boring example, and save a more interesting one for the end of the class.

Example: $\mathcal{O}_X$ is a quasicoherent sheaf. Over each affine open $\operatorname{Spec} R$, it is isomorphic the module $M = R$. This is not yet enough to specify what the sheaf is! We need also to describe the distinguished restriction maps, which are given by $R \to R_f$, where these are the "natural" ones. (This is confusing because this sheaf is too simple!) A variation on this theme is $\mathcal{O}_X^{\oplus n}$ (interpreted in the obvious way). This is called a *rank* n *free sheaf*. It corresponds to a rank n trivial vector bundle.

Joe mentioned an example of an $\mathcal{O}_X$-module that is not a quasicoherent sheaf last day.

1.2. Exercise. (a) Suppose $X = \operatorname{Spec} k[t]$. Let $\mathcal{F}$ be the skyscraper sheaf supported at the origin $[(t)]$, with group $k(t)$. Give this the structure of an $\mathcal{O}_X$-module. Show that this is not a quasicoherent sheaf. (More generally, if X is an integral scheme, and $p \in X$ that is not the generic point, we could take the skyscraper sheaf at p with group the function field of X. Except in a silly circumstances, this sheaf won't be quasicoherent.)
(b) Suppose $X = \operatorname{Spec} k[t]$. Let $\mathcal{F}$ be the skyscraper sheaf supported at the generic point $[(0)]$, with group $k(t)$. Give this the structure of an $\mathcal{O}_X$-module. Show that this *is* a quasicoherent sheaf. Describe the restriction maps in the distinguished topology of X. (Joe remarked that this is a constant sheaf!)

1.3. *Important Exercise for later.* Suppose X is a Noetherian scheme. Suppose $\mathcal{F}$ is a quasicoherent sheaf on X, and let $f \in \Gamma(X, \mathcal{O}_X)$ be a function on X. Let $R = \Gamma(X, \mathcal{O}_X)$ for convenience. Show that the restriction map $\operatorname{res}_{X_f \subset X} : \Gamma(X, \mathcal{F}) \to \Gamma(X_f, \mathcal{F})$ (here X_f is the open subset of X where f doesn't vanish) is precisely localization. In other words show that there is an isomorphism $\Gamma(X, \mathcal{F})_f \to \Gamma(X_f, \mathcal{F})$ making the following diagram commute.

$$\begin{array}{ccc} \Gamma(X, \mathcal{F}) & \xrightarrow{\operatorname{res}_{X_f \subset X}} & \Gamma(X_f, \mathcal{F}) \\ & {\scriptstyle \otimes_R R_f} \searrow \quad \nearrow {\scriptstyle \sim} & \\ & \Gamma(X, \mathcal{F})_f & \end{array}$$

All that you should need in your argument is that X admits a cover by a finite number of open sets, and that their pairwise intersections are each quasicompact. We will later rephrase this as saying that X is quasicompact and quasiseparated. (Hint: cover by affine open sets. Use the sheaf property. A nice way to formalize this is the following. Apply the exact functor $\otimes_R R_f$ to the exact sequence

$$0 \to \Gamma(X, \mathcal{F}) \to \oplus_i \Gamma(U_i, \mathcal{F}) \to \oplus \Gamma(U_{ijk}, \mathcal{F})$$

where the U_i form a finite cover of X and U_{ijk} form an affine cover of $U_i \cap U_j$.)

1.4. *Less important exercise.* Give a counterexample to show that the above statement need not hold if X is not quasicompact. (Possible hint: take an infinite disjoint union of affine schemes.)

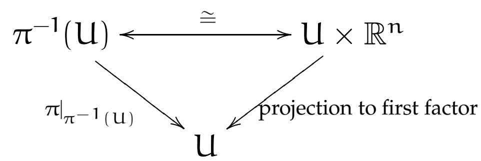
For the experts: I don't know a counterexample to this when the quasiseparated hypothesis is removed. Using the exact sequence above, I can show that there is a map $\Gamma(X_f, \mathcal{F}) \to \Gamma(X, \mathcal{F})_f$.

2. Locally free sheaves

I want to show you how that quasicoherent sheaves somehow generalize the notion of vector bundles.

(For arithmetic people: don't tune out! Fractional ideals of the ring of integers in a number field will turn out to be an example of a "line bundle on a smooth curve".)

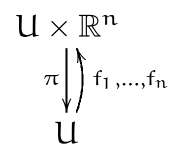
Since this is motivation, I won't make this precise, so you should feel free to think of this in the differentiable category (i.e. the category of differentiable manifolds). A rank n vector bundle on a manifold M is a fibration $\pi : V \to M$ that locally looks like the product with n-space: every point of M has a neighborhood U such that $\pi^{-1}(U) \cong U \times \mathbb{R}^n$, where the projection map is the obvious one, i.e. the following diagram commutes.



This is called a *trivialization over* U. We also want a "consistent vector space structure". Thus given trivializations over U_1 and U_2, over their intersection, the two trivializations should be related by an element of $GL(n)$ with entries consisting of functions on $U_1 \cap U_2$.

Examples of this include for example the tangent bundle on a sphere, and the moebius strip over $\mathbb{R}^1$.

Pick your favorite vector bundle, and consider its sheaf of sections $\mathcal{F}$. Then the sections over any open set form a real vector space. Moreover, given a U and a trivialization, the sections are naturally n-tuples of functions of U. [If I can figure out how to do curly arrows in xymatrix, I'll fix this.]



The open sets over which V is trivial forms a nice base of the topology.

Motivated by this, we define a *locally free sheaf of rank* n on a scheme X as follows. It is a quasicoherent sheaf that is locally, well, free of rank n. It corresponds to a vector bundle. It is determined by the following data: a cover U_i of X, and for each i, j transition functions T_{ij} lying in $GL(n, \Gamma(U_i \cap U_j, \mathcal{O}_X))$ satisfying

$$T_{ii} = Id_n, T_{ij}T_{jk} = T_{ik}$$

(which implies $T_{ij} = T_{ji}^{-1}$). Given this data, we can find the sections over any open set U as follows. Informally, they are sections of the free sheaves over each $U \cap U_i$ that agree on overlaps. More formally, for each i, they are $\vec{s}^i = \begin{pmatrix} s_1^i \\ \vdots \\ s_n^i \end{pmatrix} \in \Gamma(U \cap U_i, \mathcal{O}_X)^n$, satisfying $T_{ij}\vec{s}^i = \vec{s}^j$ on $U \cap U_i \cap U_j$.

In the differentiable category, locally free sheaves correspond precisely to vector bundles (for example, you can describe them with the same transition functions). So you should really think of these "as" vector bundles, but just keep in mind that they are not the "same", just equivalent notions.

A rank 1 vector bundle is called a *line bundle*. Similarly, a rank 1 locally free sheaf is called an *invertible sheaf*. I'll later explain why it is called invertible; but it is still a somewhat heinous term for something so fundamental.

Caution: Not every quasicoherent sheaf is locally free.

In a few sections, we will define some operations on quasicoherent sheaves that generate natural operations on vector bundles (such as dual, Hom, tensor product, etc.). The constructions will behave particularly well for locally free sheaves. We will see that the invertible sheaves on X will form a group under tensor product, called the *Picard group* of X.

We first make precise our discussion of transition functions. Given a rank n locally free sheaf $\mathcal{F}$ on a scheme X, we get transition functions as follows. Choose an open cover U_i of X so that $\mathcal{F}$ is a free rank n sheaf on each U_i. Choose a basis $e_{i,1}, \dots, e_{i,n}$ of $\mathcal{F}$ over U_i. Then over $U_i \cap U_j$, for each k, $e_{i,k}$ can be written as a $\Gamma(U_i \cap U_j, \mathcal{O}_X)$-linear combination of the $e_{j,l}$ ($1 \leq l \leq n$), so we get an $n \times n$ "transition matrix" T_{ji} with entries in $\Gamma(U_i \cap U_j, \mathcal{O}_X)$. Similarly, we get T_{ij}, and $T_{ij}T_{ji} = T_{ji}T_{ij} = I_n$, so T_{ij} and T_{ji} are invertible. Also, on $U_i \cap U_j \cap U_k$, we readily have $T_{ik} = T_{ij}T_{jk}$: both give the matrix that expresses the basis vectors of $e_{i,q}$ in terms of $e_{k,q}$. [Make sure this is right!]

2.1. Exercise. Conversely, given transition functions $T_{ij} \in GL(n, \Gamma(U_i \cap U_j, \mathcal{O}_X))$ satisfying the cocycle condition $T_{ij}T_{jk} = T_{ik}$ "on $U_i \cap U_j \cap U_k$", describe the corresponding rank n locally free sheaf.

We end this section with a few stray comments.

Caution: there are new morphisms between locally free sheaves, compared with what people usually say for vector bundles. Give example on $\mathbb{A}^1$:

$$0 \to tk[t] \to k[t] \to k[t]/(t) \to 0.$$

For vector bundle people: the thing on the left isn't a morphism of vector bundles (at least according to some definitions). (If you think it is a morphism of vector bundles, then you should still be disturbed, because its cokernel is not a vector bundle!)

2.2. Remark. Based on your intuition for line bundles on manifolds, you might hope that every point has a "small" open neighborhood on which all invertible sheaves (or locally free sheaves) are trivial. Sadly, this is not the case. We will eventually see that for the curve $y^2 - x^3 - x = 0$ in $\mathbb{A}^2_{\mathbb{C}}$, every nonempty open set has nontrivial invertible sheaves. (This will use the fact that it is an open subset of an *elliptic curve*.)

2.3. Exercise (for arithmetically-minded people only — I won't define my terms). Prove that a fractional ideal on a ring of integers in a number field yields an invertible sheaf. Show that any two that differ by a principal ideal yield the same invertible sheaf.

Thus we have described a map from the class group of the number field to the Picard group of its ring of integers. It turns out that this is an isomorphism. So strangely the number theorists in this class are the first to have an example of a nontrivial line bundle.

2.4. Exercise (for those familiar with Hartogs' Theorem for Noetherian normal schemes). Show that locally free sheaves on Noetherian normal schemes satisfy "Hartogs' theorem": sections defined away from a set of codimension at least 2 extend over that set.

3. Quasicoherent sheaves form an abelian category

The category of R-modules is an abelian category. (Indeed, this is our motivating example of our notion of abelian category.) Similarly, quasicoherent sheaves form an abelian category. I'll explain how.

When you show that something is an abelian category, you have to check many things, because the definition has many parts. However, if the objects you are considering lie in some ambient abelian category, then it is much easier. As a metaphor, there are several things you have to do to check that something is a group. But if you have a subset of group elements, it is much easier to check that it is a subgroup.

You can look at back at the definition of an abelian category, and you'll see that in order to check that a subcategory is an abelian subcategory, you need to check only the following things:

(i) 0 is in your subcategory
(ii) your subcategory is closed under finite sums
(iii) your subcategory is closed under kernels and cokernels

In our case of {quasicoherent sheaves} $\subset$ {$\mathcal{O}_X$-modules}, the first two are cheap: 0 is certainly quasicoherent, and the subcategory is closed under finite sums: if $\mathcal{F}$ and $\mathcal{G}$ are sheaves on X, and over $\operatorname{Spec} R$, $\mathcal{F} \cong \tilde{M}$ and $\mathcal{G} \cong \tilde{N}$, then $\mathcal{F} \oplus \mathcal{G} = \widetilde{M \oplus N}$, so $\mathcal{F} \oplus \mathcal{G}$ is a quasicoherent sheaf.

We now check (iii). Suppose $\alpha : \mathcal{F} \to \mathcal{G}$ is a morphism of quasicoherent sheaves. Then on any affine open set U, where the morphism is given by $\beta : M \to N$, define $(\ker \alpha)(U) = \ker \beta$ and $(\operatorname{coker} \alpha)(U) = \operatorname{coker} \beta$. Then these behave well under inversion of a single element: if

$$0 \to K \to M \to N \to P \to 0$$

is exact, then so is

$$0 \to K_f \to M_f \to N_f \to P_f \to 0,$$

from which $(\ker \beta)_f \cong \ker(\beta_f)$ and $(\operatorname{coker} \beta)_f \cong \operatorname{coker}(\beta_f)$. Thus both of these define quasicoherent sheaves. Moreover, by checking stalks, they are indeed the kernel and cokernel of α. Thus the quasicoherent sheaves indeed form an abelian category.

As a side benefit, we see that we may check injectivity, surjectivity, or exactness of a morphism of quasicoherent sheaves by checking on an affine cover.

Warning: If $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$ is an exact sequence of quasicoherent sheaves, then for any open set

$$0 \to \mathcal{F}(U) \to \mathcal{G}(U) \to \mathcal{H}(U)$$

is exact, and we have exactness on the right is guaranteed to hold only if U is affine. (To set you up for cohomology: whenever you see left-exactness, you expect to eventually interpret this as a start of a long exact sequence. So we are expecting H^1's on the right, and now we expect that $H^1(\operatorname{Spec} R, \mathcal{F}) = 0$. This will indeed be the case.)

3.1. *Exercise.* Show that you can check exactness of a sequence of quasicoherent sheaves on an affine cover. (In particular, taking sections over an affine open $\operatorname{Spec} R$ is an exact functor from the category of quasicoherent sheaves on X to the category of R-modules. Recall that taking sections is only left-exact in general.) Similarly, you can check surjectivity on an affine cover (unlike sheaves in general).

4. Module-like constructions on quasicoherent sheaves

In a similar way, basically any nice construction involving modules extends to quasicoherent sheaves.

As an important example, we consider tensor products. **Exercise.** If $\mathcal{F}$ and $\mathcal{G}$ are quasicoherent sheaves, show that $\mathcal{F} \otimes \mathcal{G}$ is given by the following information: If $\operatorname{Spec} R$ is an affine open, and $\Gamma(\operatorname{Spec} R, \mathcal{F}) = M$ and $\Gamma(\operatorname{Spec} R, \mathcal{G}) = N$, then $\Gamma(\operatorname{Spec} R, \mathcal{F} \otimes \mathcal{G}) = M \otimes N$, and the restriction map $\Gamma(\operatorname{Spec} R, \mathcal{F} \otimes \mathcal{G}) \to \Gamma(\operatorname{Spec} R_f, \mathcal{F} \otimes \mathcal{G})$ is precisely the localization map $M \otimes_R N \to (M \otimes_R N)_f \cong M_f \otimes_{R_f} N_f$. (We are using the algebraic fact that that $(M \otimes_R N)_f \cong M_f \otimes_{R_f} N_f$. You can prove this by universal property if you want, or by using the explicit construction.)

Note that thanks to the machinery behind the distinguished affine base, sheafification is taken care of.

For category-lovers: this makes the category of quasicoherent sheaves into a monoid.

4.1. Exercise. If $\mathcal{F}$ and $\mathcal{G}$ are locally free sheaves, show that $\mathcal{F} \otimes \mathcal{G}$ is locally free. (Possible hint for this, and later exercises: check on sufficiently small affine open sets.)

4.2. *Exercise.* (a) Tensoring by a quasicoherent sheaf is right-exact. More precisely, if $\mathcal{F}$ is a quasicoherent sheaf, and $\mathcal{G}' \to \mathcal{G} \to \mathcal{G}'' \to 0$ is an exact sequence of quasicoherent sheaves, then so is $\mathcal{G}' \otimes \mathcal{F} \to \mathcal{G} \otimes \mathcal{F} \to \mathcal{G}'' \otimes \mathcal{F} \to 0$ is exact.
(b) Tensoring by a locally free sheaf is exact. More precisely, if $\mathcal{F}$ is a locally free sheaf, and $\mathcal{G}' \to \mathcal{G} \to \mathcal{G}''$ is an exact sequence of quasicoherent sheaves, then then so is $\mathcal{G}' \otimes \mathcal{F} \to \mathcal{G} \otimes \mathcal{F} \to \mathcal{G}'' \otimes \mathcal{F}$.
(c) The stalk of the tensor product of quasicoherent sheaves at a point is the tensor product of the stalks.

Note: if you have a section s of $\mathcal{F}$ and a section t of $\mathcal{G}$, you get a section $s \otimes t$ of $\mathcal{F} \otimes \mathcal{G}$. If either $\mathcal{F}$ or $\mathcal{G}$ is an invertible sheaf, this section is denoted st.

We now describe other constructions.

4.3. *Exercise.* Sheaf Hom, $\underline{\mathrm{Hom}}$, is quasicoherent, and is what you think it might be. (Describe it on affine opens, and show that it behaves well with respect to localization with respect to f. To show that $\mathrm{Hom}_A(M, N)_f \cong \mathrm{Hom}_{A_f}(M_f, N_f)$, take a "partial resolution" $A^q \to A^p \to M \to 0$, and apply $\mathrm{Hom}(\cdot, N)$ and localize.) ($\underline{\mathrm{Hom}}$ was defined earlier, and was the subject of a homework problem.) Show that $\underline{\mathrm{Hom}}$ is a left-exact functor in both variables.

Definition. $\underline{\mathrm{Hom}}(\mathcal{F}, \mathcal{O}_X)$ is called the *dual* of $\mathcal{F}$, and is denoted $\mathcal{F}^\vee$.

4.4. *Exercise.* The direct sum of quasicoherent sheaves is what you think it is.

5. Some notions especially relevant for locally free sheaves

Exercise. Show that if $\mathcal{F}$ is locally free then $\mathcal{F}^\vee$ is locally free, and that there is a canonical isomorphism $(\mathcal{F}^\vee)^\vee \cong \mathcal{F}$. (Caution: your argument showing that if there is a canonical isomorphism $(\mathcal{F}^\vee)^\vee \cong \mathcal{F}$ better not also show that there is a canonical isomorphism $\mathcal{F}^\vee \cong \mathcal{F}$! We'll see an example soon of a locally free $\mathcal{F}$ that is not isomorphic to its dual. The example will be the line bundle $\mathcal{O}(1)$ on $\mathbb{P}^1$.)

Remark. This is not true for quasicoherent sheaves in general, although your argument will imply that there is always a natural morphism $\mathcal{F} \to (\mathcal{F}^\vee)^\vee$. Quasicoherent sheaves for which this is true are called *reflexive sheaves*. We will not be using this notion. Your argument may also lead to a canonical map $\mathcal{F} \otimes \mathcal{F}^\vee \to \mathcal{O}_X$. This could be called the *trace* map — can you see why?

5.1. *Exercise.* Given transition functions for the locally free sheaf $\mathcal{F}$, describe the transition functions for the locally free sheaf $\mathcal{F}^\vee$. Note that if $\mathcal{F}$ is rank 1 (i.e. locally free), the transition functions of the dual are the inverse of the transition functions of the original; in this case, $\mathcal{F} \otimes \mathcal{F}^\vee \cong \mathcal{O}_X$.

5.2. *Exercise.* If $\mathcal{F}$ and $\mathcal{G}$ are locally free sheaves, show that $\mathcal{F} \otimes \mathcal{G}$ and $\underline{\mathrm{Hom}}(\mathcal{F}, \mathcal{G})$ are both locally free.

5.3. *Exercise.* Show that the invertible sheaves on X, up to isomorphism, form an abelian group under tensor product. This is called the *Picard group* of X, and is denoted $\mathrm{Pic}\, X$. (For arithmetic people: this group, for the Spec of the ring of integers R in a number field, is the class group of R.)

For the next exercises, recall the following. If M is an A-module, then the *tensor algebra* $T^*(M)$ is a non-commutative algebra, graded by $\mathbb{Z}^{\geq 0}$, defined as follows. $T^0(M) = A$, $T^n(M) = M \otimes_A \cdots \otimes_A M$ (where n terms appear in the product), and multiplication is what you expect. The *symmetric algebra* $\operatorname{Sym}^* M$ is a symmetric algebra, graded by $\mathbb{Z}^{\geq 0}$, defined as the quotient of $T^*(M)$ by the (two-sided) ideal generated by all elements of the form $x \otimes y - y \otimes x$ for all $x, y \in M$. Thus $\operatorname{Sym}^n M$ is the quotient of $M \otimes \cdots \otimes M$ by the relations of the form $m_1 \otimes \cdots \otimes m_n - m_1' \otimes \cdots \otimes m_n'$ where $(m_1', \dots, m_n')$ is a rearrangement of $(m_1, \dots, m_n)$. The *exterior algebra* $\wedge^* M$ is defined to be the quotient of $T^* M$ by the (two-sided) ideal generated by all elements of the form $x \otimes y + y \otimes x$ for all $x, y \in M$. Thus $\wedge^n M$ is the quotient of $M \otimes \cdots \otimes M$ by the relations of the form $m_1 \otimes \cdots \otimes m_n - (-1)^{\operatorname{sgn}} m_1' \otimes \cdots \otimes m_n'$ where $(m_1', \dots, m_n')$ is a rearrangement of $(m_1, \dots, m_n)$, and the sgn is even if the rearrangement is an even permutation, and odd if the rearrangement is an odd permutation. (It is a "skew-commutative" A-algebra.) It is most correct to write $T_A^*(M)$, $\operatorname{Sym}_A^*(M)$, and $\wedge_A^*(M)$, but the "base ring" is usually omitted for convenience.

5.4. *Exercise.* If $\mathcal{F}$ is a quasicoherent sheaf, then define the quasicoherent sheaves $T^n \mathcal{F}$, $\operatorname{Sym}^n \mathcal{F}$, and $\wedge^n \mathcal{F}$. If $\mathcal{F}$ is locally free of rank m, show that $T^n \mathcal{F}$, $\operatorname{Sym}^n \mathcal{F}$, and $\wedge^n \mathcal{F}$ are locally free, and find their ranks.

You can also define the sheaf of non-commutative algebras $T^* \mathcal{F}$, the sheaf of algebras $\operatorname{Sym}^* \mathcal{F}$, and the sheaf of skew-commutative algebras $\wedge^* \mathcal{F}$.

5.5. Important exercise. If $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ is an exact sequence of locally free sheaves, then for any r, there is a filtration of $\operatorname{Sym}^r \mathcal{F}$:

$$\operatorname{Sym}^r \mathcal{F} = F^0 \supseteq F^1 \supseteq \cdots \supseteq F^r \supset F^{r+1} = 0$$

with quotients

$$F^p / F^{p+1} \cong (\operatorname{Sym}^p \mathcal{F}') \otimes (\operatorname{Sym}^{r-p} \mathcal{F}'')$$

for each p.

5.6. Exercise. Suppose $\mathcal{F}$ is locally free of rank n. Then $\wedge^n \mathcal{F}$ is called the *determinant (line) bundle*. Show that $\wedge^r \mathcal{F} \times \wedge^{n-r} \mathcal{F} \to \wedge^n \mathcal{F}$ is a perfect pairing for all r.

5.7. Exercise. If $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ is an exact sequence of locally free sheaves, then for any r, there is a filtration of $\wedge^r \mathcal{F}$:

$$\wedge^r \mathcal{F} = F^0 \supseteq F^1 \supseteq \cdots \supseteq F^r \supset F^{r+1} = 0$$

with quotients

$$F^p / F^{p+1} \cong (\wedge^p \mathcal{F}') \otimes (\wedge^{r-p} \mathcal{F}'')$$

for each p. In particular, $\det \mathcal{F} = (\det \mathcal{F}') \otimes (\det \mathcal{F}'')$.

5.8. Exercise (torsion-free sheaves). An R-module M is torsion-free if $rm = 0$ implies $r = 0$ or $m = 0$. Show that this satisfies the hypotheses of the affine communication lemma. Hence we make a definition: a quasicoherent sheaf is *torsion-free* if for one (or by the affine communication lemma, for any) affine cover, the sections over each affine open are torsion-free. By definition, "torsion-freeness is affine-local". Show that a quasicoherent sheaf is torsion-free if all its stalks are torsion-free. Hence "torsion-freeness" is "stalk-local." [This exercise is wrong! "Torsion-freeness" is should be defined as "torsion-free stalks" — it is (defined as) a "stalk-local" condition. Here is a better exercise. Show that if M is torsion-free, then so is any localization of M. In particular, M_f is torsion-free, so this notion satisfies half the hypotheses of the affine communication lemma. Also, $M_\mathfrak{p}$ is torsion-free, so this implies that $\tilde{M}$ is torsion-free. Find an example on a two-point space showing that R might not be torsion-free even though $\mathcal{O}_{\operatorname{Spec} R} = \tilde{R}$ is torsion-free.]

6. Quasicoherent sheaves of ideals, and closed subschemes

I then defined quasicoherent sheaves of ideals, and closed subschemes. But I'm happier with the definition I gave in class 15, so I'll leave the discussion until then.

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FOUNDATIONS OF ALGEBRAIC GEOMETRY CLASS 15

RAVI VAKIL

CONTENTS

Last day: quasicoherence is affine-local, (locally) free sheaves and vector bundles, invertible sheaves and line bundles, torsion-free sheaves, quasicoherent sheaves of ideals and closed subschemes.

Today: Quasicoherent sheaves form an abelian category; finite type and coherent sheaves; support; rank; quasicoherent sheaves of ideals and closed subschemes.

I'd like to start by restating some of the definitions and arguments from last day.

Suppose X is a scheme. Recall that $\mathcal{O}_X$-module $\mathcal{F}$ is a quasicoherent sheaf if one of two equivalent things is true.
(i) For every affine open $\operatorname{Spec} R$ and distinguished affine open $\operatorname{Spec} R_f$ thereof, the restriction map $\phi : \Gamma(\operatorname{Spec} R, \mathcal{F}) \to \Gamma(\operatorname{Spec} R_f, \mathcal{F})$ factors as:

$$\phi : \Gamma(\operatorname{Spec} R, \mathcal{F}) \to \Gamma(\operatorname{Spec} R, \mathcal{F})_f \cong \Gamma(\operatorname{Spec} R_f, \mathcal{F}).$$

(ii) For any affine open set $\operatorname{Spec} R$, $\mathcal{F}|_{\operatorname{Spec} R} \cong \tilde{M}$ for some R-module M.

I will use both definitions today.

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1. QUASICOHERENT SHEAVES FORM AN ABELIAN CATEGORY

Last day, I showed that the quasicoherent sheaves on X form an abelian category, and in fact an abelian subcategory of $\mathcal{O}_X$-modules. I restated the argument in a better way today. I've moved this exposition back into the Class 14 notes.

2. FINITENESS CONDITIONS ON QUASICOHERENT SHEAVES: FINITELY GENERATED QUASICOHERENT SHEAVES, AND COHERENT SHEAVES

There are some natural finiteness conditions on an A-module M. I will tell you three. In the case when A is a Noetherian ring, which is the case that almost all of you will ever care about, they are all the same.

The first is the most naive: a module could be *finitely generated*. In other words, there is a surjection $A^p \to M \to 0$.

The second is reasonable too: it could be finitely presented. In other words, it could have a finite number of generators with a finite number of relations: there exists a *finite presentation*

$$A^q \to A^p \to M \to 0.$$

The third is frankly a bit surprising, and I'll justify it soon. We say that an A-module M is *coherent* if (i) it is finitely generated, and (ii) whenenver we have a map $A^p \to M$ (not necessarily surjective!), the kernel is finitely generated.

Clearly coherent implies finitely presented, which in turn implies finitely generated.

2.1. *Proposition. — If A is Noetherian, then these three definitions are the same.*

Preparatory facts. If R is any ring, not necessarily Noetherian, we say an R-module is Noetherian if it satisfies the ascending chain condition for submodules. **Exercise.** M Noetherian implies that any submodule of M is a finitely generated R-module. Hence for example if R is a Noetherian ring then finitely generated = Noetherian. **Exercise.** If $0 \to M' \to M \to M'' \to 0$ is exact, then M' and M'' are Noetherian if and only if M is Noetherian. (Hint: Given an ascending chain in M, we get two simultaneous ascending chains in M' and M''.) **Exercise.** A Noetherian as an A-module implies A^n is a Noetherian A-module. **Exercise.** If A is a Noetherian ring and M is a finitely generated A-module, then any submodule of M is finitely generated. (Hint: suppose $M' \hookrightarrow M$ and $A^n \twoheadrightarrow M$. Construct N with

$$\begin{array}{ccc} N & \hookrightarrow & A^n \\ \downarrow & & \downarrow \\ M' & \hookrightarrow & M \end{array}$$

.)

Proof. Clearly both finitely presented and coherent imply finitely generated.

Suppose M is finitely generated. Then take any $A^p \xrightarrow{\alpha} M$. $\ker \alpha$ is a submodule of a finitely generated module over A, and is thus finitely generated. (Here's why submodules of finitely generated modules over Noetherian rings are also finitely generated: Show it is true for $M = A^n$ — this takes some inspiration. Then given $N \subset N$, consider $A^n \twoheadrightarrow M$, and take the submodule corresponding to N.) Thus we have shown coherence. By choosing a surjective $A^p \to M$, we get finite presentation. □

Hence almost all of you can think of these three notions as the same thing.

2.2. *Lemma. — The coherent* A*-modules form an abelian subcategory of the category of* A*-modules.*

I will prove this in the case where A is Noetherian, but I'll include a series of short exercises in the notes that will show it in general.

Proof if A *is Noetherian.* Recall that we have four things to check (see our discussion earlier today). We quickly check that 0 is finitely generated (=coherent), and that if M and N are finitely generated, then $M \oplus N$ is finitely generated. Suppose now that $f : M \to N$ is a map of finitely generated modules. Then $\operatorname{coker} f$ is finitely generated (it is the image of N), and $\ker f$ is too (it is a submodule of a finitely generated module over a Noetherian ring). □

Easy Exercise (only important for non-Noetherian people). Show A is coherent (as an A-module) if and only if the notion of finitely presented agrees with the notion of coherent.

I want to say a few words on the notion of coherence. There is a good reason for this definition — because of this lemma. There are two sorts of people who should care. Complex geometers should care. They consider complex-analytic spaces with the classical topology. One can define the notion of coherent $\mathcal{O}_X$-module in a way analogous to this. You can then show that the structure sheaf is coherent, and this is very hard. (It is called Oka's theorem, and takes a lot of work to prove.) I believe the notion of coherence may have come originally from complex geometry.

The second sort of people who should care are the sort of arithmetic people who sometimes are forced to consider non-Noetherian rings. (For example, for people who know what they are, the ring of adeles is non-Noetherian.)

Warning: it is common in the later literature to define coherent as finitely generated. It's possible that Hartshorne does this. Please don't do this, as it will only cause confusion. (In fact, if you google the notion of coherent sheaf, you'll get this faulty definition repeatedly.) I will try to be scrupulous about this. Besides doing this for the reason of honesty, it will also help you see what hypotheses are actually necessary to prove things — and that always helps me remember what the proofs are.

2.3. Exercise. If $f \in A$, show that if M is a finitely generated (resp. finitely presented, coherent) A-module, then M_f is a finitely generated (resp. finitely presented, coherent) A_f-module.

Exercise. If $(f_1, \dots, f_n) = A$, and M_{f_i} is finitely generated (resp. coherent) A_{f_i}-module for all i, then M is a finitely generated (resp. coherent) A-module.

I'm not sure if that exercise is even true for finitely presented. That's one of several reasons why I think that "finitely presented" is a worse notion than coherence.

Definition. A quasicoherent sheaf $\mathcal{F}$ is *finite type* (resp. *coherent*) if for every affine open $\operatorname{Spec} R$, $\Gamma(\operatorname{Spec} R, \mathcal{F})$ is a finitely generated (resp. coherent) R-module.

Thanks to the affine communication lemma, and the two previous exercises, it suffices to check this on the opens in a single affine cover.

3. Coherent modules over non-Noetherian rings

Here are some notes on coherent modules over a general ring. Read this only if you really want to! I did not discuss this in class, but promised it in the notes.

Suppose A is a ring. We say an A-module M is *finitely generated* if there is a surjection $A^n \to M \to 0$. We say it is *finitely presented* if there is a presentation $A^m \to A^n \to M \to 0$. We say M is *coherent* if (i) M is finitely generated, and (ii) every map $A^n \to M$ has a finitely generated kernel. The reason we like this third definition is that coherent modules form an abelian category.

Here are some quite accessible problems working out why these notions behave well.

1. Show that coherent implies finitely presented implies finitely generated.

2. Show that 0 is coherent.

Suppose for problems 3–9 that

$$0 \to M \to N \to P \to 0 \tag{1}$$

is an exact sequence of A-modules.

Hint $\star$**.** Here is a *hint* which applies to several of the problems: try to write

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A^p & \longrightarrow & A^{p+q} & \longrightarrow & A^q & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & M & \longrightarrow & N & \longrightarrow & P & \longrightarrow & 0 \end{array}$$

and possibly use the snake lemma.

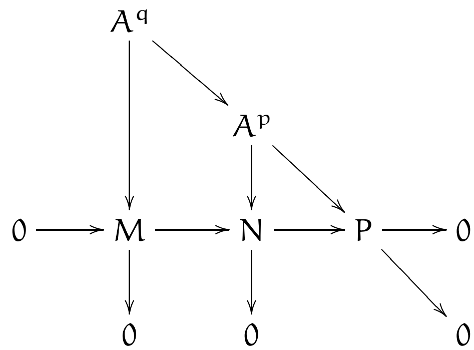
3. Show that N finitely generated implies P finitely generated. (You will only need right-exactness of (1).)

4. Show that M, P finitely generated implies N finitely generated. (Possible hint: $\star$.) (You will only need right-exactness of (1).)

5. Show that N, P finitely generated need not imply M finitely generated. (Hint: if I is an ideal, we have $0 \to I \to A \to A/I \to 0$.)

6. Show that N coherent, M finitely generated implies M coherent. (You will only need left-exactness of (1).)

7. Show that N, P coherent implies M coherent. Hint for (i):



(You will only need left-exactness of (1).)

8. Show that M finitely generated and N coherent implies P coherent. (Hint for (ii): $\star$.)

9. Show that M, P coherent implies N coherent. (Hint: $\star$.) We don't need exactness on the left for this.

At this point, we have shown that if two of (1) are coherent, the third is as well.

10. Show that a finite direct sum of coherent modules is coherent.

11. Suppose M is finitely generated, N coherent. Then if $\phi : M \to N$ is any map, then show that $\operatorname{Im} \phi$ is coherent.

12. Show that the kernel and cokernel of maps of coherent modules are coherent.

At this point, we have verified that coherent A-modules form an abelian subcategory of the category of A-modules. (Things you have to check: 0 should be in this set; it should be closed under finite sums; and it should be closed under taking kernels and cokernels.)

13. Suppose M and N are coherent submodules of the coherent module P. Show that $M + N$ and $M \cap N$ are coherent. (Hint: consider the right map $M \oplus N \to P$.)

14. Show that if A is coherent (as an A-module) then finitely presented modules are coherent. (Of course, if finitely presented modules are coherent, then A is coherent, as A is finitely presented!)

15. If M is finitely presented and N is coherent, show that $\operatorname{Hom}(M, N)$ is coherent. (Hint: Hom is left-exact in its first entry.)

16. If M is finitely presented, and N is coherent, show that $M \otimes N$ is coherent.

17. If $f \in A$, show that if M is a finitely generated (resp. finitely presented, coherent) A-module, then M_f is a finitely generated (resp. finitely presented, coherent) A_f-module. Hint: localization is exact. (This problem appears earlier as well, as Exercise 2.3.)

18. Suppose $(f_1, \dots, f_n) = A$. Show that if M_{f_i} is finitely generated for all i, then M is too. (Hint: Say M_{f_i} is generated by $m_{ij} \in M$ as an A_{f_i}-module. Show that the m_{ij} generate M. To check surjectivity $\oplus_{i,j} A \to M$, it suffices to check "on $D(f_i)$" for all i.)

19. Suppose $(f_1, \dots, f_n) = A$. Show that if M_{f_i} is coherent for all i, then M is too. (Hint from Rob Easton: if $\phi : A^2 \to M$, then $(\ker \phi)_{f_i} = \ker(\phi_{f_i})$, which is finitely generated for all i. Then apply the previous exercise.)

20. Show that the ring $A := k[x_1, x_2, \dots]$ is not coherent over itself. (Hint: consider $A \to A$ with $x_1, x_2, \dots \mapsto 0$.) Thus we have an example of a finitely presented module that is not coherent; a surjection of finitely presented modules whose kernel is not even finitely generated; hence an example showing that finitely presented modules don't form an abelian category.

4. Support of a sheaf

Suppose $\mathcal{F}$ is a sheaf of abelian groups (resp. sheaf of $\mathcal{O}_X$-modules) on a topological space X (resp. ringed space $(X, \mathcal{O}_X)$). Define the *support* of a section s of $\mathcal{F}$ to be

$$\operatorname{Supp} s = \{p \in X : s_p \neq 0 \text{ in } \mathcal{F}_p\}.$$

I think of this as saying where s "lives". Define the *support* of $\mathcal{F}$ as

$$\operatorname{Supp} \mathcal{F} = \{p \in X : \mathcal{F}_p \neq 0\}.$$

It is the union of "all the supports of sections on various open sets". I think of this as saying where $\mathcal{F}$ "lives".

4.1. *Exercise.* The support of a finite type quasicoherent sheaf on a scheme is a closed subset. (Hint: Reduce to an affine open set. Choose a finite set of generators of the corresponding module.) Show that the support of a quasicoherent sheaf need not be closed. (Hint: If $A = \mathbb{C}[t]$, then $\mathbb{C}[t]/(t-a)$ is an A-module supported at a. Consider $\oplus_{a \in \mathbb{C}} \mathbb{C}[t]/(t-a)$.)

5. Rank of a finite type sheaf at a point

The *rank* $\mathcal{F}$ of a finite type sheaf at a point p is $\dim \mathcal{F}_p/\mathfrak{m}\mathcal{F}_p$ where $\mathfrak{m}$ is the maximal ideal corresponding to p. More explicitly, on any affine set $\operatorname{Spec} A$ where $p = [\mathfrak{p}]$ and $\mathcal{F}(\operatorname{Spec} A) = M$, then the rank is $\dim_{A/\mathfrak{p}} M_\mathfrak{p}/\mathfrak{p}M_\mathfrak{p}$. By Nakayama's lemma, this is the minimal number of generators of $M_\mathfrak{p}$ as an $A_\mathfrak{p}$-module.

Note that this definition is consistent with the notion of rank of a locally free sheaf. In that case, the rank is a (locally) constant function of the point. The converse is sometimes true, as is shown in Exercise 5.2 below.

If $\mathcal{F}$ is quasicoherent (not necessarily finite type), then $\mathcal{F}_p/\mathfrak{m}\mathcal{F}_p$ can be interpreted as the fiber of the sheaf at the point. A section of $\mathcal{F}$ over an open set containing p can be said to take on a value at that point, which is an element of $\mathcal{F}_p/\mathfrak{m}\mathcal{F}_p$.

5.1. *Exercise.*

(a) If $m_1, \dots, m_n$ are generators at P, they are generators in an open neighborhood of P. (Hint: Consider $\operatorname{coker} A^n \xrightarrow{(f_1,\dots,f_n)} M$ and Exercise 4.1.)
(b) Show that at any point, $\operatorname{rank}(\mathcal{F} \oplus \mathcal{G}) = \operatorname{rank}(\mathcal{F}) + \operatorname{rank}(\mathcal{G})$ and $\operatorname{rank}(\mathcal{F} \otimes \mathcal{G}) = \operatorname{rank} \mathcal{F} \operatorname{rank} \mathcal{G}$ at any point. (Hint: Show that direct sums and tensor products commute with ring quotients and localizations, i.e. $(M \oplus N) \otimes_R (R/I) \cong M/IM \oplus N/IN$, $(M \otimes_R N) \otimes_R (R/I) \cong (M \otimes_R R/I) \otimes_{R/I} (N \otimes_R R/I) \cong M/IM \otimes_{R/I} N/IM$, etc.) Thanks to Jack Hall for improving this problem.
(c) Show that rank is an upper semicontinuous function on X. (Hint: Generators at P are generators nearby.)

5.2. *Important Exercise.* If X is reduced, $\mathcal{F}$ is coherent, and the rank is constant, show that $\mathcal{F}$ is locally free. (Hint: choose a point $p \in X$, and choose generators of the stalk $\mathcal{F}_p$. Let U be an open set where the generators are sections, so we have a map $\phi : \mathcal{O}_U^{\oplus n} \to \mathcal{F}|_U$. The cokernel and kernel of ϕ are supported on closed sets by Exercise 4.1. Show that these closed subsets don't include p. Make sure you use the reduced hypothesis!) Thus coherent sheaves are locally free on a dense open set. Show that this can be false if X is not reduced. (Hint: $\operatorname{Spec} k[x]/x^2$, $M = k$.)

You can use the notion of rank to help visualize finite type sheaves, or even quasicoherent sheaves. (We discussed first finite type sheaves on reduced schemes. We then generalized to quasicoherent sheaves, and to nonreduced schemes.)

5.3. Exercise: Geometric Nakayama. Suppose X is a scheme, and $\mathcal{F}$ is a finite type quasicoherent sheaf. Show that if $\mathcal{F}_x \otimes k(x) = 0$, then there exists V such that $\mathcal{F}|_V = 0$. Better: if I have a set that generates the fiber, it generates the stalk.

5.4. *Less important Exercise.* Suppose $\mathcal{F}$ and $\mathcal{G}$ are finite type sheaves such that $\mathcal{F} \otimes \mathcal{G} \cong \mathcal{O}_X$. Then $\mathcal{F}$ and $\mathcal{G}$ are both invertible (Hint: Nakayama.) This is the reason for the adjective "invertible" these sheaves are the invertible elements of the "monoid of finite type sheaves".

6. Quasicoherent sheaves of ideals, and closed subschemes

This section is important, and short only because we have built up some machinery.

We now define closed subschemes, and what it means for some functions on a scheme to "cut out" another scheme. The intuition we want to make precise is that a closed subscheme of X is something that on each affine looks like $\operatorname{Spec} R/I$ "$\hookrightarrow$" $\operatorname{Spec} R$.

Suppose $\mathcal{I} \hookrightarrow \mathcal{O}_X$ is a *quasicoherent sheaf of ideals*. (Quasicoherent sheaves of ideals are, not suprisingly, sheaves of ideals that are quasicoherent.) Not all sheaves of ideals are quasicoherent.

6.1. Exercise. *(A non-quasicoherent sheaf of ideals)* Let $X = \operatorname{Spec} k[x]_{(x)}$, the germ of the affine line at the origin, which has two points, the closed point and the generic point η. Define $\mathcal{I}(X) = \{0\} \subset \mathcal{O}_X(X) = k[x]_{(x)}$, and $\mathcal{I}(\eta) = k(x) = \mathcal{O}_X(\eta)$. Show that $\mathcal{I}$ is not a quasicoherent sheaf of ideals.

The cokernel of $\mathcal{I} \to \mathcal{O}_X$ is also quasicoherent, so we have an exact sequence of quasicoherent sheaves

$$0 \to \mathcal{I} \to \mathcal{O}_X \to \mathcal{O}_X/\mathcal{I} \to 0. \tag{2}$$

(This exact sequence will come up repeatedly. We could call it the *closed subscheme exact sequence*.) Now $\mathcal{O}_X/\mathcal{I}$ is finite tyupe (as over any affine open set, the corresponding module is generated by a single element), so $\operatorname{Supp} \mathcal{O}_X/\mathcal{I}$ is a closed subset. Also, $\mathcal{O}_X/\mathcal{I}$ is a sheaf of rings. Thus we have a topological space $\operatorname{Supp} \mathcal{O}_X/\mathcal{I}$ with a sheaf of rings. I claim this is a scheme. To see this, we look over an affine open set $\operatorname{Spec} R$. Here $\Gamma(\operatorname{Spec} R, \mathcal{I})$ is an ideal I of R. Then $\Gamma(\operatorname{Spec} R, \mathcal{O}_X/\mathcal{I}) = R/I$ (because quotients behave well on affine open sets).

I claim that on this open set, $\operatorname{Supp} \mathcal{O}_X/\mathcal{I}$ is the closed subset $V(I)$, which I can identify with the topological space $\operatorname{Spec} R/I$. Reason: $[\mathfrak{p}] \in \operatorname{Supp}(\mathcal{O}_X/\mathcal{I})$ if and only if $(R/I)_{\mathfrak{p}} \neq 0$ if and only if $\mathfrak{p}$ contains I if and only if $[\mathfrak{p}] \in \operatorname{Spec} R/I$.

(Remark for experts: when you have a sheaf supported in a closed subset, you can interpret it as a sheaf *on* that closed subset. More precisely, suppose X is a topological space, $i : Z \hookrightarrow X$ is an inclusion of a closed subset, and $\mathcal{F}$ is a sheaf on X with $\operatorname{Supp} \mathcal{F} \subset Z$. Then we have a natural map $\mathcal{F} \to i_* i^{-1} \mathcal{F}$ (corresponding to the map $i^{-1}\mathcal{F} \to i^{-1}\mathcal{F}$, using the adjointness of i^{-1} and i_*). You can check that this is an isomorphism on stalks, and hence an isomorphism, so $\mathcal{F}$ can be interpreted as the pushforward of a sheaf on the closed subset. Thanks to Jarod and Joe for this comment.)

I next claim that on the distinguished open set $D(f)$ of $\operatorname{Spec} R$, the sections of $\mathcal{O}_X/\mathcal{I}$ are precisely $(R/I)_f \cong R_f/I_f$. (Reason that $(R/I)_f \cong R_f/I_f$: take the exact sequence $0 \to I \to R \to R/I \to 0$ and tensor with R_f, which preserves exactness.) Reason: On $\operatorname{Spec} R$, the sections of $\mathcal{O}_X/\mathcal{I}$ are R/I, and $\mathcal{O}_X/\mathcal{I}$ is quasicoherent, hence the sections over $D(f)$ are $(R/I)_f$.

That's it!

We say that a *closed subscheme* of X is anything arising in this way from a quasicoherent sheaf of ideals. In other words, there is a tautological correspondence between quasicoherent sheaves of ideals and closed subschemes.

Important remark. Note that closed subschemes of affine schemes are affine. (This is tautological using our definition, but trickier using other definitions.)

Exercise. Suppose $\mathcal{F}$ is a locally free sheaf on a scheme X, and s is a section of $\mathcal{F}$. Describe how $s = 0$ "cuts out" a closed subscheme. (A picture is very useful here!)

6.2. Reduction of a scheme. The *reduction* of a scheme is the "reduced version" of the scheme. If R is a ring, then the nilradical behaves well with respect to localization with respect to an element of the ring: $\mathfrak{N}(R)_f$ is naturally isomorphic to $\mathfrak{N}(R_f)$ (check this!). Thus on any scheme, we have an ideal sheaf of nilpotents, and the corresponding closed subscheme is called the *reduction* of X, and is denoted X^{red}. We will soon see that X^{red} satisfies a universal property; we will need the notion of a morphism of schemes to say what this universal property is.

6.3. *Unimportant exercise.*

(a) X^{red} has the same underlying topological space as X: there is a natural homeomorphism of the underlying topological spaces $X^{red} \cong X$. Picture: taking the reduction may be interpreted as shearing off the fuzz on the space.

(b) Give an example to show that it is *not* true that $\Gamma(X^{red}, \mathcal{O}_{X^{red}}) = \Gamma(X, \mathcal{O}_X)/\sqrt{\Gamma(X, \mathcal{O}_X)}$. (Hint: $\coprod_{n>0} \operatorname{Spec} k[t]/(t^n)$ with global section $(t, t, t, \dots)$.) Motivation for this exercise: this *is* true on each affine open set.

By Exercise 4.1, we have that the reduced locus of a locally Noetherian scheme is open. More precisely: Let $\mathcal{I}$ be the ideal sheaf of X^{red}, so on X we have an exact sequence

$$0 \to \mathcal{I} \to \mathcal{O}_X \to \mathcal{O}_{X^{red}} \to 0$$

of quasicoherent sheaves on X. Then $\mathcal{I}$ is coherent as X is locally Noetherian. Hence the support of I is closed. The complement of the support of I is the **reduced locus**. Geometrically, this says that "the fuzz is on a closed subset". (A picture is really useful here!)

6.4. Important exericse (the reduced subscheme induced by a closed subset). Suppose X is a scheme, and K is a closed subset of X. Show that the following construction determines a closed subscheme Y: on any affine open subset $\operatorname{Spec} R$ of X, consider the ideal $I(K \cap \operatorname{Spec} R)$. This is called the *reduced subscheme induced by* K. Show that Y is reduced.

7. Discussion of future topics

I then discussed the notion of when a sheaf is *generated by global sections*, and gave a preview of quasicoherent sheaves on projective A-schemes. These ideas will appear in the notes for class 16.

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FOUNDATIONS OF ALGEBRAIC GEOMETRY CLASS 16

CONTENTS

Last day: much more on quasicoherence. Quasicoherent sheaves form an abelian category. Finite type (= "locally finitely generated") and coherent sheaves. Support of a sheaf. Rank of a finite type sheaf at a point. Closed subschemes at a point.

Today: effective Cartier divisors; quasicoherent sheaves on projective A-schemes corresponding to graded modules, line bundles on projective A-schemes, O(n), generated by global sections, Serre's theorem, the adjoint functors $\sim$ and Γ_*.

1. Yet more on closed subschemes

Here are few more notions about closed subschemes.

In analogy with closed subsets of a topological space, we can define finite unions and arbitrary intersections of closed subschemes. On affine open set $\operatorname{Spec} R$, if for each i in an index set, I_i corresponds to a closed subscheme, the scheme-theoretic intersection of the closed subschemes corresponds to the ideal generated by the I_i (here the index set may be infinite), and the scheme-theoretic union corresponds to the intersection of by all I_i (here the index should be finite).

Exercise: Describe the scheme-theoretic intersection of $(y - x^2)$ and y in $\mathbb{A}^2$. Describe the scheme-theoretic union. Draw a picture.

Exercise: Prove some fact of your choosing showing that closed subschemes behave similarly to closed subsets. For example, if X, Y, and Z are closed subschemes of W, show that $(X \cap Z) \cup (Y \cap Z) = (X \cup Y) \cap Z$.

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1.1. From closed subschemes to effective Cartier divisors. There is a special name for a closed subscheme locally cut out by one equation that is not a zero-divisor. More precisely, it is a closed subscheme such that there exists an affine cover such that on each one it is cut out by a single equation, not a zero-divisor. (This does not mean that on *any* affine it is cut out by a single equation — this notion doesn't satisfy the "gluability" hypothesis of the Affine Communication Theorem. If $I \subset R$ is generated by a non-zero divisor, then $I_f \subset R_f$ is too. But "not conversely". I might give an example later.) We call this an *effective Cartier divisor*. (This admittedly unwieldy terminology! But there is a reason for it.) By Krull, it is pure codimension 1.

Remark: if $I = (u) = (v)$, and u is not a zero-divisor, then u and v differ by a unit in R. Proof: $u \in (v)$ implies $u = av$. Similarly $v = bu$. Thus $u = abu$, from which $u(1-ab) = 0$. As u is not a zero-divisor, $1 = ab$, so a and b are units.

Reason we care: effective Cartier divisors give invertible sheaves. If $\mathcal{I}$ is an effective Cartier divisor on X, then $\mathcal{I}$ is an invertible sheaf. Reason: locally, sections are multiples of a single generator u, and there are no "relations".

The invertible sheaf corresponding to an effective Cartier divisor is for various reasons defined to be the dual of the ideal sheaf. This line bundle has a canonical section: Dualizing $0 \to \mathcal{I} \to \mathcal{O}$ gives us $\mathcal{O} \to \mathcal{I}^*$. **Exercise.** This section vanishes along our actual effective Cartier divisor.

1.2. Exercise. Describe the invertible sheaf in terms of transition functions. More precisely, on any affine open set where the effective Cartier divisor is cut out by a single equation, the invertible sheaf is trivial. Determine the transition functions between two such overlapping affine open sets. Verify that there is indeed a canonical section of this invertible sheaf, by describing it.

To describe the tensor product of such invertible sheaves: if $I = (u)$ (locally) and $J = (v)$, then the tensor product corresponds to (uv).

We get a monoid of effective Cartier divisors, with unit $\mathcal{I} = \mathcal{O}$. Notation: D is an effective Cartier divisor. $\mathcal{O}(D)$ is the corresponding line bundle. $\mathcal{O}(-D)$ is the ideal sheaf.

$$0 \to \mathcal{O}(-D) \to \mathcal{O} \to \mathcal{O}_D \to 0.$$

D is associated to the closed subscheme.

Hence we can get a bunch of invertible sheaves, by taking differences of these two. Surprising fact: we almost get them all! (True for all nonsingular schemes. This is true for all projective schemes. It is very hard to describe an invertible sheaf that is not describable in such a way.)

We now describe quasicoherent sheaves on a projective A-scheme. Recall that a projective A-scheme is produced from the data of $\mathbb{Z}^{\geq 0}$-graded ring S_*, with $S_0 = A$, and S_+ finitely generated as an A-module. The resulting scheme is denoted $\operatorname{Proj} S_*$.

Let $X = \operatorname{Proj} S_*$. Suppose M_* is a graded S_* module, *graded by* $\mathbb{Z}$. We define the quasicoherent sheaf $\widetilde{M_*}$ as follows. (I will avoid calling it $\tilde{M}$, as this might cause confusion with the affine case.) On the distinguished open $D(f)$, we let

$$\widetilde{M_*}(D(f)) \cong \widetilde{(M_f)_0}.$$

(More correctly: we define a sheaf $\widetilde{M_*(f)}$ on $D(f)$ for each f. We give identifications of the restriction of $\widetilde{M_*(f)}$ and $\widetilde{M_*(g)}$ to $D(fg)$. Then by an earlier problem set problem telling how to glue sheaves, these sheaves glue together to a single sheaf on $\widetilde{M_*}$ on X. We then discard the temporary notation $\widetilde{M_*(f)}$.)

This is clearly quasicoherent, because it is quasicoherent on each $D(f)$. If M_* is finitely generated over S_*, then so $\tilde{M}_*$ is a finite type sheaf.

I will now give some straightforward facts.

If M_* and M'_* agree in high enough degrees, then $\widetilde{M_*} \cong \widetilde{M'_*}$. Thus the map from graded S_*-modules to quasicoherent sheaves on $\operatorname{Proj} S_*$ is not a bijection.

Given a map of graded modules $\phi : M_* \to N_*$, we we get an induced map of sheaves $\widetilde{M_*} \to \widetilde{N_*}$. Explicitly, over $D(f)$, the map $M_* \to N_*$ induces $M_*[1/f] \to N_*[1/f]$ which induces $\phi_f : (M_*[1/f])_0 \to (N_*[1/f])_0$; and this behaves well with respect to restriction to smaller distinguished open sets, i.e. the following diagram commutes.

$$\begin{array}{ccc} (M_*[1/f])_0 & \xrightarrow{\phi_f} & (N_*[1/f])_0 \\ \downarrow & & \downarrow \\ (M_*[1/(fg)])_0 & \xrightarrow{\phi_{fg}} & (N_*[1/(fg)])_0. \end{array}$$

In fact $\sim$ is a functor from the category of graded S_*-modules to the category of quasicoherent sheaves on $\operatorname{Proj} S_*$. This isn't quite an isomorphism, but it is close. The relationship is akin to that between presheaves and sheaves, and the sheafification functor, as we will see before long.

2.1. Exercise. Show that $\widetilde{M_*} \otimes \widetilde{N_*} \cong \widetilde{M_* \otimes_{S_*} N_*}$. (Hint: describe the isomorphism of sections over each $D(f)$, and show that this isomorphism behaves well with respect to smaller distinguished opens.)

If $I_* \subset S_*$ is a graded ideal, we get a closed subscheme. Example: Suppose $S_* = k[w, x, y, z]$, so $\operatorname{Proj} S_* \cong \mathbb{P}^3$. Suppose $I_* = (wx - yz, x^2 - wy, y^2 - xz)$. Then we get the exact sequence of graded S_*-modules

$$0 \to I_* \to S_* \to S_*/I_* \to 0.$$

Which closed subscheme of $\mathbb{P}^3$ do we get? The twisted cubic!

2.2. Exercise. Show that if I_* is a graded ideal of S_*, show that we get a closed immersion $\operatorname{Proj} S_*/I_* \hookrightarrow \operatorname{Proj} S_*$.

3. Invertible sheaves (line bundles) on projective A-schemes

We now come to one of the most fundamental concepts in projective geometry.

First, I want to mention something that I should have mentioned some time ago.

3.1. Exercise. Suppose S_* is generated over S_0 by $f_1, \dots, f_n$. Suppose $d = \operatorname{lcm}(\deg f_1, \dots, \deg f_n)$. Show that S_{d*} is generated in "new" degree 1 (= "old" degree d). (Hint: I like to show this by induction on the size of the set $\{\deg f_1, \dots, \deg f_n\}$.) This is handy, because we can stick every Proj in some projective space via the construction of Exercise 2.2.

Suppose that S_* is generated in degree 1. By the previous exercise, this is not a huge assumption. Suppose M_* is a graded S_*-module. Define the graded module $M(n)_*$ so that $M(n)_m := M_{n+m}$. Thus the quasicoherent sheaf $\widetilde{M(n)_*}$ is given by

$$\Gamma(D(f), \widetilde{M(n)_*}) = (\widetilde{M_f})_n$$

where here the subscript means we take the nth graded piece. (These subscripts are admittedly confusing!)

As an incredibly important special case, define $\mathcal{O}_{\operatorname{Proj} S_*}(n) := \widetilde{S(n)_*}$. When the space is implicit, it can be omitted from the notation: $\mathcal{O}(n)$ (pronounced "oh of n").

3.2. Important exercise. If S_* is generated in degree 1, show that $\mathcal{O}_{\operatorname{Proj} S_*}(n)$ is an invertible sheaf.

3.3. Essential exercise. Calculate $\dim_k \Gamma(\mathbb{P}^m_k, \mathcal{O}_{\mathbb{P}^m_k}(n))$.

I'll get you started on this. As always, consider the "usual" affine cover. Consider the $n = 1$ case. Say $S_* = k[x_0, \dots, x_m]$. Suppose we have a global section of $\mathcal{O}(1)$. On $D(x_0)$, the sections are of the form $f(x_0, \dots, x_n)/x_0^{\deg f - 1}$. On $D(x_1)$, the sections are of the form $g(x_0, \dots, x_n)/x_1^{\deg g - 1}$. They are supposed to agree on the overlap, so

$$x_0^{\deg f - 1} g(x_0, \dots, x_n) = x_1^{\deg g - 1} f(x_0, \dots, x_n).$$

How is this possible? Well, we must have that $g = x_1^{\deg g - 1} \times$ some linear factor, and $f = x_0^{\deg f - 1} \times$ the same linear factor. Thus on $D(x_0)$, this section must be some linear form. On $D(x_1)$, this section must be the same linear form. By the same argument, on each

$D(x_i)$, the section must be the same linear form. Hence (with some argumentation), the global sections of $\mathcal{O}(1)$ correspond to the linear forms in $x_0, \ldots, x_m$, of which there are $m+1$.

Thus $x+y+2z$ is a section of $\mathcal{O}(1)$ on $\mathbb{P}^2$. It isn't a function, but I can say when this section vanishes — precisely where $x+y+2z=0$.

3.4. Exercise. Show that $\mathcal{F}(n) \cong \mathcal{F} \otimes \mathcal{O}(n)$.

3.5. Exercise. Show that $\mathcal{O}(m+n) \cong \mathcal{O}(m) \otimes \mathcal{O}(n)$.

3.6. Exercise. Show that if $m \neq n$, then $\mathcal{O}_{\mathbb{P}^l_k}(m)$ is not isomorphic to $\mathcal{O}_{\mathbb{P}^l_k}(n)$ if $l > 0$. (Hence we have described a countable number of invertible sheaves (line bundles) that are non-isomorphic. We will see later that these are *all* the line bundles on projective space $\mathbb{P}^n_k$.)

4. Generation by global sections, and Serre's Theorem

(I discussed this in class 15, but should have discussed them here. Hence they are in the class 16 notes.)

Suppose $\mathcal{F}$ is a sheaf on X. We say that $\mathcal{F}$ is *generated by global sections at a point* p if we can find $\phi : \mathcal{O}^{\oplus I} \to \mathcal{F}$ that is surjective at the stalk of p: $\phi_p : \mathcal{O}_p^{\oplus I} \to \mathcal{F}_p$ is surjective. (Some what more precisely, the stalk of $\mathcal{F}$ at p is generated by global sections of $\mathcal{F}$. The global sections in question are the images of the $|I|$ factors of $\mathcal{O}_p^{\oplus I}$.) We say that $\mathcal{F}$ is *generated by global sections* if it is generated by global sections at all p, or equivalently, if we can we can find $\mathcal{O}^{\oplus I} \to \mathcal{F}$ that is surjective. (By our earlier result that we can check surjectivity at stalks, this is the same as saying that it is surjective at all stalks.) If I can be taken to be finite, we say that $\mathcal{F}$ is generated by a finite number of global sections. We'll see soon why we care.

4.1. *Exercise.* If quasicoherent sheaves $\mathcal{L}$ and $\mathcal{M}$ are generated by global sections at a point p, then so is $\mathcal{L} \otimes \mathcal{M}$. (This exercise is less important, but is good practice for the concept.)

4.2. *Easy exercise.* An invertible sheaf $\mathcal{L}$ on X is generated by global sections if and only if for any point $x \in X$, there is a section of $\mathcal{L}$ not vanishing at x. (Hint: Nakayama.)

4.3. *Lemma. — Suppose $\mathcal{F}$ is a finite type sheaf on* X*. Then the set of points where $\mathcal{F}$ is generated by global sections is an open set.*

Proof. Suppose $\mathcal{F}$ is generated by global sections at a point p. Then it is generated by a finite number of global sections, say m. This gives a morphism $\phi : \mathcal{O}^{\oplus m} \to \mathcal{F}$, hence

$\text{im}\phi \hookrightarrow \mathcal{F}$. The support of the (finite type) cokernel sheaf is a closed subset not containing p. □

(Back to class 16!)

4.4. Important Exercise (an important theorem of Serre). Suppose S_0 is a Noetherian ring, and S_* is generated in degree 1. Let $\mathcal{F}$ be any finite type sheaf on $\operatorname{Proj} S_*$. Then for some integer n_0, for all $n \geq n_0$, $\mathcal{F}(n)$ can be generated by a finite number of global sections.

I'm going to sketch how you should tackle this exercise, after first telling you the reason we will care.

4.5. *Corollary.* — Thus any coherent sheaf $\mathcal{F}$ on $\operatorname{Proj} S_*$ can be presented as:

$$\oplus_{\text{finite}} \mathcal{O}(-n) \to \mathcal{F} \to 0.$$

We're going to use this a lot!

Proof. Suppose we have m global sections $s_1, \ldots, s_m$ of $\mathcal{F}(n)$ that generate $\mathcal{F}(n)$. This gives a map

$$\oplus_m \mathcal{O} \longrightarrow \mathcal{F}(n)$$

given by $(f_1, \ldots, f_m) \mapsto f_1 s_1 + \cdots + f_m s_m$ on any open set. Because these global sections generate $\mathcal{F}$, this is a surjection. Tensoring with $\mathcal{O}(-n)$ (which is exact, as tensoring with any locally free is exact) gives the desired result. □

Here is now a hint/sketch for the Serre exercise 4.4.

We can assume that S_* is generated in degree 1; we can do this thanks to Exercise 3.1. Suppose $\deg f = 1$. Say $\mathcal{F}|_{D(f)} = \tilde{M}$, where M is a $(S_*[1/f])_0$-module, generated by $m_1, \ldots, m_n$. These elements generate all the stalks over all the points of $D(f)$. They are sections over this big distinguished open set. It would be wonderful if we knew that they had to be restrictions of global sections, i.e. that there was a global section m_i' that restricted to m_i on $D(f)$. If that were always true, then we would cover X with a finite number of each of these $D(f)$'s, and for each of them, we would take the finite number of generators of the corresponding module. Sadly this is not true.

However, we will see that $f^N m$ "extends", where m is any of the m_i's, and N is sufficiently large. We will see this by (easily) checking first that $f^N m$ extends over another distinguished open $D(g)$ (i.e. that there is a section of $\mathcal{F}(N)$ over $D(g)$ that restricts to $f^n m$ on $D(g) \cap D(f) = D(fg)$). But we're still not done, because we don't know that the extension over $D(g)$ and over some other $D(g')$ agree on the overlap $D(g) \cap D(g') = D(gg')$ — in fact, they need not agree! But after multiplying both extensions by $f^{N'}$ for large enough N', we will see that they agree on the overlap. By quasicompactness, we need to to extend over only a finite number of $D(g)$'s, and to make sure extensions agree over the finite number of pairs of such $D(g)$'s, so we will be done.

Great, let's make this work. Let's investigate this on $D(g) = \operatorname{Spec} A$, where the degree of g is also 1. Say $\mathcal{F}|D(g) \cong \tilde{N}$. Let $f' = f/g$ be "the function corresponding to f on $D(g)$". We have a section over $D(f')$ on the affine scheme $D(g)$, i.e. an element of $N_{f'}$, i.e. something of the form $n/(f')^N$ for some $n \in N$. So then if we multiply it by f'^N, we can certainly extend it! So if we multiply by a big enough power of f, m certainly extends over any $D(g)$.

As described earlier, the only problem is, we can't guarantee that the extensions over $D(g)$ and $D(g')$ agree on the overlap (and hence glue to a single extensions). Let's check on the intersection $D(g) \cap D(g') = D(gg')$. Say $m = n/(f')^N = n'/(f')^{N'}$ where we can take $N = N'$ (by increasing N or N' if necessary). We certainly may not have $n = n'$, but by the (concrete) definition of localization, after multiplying with enough f''s, they become the same.

In conclusion after multiplying with enough f's, our sections over $D(f)$ extend over each $D(g)$. After multiplying by even more, they will all agree on the overlaps of any two such distinguished affine. Exercise 4.4 is to make this precise.

5. Every quasicoherent sheaf on a projective A-scheme arises from a graded module

We have gotten lots of quasicoherent sheaves on $\operatorname{Proj} S_*$ from graded S_*-modules. We'll now see that we can get them all in this way.

We want to figure out how to "undo" the $\tilde{M}$ construction. When you do the essential exercise 3.3, you'll suspect that in good situations,

$$M_n \cong \Gamma(\operatorname{Proj} S_*, \tilde{M}(n)).$$

Motivated by this, we define

$$\Gamma_n(\mathcal{F}) := \Gamma(\operatorname{Proj} S_*, \mathcal{F}_n).$$

Then $\Gamma_*(\mathcal{F})$ is a graded S_*-module, and we can dream that $\Gamma_*(\mathcal{F})^\sim \cong \mathcal{F}$. We will see that this is indeed the case!

5.1. Exercise. Show that Γ_* gives a functor from the category of quasicoherent sheaves on $\operatorname{Proj} S_*$ to the category of graded S_*-modules. (In other words, show that if $\mathcal{F} \to \mathcal{G}$ is a morphism of quasicoherent sheaves on $\operatorname{Proj} S_*$, describe the natural map $\Gamma_*(\mathcal{F}) \to \Gamma_*(\mathcal{G})$, and show that such natural maps respect the identity and composition.)

Note that $\sim$ and Γ_* cannot be inverses, as $\sim$ can turn two different graded modules into the same quasicoherent sheaf.

Our initial goal is to show that there is a natural isomorphism $\widetilde{\Gamma_*(\mathcal{F})} \to \mathcal{F}$, and that there is a natural map $M_* \to \Gamma_*(\widetilde{M_*})$. We will show something better: that $\sim$ and Γ_* are adjoint.

We start by describing the natural map $M_* \to \Gamma_* \widetilde{M_*}$. We describe it in degree n. Given an element m_n, we seek an element of $\Gamma(\operatorname{Proj} S_*, \widetilde{M_*}(n)) = \Gamma(\operatorname{Proj} S_*, \widetilde{M_{(n+*)}})$. By shifting the grading of M_* by n, we can assume $n = 0$. For each $D(f)$, we certainly have an element of $(M[1/f])_0$ (namely m), and they agree on overlaps, so the map is clear.

5.2. Exercise. Show that this canonical map need not be injective, nor need it be surjective. (Hint: $S_* = k[x]$, $M_* = k[x]/x^2$ or $M_* = \{$ polynomials with no constant terms $\}$.)

The natural map $\widetilde{\Gamma_* \mathcal{F}} \to \mathcal{F}$ is more subtle (although it will have the advantage of being an isomorphism).

5.3. Exercise. Describe the natural map $\widetilde{\Gamma_* \mathcal{F}} \to \mathcal{F}$ as follows. First describe it over $D(f)$. Note that sections of the left side are of the form m/f^n where $m \in \Gamma_{n \deg f} \mathcal{F}$, and $m/f^n = m'/f^{n'}$ if there is some N with $f^N(f^{n'}m - f^n m') = 0$. Show that your map behaves well on overlaps $D(f) \cap D(g) = D(fg)$.

5.4. Longer Exercise. Show that the natural map $\widetilde{\Gamma_* \mathcal{F}} \to \mathcal{F}$ is an isomorphism, by showing that it is an isomorphism over $D(f)$ for any f. Do this by first showing that it is surjective. This will require following some of the steps of the proof of Serre's theorem (Exercise 4.4). Then show that it is injective.

5.5. *Corollary. — Every quasicoherent sheaf arises from this tilde construction. Each closed subscheme of* $\operatorname{Proj} S_*$ *arises from a graded ideal* $I_* \subset S_*$.

In particular, let $x_0, \dots, x_n$ be generators of S_1 as an A-module. Then we have a surjection of graded rings

$$A[t_0, \dots, t_n] \to S_*$$

where $t_i \mapsto x_i$. Then this describes $\operatorname{Proj} S_*$ as a closed subscheme of $\mathbb{P}^n_A$.

5.6. Exercise (Γ_* and $\sim$ are adjoint functors). Prove part of the statement that Γ_* and $\sim$ are adjoint functors, by describing a natural bijection $\operatorname{Hom}(M_*, \Gamma_*(\mathcal{F})) \cong \operatorname{Hom}(\widetilde{M_*}, \mathcal{F})$. For the map from left to right, start with a morphism $M_* \to \Gamma_*(\mathcal{F})$. Apply $\sim$, and postcompose with the isomorphism $\widetilde{\Gamma_* \mathcal{F}} \to \mathcal{F}$, to obtain

$$\widetilde{M_*} \to \widetilde{\Gamma_* \mathcal{F}} \to \mathcal{F}.$$

Do something similar to get from right to left. Show that "both compositions are the identity in the appropriate category". (Is there a clever way to do that?)

5.7. Saturated S_*-modules. We end with a remark: different graded S_*-modules give the same quasicoherent sheaf on $\operatorname{Proj} S_*$, but the results of this section show that there is a "best" graded module for each quasicoherent sheaf, and there is a map form each graded module to its "best" version, $M_* \to \Gamma_*(\widetilde{M_*})$. A module for which this is an isomorphism (a "best" module) is called *saturated*. I don't think we'll use this notation, but other people do.

This "saturation" map $M_* \to \Gamma_*(\widetilde{M_*})$ should be seen as analogous to the sheafification map, taking presheaves to sheaves.

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FOUNDATIONS OF ALGEBRAIC GEOMETRY CLASS 17

CONTENTS

Last day: effective Cartier divisors; quasicoherent sheaves on projective A-schemes corresponding to graded modules, line bundles on projective A-schemes, O(n), generated by global sections, Serre's theorem, the adjoint functors $\sim$ and Γ_*.

Today: Associated points; more on normality; invertible sheaves and divisors take 1.

Our goal for today and part of next day is to develop tools to understand what invertible sheaves there can be on a scheme. As a key motivating example, we will show (by next day) that the only invertible sheaves on $\mathbb{P}^n_k$ are $\mathcal{O}(m)$.

But first, I want to tell you about *associated points* and the *ring of fractions* of a scheme. This topic isn't logically needed, but it is a description of the "most interesting points" of a scheme, where "all the action is".

1. ASSOCIATED POINTS

Recall that for an integral (= irreducible + reduced, by an earlier homework problem) scheme X, we have the notion of the *function field*, which is the stalk at the generic point. For any affine open subset $\operatorname{Spec} R$, we have that R is a subring of the function field.

It would be nice to generalize this to more general schemes, with possibly many components, and with nonreduced behavior.

The answer to this question is that on a "nice" (Noetherian) scheme, there are a finite number of points that will have similar information. (On a locally Noetherian scheme, we'll also have the notion of associated points, but there could be an infinite number of them.) I then drew a picture of a scheme with two components, one of which looked like a (reduced) line, and one of which was a plane, with some nonreduced behavior ("fuzz") along a line of it, and even more nonreduced behavior ("more fuzz") at a point of the line.

Date: Wednesday, November 30, 2005. Very minor update January 31, 2007.

I stated that the associated points are the generic points of the two components, plus the generic point of the line where this is fuzz, and the point where there is more fuzz.

We will define associated points of locally Noetherian schemes, and show the following important properties. You can skip the proofs if you want, but you should remember these properties.

(1) The generic points of the irreducible components are associated points. The other associated points are called *embedded points*.

(2) If X is reduced, then X has no embedded points. (This jibes with the intuition of the picture of associated points I described earlier.)

(3) If X is affine, say $X = \operatorname{Spec} R$ affine, then the natural map

$$R \to \prod_{\text{associated } \mathfrak{p}} R_{\mathfrak{p}} \tag{1}$$

is an injection. The primes corresponding to the associated points of R will be called *associated primes*. (In fact this is backwards; we will define associated primes first, and then define associated points.) The ring on the right of (1) is called the *ring of fractions*. If X is a locally Noetherian scheme, then the products of the stalks at the associated points will be called the *ring of fractions* of X. Note that if X is integral, this is the function field.

We define a *rational function* on a locally Noetherian scheme: it is an equivalence class. Any function defined on an open set containing all associated points is a rational function. Two such are considered the same if they agree on an open subset containing all associated points. If X is reduced, this is the same as requiring that they are defined on an open set of each of the irreducible components. A rational function has a maximal domain of definition, because any two actual functions on an open set (i.e. sections of the structure sheaf over that open set) that agree as "rational functions" (i.e. on small enough open sets containing associated points) must be the same function, by this fact **(3)**. We say that a rational function f is *regular* at a point p if p is contained in this maximal domain of definition (or equivalently, if there is some open set containing p where f is defined).

We similarly define *rational and regular sections of an invertible sheaf $\mathcal{L}$* on a scheme X.

(4) A function is a zero divisor if and only if it vanishes at an associated point of $\operatorname{Spec} R$.

Okay, let's get down to business.

An ideal $I \subset A$ is *primary* if $I \neq A$ and if $xy \in I$ implies either $x \in I$ or $y^n \in I$ for some $n > 0$.

I like to interpret maximal ideals as "the quotient is a field", and prime ideals as "the quotient is an integral domain". We can interpret primary ideals similarly as "the quotient is not 0, and every zero-divisor is nilpotent".

1.1. Exercise. Show that if $\mathfrak{q}$ is primary, then $\sqrt{\mathfrak{q}}$ is prime. If $\mathfrak{p} = \sqrt{\mathfrak{q}}$, we say that $\mathfrak{q}$ is $\mathfrak{p}$-*primary*.

1.2. Exercise. Show that if $\mathfrak{q}$ and $\mathfrak{q}'$ are $\mathfrak{p}$-primary, then so is $\mathfrak{q} \cap \mathfrak{q}'$.

1.3. Exercise (reality check). Find all the primary ideals in $\mathbb{Z}$. (Answer: (0) and (p^n).)

(Here is an unimportant side remark for experts; everyone else should skip this. Warning: a prime power need not be primary. An example is given in Atiyah-Macdonald, p. 51. $A = k[x,y,z]/(xy - z^2)$. then $\mathfrak{p} = (x,y)$ is prime but $\mathfrak{p}^2$ is not primary. Geometric hint that there is something going on: this is a ruling of a cone.)

A *primary decomposition* of an ideal $I \subset A$ is an expression of the ideal as a finite intersection of primary ideals.

$$I = \cap_{i=1}^n \mathfrak{q}_i$$

If there are "no redundant elements" (i.e. the $\sqrt{\mathfrak{q}_i}$ are all distinct, and for no i is $\mathfrak{q}_i \supset \cap_{j \neq i} \mathfrak{q}_j$), we say that the decomposition is *minimal*. Clearly any ideal with a primary decomposition has a minimal primary decomposition (using Exercise 1.2).

1.4. Exercise. Suppose A is a Noetherian ring. Show that every proper ideal $I \neq A$ has a primary decomposition. (Hint: Noetherian induction.)

1.5. Important Example. Find a minimal primary decomposition of (x^2, xy). (Answer: $(x) \cap (x^2, xy, y^n)$.)

In order to study these objects, we'll need a definition and a useful fact.

If $I \subset A$ is an ideal, and $x \in A$, then $(I : x) := \{a \in A : ax \in I\}$. (We will use this terminology only for this section.) For example, x is a *zero-divisor* if $(0 : x) \neq 0$.

1.6. Useful Exercise. (a) If $\mathfrak{p}$, $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ are prime ideals, and $\mathfrak{p} = \cap \mathfrak{p}_i$, show that $\mathfrak{p} = \mathfrak{p}_i$ for some i. (Hint: assume otherwise, choose $f_i \in \mathfrak{p}_i - \mathfrak{p}$, and consider $\prod f_i$.)
(b) If $\mathfrak{p} \supset \cap \mathfrak{p}_i$, then $\mathfrak{p} \supset \mathfrak{p}_i$ for some i.
(c) Suppose $I \subseteq \cup^n \mathfrak{p}_i$. Show that $I \subset \mathfrak{p}_i$ for some i. (Hint: by induction on n.)

1.7. *Theorem (Uniqueness of primary decomposition). — Suppose I has a minimal primary decomposition*

$$I = \cap_{i=1}^n \mathfrak{q}_i.$$

Then the $\sqrt{\mathfrak{q}_i}$ are precisely the prime ideals that are of the form

$$\sqrt{(I : x)}$$

for some $x \in A$. Hence this list of primes is independent of the decomposition.

These primes are called the *associated primes* of the ideal.

Proof. We make a very useful observation: for any $x \in A$,

$$(I : x) = (\cap \mathfrak{q}_i : x) = \cap (\mathfrak{q}_i : x),$$

from which

(2) $$\sqrt{(I : x)} = \cap \sqrt{(\mathfrak{q}_i : x)} = \cap_{x \notin \mathfrak{q}_j} \mathfrak{p}_j.$$

Now we prove the result.

Suppose first that $\sqrt{(I : x)}$ is prime, say $\mathfrak{p}$. Then $\mathfrak{p} = \cap_{x \notin \mathfrak{q}_j} \mathfrak{p}_j$ by (2), and by Exercise 1.6(a), $\mathfrak{p} = \mathfrak{p}_j$ for some j.

Conversely, we find an x such that $\sqrt{(I : x)} = \sqrt{\mathfrak{q}_i}$ $(= \mathfrak{p}_i)$. Take $x \in \cap_{j \neq i} \mathfrak{q}_j - \mathfrak{q}_i$ (which is possible by minimality of the primary decomposition). Then by (2), we're done. ☐

If A is a ring, the *associated primes* of A are the associated primes of 0.

1.8. Exercise. Show that these associated primes behave well with respect to localization. In other words if A is a Noetherian ring, and S is a multiplicative subset (so, as we've seen, there is an inclusion-preserving correspondence between the primes of $S^{-1}A$ and those primes of A not meeting S), then the associated primes of $S^{-1}A$ are just the associated primes of A not meeting S.

We then define the *associated points* of a locally Noetherian scheme X to be those points $p \in X$ such that, on any affine open set $\operatorname{Spec} A$ containing p, p corresponds to an associated prime of A. If furthermore X is quasicompact (i.e. X is a Noetherian scheme), then there are a finite number of associated points.

1.9. Exercise. Show that the minimal primes of 0 are associated primes. (We have now proved important fact **(1)**.) (Hint: suppose $\mathfrak{p} \supset \cap_{i=1}^n \mathfrak{q}_i$. Then $\mathfrak{p} = \sqrt{\mathfrak{p}} \supset \sqrt{\cap_{i=1}^n \mathfrak{q}_i} = \cap_{i=1}^n \sqrt{\mathfrak{q}_i} = \cap_{i=1}^n \mathfrak{p}_i$, so by Exercise 1.6(b), $\mathfrak{p} \supset \mathfrak{p}_i$ for some i. If $\mathfrak{p}$ is minimal, then as $\mathfrak{p} \supset \mathfrak{p}_i \supset (0)$, we must have $\mathfrak{p} = \mathfrak{p}_i$.) Show that there can be other associated primes that are not minimal. (Hint: Exercise 1.5.)

1.10. Exercise. Show that if A is reduced, then the only associated primes are the minimal primes. (This establishes **(2)**.)

The $\mathfrak{q}_i$ corresponding to minimal primes are unique, but the $\mathfrak{q}_i$ corresponding to other associated primes are not unique, but we will not need this fact, and hence won't prove it.

1.11. *Proposition. — The natural map* $R \to \prod R_{\mathfrak{p}}$ *is an inclusion.*

This establishes **(3)**.

Proof. Suppose $r \mapsto 0$. Thus there are $s_i \in R - \mathfrak{p}$ with $s_i r = 0$. Then $I := (s_1, \dots, s_n)$ is an ideal consisting only of zero-divisors. Hence $I \subseteq \cap \mathfrak{p}_i$. Then $I \subset \mathfrak{p}_i$ for some i by Exercise 1.6(c), contradicting $s_i \notin \mathfrak{p}_i$. □

1.12. *Proposition. — The set of zero-divisors is precisely the union of the associated primes.*

This establishes **(4)**: a function is a zero-divisor if and only if it vanishes at an associated point. Thus (for a Noetherian scheme) a function is a zero divisor if and only if its zero locus contains one of a finite set of points.

You may wish to try this out on the example of Exercise 1.5.

Proof. If $\mathfrak{p}_i$ is an associated prime, then $\mathfrak{p}_i = \sqrt{(0 : x)}$ from the proof of Theorem 1.7, so $\cup \mathfrak{p}_i$ is certainly contained in the set D of zero-divisors.

For the converse, verify the inclusions and equalities (**Exercise**)

$$D = \cup_{x \neq 0}(0 : x) \subseteq \cup_{x \neq 0} \sqrt{(0 : x)} \subseteq D.$$

Hence

$$D = \cup_{x \neq 0} \sqrt{(0 : x)} = \cup_x \left(\cap_{x \notin \mathfrak{q}_j} \mathfrak{p}_j \right) \subseteq \cup \mathfrak{p}_j$$

using (2). □

(Note for experts from Kirsten and Joe: Let X be a locally Noetherian scheme, $x \in X$. Then x is an associated point of X if and only if every nonunit of $\mathcal{O}_{X,x}$ is a zero-divisor. Proof: We must show that a prime ideal $\mathfrak{p}$ of a Noetherian ring A is associated if and only if every nonunit of $A_\mathfrak{p}$ is a zero-divisor, i.e., if and only if $\mathfrak{p}A_\mathfrak{p}$ is an associated prime in $A_\mathfrak{p}$. But this is obvious since primary decompositions respect localization.)

2. Invertible sheaves and divisors

We want to understand invertible sheaves (line bundles) on a given sheaf X. How can we describe many of them? How can we describe them all?

In order to answer this question, I should tell you a bit more about normality.

2.1. **A bit more on normality.**

I earlier defined normality in the wrong way, only for integral schemes: I said that an integral scheme X is normal if and only if for every affine open set $\operatorname{Spec} R$, R is integrally closed in its fraction field.

Here is the right definition: we say a scheme X is normal if all of its stalks $\mathcal{O}_{X,x}$ are normal. (In particular, all stalks are necessarily domains.) This is clearly a local property: if $\cup U_i$ is an open cover of X, then X is normal if and only if each U_i is normal.

Note that for Noetherian schemes, normality can be checked at closed points, as integral closure behaves well under localization (we've checked that), and every open set contains closed points of the scheme (we've checked that), so any point is a generization of a closed point.

As reducedness is a stalk-local property (we've checked that X is reduced if and only if all its stalks are reduced), a normal scheme is necessarily reduced. It is not true however that normal schemes are integral. For example, the disjoint union of two normal schemes is normal. So for example $\operatorname{Spec} k \coprod \operatorname{Spec} k \cong \operatorname{Spec}(k \times k) \cong \operatorname{Spec} k[x]/(x(x-1))$ is normal, but its ring of global sections is not a domain.

Unimportant remark. Normality satisfies the hypotheses of the Affine Covering Lemma, fairly tautologically, because it is a stalk-local property. We can say more explicitly and ring-theoretically what it means for $\operatorname{Spec} A$ to be normal, at least when A is Noetherian. It is that $\operatorname{Spec} A$ is normal if and only if A is reduced, and it is integrally closed in its ring of fractions. (The ring of fractions was defined earlier today in the discussion on associated points. It is the product of the localizations at the associated points. In this case, as A is reduced, it is the product of the localizations at the minimal primes.) Basically, most constructions that make sense for domains and involve function fields should be generalized to Noetherian rings in general, and the role of "function field" should be replaced by "ring of fractions".

I should finally state "Hartogs' theorem" explicitly and rigorously. (Caution: No one else calls this Hartogs' Theorem. I've called it this because of the metaphor to complex geometry.)

2.2. *"Hartogs' theorem". — Suppose A is a Noetherian normal domain. Then in $\operatorname{Frac}(A)$,*

$$A = \cap_{\mathfrak{p} \text{ height } 1} A_{\mathfrak{p}}.$$

More generally, if A is a product of Noetherian normal domains (i.e. $\operatorname{Spec} A$ is Noetherian normal scheme), then in the ring of fractions of A,

$$A = \cap_{\mathfrak{p} \text{ height } 1} A_{\mathfrak{p}}.$$

I stated the special case first so as to convince you that this isn't scary.

To show you the power of this result, let me prove Krull's Principal Ideal Theorem in the case of Noetherian normal domains. (Eventually, I hope to add to the notes a proof of Krull's Principal Ideal Theorem in general, as well as "Hartogs' Theorem".)

2.3. *Theorem (Krull's Principal Ideal Theorem for Noetherian normal domains). — Suppose A is a Noetherian normal domains, and $f \in A$. Then the minimal primes containing f are all of height precisely 1.*

Proof. The first statement implies the second: because A is a domain, the associated primes of $\operatorname{Spec} A$ are precisely the minimal (i.e. height 0) primes. If f is a not a zero-divisor, then f is not an element of any of these primes, by Proposition 1.12.

So we will now prove the first statement.

Suppose $f \in \mathrm{Frac}(A)$. We wish to show that the minimal primes containing f are all height 1. If there is one which is height greater than 1, then after localizing at this prime, we may assume that A is a local ring with maximal ideal $\mathfrak{m}$ of height at least 2, and that the only prime containing f is $\mathfrak{m}$. Let $g = 1/f \in \mathrm{Frac}(A)$. Then $g \in A_{\mathfrak{p}}$ for all height 1 primes $\mathfrak{p}$, so by "Hartogs' Theorem", $g \in A$. Thus $gf = 1$. But $g, f \in A$, and $f \in \mathfrak{m}$, so we have a contradiction.

Exercise. Suppose f and g are two global sections of a Noetherian normal scheme with the same poles and zeros. Show that each is a unit times the other.

I spent the rest of the class discussing Cartier divisors. I've put these notes with the class 18 notes.

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FOUNDATIONS OF ALGEBRAIC GEOMETRY CLASS 18

CONTENTS

Last day: Associated points; more on normality; invertible sheaves and divisors take 1.

Today: Invertible sheaves and divisors. Morphisms of schemes.

1. INVERTIBLE SHEAVES AND DIVISORS

We next develop some mechanism of understanding invertible sheaves (line bundles) on a given scheme X. Define $\operatorname{Pic} X$ to be the group of invertible sheaves on X. How can we describe many of them? How can we describe them all? Our goal for the first part of today will be to partially address this question. As an important example, we'll show that we have already found all the invertible sheaves on projective space $\mathbb{P}^n_k$ — they are the $\mathcal{O}(m)$.

One moral of this story will be that invertible sheaves will correspond to "codimension 1 information".

Recall one way of getting invertible sheaves, by way of *effective Cartier divisors*. Recall that an effective Cartier divisor is a closed subscheme such that there exists an affine cover such that on each one it is cut out by a single equation, not a zero-divisor. (This does not mean that on *any* affine it is cut out by a single equation — this notion doesn't satisfy the "gluability" hypothesis of the Affine Communication Lemma. If $I \subset R$ is generated by a non-zero divisor, then $I_f \subset R_f$ is too. But "not conversely". I might give an example later, involving an elliptic curve.) By Krull's Principal Ideal Theorem, it is pure codimension 1.

Remark: if $I = (u) = (v)$, and u is not a zero-divisor, then u and v differ multiplicatively by a unit in R. Proof: $u \in (v)$ implies $u = av$. Similarly $v = bu$. Thus $u = abu$, from which $u(1 - ab) = 0$. As u is not a zero-divisor, $1 = ab$, so a and b are units. In other words, the generator of such an ideal is well-defined up to a unit.

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The reason we care: effective Cartier divisors give invertible sheaves. If $\mathcal{I}$ is an effective Cartier divisor on X, then $\mathcal{I}$ is an invertible sheaf. Reason: locally, sections are multiples of a single generator u, and there are no "relations".

Recall that the invertible sheaf $\mathcal{O}(D)$ corresponding to an effective Cartier divisor is defined to be the *dual* of the ideal sheaf $\mathcal{I}_D$. The ideal sheaf itself is sometimes denoted $\mathcal{O}(-D)$. We have an exact sequence

$$0 \to \mathcal{O}(-D) \to \mathcal{O} \to \mathcal{O}_D \to 0.$$

The invertible sheaf $\mathcal{O}(D)$ has a canonical section: Dualizing $0 \to \mathcal{I} \to \mathcal{O}$ gives us $\mathcal{O} \to \mathcal{I}^*$.

Exercise. This section vanishes along our actual effective Cartier divisor.

Exercise. Conversely, if $\mathcal{L}$ is an invertible sheaf, and s is a section that is not locally a zero divisor (make sense of this!), then $s = 0$ cuts out an effective Cartier divisor D, and $\mathcal{O}(D) \cong \mathcal{L}$. (If X is locally Noetherian, "not locally a zero divisor" translate to "does not vanish at an associated point".)

Define the *sum* of two effective Cartier divisors as follows: if $I = (u)$ (locally) and $J = (v)$, then the sum corresponds to (uv) locally. (Verify that this is well-defined!)

Exercise. Show that $\mathcal{O}(D + E) \cong \mathcal{O}(D) \otimes \mathcal{O}(E)$.

Thus we have a map of semigroups, from effective Cartier divisors to invertible sheaves with sections not locally zero-divisors (and hence also to the Picard group of invertible sheaves).

Hence we can get a bunch of invertible sheaves, by taking differences of these two. The surprising fact: we "usually get them all"! In fact it is very hard to describe an invertible sheaf on a finite type k-scheme that is not describable in such a way (we will see later today that there are none if the scheme is nonsingular or even factorial; and we might see later in the year that there are none if the scheme is quasiprojective).

Instead, I want to take another tack. Some of what we do will generalize to the non-normal case, which is certainly important, and experts are invited to think about this.

Define a *Weil divisor* as a formal sum of height 1 irreducible closed subsets of X. (This makes sense more generally on any pure dimensional, or even locally equidimensional, scheme.) In other words, a Weil divisor is defined to be an object of the form

$$\sum_{Y \subset X \text{ height } 1} n_Y [Y]$$

the n_Y are integers, all but a finite number of which are zero. Weil divisors obviously form an abelian group, denoted $\operatorname{Weil} X$.

A Weil divisor is said to be *effective* if $n_Y \geq 0$ for all Y. In this case we say $D \geq 0$, and by $D_1 \geq D_2$ we mean $D_1 - D_2 \geq 0$. The *support* of a Weil divisor D is the subset

$\cup_{n_Y \neq 0} Y$. If $U \subset X$ is an open set, there is a natural restriction map $\mathrm{Weil}\, X \to \mathrm{Weil}\, U$, where $\sum n_Y[Y] \mapsto \sum_{Y \cap U \neq \emptyset} n_Y [Y \cap U]$.

Suppose now that X is a Noetherian scheme, regular in codimension 1. We add this hypothesis because we will use properties of discrete valuation rings. Suppose that $\mathcal{L}$ is an invertible sheaf, and s a rational section not vanishing on any irreducible component of X. Then s determines a Weil divisor

$$\mathrm{div}(s) := \sum_Y \mathrm{val}_Y(s)[Y].$$

(Recall that $\mathrm{val}_Y(s) = 0$ for all but finitely many Y, by problem 46 on problem set 5.) This is the "divisor of poles and zeros of s". (To determine the valuation $\mathrm{val}_Y(s)$ of s along Y, take any open set U containing the generic point of Y where $\mathcal{L}$ is trivializable, along with any trivialization over U; under this trivialization, s is a function on U, which thus has a valuation. Any two such trivializations differ by a unit, so this valuation is well-defined.) This map gives a group homomorphism
(1)
$$\mathrm{div} : \{(\text{invertible sheaf } \mathcal{L}, \text{rational section } s \text{ not vanishing at any minimal prime})\}/\Gamma(X, \mathcal{O}_X^*) \to \mathrm{Weil}\, X.$$

1.1. *Exercise.* (a) (divisors of rational functions) Verify that on $\mathbb{A}^1_k$, $\mathrm{div}(x^3/(x+1)) = 3[(x)] - [(x+1)] = 3[0] - [-1]$.
(b) (divisor of a rational sections of a nontrivial invertible sheaf) Verify that on $\mathbb{P}^1_k$, there is a rational section of $\mathcal{O}(1)$ "corresponding to" x^2/y. Calculate $\mathrm{div}(x^2/y)$.

We want to classify all invertible sheaves on X, and this homomorphism (1) will be the key. Note that any invertible sheaf will have such a rational section (for each irreducible component, take a non-empty open set not meeting any other irreducible component; then shrink it so that $\mathcal{L}$ is trivial; choose a trivialization; then take the union of all these open sets, and choose the section on this union corresponding to 1 under the trivialization). We will see that in reasonable situations, this map div will be injective, and often even an isomorphism. Thus by forgetting the rational section (taking an appropriate quotient), we will have described the Picard group. Let's put this strategy into action. *Suppose from now on that* X *is normal.*

1.2. *Proposition. — The map* div *is injective.*

Proof. Suppose $\mathrm{div}(\mathcal{L}, s) = 0$. Then s has no poles. Hence by Hartogs' theorem, s is a regular section. Now s vanishes nowhere, so s gives an isomorphism $\mathcal{O}_X \to \mathcal{L}$ (given by $1 \mapsto s$). □

Motivated by this, we try to find the inverse map to div.

Definition. Suppose D is a Weil divisor. If $U \subset X$ is an open subscheme, define $\mathrm{Frac}(U)$ to be the field of total fractions of U, i.e. the product of the stalks at the minimal primes of U. (As described earlier, if U is irreducible, this is the function field.) Define $\mathrm{Frac}(U)^*$ to be those rational functions not vanishing at any generic point of U (i.e. not vanishing on any irreducible component of U). Define the sheaf $\mathcal{O}(D)$ by

$$\Gamma(U, \mathcal{O}(D)) := \{s \in \mathrm{Frac}(U)^* : \operatorname{div} s + D|_U \geq 0\}.$$

Note that the sheaf $\mathcal{O}(D)$ has a canonical rational section, corresponding to $1 \in \mathrm{Frac}(U)^*$.

1.3. *Proposition. — Suppose $\mathcal{L}$ is an invertible sheaf, and s is a rational section not vanishing on any irreducible component of X. Then there is an isomorphism $(\mathcal{L}, s) \cong (\mathcal{O}(\operatorname{div} s), t)$, where t is the canonical section described above.*

Proof. We first describe the isomorphism $\mathcal{O}(\operatorname{div} s) \cong \mathcal{L}$. Over open subscheme $U \subset X$, we have a bijection $\Gamma(U, \mathcal{L}) \to \Gamma(U, \mathcal{O}(\operatorname{div} s))$ given by $s' \mapsto s'/s$, with inverse obviously given by $t' \mapsto st'$. Clearly under this bijection, s corresponds to the section 1 in $\mathrm{Frac}(U)^*$; this is the section we are calling t. $\square$

We denote the subgroup of $\mathrm{Weil}\, X$ corresponding to divisors of rational *functions* the subgroup of *principal divisors*, which we denote $\mathrm{Prin}\, X$. Define the *class group* of X, $\mathrm{Cl}\, X$, by $\mathrm{Weil}\, X / \mathrm{Prin}\, X$. By taking the quotient of the inclusion (1) by $\mathrm{Prin}\, X$, we have the inclusion

$$\mathrm{Pic}\, X \hookrightarrow \mathrm{Cl}\, X.$$

We're now ready to get a hold of $\mathrm{Pic}\, X$ rather explicitly!

First, some algebraic preliminaries.

1.4. *Exercise.* Suppose that A is a Noetherian domain. Show that A is a Unique Factorization Domain if and only if all height 1 primes are principal. You can use this to answer that homework problem, about showing that $k[w, x, y, z]/(wz - xy)$ is not a Unique Factorization Domain.

1.5. *Exercise.* Suppose that A is a Noetherian domain. Show that A is a Unique Factorization Domain if and only if A is integrally closed and $\mathrm{Cl}\,\mathrm{Spec}\, A = 0$. (One direction is easy: we have already shown that Unique Factorization Domains are integrally closed in their fraction fields. Also, the previous exercise shows that all height 1 primes are principal, so that implies that $\mathrm{Cl}\,\mathrm{Spec}\, A = 0$. It remains to show that if A is integrally closed and $\mathrm{Cl}\, X = 0$, then all height 1 prime ideals are principal. "Hartogs" may arise in your argument.)

Hence $\mathrm{Cl}(\mathbb{A}^n_k) = 0$, so $\boxed{\mathrm{Pic}(\mathbb{A}^n_k) = 0}$. (Geometers will find this believable: "$\mathbb{C}^n$ is a contractible manifold, and hence should have no nontrivial line bundles".)

Another handy trick is the following. Suppose Z is an irreducible codimension 1 subset of X. Then we clearly have an exact sequence:

$$0 \longrightarrow \mathbb{Z} \xrightarrow{1 \mapsto [Z]} \mathrm{Weil}\, X \longrightarrow \mathrm{Weil}(X - Z) \longrightarrow 0.$$

When we take the quotient by principal divisors, we get:

$$\mathbb{Z} \xrightarrow{1 \mapsto [Z]} \operatorname{Cl} X \longrightarrow \operatorname{Cl}(X - Z) \longrightarrow 0.$$

For example, let $X = \mathbb{P}^n_k$, and Z be the hyperplane $x_0 = 0$. We have

$$\mathbb{Z} \to \operatorname{Cl} \mathbb{P}^n_k \to \operatorname{Cl} \mathbb{A}^n_k \to 0$$

from which $\operatorname{Cl} \mathbb{P}^n_k = \mathbb{Z}[Z]$ (which is $\mathbb{Z}$ or 0), and $\operatorname{Pic} \mathbb{P}^n_k$ is a subgroup of this.

1.6. *Important exercise.* Verify that $[Z] \to \mathcal{O}(1)$. In other words, let Z be the Cartier divisor $x_0 = 0$. Show that $\mathcal{O}(Z) \cong \mathcal{O}(1)$. (For this reason, people sometimes call $\mathcal{O}(1)$ the *hyperplane class* in $\operatorname{Pic} X$.)

Hence $\operatorname{Pic} \mathbb{P}^n_k \hookrightarrow \operatorname{Cl} \mathbb{P}^n_k$ is an isomorphism, and $\boxed{\operatorname{Pic} \mathbb{P}^n_k \cong \mathbb{Z}}$, with generator $\mathcal{O}(1)$. The *degree* of an invertible sheaf on $\mathbb{P}^n$ is defined using this: the degree of $\mathcal{O}(d)$ is of course d.

More generally, if X is *factorial* — all stalks are Unique Factorization Domains — then for any Weil divisor D, $\mathcal{O}(D)$ is invertible, and hence the map $\operatorname{Pic} X \to \operatorname{Cl} X$ is an isomorphism. (Proof: It will suffice to show that $[Y]$ is Cartier if Y is any irreducible codimension 1 set. Our goal is to cover X by open sets so that on each open set U there is a function whose divisor is $[Y \cap U]$. One open set will be $X - Y$, where we take the function 1. Next, suppose $x \in Y$; we will find an open set $U \subset X$ containing x, and a function on it. As $\mathcal{O}_{X,x}$ is a unique factorization domain, the prime corresponding to 1 is height 1 and hence principal (by Exercise 1.4). Let $f \in \operatorname{Frac} A$ be a generator. Then f is regular at x. f has a finite number of zeros and poles, and through x there is only one 0, notably $[Y]$. Let U be X minus all the others zeros and poles.)

I will now mention a bunch of other examples of class groups and Picard groups you can calculate.

For the first, I want to note that you can restrict invertible sheaves on X to any subscheme Y, and this can be a handy way of checking that an invertible sheaf is not trivial. For example, if X is something crazy, and $Y \cong \mathbb{P}^1$, then we're happy, because we understand invertible sheaves on $\mathbb{P}^1$. Effective Cartier divisors sometimes restrict too: if you have effective Cartier divisor on X, then it restricts to a closed subscheme on Y, locally cut out by one equation. If you are fortunate that this equation doesn't vanish on any associated point of Y, then you get an effective Cartier divisor on Y. You can check that the restriction of effective Cartier divisors corresponds to restriction of invertible sheaves.

1.7. *Exercise: a torsion Picard group.* Show that Y is an irreducible degree d hypersurface of $\mathbb{P}^n$. Show that $\operatorname{Pic}(\mathbb{P}^n - Y) \cong \mathbb{Z}/d$. (For differential geometers: this is related to the fact that $\pi_1(\mathbb{P}^n - Y) \cong \mathbb{Z}/d$.)

1.8. *Exercise.* Let $X = \operatorname{Proj} k[w, x, y, z]/(wz - xy)$, a smooth quadric surface. Show that $\operatorname{Pic} X \cong \mathbb{Z} \oplus \mathbb{Z}$ as follows: Show that if L and M are two lines in different rulings (e.g. $L = V(w, x)$ and $M = V(w, y)$), then $X - L - M \cong \mathbb{A}^2$. This will give you a surjection

$\mathbb{Z} \oplus \mathbb{Z} \twoheadrightarrow \operatorname{Cl} X$. Show that $\mathcal{O}(L)$ restricts to $\mathcal{O}$ on L and $\mathcal{O}(1)$ on M. Show that $\mathcal{O}(M)$ restricts to $\mathcal{O}$ on M and $\mathcal{O}(1)$ on L. (This is a bit longer to do, but enlightening.)

1.9. *Exercise.* Let $X = \operatorname{Spec} k[w,x,y,z]/(xy - z^2)$, a cone. show that $\operatorname{Pic} X = 1$, and $\operatorname{Cl} X \cong \mathbb{Z}/2$. (Hint: show that the ruling $Z = \{x = z = 0\}$ generates $\operatorname{Cl} X$ by showing that its complement is isomorphic to $\mathbb{A}^2_k$. Show that $2[Z] = \operatorname{div}(x)$ (and hence principal), and that Z is not principal (an example we did when discovering the power of the Zariski tangent space).

Note: on curves, the invertible sheaves correspond to formal sums of points, modulo equivalence relation.

Number theorists should note that we have recovered a common description of the class group: formal sums of primes, modulo an equivalence relation.

Remark: Much of this discussion works without the hypothesis of normality, and indeed because non-normal schemes come up all the time, we need this additional generality. Think through this if you like.

2. Morphisms of schemes

Here are two motivations that will "glue together".

(a) We'll want morphisms of affine schemes $\operatorname{Spec} R \to \operatorname{Spec} S$ to be precisely the ring maps $S \to R$. Then we'll want maps of schemes to be things that "look like this". "the category of affine schemes is opposite to the category of rings". More correctly there is an equivalence of categories...

(b) We are also motivated by the theory of differentiable manifolds. We'll want a continuous maps from the underlying topological spaces $f : X \to Y$, along with a "pullback morphism" $f^\# : \mathcal{O}_S \to f_* \mathcal{O}_X$. There are many things we'll want to be true, that seem make a tall order; a clever idea will give us all of this for free. (i) Certainly values at points should map. They can't be the same: $\operatorname{Spec} \mathbb{C} \to \operatorname{Spec} \mathbb{R}$. (ii) $\operatorname{Spec} k[\epsilon]/\epsilon^2 \to \operatorname{Spec} k[\delta]/\delta^2$ is given by a map $\delta \mapsto q\epsilon$. These aren't distinguished by maps on points. (iii) Suppose you have a function σ on Y (i.e. $\sigma \in \Gamma(Y, \mathcal{O}_Y)$. Then it will pull back to a function $f^{-1}(\sigma)$ on X. However we make sense of pullbacks of functions (i) and (ii), certainly the locus where $f^{-1}(\sigma)$ vanishes on X should be the pullback of the locus where σ vanishes on Y. This will imply that the maps on stalks will be a local map (if $f(p) = q$ then $f^\# : \mathcal{O}_{Y,q} \to \mathcal{O}_{X,p}$ sends the maximal ideal. translating to: then germs of functions vanishing at q pullback to germs of functions vanishing at p). This last thing does it for us.

3. Ringed spaces and their morphisms

A ringed space is a topological space X along with a sheaf $\mathcal{O}_X$ of rings (called the *structure sheaf*. Our central example is a scheme. Another example is a differentiable manifold with the analytic topology and the sheaf of differentiable functions.

A **morphism of ringed spaces** $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a continuous map $f : X \to Y$ (also sloppily denoted by the same name "f") along with a morphism $f^\# : \mathcal{O}_Y \to f_*\mathcal{O}_X$ of sheaves on Y (or equivalently but less usefully $f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$ of sheaves on X, by adjointness). The morphism is often denoted $X \to Y$ when the structure sheaves and morphisms "between" them are clear from the context. There is an obvious notion of composition of morphisms; hence there is a category of ringed spaces. Hence we have notion of automorphisms and isomorphisms.

Slightly unfortunate notation: $f : X \to Y$ often denotes everything. Also used for maps of underlying sets, or underlying topological spaces. Usually clear from context.

3.1. *Exercise.* If $W \subset X$ and $Y \subset Z$ are both open immersions of ringed spaces, show that any morphism of ringed spaces $X \to Y$ induces a morphism of ringed spaces $W \to Z$.

3.2. *Exercise.* Show that morphisms of ringed spaces glue. In other words, suppose X and Y are ringed spaces, $X = \cup_i U_i$ is an open cover of X, and we have morphisms of ringed spaces $f_i : U_i \to Y$ that "agree on the overlaps", i.e. $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$. Show that there is a unique morphism of ringed spaces $f : X \to Y$ such that $f|_{U_i} = f_i$. (Long ago we had an exercise proving this for topological spaces.)

3.3. *Easy important exercise.* Given a morphism of ringed spaces $f : X \to Y$ with $f(p) = q$, show that there is a map of stalks $(\mathcal{O}_Y)_q \to (\mathcal{O}_X)_p$.

3.4. Important Example. Suppose $f^\# : S \to R$ is a morphism of rings. Define a morphism of ringed spaces as follows. $f : sp(\operatorname{Spec} R) \to sp(\operatorname{Spec} S)$. First as sets. $\mathfrak{p}$ prime in S, then ${f^\#}^{-1}(\mathfrak{p})$ is prime in R.

We interrupt this definition for a picture: $R = \operatorname{Spec} k[x, y]$, $S = \operatorname{Spec} k[t]$, $t \mapsto x$. Draw picture. Look at primes $(x - 2, y - 3)$. Look at (0). Look at $(x - 3)$. $(y - x^2)$.

It's a continuous map of topological spaces: $D(s)$ pulls back to $D(f^\# s)$. Now the map on sheaves. If $s \in S$, then show that $\Gamma(D(s), f_*\mathcal{O}_R) = R_{f^\# s} \cong R \otimes_S S_s$. (**Exercise.** Verify that $R_{f^\# s} \cong R \otimes_S S_s$ if you haven't seen this before.) Show that $f_* : \Gamma(D(s), \mathcal{O}_S) = S_s \to \Gamma(D(s), f_*\mathcal{O}_R) = R \otimes_S S_s$ given by $s' \mapsto 1 \otimes s'$ is a morphism of sheaves on the distinguished base of S, and hence defines a morphism of sheaves $f_*\mathcal{O}_R \to \mathcal{O}_S$.

4. Definition of morphisms of schemes

A morphism $f : X \to Y$ of schemes is a morphism of ringed spaces. Sadly, if X and Y are schemes, then there are morphisms $X \to Y$ *as ringed spaces* that are not morphisms as schemes. (See Example II.2.3.2 in Hartshorne for an example.)

The idea behind definition of morphisms is as follows. We define morphisms of affine schemes as in Important Example 3.4. (Note that the category of affine schemes is "opposite to the category of rings": given a morphisms of schemes, we get a map of rings in the opposite direction, and vice versa.)

4.1. *Definition/Proposition.* — A morphism of schemes $f : X \to Y$ is a morphism of ringed spaces that looks locally like morphisms of affines. In other words, if $\operatorname{Spec} A$ is an affine open subset of X and $\operatorname{Spec} B$ is an affine open subset of Y, and $f(\operatorname{Spec} A) \subset \operatorname{Spec} B$, then the induced morphism of ringed spaces (Exercise 3.1) is a morphism of affine schemes. It suffices to check on a set $(\operatorname{Spec} A_i, \operatorname{Spec} B_i)$ where the $\operatorname{Spec} A_i$ form an open cover X.

We could prove the proposition using the affine communication theorem, but there's a clever trick. For this we need a digression on locally ringed spaces. They will not be used hereafter.

A *locally ringed space* is a ringed space $(X, \mathcal{O}_X)$ such that the stalks $\mathcal{O}_{X,x}$ are all local rings. A *morphism of locally ringed spaces* $f : X \to Y$ is a morphism of ringed spaces such that the induced map of stalks (Exercise 3.3) $\mathcal{O}_{Y,q} \to \mathcal{O}_{X,p}$ sends the maximal ideal of the former to the maximal ideal of the latter. (This is sometimes called a "local morphism of local rings".) This means something rather concrete and intuitive: "if $p \mapsto q$, and g is a function vanishing at q, then it will pull back to a function vanishing at p." Note that locally ringed spaces form a category.

4.2. *Exercise.* Show that morphisms of locally ringed spaces glue (cf. Exercise 3.2). (Hint: Basically, the proof of Exercise 3.2 works.)

4.3. *Easy important exercise.* (a) Show that $\operatorname{Spec} R$ is a locally ringed space. (b) The morphism of ringed spaces $f : \operatorname{Spec} R \to \operatorname{Spec} S$ defined by a ring morphism $f^\# : S \to R$ (Exercise 3.4) is a morphism of locally ringed spaces.

Proposition 4.1 now follows from:

4.4. *Key Proposition.* — *If* $f : \operatorname{Spec} R \to \operatorname{Spec} S$ *is a morphism of locally ringed spaces then it is the morphism of locally ringed spaces induced by the map* $f^\# : S = \Gamma(\operatorname{Spec} S, \mathcal{O}_{\operatorname{Spec} S}) \to \Gamma(\operatorname{Spec} R, \mathcal{O}_{\operatorname{Spec} R}) = R$. .

Proof. Suppose $f : \operatorname{Spec} R \to \operatorname{Spec} S$ is a morphism of locally ringed spaces. Then we wish to show that $f^\# : \mathcal{O}_{\operatorname{Spec} S} \to f_* \mathcal{O}_{\operatorname{Spec} R}$ is the morphism of sheaves given by Exercise 3.4 (cf. Exercise 4.3(b)). It suffices to checked this on the distinguished base.

Note that if $s \in S$, $f^{-1}(D(s)) = D(f^\# s)$; this is where we use the hypothesis that f is a morphism of locally ringed spaces.

The commutative diagram

$$\begin{array}{ccc} \Gamma(\operatorname{Spec} S, \mathcal{O}_{\operatorname{Spec} S}) & \xrightarrow{f^{\#}_{\operatorname{Spec} S}} & \Gamma(\operatorname{Spec} R, \mathcal{O}_{\operatorname{Spec} R}) \\ \downarrow & & \downarrow \otimes_S S_s \\ \Gamma(D(s), \mathcal{O}_{\operatorname{Spec} S}) & \xrightarrow{f^{\#}_{D(s)}} & \Gamma(D(f^{\#}s), \mathcal{O}_{\operatorname{Spec} R}) \end{array}$$

may be written as

$$\begin{array}{ccc} S & \xrightarrow{f^{\#}_{\operatorname{Spec} S}} & R \\ \downarrow & & \downarrow \otimes_S S_s \\ S_s & \xrightarrow{f^{\#}_{D(s)}} & R_{f^{\#}s}. \end{array}$$

We want that $f^{\#}_{D(s)} = (f^{\#}_{\operatorname{Spec} S})_s$. This is clear from the commutativity of that last diagram. □

In particular, we can check on an affine cover, and then we'll have it on all affines. Also, morphisms glue (Exercise 4.2). And: the composition of two morphisms is a morphism.

4.5. *Exercise.* Make sense of the following sentence: "$\mathbb{A}^{n+1} - \vec{0} \to \mathbb{P}^n$ given by

$$(x_0, x_1, \ldots, x_{n+1}) \mapsto [x_0; x_1; \ldots; x_n]$$

is a morphism of schemes." Caution: you can't just say where points go; you have to say where functions go. So you'll have to divide these up into affines, and describe the maps, and check that they glue.

4.6. The category of schemes (or k-schemes, or R-schemes, or Z-schemes).

We have thus defined a *category* of schemes. We then have notions of **isomorphism** and **automorphism**. It is often convenient to consider subcategories. For example, the category of k-schemes (where k is a field) is defined as follows. The objects are morphisms of the form $\begin{array}{c} X \\ \downarrow \\ \operatorname{Spec} k \end{array}$. (This is definition is identical to the one we gave earlier, but in a more satisfactory form.) The morphism (in the category of schemes, not in the category of k-schemes) $X \to \operatorname{Spec} k$ is called the **structure morphism**. The morphisms in the category of k-schemes are commutative diagrams

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow & & \downarrow \\ \operatorname{Spec} k & \xrightarrow{=} & \operatorname{Spec} k \end{array}$$

which is more conveniently written as a commutative diagram

$$\begin{array}{ccc} X & \longrightarrow & Y \\ & \searrow \quad \swarrow & \\ & \operatorname{Spec} k. & \end{array}$$

For example, complex geometers may consider the category of $\mathbb{C}$-schemes.

When there is no confusion, simply the top row of the diagram is given. More generally, if R is a ring, the category of R-schemes is defined in the same way, with R replacing k. And if Z is a scheme, the category of Z-schemes is defined in the same way, with Z replacing $\operatorname{Spec} k$.

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FOUNDATIONS OF ALGEBRAIC GEOMETRY CLASS 19

CONTENTS

Last day: Associated points; more on normality; invertible sheaves and divisors take 1.

Today: Maps to affine schemes; surjective, open immersion, closed immersion, quasicompact, locally of finite type, finite type, affine morphism, finite, quasifinite. Images of morphisms: constructible sets, and Chevalley's theorem (finite type morphism of Noetherian schemes sends constructibles to constructibles).

Last day, I defined a morphism of schemes $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ as follows.

I first defined the notion of a morphism of ringed spaces $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$, which is a continuous map of topological spaces $f : X \to Y$ along with a map of sheaves of rings (on Y) $\mathcal{O}_Y \to f_*\mathcal{O}_X$, or equivalently (by adjointness of inverse image and pushfoward) $f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$ (a map of sheaves of rings on X). This should be seen as a description of how to pull back functions on Y to get functions on X.

An example is a morphism of affine schemes $\operatorname{Spec} A \to \operatorname{Spec} B$. These correspond to morphisms of rings $B \to A$.

Then a morphism of schemes $X \to Y$ can be defined as a morphism of these ringed spaces, that locally looks like a morphism of affine schemes. In other words, X can be covered by affine open sets, such that for each such $\operatorname{Spec} R$, there is an affine open set $\operatorname{Spec} S$ of Y containing its image, such that the map $\operatorname{Spec} R \to \operatorname{Spec} S$ is of the form described in the primordial example.

We proved this by temporarily introducing a new concept, that of a *locally ringed space*. Then a morphism of schemes $X \to Y$ is just the same as a morphism of locally ringed spaces; we showed this by showing this for affine schemes.

I encouraged you to get practice with this in the following exercise, to make sense of the map $\mathbb{A}^{n+1} - 0 \to \mathbb{P}^n$ "given by" $(x_0, \dots, x_n) \mapsto [x_0; \dots; x_n]$.

We thus have described the *category of schemes*. The notion of an isomorphism of schemes subsumes our earlier definition.

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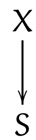
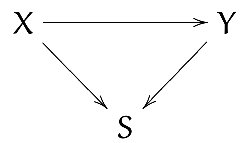
I described the category of k-schemes, or more generally A-schemes where A is a ring. More generally, if S is a scheme, we have the category of S-schemes. The objects are diagrams of the form



and morphisms are commutative diagrams of the form



The category of k-schemes corresponds to the case $S = \operatorname{Spec} k$, and the category of A-schemes correspond to the case $S = \operatorname{Spec} A$.

We now give some examples.

0.1. *Exercise.* Show that morphisms $X \to \operatorname{Spec} A$ are in natural bijection with ring morphisms $A \to \Gamma(X, \mathcal{O}_X)$. (Hint: Show that this is true when X is affine. Use the fact that morphisms glue.)

In particular, there is a canonical morphism from a scheme to Spec of its space of global sections. (Warning: Even if X is a finite-type k-scheme, the ring of global sections might be nasty! In particular, it might not be finitely generated.)

Example: Suppose S_* is a graded ring, with $S_0 = A$. Then we get a natural morphism $\operatorname{Proj} S_* \to \operatorname{Spec} A$. For example, we have a natural map $\mathbb{P}^n_A \to \operatorname{Spec} A$

0.2. *Exercise.* Show that $\operatorname{Spec} \mathbb{Z}$ is the final object in the category of schemes. In other words, if X is any scheme, there exists a unique morphism to $\operatorname{Spec} \mathbb{Z}$. (Hence the category of schemes is isomorphic to the category of $\mathbb{Z}$-schemes.)

0.3. *Exercise.* Show that morphisms $X \to \operatorname{Spec} \mathbb{Z}[t]$ correspond to global sections of the structure sheaf.

This is one of our first explicit examples of an important idea, that of representable functors! This is a very useful idea, whose utility isn't immediately apparent. We have a contravariant functor from schemes to sets, taking a scheme to its set of global sections. We have another contravariant functor from schemes to sets, taking X to $\operatorname{Hom}(X, \operatorname{Spec} \mathbb{Z}[t])$. This is describing an "isomorphism" of the functors. More precisely, we are describing an isomorphism $\Gamma(X, \mathcal{O}_X) \cong \operatorname{Hom}(X, \operatorname{Spec} \mathbb{Z}[t])$ that behaves well with respect to morphisms of schemes: given any morphism $f : X \to Y$, the diagram

$$\begin{array}{ccc} \Gamma(Y, \mathcal{O}_Y) & \xrightarrow{\sim} & \mathrm{Hom}(Y, \mathrm{Spec}\,\mathbb{Z}[t]) \\ \downarrow f^* & & \downarrow f\circ \\ \Gamma(X, \mathcal{O}_X) & \xrightarrow{\sim} & \mathrm{Hom}(X, \mathrm{Spec}\,\mathbb{Z}[t]) \end{array}$$

commutes. *Given a contravariant functor from schemes to sets, by Yoneda's lemma, there is only one possible scheme Y, up to isomorphism, such that there is a natural isomorphism between this functor and* $\mathrm{Hom}(\cdot, Y)$. If there is such a Y, we say that the functor is *representable*.

Here are a couple of examples of something you've seen to put it in context. (i) The contravariant functor $\mathrm{Hom}(\cdot, Y)$ (i.e. $X \mapsto \mathrm{Hom}(X, Y)$) is representable by Y. (ii) Suppose we have morphisms $X, Y \to Z$. The contravariant functor $\mathrm{Hom}(\cdot, X) \times_{\mathrm{Hom}(\cdot, Z)} \mathrm{Hom}(\cdot, Y)$ is representable if and only if the fibered product $X \times_Z Y$ exists (and indeed then the contravariant functor is represented by $\mathrm{Hom}(\cdot, X \times_Z Y)$). This is purely a translation of the definition of the fibered product — you should verify this yourself.

Remark for experts: The global sections form something better than a set — they form a scheme. You can define the notion of ring scheme, and show that if a contravariant functor from schemes to rings is representable (as a contravariant functor from schemes to sets) by a scheme Y, then Y is guaranteed to be a ring scheme. The same is true for group schemes.

0.4. *Related Exercise.* Show that global sections of $\mathcal{O}_X^*$ correspond naturally to maps to $\mathrm{Spec}\,\mathbb{Z}[t, t^{-1}]$. ($\mathrm{Spec}\,\mathbb{Z}[t, t^{-1}]$ is a *group scheme*. We will discuss group schemes more in class 36.)

Morphisms and tangent spaces. Suppose $f : X \to Y$, and $f(p) = q$. Then if we were in the category of manifolds, we would expect a tangent map, from the tangent space of p to the tangent space at q. Indeed that is the case; we have a map of stalks $\mathcal{O}_{Y,q} \to \mathcal{O}_{X,p}$, which sends the maximal ideal of the former $\mathfrak{n}$ to the maximal ideal of the latter $\mathfrak{m}$ (we have checked that this is a "local morphism" when we briefly discussed locally ringed spaces). Thus $\mathfrak{n}^2 \to \mathfrak{m}^2$, from which $\mathfrak{n}/\mathfrak{n}^2 \to \mathfrak{m}/\mathfrak{m}^2$, from which we have a natural map $(\mathfrak{m}/\mathfrak{m}^2)^\vee \to (\mathfrak{n}/\mathfrak{n}^2)^\vee$. This is the map from the tangent space of p to the tangent space at q that we sought.

0.5. *Exercise.* Suppose X is a finite type k-scheme. Describe a natural bijection $\mathrm{Hom}(\mathrm{Spec}\, k[\epsilon]/\epsilon^2, X)$ to the data of a k-valued point (a point whose residue field is k, necessarily closed) and a tangent vector at that point.

1. Properties of morphisms of schemes

I'm going to define a lot of useful notions.

The notion of **surjective** will have the same meaning as always: $X \to Y$ is surjective if the map of sets is surjective.

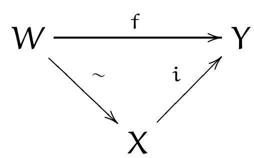
1.1. *Unimportant Exercise.* Show that integral ring extensions induces a surjective map of spectra. (Hint: Recall the Cohen-Seidenberg Going-up Theorem: Suppose $B \subset A$ is an inclusion of rings, with A integrally dependent on B. For any prime $\mathfrak{q} \subset B$, there is a prime $\mathfrak{p} \subset A$ such that $\mathfrak{p} \cap B = \mathfrak{q}$.)

Definition. If U is an open subscheme of Y, then there is a natural morphism $U \to Y$. We say that $f : X \to Y$ is an *open immersion* if f gives an isomorphism from X to an open subscheme of Y. (Really, we want to say that X "is" an open subscheme of Y.) Observe that if f is an open immersion, then $f^{-1}\mathcal{O}_Y \cong \mathcal{O}_X$.

1.2. *Exercise.* Suppose $i : U \to Z$ is an open immersion, and $f : Y \to Z$ is any morphism. Show that $U \times_Z Y$ exists. (Hint: I'll even tell you what it is: $(f^{-1}(U), \mathcal{O}_Y|_{f^{-1}(U)})$.)

1.3. *Easy exercise.* Show that open immersions are monomorphisms.

Suppose X is a closed subscheme of Y. Then there is a natural morphism $i : X \to Y$: on the affine open set $\operatorname{Spec} R$ of Y, where X is "cut out" by the ideal $I \subset R$, this corresponds to the ring map $R \to R/I$. A morphism $f : W \to Y$ is a **closed immersion** if it can be factored as



where $i : X \to Y$ is a closed subscheme. (Really, we want to say that W "is" a closed subscheme of Y.)

(Example: If X is a scheme and X^{red} is its reduction, then there is a natural closed immersion $X^{\text{red}} \to X$.)

1.4. *Proposition (the property of being a closed immersion is affine-local on the target). — Suppose* $f : X \to Y$ *is a morphism of schemes. It suffices to check that* f *is a closed immersion on an affine open cover of* Y.

Reason: The way in which closed subschemes are defined is local on the target.

(In particular, a morphism of affine schemes is a closed immersion if and only if it corresponds to a surjection of rings.)

1.5. Exercise. Suppose $Y \to Z$ is a closed immersion, and $X \to Z$ is any morphism. Show that the fibered product $X \times_Z X$ exists, by explicitly describing it. Show that the projection $X \times_Z Y \to Y$ is a closed immersion. We say that "closed immersions are preserved by base change" or "closed immersions are preserved by fibered product". (Base change is another word for fibered products.)

1.6. Less important exercise. Show that closed immersions are monomorphisms.

Definition. A morphism $X \to Y$ is a *locally closed immersion* if it factors into $X \xrightarrow{f} Z \xrightarrow{g} Y$ where f is a closed immersion and g is an open immersion. Example: $\operatorname{Spec} k[t, t^{-1}] \to \operatorname{Spec} k[x, y]$ where $x \mapsto t$, $y \mapsto 0$. (Unimportant fact: as the composition of monomorphisms are monomorphisms, so locally closed immersions are monomorphisms. Clearly open immersions and closed immersions are locally closed immersions.)

(Interesting question: is this the same as defining locally closed immersions as open immersions of closed immersions? In other words, can the roles of open and closed immersions in the definition be reversed?)

A morphism $f : X \to Y$ is **quasicompact** if for every open affine subset U of Y, $f^{-1}(U)$ is quasicompact.

1.7. *Exercise (quasicompactness is affine-local on the target).* Show that a morphism $f : X \to Y$ is quasicompact if there is cover of Y by open affine sets U_i such that $f^{-1}(U_i)$ is quasicompact. (Hint: easy application of the affine communication lemma!)

1.8. *Exercise.* Show that the composition of two quasicompact morphisms is quasicompact.

A morphism $f : X \to Y$ is **locally of finite type** if for every affine open set $\operatorname{Spec} B$ of Y, $f^{-1}(\operatorname{Spec} B)$ can be covered with open sets $\operatorname{Spec} A_i$ so that the induced morphism $B \to A_i$ expresses A_i as a finitely generated B-algebra.

A morphism is **of finite type** if it is locally of finite type and quasicompact. Translation: for every affine open set $\operatorname{Spec} B$ of Y, $f^{-1}(\operatorname{Spec} B)$ can be covered with *a finite number of* open sets $\operatorname{Spec} A_i$ so that the induced morphism $B \to A_i$ expresses A_i as a finitely generated B-algebra.

1.9. *Exercise (the notions "locally of finite type" and "finite type" are affine-local on the target).* Show that a morphism $f : X \to Y$ is locally of finite type if there is a cover of Y by open affine sets $\operatorname{Spec} R_i$ such that $f^{-1}(\operatorname{Spec} R_i)$ is locally of finite type over R_i.

1.10. *Exercise.* Show that a morphism $f : X \to Y$ is locally of finite type if for *every* affine open subsets $\operatorname{Spec} A \subset X$, $\operatorname{Spec} B \subset Y$, with $f(\operatorname{Spec} A) \subset \operatorname{Spec} B$, A is a finitely generated B-algebra. (Hint: use the affine communication lemma on $f^{-1}(\operatorname{Spec} B)$.)

Example: the "structure morphism" $\mathbb{P}^n_A \to \operatorname{Spec} A$ is of finite type, as $\mathbb{P}^n_A$ is covered by $n + 1$ open sets of the form $\operatorname{Spec} A[x_1, \dots, x_n]$. More generally, $\operatorname{Proj} S_* \to \operatorname{Spec} A$ (where $S_0 = A$) is of finite type.

More generally still: our earlier definition of schemes of "finite type over k" (or "finite type k-schemes") is now a special case of this more general notion: a scheme X is of finite type over k means that we are given a morphism $X \to \operatorname{Spec} k$ (the "structure morphism") that is of finite type.

Here are some properties enjoyed by morphisms of finite type.

1.11. Exercises. These exercises are important and not hard.

- Show that a closed immersion is a morphism of finite type.
- Show that an open immersion is locally of finite type. Show that an open immersion into a Noetherian scheme is of finite type. More generally, show that a quasicompact open immersion is of finite type.
- Show that a composition of two morphisms of finite type is of finite type.
- Suppose we have a composition of morphisms $X \xrightarrow{f} Y \xrightarrow{g} Z$, where f is quasicompact, and $g \circ f$ is finite type. Show that f is finite type.
- Suppose $f : X \to Y$ is finite type, and Y is Noetherian. Show that X is also Noetherian.

A morphism $f : X \to Y$ is **affine** if for every affine U of Y, $f^{-1}(U)$ is an affine scheme. Clearly affine morphisms are quasicompact. Also, clearly closed immersions are affine: if $X \to Y$ is a closed immersion, then the preimage of an affine open set $\operatorname{Spec} R$ of Y is (isomorphic to) some $\operatorname{Spec} R/I$, by the definition of closed subscheme.

1.12. *Proposition (the property of "affineness" is affine-local on the target).* A morphism $f : X \to Y$ is affine if there is a cover of Y by open affine sets U such that $f^{-1}(U)$ is affine.

Proof. As usual, we use the Affine Communication Theorem. We check our two criteria. First, suppose $f : X \to Y$ is affine over $\operatorname{Spec} S$, i.e. $f^{-1}(\operatorname{Spec} S) = \operatorname{Spec} R$. Then $f^{-1}(\operatorname{Spec} S_s) = \operatorname{Spec} R_{f\# s}$.

Second, suppose we are given $f : X \to \operatorname{Spec} S$ and $(f_1, \dots, f_n) = S$ with X_{f_i} affine ($\operatorname{Spec} R_i$, say). We wish to show that X is affine too. Define R as the kernel of S-modules

$$R_1 \times \cdots \times R_n \to R_{12} \times \cdots R_{(n-1)n}$$

where $X_{f_i f_j} = \operatorname{Spec} R_{ij}$. Then R is clearly an S-module, and has a ring structure. We define a morphism $\operatorname{Spec} R \to \operatorname{Spec} S$. Note that $R_{f_i} = R_i$. Then we define $\operatorname{Spec} R \to \operatorname{Spec} S$ via $\operatorname{Spec} R_i \to \operatorname{Spec} R_{f_i} \hookrightarrow \operatorname{Spec} S$. The morphisms glue. ∎

This has some non-obvious consequences, as shown in the next exercise.

1.13. *Exercise.* Suppose X is an affine scheme, and Y is a closed subscheme locally cut out by one equation (e.g. if Y is an effective Cartier divisor). Show that $X - Y$ is affine. (This is clear if Y is globally cut out by one equation f; then if $X = \operatorname{Spec} R$ then $Y = \operatorname{Spec} R_f$. However, Y is not always of this form.)

1.14. Example. Here is an explicit consequence. We showed earlier that on the cone over the smooth quadric surface $\operatorname{Spec} k[w, x, y, z]/(wz - xy)$, the cone over a ruling $w = x = 0$ is not cut out scheme-theoretically by a single equation, by considering Zariski-tangent spaces. We now show that it isn't even cut out set-theoretically by a single equation.

For if it were, its complement would be affine. But then the closed subscheme of the complement cut out by $y = z = 0$ would be affine. But this is the scheme $y = z = 0$ (also known as the wx-plane) minus the point $w = x = 0$, which we've seen is non-affine. (For comparison, on the cone $\operatorname{Spec} k[x, y, z]/(xy - z^2)$, the ruling $x = z = 0$ is not cut out scheme-theoretically by a single equation, but it *is* cut out set-theoretically by $x = 0$.) Verify all this!

We remark here that we have shown that if $f : X \to Y$ is an affine morphism, then $f_* \mathcal{O}_X$ is a quasicoherent sheaf of algebras (a quasicoherent sheaf with the structure of an algebra "over $\mathcal{O}_X$"). We'll soon reverse this process to obtain Spec of a quasicoherent sheaf of algebras.

A morphism $f : X \to Y$ is **finite** if for every affine $\operatorname{Spec} R$ of Y, $f^{-1}(\operatorname{Spec} R)$ is the spectrum of an R-algebra that is a finitely-generated R-*module*. Clearly finite morphisms are affine. Note that $f_* \mathcal{O}_X$ is a finite type quasicoherent sheaf of algebras (= coherent if X is locally Noetherian).

1.15. *Exercise (the property of finiteness is affine-local on the target).* Show that a morphism $f : X \to Y$ is finite if there is a cover of Y by open affine sets $\operatorname{Spec} R$ such that $f^{-1}(\operatorname{Spec} R)$ is the spectrum of a finite R-algebra.

(Hint: Use Exercise 1.12, and that $f_* \mathcal{O}_X$ is finite type.)

1.16. *Easy exercise.* Show that closed immersions are finite morphisms.

Degree of a finite morphism at a point. Suppose $f : X \to Y$ is a finite morphism. $f_* \mathcal{O}_X$ is a finite type (quasicoherent) sheaf on Y, and the rank of this sheaf at a point p is called the *degree* of the finite morphism at p. This is a upper semicontinuous function (we've shown that the rank of a finite type sheaf is uppersemicontinuous in an exercise when we discussed rank).

1.17. *Exercise.* Show that the rank at p is non-zero if and only if $f^{-1}(p)$ is non-empty.

1.18. *Exercise.* Show that finite morphisms are *closed*, i.e. the image of any closed subset is closed.

A morphism is **quasifinite** if it is of finite type, and for all $y \in Y$, the scheme $X_y = f^{-1}(y)$ is finite over y.

1.19. *Exercise.* (a) Show that if a morphism is finite then it is quasifinite. (b) Show that the converse is not true. (Hint: $\mathbb{A}^1 - \{0\} \to \mathbb{A}^1$.)

1.20. Images of morphisms. I want to go back to the point that the image of a finite morphism is closed. Something more general is true. We answer the question: what can the image of a morphism look like? We know it can be open (open immersion), and closed

(closed immersions), locally closed (locally closed immersions). But it can be weirder still: Consider $\mathbb{A}^2 \to \mathbb{A}^2$ given by $(x, y) \mapsto (x, xy)$. then the image is the plane, minus the y-axis, plus the origin. It can be stranger still, and indeed if S is *any* subset of a scheme Y, it can be the image of a morphism: let X be the disjoint union of spectra of the residue fields of all the points of S, and let $f : X \to Y$ be the natural map. This is quite pathological (e.g. likely horribly noncompact), and we will show that if we are in any reasonable situation, the image is essentially no worse than arose in the previous example.

We define a *constructible subset* of a scheme to be a subset which belongs to the smallest family of subsets such that (i) every open set is in the family, (ii) a finite intersection of family members is in the family, and (iii) the complement of a family member is also in the family. So for example the image of $(x, y) \mapsto (x, xy)$ is constructible.

Note that if $X \to Y$ is a morphism of schemes, then the preimage of a constructible set is a constructible set.

1.21. Exercise. Suppose X is a Noetherian scheme. Show that a subset of X is constructible if and only if it is the finite disjoint union of locally closed subsets.

Chevalley's Theorem. Suppose $f : X \to Y$ is a morphism of finite type of Noetherian schemes. Then the image of any constructible set is constructible.

I might give a proof in the notes eventually. See Atiyah-Macdonald, Exercise 7.25 for the key algebraic argument. Next quarter, we will see that in good situations (e.g. if the source is projective over k and the target is quasiprojective) then the image is closed.

We end with a useful fact about images of schemes that didn't naturally fit in anywhere in the previous exposition.

1.22. *Fast important exercise.* Show that the image of an irreducible scheme is irreducible.

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FOUNDATIONS OF ALGEBRAIC GEOMETRY CLASS 20

CONTENTS

Last day: Maps to affine schemes; surjective, open immersion, closed immersion, quasicompact, locally of finite type, finite type, affine morphism, finite, quasifinite. Images of morphisms: constructible sets, and Chevalley's theorem (finite type morphism of Noetherian schemes sends constructibles to constructibles).

Today: Pushforwards and pullbacks of quasicoherent sheaves.

This is the last class of the first quarter of this three-quarter sequence. Last day, I defined a large number of classes of morphisms. Today, I will talk about how quasicoherent sheaves push forward or pullback. I'll then sum up what's happened in this class, and give you some idea of what will be coming in the next quarter.

1. PUSHFORWARDS AND PULLBACKS OF QUASICOHERENT SHEAVES

There are two things you can do with modules and a ring homomorphism $B \to A$. If M is an A-module, you can create an B-module M_B by simply treating it as an B-module. If N is an B-module, you can create an A-module $N \otimes_B A$.

These notions behave well with respect to localization (in a way that we will soon make precise), and hence work (often) in the category of quasicoherent sheaves. The two functors are adjoint:

$$\operatorname{Hom}_A(N \otimes_B A, M) \cong \operatorname{Hom}_B(N, M_B)$$

(where this isomorphism of groups is functorial in both arguments), and we will see that this remains true on the scheme level.

One of these constructions will turn into our old friend pushforward. The other will be a relative of pullback, whom I'm reluctant to call an "old friend".

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The main message of this section is that in "reasonable" situations, the pushforward of a quasicoherent sheaf is quasicoherent, and that this can be understood in terms of one of the module constructions defined above. We begin with a motivating example:

2.1. *Exercise.* Let $f : \operatorname{Spec} A \to \operatorname{Spec} B$ be a morphism of affine schemes, and suppose M is an A-module, so $\tilde{M}$ is a quasicoherent sheaf on $\operatorname{Spec} A$. Show that $f_*\tilde{M} \cong \widetilde{M_B}$. (Hint: There is only one reasonable way to proceed: look at distinguished opens!)

In particular, $f_*\tilde{M}$ is quasicoherent. Perhaps more important, this implies that the pushforward of a quasicoherent sheaf under an affine morphism is also quasicoherent. The following result doesn't quite generalize this statement, but the argument does.

2.2. *Theorem. — Suppose $f : X \to Y$ is a morphism, and X is a Noetherian scheme. Suppose $\mathcal{F}$ is a quasicoherent sheaf on X. Then $f_*\mathcal{F}$ is a quasicoherent sheaf on Y.*

The fact about f that we will use is that the preimage of any affine open subset of Y is a finite union of affine sets (f is quasicompact), and the intersection of any two of these affine sets is also a finite union of affine sets (this is a definition of the notion of a *quasiseparated morphism*). Thus the "correct" hypothesis here is that f is quasicompact and quasiseparated.

Proof. By the first definition of quasicoherent sheaves, it suffices to show the following: if $\mathcal{F}$ is a quasicoherent sheaf on X, and $f : X \to \operatorname{Spec} R$, then the following diagram commutes:

$$\begin{array}{ccc} \Gamma(X,\mathcal{F}) & \xrightarrow{\operatorname{res}_{D(g)\subset \operatorname{Spec} R}} & \Gamma(X_g,\mathcal{F}) \\ & {\scriptstyle \otimes_R R_g}\searrow \quad \nearrow{\scriptstyle \sim} & \\ & \Gamma(X,\mathcal{F})_g & \end{array}$$

This was a homework problem (# 18 on problem set 6)! □

2.3. *Exercise.* Give an example of a morphism of schemes $\pi : X \to Y$ and a quasicoherent sheaf $\mathcal{F}$ on X such that $\pi_*\mathcal{F}$ is not quasicoherent. (Answer: $Y = \mathbb{A}^1$, $X =$ countably many copies of $\mathbb{A}^1$. Then let $f = t$. X_t has a global section $(1/t, 1/t^2, 1/t^3, \dots)$. The key point here is that infinite direct sums do not commute with localization.)

Coherent sheaves don't always push forward to coherent sheaves. For example, consider the structure morphism $f : \mathbb{A}^1_k \to \operatorname{Spec} k$, given by $k \mapsto k[t]$. Then $f_*\mathcal{O}_{\mathbb{A}^1_k}$ is the $k[t]$, which is not a finitely generated k-module. Under especially good situations, coherent sheaves do push forward. For example:

2.4. *Exercise.* Suppose $f : X \to Y$ is a finite morphism of Noetherian schemes. If $\mathcal{F}$ is a coherent sheaf on X, show that $f_*\mathcal{F}$ is a coherent sheaf. (Hint: Show first that $f_*\mathcal{O}_X$ is finite type = locally finitely generated.)

Once we define cohomology of quasicoherent sheaves, we will quickly prove that if $\mathcal{F}$ is a coherent sheaf on $\mathbb{P}^n_k$, then $\Gamma(\mathbb{P}^n_k)$ is a finite-dimensional k-module, and more generally if $\mathcal{F}$ is a coherent sheaf on $\operatorname{Proj} S_*$, then $\Gamma(\operatorname{Proj} S_*)$ is a coherent A-module (where $S_0 = A$). This is a special case of the fact the "pushforwards of coherent sheaves by projective morphisms are also coherent sheaves". We will first need to define "projective morphism"! This notion is a generalization of $\operatorname{Proj} S_* \to \operatorname{Spec} A$.

3. Pullback of quasicoherent sheaves

(Note added in February: I will try to reserve the phrase "pullback of a sheaf" for pullbacks of quasicoherent sheaves f^*, and "inverse image sheaf" for f^{-1}, which applies in a more general situation, in the category of sheaves on topological spaces.)

I will give four definitions of the pullback of a quasicoherent sheaf. The first one is the most useful in practice, and is in keeping with our emphasis of quasicoherent sheaves as just "modules glued together". The second is the "correct" definition, as an adjoint of pushforward. The third, which we mention only briefly, is *more* correct, as adjoint in the category of $\mathcal{O}_X$-modules. And we end with a fourth definition.

We note here that pullback to a closed subscheme or an open subscheme is often called **restriction**.

3.1. Construction/description of the pullback. Let us now define the pullback functor precisely. Suppose $X \to Y$ is a morphism of schemes, and $\mathcal{G}$ is a quasicoherent sheaf on Y. We will describe the pullback quasicoherent sheaf $f^*\mathcal{G}$ on X by describing it as a sheaf on the distinguished affine base. In our base, we will permit only those affine open sets $U \subset X$ such that $f(U)$ is contained in an affine open set of Y. The distinguished restriction map will force this sheaf to be quasicoherent.

Suppose $U \subset X$, $V \subset Y$ are affine open sets, with $f(U) \subset V$, $U \cong \operatorname{Spec} A$, $V \cong \operatorname{Spec} B$. Suppose $\mathcal{F}|_V \cong \tilde{N}$. Then define $\Gamma(f^*_V\mathcal{F}, U) := N \otimes_B A$. Our main goal will be to show that this is independent of our choice of V.

We begin as follows: we fix an affine open subset $V \subset Y$, and use it to define sections over any affine open subset $U \subset f^{-1}(V)$. We show that this gives us a quasicoherent sheaf $f^*_V\mathcal{G}$ on $f^{-1}(V)$, by showing that these sections behave well with respect to distinguished restrictions. First, note that if $D(f) \subset U$ is a distinguished open set, then

$$\Gamma(f^*_V\mathcal{F}, D(f)) = N \otimes_B A_f \cong (N \otimes_B A) \otimes_A A_f = \Gamma(f^*_V\mathcal{F}, U) \otimes_A A_f.$$

Define the restriction map $\Gamma(f^*_V\mathcal{F}, U) \to \Gamma(f^*_V\mathcal{F}, D(f))$ by

$$\Gamma(f^*_V\mathcal{F}, U) \to \Gamma(f^*_V\mathcal{F}, U) \otimes_A A_f \tag{1}$$

(with $\alpha \mapsto \alpha \otimes 1$ of course). Thus on the *distinguished affine topology of* $\operatorname{Spec} A$ we have defined a quasicoherent sheaf.

Finally, we show that if $f(U)$ is contained in *two* affine open sets V_1 and V_2, then the alleged sections of the pullback we have described do not depend on whether we use V_1 or V_2. More precisely, we wish to show that

$$\Gamma(f^*_{V_1}\mathcal{F}, U) \quad \text{and} \quad \Gamma(f^*_{V_2}\mathcal{F}, U)$$

have a canonical isomorphism, which commutes with the restriction map (1).

Let $\{W_i\}_{i\in I}$ be an affine cover of $V_1 \cap V_2$ by sets that are distinguished in *both* V_1 and V_2 (possible by the Proposition we used in the proof of the Affine Communication Lemma). Then by the previous paragraph, as $f^*_{V_1}\mathcal{F}$ is a sheaf on the distinguished base of V_1,

$$\Gamma(f^*_{V_1}\mathcal{F}, U) = \ker\left(\oplus_i \Gamma(f^*_{V_1}\mathcal{F}, f^{-1}(W_i)) \to \oplus_{i,j}\Gamma(f^*_{V_1}\mathcal{F}, f^{-1}(W_i \cap W_j))\right).$$

If $V_1 = \operatorname{Spec} B_1$ and $W_i = D(g_i)$, then

$$\Gamma(f^*_{V_1}\mathcal{F}, f^{-1}(W_i)) = N \otimes_{B_1} A_{f^\# g_i} \cong N \otimes_{(B_1)_{g_i}} A_{f^\# g_i} = \Gamma(f^*_{W_i}\mathcal{F}, f^{-1}(W_i)),$$

so

$$\Gamma(f^*_{V_1}\mathcal{F}, U) = \ker\left(\oplus_i \Gamma(f^*_{W_i}\mathcal{F}, f^{-1}(W_i)) \to \oplus_{i,j}\Gamma(f^*_{W_i}\mathcal{F}, f^{-1}(W_i \cap W_j))\right). \tag{2}$$

The same argument for V_2 yields

$$\Gamma(f^*_{V_2}\mathcal{F}, U) = \ker\left(\oplus_i \Gamma(f^*_{W_i}\mathcal{F}, f^{-1}(W_i)) \to \oplus_{i,j}\Gamma(f^*_{W_i}\mathcal{F}, f^{-1}(W_i \cap W_j))\right). \tag{3}$$

But the right sides of (2) and (3) are the same, so the left sides are too. Moreover, (2) and (3) behave well with respect to restricting to a distinguished open $D(g)$ of $\operatorname{Spec} A$ (just apply $\otimes_A A_g$ to the the right side) so we are done.

Hence we have described a quasicoherent sheaf $f^*\mathcal{G}$ on X whose behavior on affines mapping to affines was as promised.

3.2. *Theorem.* —

(1) *The pullback of the structure sheaf is the structure sheaf.*
(2) *The pullback of a finite type (=locally finitely generated) sheaf is finite type.*
(3) *The pullback of a finitely presented sheaf is finitely presented. Hence if* $f : X \to Y$ *is a morphism of locally Noetherian schemes, then the pullback of a coherent sheaf is coherent. (It is not always true that the pullback of a coherent sheaf is coherent, and the interested reader can think of a counterexample.)*
(4) *The pullback of a locally free sheaf of rank* r *is another such. (In particular, the pullback of an invertible sheaf is invertible.)*
(5) *(functoriality in the morphism)* $\pi_1^*\pi_2^*\mathcal{F} \cong (\pi_2 \circ \pi_1)^*\mathcal{F}$
(6) *(functoriality in the quasicoherent sheaf)* $\mathcal{F}_1 \to \mathcal{F}_2$ *induces* $\pi^*\mathcal{F}_1 \to \pi^*\mathcal{F}_2$
(7) *If* s *is a section of* $\mathcal{F}$ *then there is a natural section* π^*s *that is a section of* $\pi^*\mathcal{F}$.
(8) *(stalks) If* $\pi : X \to Y$, $\pi(x) = y$, *then* $(\pi^*\mathcal{F})_x \cong \mathcal{F}_y \otimes_{\mathcal{O}_{Y,y}} \mathcal{O}_{X,x}$. *The previous map, restricted to the stalks, is* $f \mapsto f \otimes 1$. *(In particular, the locus where the section on the target vanishes pulls back to the locus on the source where the pulled back section vanishes.)*

(9) *(fibers) Pullbacks of fibers are given as follows: if* $\pi : X \to Y$*, where* $\pi(x) = y$*, then* $\pi^*\mathcal{F}/\mathfrak{m}_{X,x}\pi^*\mathcal{F} \cong (\mathcal{F}/\mathfrak{m}_{Y,y}\mathcal{F}) \otimes_{\mathcal{O}_{Y,y}/\mathfrak{m}_{Y,y}} \mathcal{O}_{X,x}/\mathfrak{m}_{X,x}$

(10) *(tensor product)* $\pi^*(\mathcal{F} \otimes \mathcal{G}) = \pi^*\mathcal{F} \otimes \pi^*\mathcal{G}$

(11) *pullback is a right-exact functor*

All of the above are interconnected in obvious ways. For example, given $\mathcal{F}_1 \to \mathcal{F}_2$ and a section s of $\mathcal{F}_1$, then we can pull back the section and then send it to $\pi^*\mathcal{F}_2$, or vice versa, and we get the same thing.

I used some of these results to help give an intuitive picture of the pullback.

Proof. Most of these are left to the reader. It is convenient to do right-exactness early (e.g. before showing that finitely presented sheaves pull back to finitely presented sheaves). For the tensor product fact, show that $(M \otimes_S R) \otimes (N \otimes_S R) \cong (M \otimes N) \otimes_S R$, and that this behaves well with respect to localization. The proof of the fiber fact is as follows. $(S, \mathfrak{n}) \to (R, \mathfrak{m})$.

$$\begin{array}{ccc} S & \longrightarrow & R \\ \downarrow & & \downarrow \\ S/\mathfrak{n} & \longrightarrow & R/\mathfrak{m} \end{array}$$

$(N \otimes_S R) \otimes_R (R/\mathfrak{m}) \cong (N \otimes_S (S/\mathfrak{n})) \otimes_{S/\mathfrak{n}} (R/\mathfrak{m})$ as both sides are isomorphic to $N \otimes_S (R/\mathfrak{m})$. □

3.3. *Unimportant Exercise.* Verify that the following is an example showing that pullback is not left-exact: consider the exact sequence of sheaves on $\mathbb{A}^1$, where p is the origin:

$$0 \to \mathcal{O}_{\mathbb{A}^1}(-p) \to \mathcal{O}_{\mathbb{A}^1} \to \mathcal{O}_p \to 0.$$

(This is a closed subscheme exact sequence; also an effective Cartier exact sequence. Algebraically, we have $k[t]$-modules $0 \to tk[t] \to k[t] \to k \to 0$.) Restrict to p.

3.4. Pulling back closed subschemes. Suppose $Z \hookrightarrow Y$ is a closed immersion, and $X \to Y$ is any morphism. Then we define the pullback of the closed subscheme Z to X as follows. We pullback the quasicoherent sheaf of ideals on Y defining Z to get a quasicoherent sheaf of ideals on X (which we take to define W). Equivalently, on any affine open Y, Z is cut out by some functions; we pull back those functions to X, and denote the scheme cut out by them by W.

Exercise. Let W be the pullback of the closed subscheme Z to X. Show that $W \cong Z \times_Y X$. In other words, the fibered product with a closed immersion always exists, and closed immersions are preserved by fibered product (or by pullback), i.e. if

$$\begin{array}{ccc} W & \xrightarrow{g'} & X \\ \downarrow & & \downarrow \\ Z & \xrightarrow{g} & Y \end{array}$$

is a fiber diagram, and g is a closed immersion, then so is g'. (This is actually a repeat of an exercise in class 19 — sorry!)

3.5. Three more "definitions". Pullback is left-adjoint of the pushforward. This is a theorem (which we'll soon prove), but it is actually a pretty good definition. If it exists, then it is unique up to unique isomorphism by Yoneda nonsense.

The problem is this: pushforwards don't always exist (in the category of quasicoherent sheaves); we need the quasicompact and quasiseparated hypotheses. However, pullbacks always exist. So we need to motivate our definition of pullback even without the quasicompact and quasiseparated hypothesis. (One possible motivation will be given in Remark 3.7.)

3.6. *Theorem. — Suppose* $\pi : X \to Y$ *is a quasicompact, quasiseparated morphism. Then pullback is left-adjoint to pushforward. More precisely,* $\mathrm{Hom}(f^*\mathcal{G}, \mathcal{F}) \cong \mathrm{Hom}(\mathcal{G}, f_*\mathcal{F})$.

(The quasicompact and quasiseparated hypothesis is required to ensure that the pushforward exists, not because it is needed in the proof.)

More precisely still, we describe natural homomorphisms that are functorial in both arguments. We show that it is a bijection of sets, but it is fairly straightforward to verify that it is an isomorphism of groups. Not surprisingly, we will use adjointness for modules.

Proof. Let's unpack the right side. What's an element of $\mathrm{Hom}(\mathcal{G}, f_*\mathcal{F})$? For every affine V in Y, we get an element of $\mathrm{Hom}(\mathcal{G}(V), \mathcal{F}(f^{-1}(V)))$, and this behaves well with respect to distinguished opens. Equivalently, for every affine V in Y and U in $f^{-1}(V) \subset X$, we have an element $\mathrm{Hom}(\mathcal{G}(V), \mathcal{F}(U))$, that behaves well with respect to localization to distinguished opens on both affines. By the adjoint property, this corresponds to elements of $\mathrm{Hom}(\mathcal{G}(V) \otimes_{\mathcal{O}_Y(V)} \mathcal{O}_X(U), \mathcal{F}(U))$, which behave well with respect to localization. And that's the left side. □

3.7. *Pullback for ringed spaces.* (This is actually conceptually important but distracting for our exposition; we encourage the reader to skip this, at least on the first reading.) Pullbacks and pushforwards may be defined in the category of modules over ringed spaces. We define pushforward in the usual way, and then define the pullback of an $\mathcal{O}_Y$-module using the adjoint property. Then one must show that (i) it exists, and (ii) the pullback of a quasicoherent sheaf is quasicoherent. The fourth definition is as follows: suppose we have a morphism of ringed spaces $\pi : X \to Y$, and an $\mathcal{O}_Y$-module $\mathcal{G}$. Then we define $f^*\mathcal{G} = f^{-1}\mathcal{G} \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X$. We will not show that this definition is equivalent to ours, but the interested reader is welcome to try this as an exercise. There is probably a proof in Hartshorne. The statements of Theorem 3.6 apply in this more general setting. (Really the third definition "requires" the fourth.)

Here is a hint as to why this definition is equivalent to ours (a hint for the exercise if you will). We need to show that $f^{-1}\mathcal{F} \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X$ ("definition 4") and $f^*\mathcal{F}$ ("definition 1") are isomorphic. You should (1) find a natural morphism from one to the other, and (2) show that it is an isomorphism at the level of stalks. The difficulty of (1) shows the disadvantages of our definition of quasicoherent sheaves.

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FOUNDATIONS OF ALGEBRAIC GEOMETRY BONUS HANDOUT: PROOFS OF "HARTOGS" AND KRULL

RAVI VAKIL

I said earlier that I hoped to give you proofs of (i) "Hartogs' Theorem" for normal Noetherian schemes, (ii) Krull's Principal Ideal Theorem, and (iii) the fact that if $(R, \mathfrak{m})$ is a Noetherian ring, then $\cap \mathfrak{m}^i = 0$ (corresponding to the fact that a function that is analytically zero at a point is zero in a neighborhood of that point).

You needn't read these; but you may appreciate the fact that the proofs aren't that long. Thus there are very few statements in this class (beyond Math 210) that we actually used, but didn't justify.

I am going to repeat the Nakayama statements, so the entire argument is in one place.

0.1. *Nakayama's Lemma version 1. — Suppose* R *is a ring,* I *an ideal of* R*, and* M *is a finitely-generated* R*-module. Suppose* $M = IM$*. Then there exists an* $a \in R$ *with* $a \equiv 1 \pmod{I}$ *with* $aM = 0$*.*

Proof. Say M is generated by $m_1, \dots, m_n$. Then as $M = IM$, we have $m_i = \sum_j a_{ij} m_j$ for some $a_{ij} \in I$. Thus

$$(\mathrm{Id}_n - A) \begin{pmatrix} m_1 \\ \vdots \\ m_n \end{pmatrix} = 0 \tag{1}$$

where Id_n is the $n \times n$ identity matrix in R, and $A = (a_{ij})$. We can't quite invert this matrix, but we almost can. Recall that any $n \times n$ matrix M has an adjoint matrix $\mathrm{adj}(M)$ such that $\mathrm{adj}(M)M = \det(M)\mathrm{Id}_n$. The coefficients of $\mathrm{adj}(M)$ are polynomials in the coefficients of M. (You've likely seen this in the form of a formula for M^{-1} when there *is* an inverse.) Multiplying both sides of (1) on the left by $\mathrm{adj}(M)$, we obtain

$$\det(\mathrm{Id}_n - A) \begin{pmatrix} m_1 \\ \vdots \\ m_n \end{pmatrix} = 0.$$

But when you expand out $\det(\mathrm{Id}_n - A)$, you get something that is $1 \pmod{I}$. □

Here is why you care: Suppose I is contained in all maximal ideals of R. (The intersection of all the maximal ideals is called the *Jacobson radical*, but I won't use this phrase. For comparison, recall that the nilradical was the intersection of the *prime ideals* of R.) Then I claim that any $a \equiv 1 \pmod{I}$ is invertible. For otherwise $(a) \neq R$, so the ideal (a) is

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contained in some maximal ideal $\mathfrak{m}$ — but $a \equiv 1 \pmod{\mathfrak{m}}$, contradiction. Then as a is invertible, we have the following.

0.2. *Nakayama's Lemma version 2.* — *Suppose R is a ring, I an ideal of R contained in all maximal ideals, and M is a finitely-generated R-module. (The most interesting case is when R is a local ring, and I is the maximal ideal.) Suppose $M = IM$. Then $M = 0$.*

0.3. Important exercise (Nakayama's lemma version 3). Suppose R is a ring, and I is an ideal of R contained in all maximal ideals. Suppose M is a *finitely generated* R-module, and $N \subset M$ is a submodule. If $N/IN \hookrightarrow M/IM$ an isomorphism, then $M = N$.

0.4. Important exercise (Nakayama's lemma version 4). Suppose $(R, \mathfrak{m})$ is a local ring. Suppose M is a finitely-generated R-module, and $f_1, \dots, f_n \in M$, with (the images of) $f_1, \dots, f_n$ generating $M/\mathfrak{m}M$. Then $f_1, \dots, f_n$ generate M. (In particular, taking $M = \mathfrak{m}$, if we have generators of $\mathfrak{m}/\mathfrak{m}^2$, they also generate $\mathfrak{m}$.)

0.5. *Important Exercise that we will use soon.* Suppose S is a subring of a ring R, and $r \in R$. Suppose there is a faithful $S[r]$-module M that is finitely generated as an S-module. Show that r is integral over S. (Hint: look carefully at the proof of Nakayama's Lemma version 1, and change a few words.)

We are ready to prove "Hartogs' Theorem".

0.6. *"Hartogs' theorem".* — *Suppose A is a Noetherian normal domain. Then in $\operatorname{Frac}(A)$,*

$$A = \cap_{\mathfrak{p} \text{ height } 1} A_{\mathfrak{p}}.$$

More generally, if A is a product of Noetherian normal domains (i.e. $\operatorname{Spec} A$ is Noetherian normal scheme), then in the ring of fractions of A,

$$A = \cap_{\mathfrak{p} \text{ height } 1} A_{\mathfrak{p}}.$$

I stated the special case first so as to convince you that this isn't scary.

Proof. Obviously the right side is contained in the left. Assume we have some x in all A_p but not in A. Let I be the "ideal of denominators":

$$I := \{r \in A : rx \in A\}.$$

(The ideal of denominators arose in an earlier discussion about normality.) We know that $I \neq A$, so choose $\mathfrak{q}$ a minimal prime containing I.

Observe that this construction behaves well with respect to localization (i.e. if $\mathfrak{p}$ is any prime, then the ideal of denominators x in $A_{\mathfrak{p}}$ is the $I_{\mathfrak{p}}$, and it again measures the failure of "'Hartogs' Theorem" for x,' this time in $A_{\mathfrak{p}}$). But Hartogs' Theorem is vacuously true for dimension 1 rings, so hence no height 1 prime contains I. Thus $\mathfrak{q}$ has height at least 2. By localizing at $\mathfrak{q}$, we can assume that A is a local ring with maximal ideal $\mathfrak{q}$, and that $\mathfrak{q}$ is the only prime containing I. Thus $\sqrt{I} = \mathfrak{q}$, so there is some n with $I \subset \mathfrak{q}^n$. Take a minimal such n, so $I \not\subset \mathfrak{q}^{n-1}$, and choose any $y \in \mathfrak{q}^{n-1} - \mathfrak{q}^n$. Let $z = yx$. Then $z \notin A$ (so $\mathfrak{q}z \notin \mathfrak{q}$), but $\mathfrak{q}z \subset A$: $\mathfrak{q}z$ is an ideal of A.

I claim $\mathfrak{q}z$ is not contained in $\mathfrak{q}$. Otherwise, we would have a finitely-generated A-module (namely $\mathfrak{q}$) with a faithful $A[z]$-action, forcing z to be integral over A (and hence in A) by Exercise 0.5.

Thus $\mathfrak{q}z$ is an ideal of A not contained in $\mathfrak{q}$, so it must be A! Thus $\mathfrak{q}z = A$ from which $\mathfrak{q} = A(1/z)$, from which $\mathfrak{q}$ is principal. But then $\operatorname{ht} Q = \dim A \leq \dim_{A/Q} Q/Q^2 \leq 1$ by Nakayama's lemma 0.4, contradicting the fact that $\mathfrak{q}$ has height at least 2. □

We now prove:

0.7. *Krull's Principal Ideal Theorem. — Suppose* A *is a Noetherian ring, and* $f \in A$*. Then every minimal prime* $\mathfrak{p}$ *containing* f *has height at most* 1*. If furthermore* f *is not a zero-divisor, then every minimal prime* $\mathfrak{p}$ *containing* f *has height precisely* 1*.*

0.8. *Lemma. — If* R *is a Noetherian ring with one prime ideal. Then* R *is Artinian, i.e., it satisfies the descending chain condition for ideals.*

The notion of Artinian rings is very important, but we will get away without discussing it much.

Proof. If R is a ring, we define more generally an *Artinian* R*-module*, which is an R-module satisfying the descending chain condition for submodules. Thus R is an Artinian ring if it is Artinian over itself as a module.

If $\mathfrak{m}$ is a maximal ideal of R, then any finite-dimensional $(R/\mathfrak{m})$-vector space (interpreted as an R-module) is clearly Artinian, as any descending chain

$$M_1 \supset M_2 \supset \cdots$$

must eventually stabilize (as $\dim_{R/\mathfrak{m}} M_i$ is a non-increasing sequence of non-negative integers).

Exercise. Show that for any n, $\mathfrak{m}^n/\mathfrak{m}^{n+1}$ is a finitely-dimensional $R/\mathfrak{m}$-vector space. (Hint: show it for $n = 0$ and $n = 1$. Use the dimension for $n = 1$ to bound the dimension for general n.) Hence $\mathfrak{m}^n/\mathfrak{m}^{n+1}$ is an Artinian R-module.

As $\sqrt{0}$ is prime, it must be $\mathfrak{m}$. As $\mathfrak{m}$ is finitely generated, $\mathfrak{m}^n = 0$ for some n. **Exercise.** Prove this. (Hint: suppose $\mathfrak{m}$ can be generated by m elements, each of which has kth power 0, and show that $\mathfrak{m}^{m(k-1)+1} = 0$.)

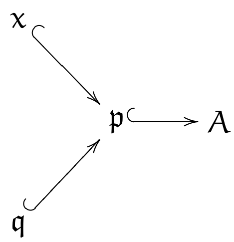
Exercise. Show that if $0 \to M' \to M \to M'' \to 0$ is an exact sequence of modules. then M is Artinian if and only if M' and M'' are Artinian.

Thus as we have a finite filtration

$$R \supset \mathfrak{m} \supset \cdots \supset \mathfrak{m}^n = 0$$

all of whose quotients are Artinian, so R is Artinian as well. □

Proof of Krull's principal ideal theorem 0.7. Suppose we are given $x \in A$, with $\mathfrak{p}$ a minimal prime containing x. By localizing at $\mathfrak{p}$, we may assume that A is a local ring, with maximal ideal $\mathfrak{p}$. Suppose $\mathfrak{q}$ is another prime strictly containing $\mathfrak{p}$.



For the first part of the theorem, we must show that $A_{\mathfrak{q}}$ has dimension 0. The second part follows from our earlier work: if any minimal primes are height 0, f is a zero-divisor, by our identification of the associated primes of a ring as the union of zero-divisors.

Now $\mathfrak{p}$ is the only prime ideal containing (x), so $A/(x)$ has one prime ideal. By Lemma 0.8, $A/(x)$ is Artinian.

We invoke a useful construction, the n*th symbolic power of a prime ideal*: if R is a ring, and $\mathfrak{q}$ is a prime ideal, then define

$$\mathfrak{q}^{(n)} := \{r \in R : rs \in \mathfrak{q}^n \text{ for some } s \in R - \mathfrak{q}\}.$$

We have a descending chain of ideals in A

$$\mathfrak{q}^{(1)} \supset \mathfrak{q}^{(2)} \supset \cdots,$$

so we have a descending chain of ideals in $A/(x)$

$$\mathfrak{q}^{(1)} + (x) \supset \mathfrak{q}^{(2)} + (x) \supset \cdots$$

which stabilizes, as $A/(x)$ is Artinian. Say $\mathfrak{q}^{(n)} + (x) = \mathfrak{q}^{(n+1)} + (x)$, so

$$\mathfrak{q}^{(n)} \subset \mathfrak{q}^{(n+1)} + (x).$$

Hence for any $f \in \mathfrak{q}^{(n)}$, we can write $f = ax + g$ with $g \in \mathfrak{q}^{(n+1)}$. Hence $ax \in \mathfrak{q}^{(n)}$. As $\mathfrak{p}$ is minimal over x, $x \notin \mathfrak{q}$, so $a \in \mathfrak{q}^{(n)}$. Thus

$$\mathfrak{q}^{(n)} = (x)\mathfrak{q}^{(n)} + \mathfrak{q}^{(n+1)}.$$

As x is in the maximal ideal $\mathfrak{p}$, the second version of Nakayama's lemma 0.2 gives $\mathfrak{q}^{(n)} = \mathfrak{q}^{(n+1)}$.

We now shift attention to the local ring $A_{\mathfrak{q}}$, which we are hoping is dimension 0. We have $\mathfrak{q}^{(n)}A_{\mathfrak{q}} = \mathfrak{q}^{(n+1)}A_{\mathfrak{q}}$ (the symbolic power construction clearly construction commutes with respect to localization). For any $r \in \mathfrak{q}^n A_{\mathfrak{q}} \subset \mathfrak{q}^{(n)}A_{\mathfrak{q}}$, there is some $s \in A_{\mathfrak{q}} - \mathfrak{q}A_{\mathfrak{q}}$ such that $rs \in \mathfrak{q}^{n+1}A_\mathfrak{q}$. As s is invertible, $r \in \mathfrak{q}^{n+1}A_\mathfrak{q}$ as well. Thus $\mathfrak{q}^n A_\mathfrak{q} \subset \mathfrak{q}^{n+1}A_\mathfrak{q}$, but as $\mathfrak{q}^{n+1}A_\mathfrak{q} \subset \mathfrak{q}^n A_\mathfrak{q}$, we have $\mathfrak{q}^n A_\mathfrak{q} = \mathfrak{q}^{n+1}A_\mathfrak{q}$. By Nakayama's Lemma version 4 (Exercise 0.4),

$$\mathfrak{q}^n A_\mathfrak{q} = 0.$$

Finally, any local ring $(R, \mathfrak{m})$ such that $\mathfrak{m}^n = 0$ has dimension 0, as $\operatorname{Spec} R$ consists of only one point: $[\mathfrak{m}] = V(\mathfrak{m}) = V(\mathfrak{m}^n) = V(0) = \operatorname{Spec} R$. □

Finally:

0.9. *Proposition. — If* $(A, \mathfrak{m})$ *is a Noetherian local ring, then* $\cap_i \mathfrak{m}^i = 0$.

It is tempting to argue that $\mathfrak{m}(\cap_i \mathfrak{m}^i) = \cap_i \mathfrak{m}^i$, and then to use Nakayama's lemma 0.4 to argue that $\cap_i \mathfrak{m}^i = 0$. Unfortunately, it is not obvious that this first equality is true: product does not commute with infinite intersections in general. I heard this argument from Kirsten Wickelgren, who I think heard it from Greg Brumfiel. We used it in showing an equivalence in that big chain of equivalent characterizations of discrete valuation rings.

Proof. Let $I = \cap_i \mathfrak{m}^i$. We wish to show that $I \subset \mathfrak{m}I$; then as $\mathfrak{m}I \subset I$, we have $I = \mathfrak{m}I$, and hence by Nakayama's Lemma 0.4, $I = 0$. Fix a primary decomposition of $\mathfrak{m}I$. It suffices to show that $\mathfrak{p}$ contains I for any $\mathfrak{p}$ in this primary decomposition, as then I is contained in all the primary ideals in the decomposition of $\mathfrak{m}I$, and hence $\mathfrak{m}I$.

Let $\mathfrak{q} = \sqrt{\mathfrak{p}}$. If $\mathfrak{q} \neq \mathfrak{m}$, then choose $x \in \mathfrak{m} - \mathfrak{q}$. Now x is not nilpotent in $R/\mathfrak{p}$, and hence is not a zero-divisor. But $xI \subset \mathfrak{p}$, so $I \subset \mathfrak{p}$.

On the other hand, if $\mathfrak{q} = \mathfrak{m}$, then as $\mathfrak{m}$ is finitely generated, and each generator is in $\sqrt{\mathfrak{p}}$, there is some a such that $\mathfrak{m}^a \subset \mathfrak{p}$. But $I \subset \mathfrak{m}^a$, so we are done. □

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FOUNDATIONS OF ALGEBRAIC GEOMETRY CLASS 21

RAVI VAKIL

CONTENTS

Today: integral extensions, Going-up theorem, Noether Normalization, proof that transcendence degree = Krull dimension, proof of Chevalley's theorem. Invertible sheaves and morphisms to (quasi)projective schemes

Welcome back everyone! This is the second quarter in a three-quarter experimental sequence on algebraic geometry.

We know what schemes are, their properties, quasicoherent sheaves on them, and morphisms between them. This quarter, we're going to talk about fancier concepts: fibered products; normalization; separatedness and the definition of a variety; rational maps; classification of curves; cohomology; differentials; and Riemann-Roch.

I'd like to start with some notions that I now think I should have done in the middle of last quarter. They are some notions that I think are easier than are usually presented.

1. Integral extensions, the going-up theorem, Noether normalization, and a proof of the big dimension theorem (that transcendence degree = Krull dimension)

Recall the maps of sets corresponding to a map of rings. If we have $\phi : B \to A$, we get a map $\operatorname{Spec} A \to \operatorname{Spec} B$ as sets (and indeed as topological spaces, and schemes), which sends $\mathfrak{p} \subset A$ to $\phi^{-1}\mathfrak{p} \subset B$. The notion behaves well under quotients and localization of both the source and target affine scheme.

A ring homomorphism $\phi : B \to A$ is *integral* if every element of A is integral over $\phi(B)$. (Thanks to Justin for pointing out that this notation is not just my invention — it is in Atiyah-Macdonald, p. 60.) In other words, if a is any element of A, then a satisfies some

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monic polynomial $a^n + \cdots = 0$ where all the coefficients lie in $\phi(B)$. We call it an *integral extension* if ϕ is an inclusion of rings.

1.1. Exercise. The notion of integral morphism is well behaved with respect to localization and quotient of B, and quotient of A (but not localization of A, witness $k[t] \to k[t]$, but $k[t] \to k[t]_{(t)}$). The notion of integral extension is well behaved with respect to localization and quotient of B, but not quotient of A (same example, $k[t] \to k[t]/(t)$).

1.2. Exercise. Show that if B is an integral extension of A, and C is an integral extension of B, then C is an integral extension of A.

1.3. *Proposition. — If A is finitely generated as a B-module, then ϕ is an integral morphism.*

Proof. (If B is Noetherian, this is easiest: suppose $a \in B$. Then A is a Noetherian B-module, and hence the ascending chain of B-submodules of A $(1) \subset (1, a) \subset (1, a, a^2) \subset (1, a, a^2, a^3) \subset \cdots$ eventually stabilizes, say $(1, a, \dots, a^{n-1}) = (1, a, \dots, a^{n-1}, a^n)$. Hence a^n is a B-linear combination of 1, …, a^{n-1}, i.e. is integral over B. So Noetherian-minded readers can stop reading.) We use a trick we've seen before. Choose a finite generating set $m_1, \dots, m_n$ of A as a B-module. Then $a m_i = \sum a_{ij} m_j$, where $a_{ij} \in B$. Thus

$$(aI_{n\times n} - [a_{ij}]_{ij}) \begin{pmatrix} m_1 \\ \vdots \\ m_n \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Multiplying this equation by the adjoint of the left side, we get

$$\det(aI_{n\times n} - [a_{ij}]_{ij}) \begin{pmatrix} m_1 \\ \vdots \\ m_n \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}.$$

(We saw this trick when discussing Nakayama's lemma.) So $\det(aI - M)$ annihilates A, i.e. $\det(aI - M) = 0$. $\square$

1.4. Exercise (cf. Exercise 1.2). Show that if B is a finite extension of A, and C is a finite extension of B, then C is an finite extension of A. (Recall that if we have a ring homomorphism $A \to B$ such that B is a finitely-generated A-module (*not* necessarily A-algebra) then we say that B is a finite extension of A.)

We now recall the Going-up theorem.

1.5. *Cohen-Seidenberg Going up theorem. — Suppose $\phi : B \to A$ is an integral extension. Then for any prime ideal $\mathfrak{q} \subset B$, there is a prime ideal $\mathfrak{p} \subset A$ such that $\mathfrak{p} \cap B = \mathfrak{q}$.*

Although this is a theorem in algebra, the name reflects its geometric motivation: the theorem asserts that the corresponding morphism of schemes is surjective, and that "above" every prime $\mathfrak{q}$ "downstairs", there is a prime $\mathfrak{q}$ "upstairs". (I drew a picture here.) For this reason, it is often said that $\mathfrak{q}$ is "above" $\mathfrak{p}$ if $\mathfrak{p} \cap B = \mathfrak{q}$. (Joe points out that my speculation on the origin of the name "going up" is wrong.)

As a reality check: note that the morphism $k[t] \to k[t]_{(t)}$ is not integral, so the conclusion of the Going-up theorem 1.5 fails. (I drew a picture again.)

Proof of the Cohen-Seidenberg Going-Up theorem 1.5. This proof is eminently readable, but could be skipped on first reading. We start with an exercise.

1.6. Exercise. Show that the special case where A is a field translates to: if $B \subset A$ is a subring with A integral over B, then B is a field. Prove this. (Hint: all you need to do is show that all nonzero elements in B have inverses in B. Here is the start: If $b \in B$, then $1/b \in A$, and this satisfies some integral equation over B.)

We're ready to prove the Going-Up Theorem 1.5.

We first make a reduction: by localizing at $\mathfrak{q}$, so we can assume that $(B, \mathfrak{q})$ is a local ring.

Then let $\mathfrak{p}$ be any *maximal* ideal of A. We will see that $\mathfrak{p} \cap B = \mathfrak{q}$. Consider the following diagram.

$$\begin{array}{ccc} A & \twoheadrightarrow & A/\mathfrak{p} \qquad \text{field} \\ \uparrow & & \uparrow \\ B & \twoheadrightarrow & B/(B \cap \mathfrak{p}) \end{array}$$

By the Exercise above, the lower right is a field too, so $B \cap \mathfrak{p}$ is a maximal ideal, hence $\mathfrak{q}$. $\square$

1.7. Important but straightforward exercise (sometimes also called the going-up theorem). Show that if $\mathfrak{q}_1 \subset \mathfrak{q}_2 \subset \cdots \subset \mathfrak{q}_n$ is a chain of prime ideals of B, and $\mathfrak{p}_1 \subset \cdots \subset \mathfrak{p}_m$ is a chain of prime ideals of A such that $\mathfrak{p}_i$ "lies over" $\mathfrak{q}_i$ (and $m < n$), then the second chain can be extended to $\mathfrak{p}_1 \subset \cdots \subset \mathfrak{p}_n$ so that this remains true.

The going-up theorem has an important consequence.

1.8. Important exercise. Show that if $f : \operatorname{Spec} A \to \operatorname{Spec} B$ corresponds to an integral *extension of rings*, then $\dim \operatorname{Spec} A = \dim \operatorname{Spec} B$.

I'd like to walk you through much of this exercise. You can show that a chain downstairs gives a chain upstairs, by the going up theorem, of the same length. Conversely, a chain upstairs gives a chain downstairs. We need to check that no two elements of the chain upstairs goes to the same element of the chain downstairs. That boils down to this: If $\phi : k \to A$ is an integral extension, then $\dim A = 0$. *Proof.* Suppose $\mathfrak{p} \subset \mathfrak{m}$ are two prime ideals of $\mathfrak{p}$. Mod out by $\mathfrak{p}$, so we can assume that A is a domain. I claim that any non-zero element is invertible. Here's why. Say $x \in A$, and $x \neq 0$. Then the minimal monic polynomial for x has non-zero constant term. But then x is invertible (recall coefficients are in a field).

We now introduce another important and ancient result, Noether's Normalization Lemma.

1.9. *Noether Normalization Lemma. — Suppose* A *is an integral domain, finitely generated over a field* k*. If* $\mathrm{tr.deg.}_k A = n$*, then there are elements* $x_1, \ldots, x_n \in A$*, algebraically independent over* k*, such that* A *is a finite (hence integral by Proposition 1.3) extension of* $k[x_1, \ldots, x_n]$*.*

The geometric content behind this result is that given any integral affine k-scheme X, we can find a surjective finite morphism $X \to \mathbb{A}^n_k$, where n is the transcendence degree of the function field of X (over k).

Proof of Noether normalization. We give Nagata's proof, following Mumford's Red Book (§1.1). Suppose we can write $A = k[y_1, \ldots, y_m]/\mathfrak{p}$, i.e. that A can be chosen to have m generators. Note that $m \geq n$. We show the result by induction on m. The base case $m = n$ is immediate.

Assume now that $m > n$, and that we have proved the result for smaller m. We will find $m - 1$ elements $z_1, \ldots, z_{m-1}$ of A such that A is finite over $A' := k[z_1, \ldots, z_{m-1}]$ (by which we mean the subring of A generated by $z_1, \ldots, z_{m-1}$). Then by the inductive hypothesis, A' is finite over some $k[x_1, \ldots, x_n]$, and A is finite over A, so by Exercise 1.4 A is finite over $k[x_1, \ldots, x_n]$.

As $y_1, \ldots, y_m$ are algebraically dependent, there is some non-zero algebraic relation $f(y_1, \ldots, y_m) = 0$ among them (where f is a polynomial in m variables).

Let $z_1 = y_1 - y_m^{r_1}, z_2 = y_2 - y_m^{r_2}, \ldots, z_{m-1} = y_{m-1} - y_m^{r_{m-1}}$, where $r_1, \ldots, r_{m-1}$ are positive integers to be chosen shortly. Then

$$f(z_1 + y_m^{r_1}, z_2 + y_m^{r_2}, \ldots, z_{m-1} + y_m^{r_{m-1}}, y_m) = 0.$$

Then upon expanding this out, each monomial in f (as a polynomial in m variables) will yield a single term in that is a constant times a power of y_m (with no z_i factors). By choosing the r_i so that $0 \ll r_1 \ll r_2 \ll \cdots \ll r_{m-1}$, we can ensure that the powers of y_m appearing are all distinct, and so that in particular there is a leading term y_m^N, and all other terms (including those with z_i-factors) are of smaller degree in y_m. Thus we have described an integral dependence of y_m on $z_1, \ldots, z_{m-1}$ as desired. □

Now we can give a proof of something we used a lot last quarter:

1.10. *Important Theorem about Dimension. — Suppose* R *is a finitely-generated domain over a field* k*. Then* $\dim \operatorname{Spec} R$ *is the transcendence degree of the fraction field* $\operatorname{Frac}(R)$ *over* k*.*

We proved this in class 9, but I think this proof is much slicker.

Proof. Suppose X is an integral affine k-scheme. We show that $\dim X$ equals the transcendence degree n of its function field, by induction on n. Fix X, and assume the result is known for all transcendence degrees less than n. The base case $n = -1$ is vacuous.

By Exercise 1.8, $\dim X = \dim \mathbb{A}^n_k$. If $n = 0$, we are done.

We now show that $\dim \mathbb{A}^n_k = n$ for $n > 0$. Clearly $\dim \mathbb{A}^n_k \geq n$, as we can describe a chain of irreducible subsets of length $n+1$: if $x_1, \ldots, x_n$ are coordinates on $\mathbb{A}^n$, consider the chain of ideals

$$(0) \subset (x_1) \subset \cdots \subset (x_1, \ldots, x_n)$$

in $k[x_1, \ldots, x_n]$. Suppose we have a chain of prime ideals of length at least n:

$$(0) = \mathfrak{p}_0 \subset \cdots \subset \mathfrak{p}_m.$$

where $\mathfrak{p}_1$ is a height 1 prime ideal. Then $\mathfrak{p}_1$ is principal (as $k[x_1, \ldots, x_n]$ is a unique factorization domain, cf. Exercises 1 and 4 on problem set 6); say $\mathfrak{p}_1 = (f(x_1, \ldots, x_n))$, where f is an irreducible polynomial. Then $k[x_1, \ldots, x_n]/(f(x_1, \ldots, x_n))$ has transcendence degree $n-1$, so by induction,

$$\dim k[x_1, \ldots, x_n]/(f) = n - 1.$$

□

2. Images of morphisms

Here are two applications of the going-up theorem, which are quite similar to each other.

2.1. *Exercise.* Show that finite morphisms are *closed*, i.e. the image of any closed subset is closed.

2.2. *Exercise.* Show that integral ring extensions induce a surjective map of spectra.

I now want to use the Noether normalization lemma to prove Chevalley's theorem. Recall that we define a *constructable subset* of a scheme to be a subset which belongs to the smallest family of subsets such that (i) every open set is in the family, (ii) a finite intersection of family members is in the family, and (iii) the complement of a family member is also in the family. So for example the image of $(x, y) \mapsto (x, xy)$ is constructable.

2.3. Exercise. Suppose X is a Noetherian scheme. Show that a subset of X is constructable if and only if it is the finite disjoint union of locally closed subsets.

Last quarter we stated the following.

2.4. *Chevalley's Theorem.* — *Suppose $f : X \to Y$ is a morphism of finite type of Noetherian schemes. Then the image of any constructable set is constructable.*

We'll now prove this using Noether normalization. (This is remarkable: Noether normalization is about finitely generated algebras over a field. There is no field in the statement of Chevalley's theorem. Hence if you prefer to work over arbitrary rings (or schemes), this shows that you still care about facts about finite type schemes over a field. Also, even if you are interested in finite type schemes over a given field (like $\mathbb{C}$), the field that comes up in the proof of Chevalley's theorem is *not* that field, so even if you prefer to work over $\mathbb{C}$, this argument shows that you still care about working over arbitrary fields, not necessarily algebraically closed.)

We say a morphism $f : X \to Y$ is *dominant* if the image of f meets every dense open subset of Y. (This is sometimes called *dominating*, but we will not use this notation.)

2.5. Exercise. Show that a dominant morphism of integral schemes $X \to Y$ induces an inclusion of function fields in the other direction.

2.6. *Exercise.* If $\phi : A \to B$ is a ring morphism, show that the corresponding morphism of affine schemes $\operatorname{Spec} B \to \operatorname{Spec} A$ is dominant iff ϕ has nilpotent kernel.

2.7. Exercise. Reduce the proof of the Going-up theorem to the following case: suppose $f : X = \operatorname{Spec} A \to Y = \operatorname{Spec} B$ is a dominant morphism, where A and B are domains, and f corresponds to $\phi : B \to B[x_1, \dots, x_n]/I \cong A$. Show that the image of f contains a dense open subset of $\operatorname{Spec} B$.

Proof. We prove the problem posed in the previous exercise. This argument uses Noether normalization 1.9 in an interesting context — even if we are interested in schemes over a field k, this argument will use a larger field, the field $K := \operatorname{Frac}(B)$. Now $A \otimes_B K$ is a localization of A with respect to B^*, so it is a domain, and it is finitely generated over K (by $x_1, \dots, x_n$), so it has finite transcendence degree r over K. Thus by Noether normalization, we can find a subring $K[y_1, \dots, y_r] \subset A \otimes_B K$, so that $A \otimes_B K$ is integrally dependent on $K[y_1, \dots, y_r]$. We can choose the y_i to be in A: each is in $(B^*)^{-1}A$ to begin with, so we can replace each y_i by a suitable K-multiple.

Sadly A is not necessarily integrally dependent on $K[y_1, \dots, y_r]$ (as this would imply that $\operatorname{Spec} A \to \operatorname{Spec} B$ is surjective). However, each x_i satisfies some integral equation

$$x_i^n + f_1(y_1, \dots, y_r)x_i^{n-1} + \cdots + f_n(y_1, \dots, y_r) = 0$$

where f_j are polynomials with coefficients in $K = \operatorname{Frac}(B)$. Let g be the product of the denominators of all the coefficients of all these polynomials (a finite set). Then A_g is integral over B_g, and hence $\operatorname{Spec} A_g \to \operatorname{Spec} B_g$ is surjective; $\operatorname{Spec} B_g$ is our open subset. $\square$

3. Important example: Morphisms to projective (and quasiprojective) schemes, and invertible sheaves

This will tell us why invertible sheaves are crucially important: they tell us about maps to projective space, or more generally to quasiprojective schemes. (And given that we have had a hard time naming any non-quasiprojective schemes, they tell us about maps to essentially all schemes that are interesting to us.)

3.1. *Important theorem. — Maps to $\mathbb{P}^n$ correspond to $n+1$ sections of a line bundle, not all vanishing at any point (= generated by global sections, by an earlier exercise, Class 16 Exercise 4.2, = Problem Set 7, Exercise 28), modulo sections of $\mathcal{O}_X^*$.*

The explanation and proof of the correspondence is in the notes for next day.

Here are some examples.

Example 1. Consider the $n+1$ functions $x_0, \dots, x_n$ on $\mathbb{A}^{n+1}$ (otherwise known as $n+1$ sections of the trivial bundle). They have no common zeros on $\mathbb{A}^n - 0$. Hence they determine a morphism $\mathbb{A}^{n+1} - 0 \to \mathbb{P}^n$. (We've talked about this morphism before. But now we don't have to worry about gluing.)

Example 2: the Veronese morphism. Consider the line bundle $\mathcal{O}_{\mathbb{P}^n}(m)$ on $\mathbb{P}^n$. We've checked that the number of sections of this line bundle are $\binom{n+m}{m}$, and they correspond to homogeneous degree m polynomials in the projective coordinates for $\mathbb{P}^n$. Also, they have no common zeros (as for example the subset of sections $x_0^m, x_1^m, \dots, x_n^m$ have no common zeros). Thus these determine a morphism $\mathbb{P}^n \to \mathbb{P}^{\binom{n+m}{m}-1}$. This is called the *Veronese morphism*. For example, if $n=2$ and $m=2$, we get a map $\mathbb{P}^2 \to \mathbb{P}^5$.

This is in fact a closed immersion. Reason: This map corresponds to a surjective map of graded rings. The first ring R_1 has one generator for each of degree m monomial in the x_i. The second ring is not $k[x_0, \dots, x_n]$, as R_1 does not surject onto it. Instead, we take $R_2 = k[x_0, \dots, x_n]_{(m)}$, i.e. we consider only those polynomials all of whose terms have degree divisible by m. Then the natural map $R_1 \to R_2$ is fairly clearly a surjection. Thus the corresponding map of projective schemes is a closed immersion by an earlier exercise.

How can you tell in general if something is a closed immersion, and not just a map? Here is one way.

3.2. *Exercise.* Let $f : X \to \mathbb{P}^n_A$ be a morphism of A-schemes, corresponding to an invertible sheaf $\mathcal{L}$ on X and sections $s_0, \dots, s_n \in \Gamma(X, \mathcal{L})$ as above. Then ϕ is a closed immersion iff (1) each open set $X_i = X_{s_i}$ is affine, and (2) for each i, the map of rings $A[y_0, \dots, y_n] \to \Gamma(X_i, \mathcal{O}_{X_i})$ given by $y_j \mapsto s_j/s_i$ is surjective.

We'll give another method of detecting closed immersions later. The intuition for this will come from differential geometry: the morphism should separate points, and also separate tangent vectors.

Example 3. The rational normal curve. The image of the Veronese morphism when $n = 1$ is called a *rational normal curve of degree* m. Our map is $\mathbb{P}^1 \to \mathbb{P}^m$ given by $[x; y] \to [x^m; x^{m-1}y; \cdots; xy^{m-1}; y^m]$. When $m = 3$, we get our old friend the *twisted cubic*. When $m = 2$, we get a smooth conic. What happens when $m = 1$?

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FOUNDATIONS OF ALGEBRAIC GEOMETRY CLASS 22

RAVI VAKIL

CONTENTS

Last day: integral extensions, Going-up theorem, Noether Normalization, proof that transcendence degree = Krull dimension, proof of Chevalley's theorem.

Today: Morphisms to (quasi)projective schemes, and invertible sheaves; fibered products; fibers.

1. IMPORTANT EXAMPLE: MORPHISMS TO PROJECTIVE (AND QUASIPROJECTIVE) SCHEMES, AND INVERTIBLE SHEAVES

1.1. *Important theorem. — Maps to* $\mathbb{P}^n$ *correspond to* $n+1$ *sections of an invertible sheaf, not all vanishing at any point (= generated by global sections), modulo sections of* $\mathcal{O}_X^*$.

Here more precisely is the correspondence. If you have $n+1$ sections, then away from the intersection of their zero-sets, we have a morphism. Conversely, if you have a map to projective space $f : X \to \mathbb{P}^n$, then we have $n+1$ sections of $\mathcal{O}_{\mathbb{P}^n}(1)$, corresponding to the hyperplane sections, $x_0, \dots, x_{n+1}$. then $f^*x_0, \dots, f^*x_{n+1}$ are sections of $f^*\mathcal{O}_{\mathbb{P}^n}(1)$, and they have no common zero.

So to prove this, we just need to show that these two constructions compose to give the identity in either direction.

Given $n+1$ sections $s_0, \dots, s_n$ of an invertible sheaf. We get trivializations on the open sets where each one vanishes. The transition functions are precisely s_i/s_j on $U_i \cap U_j$. We pull back $\mathcal{O}(1)$ by this map to projective space, This is trivial on the distinguished open sets. Furthermore, $f^*D(x_i) = D(s_i)$. Moreover, $s_i/s_j = f^*x_i/x_j$. Thus starting with the

Date: Thursday, January 12, 2006.

$n+1$ sections, taking the map to the projective space, and pulling back $\mathcal{O}(1)$ and taking the sections $x_0, \dots, x_n$, we recover the s_i's. That's one of the two directions.

Correspondingly, given a map $f : X \to \mathbb{P}^n$, let $s_i = f^* x_i$. The map $[s_0; \cdots ; s_n]$ is precisely the map f. We see this as follows. The preimage of U_i is $D(s_i) = D(f^* x_i) = f^* D(x_i)$. So the right open sets go to the right open sets. And $D(s_i) \to D(x_i)$ is precisely by $s_j/s_i = f^*(x_j/x_i)$. $\square$

1.2. *Exercise (Automorphisms of projective space).* Show that all the automorphisms of projective space $\mathbb{P}^n_k$ correspond to $(n+1) \times (n+1)$ invertible matrices over k, modulo scalars (also known as $PGL_{n+1}(k)$). (Hint: Suppose $f : \mathbb{P}^n_k \to \mathbb{P}^n_k$ is an automorphism. Show that $f^*\mathcal{O}(1) \cong \mathcal{O}(1)$. Show that $f^* : \Gamma(\mathbb{P}^n, \mathcal{O}(1)) \to \Gamma(\mathbb{P}^n, \mathcal{O}(1))$ is an isomorphism.)

This exercise will be useful later, especially for the case $n = 1$.

(A question for experts: why did I not state that previous exercise over an arbitrary base ring A? Where does the argument go wrong in that case?)

1.3. *Neat Exercise.* Show that any map from projective space to a smaller projective space is constant.

Here are some useful phrases to know.

A *linear series* on a scheme X over a field k is an invertible sheaf $\mathcal{L}$ and a finite-dimensional k-vector space V of sections. (We will not require that this vector space be a subspace of $\Gamma(X, \mathcal{L})$; in general, we just have a map $V \to \Gamma(X, \mathcal{L})$.) If the linear series is $\Gamma(X, \mathcal{L})$, we call it a *complete linear series*, and is often written $|\mathcal{L}|$. Given a linear series, any point $x \in X$ on which all elements of the linear series V vanish, we say that x is a *base-point* of V. If V has no base-points, we say that it is *base-point-free*. The union of base-points is called the *base locus*. In fact, it naturally has a scheme-structure — it is the (scheme-theoretic) intersection of the vanishing loci of the elements of V (or equivalently, of a basis of V). In this incarnation, it is called the *base scheme* of the linear series.

Then Theorem 1.1 says that each base-point-free linear series gives a morphism to projective space $X \to \mathbb{P}V^* = \operatorname{Proj} \oplus_n \mathcal{L}^{\otimes n}$. The resulting morphism is often written $X \xrightarrow{|V|} \mathbb{P}^n$. (I may not have this notation quite standard; I should check with someone. I always forget whether I should use "linear system" or "linear series".)

1.4. *Exercise.* If the image scheme-theoretically lies in a hyperplane of projective space, we say that it is *degenerate* (and otherwise, *non-degenerate*). Show that a base-point-free linear series V with invertible sheaf $\mathcal{L}$ is non-degenerate if and only if the map $V \to \Gamma(X, \mathcal{L})$ is an inclusion. Hence in particular a complete linear series is always non-degenerate.

Example: The Veronese and Segre morphisms. *Whoops! We don't know much about fibered products yet, so the Segre discussion may be a bit confusing. But fibered products are* *coming very very shortly...* The Veronese morphism can be interpreted in this way. The dth Veronese morphism on $\mathbb{P}^n$ corresponds to the complete linear series $|\mathcal{O}_{\mathbb{P}^n}(d)|$.

The Segre morphism can also be interpreted in this way. In case I haven't defined it yet, suppose $\mathcal{F}$ is a quasicoherent sheaf on a Z-scheme X, and $\mathcal{G}$ is a quasicoherent sheaf on a Z-scheme Y. Let π_X, π_Y be the projections from $X \times_Z Y$ to X and Y respectively. Then $\mathcal{F} \boxtimes \mathcal{G}$ is defined to be $\pi_X^* \mathcal{F} \otimes \pi_Y^* \mathcal{G}$. In particular, $\mathcal{O}_{\mathbb{P}^m \times \mathbb{P}^n}(a, b)$ is defined to be $\mathcal{O}_{\mathbb{P}^m}(a) \boxtimes \mathcal{O}_{\mathbb{P}^n}(b)$ (over any base Z). The Segre morphism $\mathbb{P}^m \times \mathbb{P}^n \to \mathbb{P}^{mn+m+n}$ corresponds to the complete linear system for the invertible sheaf $\mathcal{O}(1, 1)$.

Both of these complete linear systems are easily seen to be base-point-free (*exercise*). We still have to check by hand that they are closed immersions. (We will later see, in class 34, a criterion for linear series to be a closed immersion, at least in the special case where we are working over an algebraically closed field.)

2. Fibered products

We will now construct the fibered product in the category of schemes. In other words, given $X, Y \to Z$, we will show that $X \times_Z Y$ exists. (Recall that the *absolute product* in a category is the fibered product over the final object, so $X \times Y = X \times_{\mathbb{Z}} Y$ in the category of schemes, and $X \times Y = X \times_S Y$ if we are implicitly working in the category of S-schemes, for example if S is the spectrum of a field.)

Here is a notation warning: in the literature (and indeed in this class) lazy people wanting to save chalk and ink will write $\times_k$ for $\times_{\operatorname{Spec} k}$, and similarly for $\times_{\mathbb{Z}}$. In fact it already happened in the paragraph above!

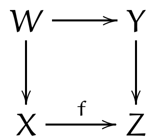
As always when showing that certain objects defined by universal properties exist, we have two ways of looking at the objects in practice: by using the universal property, or by using the details of the construction.

The key idea, roughly, is this: we cut everything up into affine open sets, do fibered products in that category (where it turns out we have seen the concept before in a different guise), and show that everything glues nicely. We can't do this too naively (e.g. by induction), as in general we won't be able to cut things into a finite number of affine open sets, so there will be a tiny bit of cleverness.

The argument will be an inspired bit of abstract nonsense, where we'll have to check almost nothing. This sort of argument is very powerful, and we will use it immediately after to construct lots of other interesting notions, so please pay attention!

Before we get started, here is a sign that something interesting happens for fibered products of schemes. Certainly you should believe that if we take the product of two affine lines (over your favorite algebraically field k, say), you should get the affine plane: $\mathbb{A}^1_k \times_k \mathbb{A}^1_k$ should be $\mathbb{A}^2_k$. But the underlying set of the latter is *not* the underlying set of the former —- we get additional points! I'll give an exercise later for you to verify this.

Let's take a break to introduce some language. Say

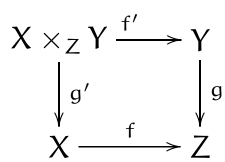


is a *fiber diagram* or *Cartesian diagram* or *base change diagram*. It is often called a *pullback diagram*, and $W \to X$ is called the *pullback* of $Y \to Z$ by f, and W is called the *pullback* of Y by f.

At this point, I drew some pictures on the blackboard giving some intuitive idea of what a pullback does. If $Y \to Z$ is a "family of schemes", then $W \to Z$ is the "pulled back family". To make this more explicit or precise, I need to tell you about fibers of a morphism. I also want to give you a bunch of examples. But before doing either of these things, I want to tell you how to compute fibered products in practice.

Okay, lets get to work.

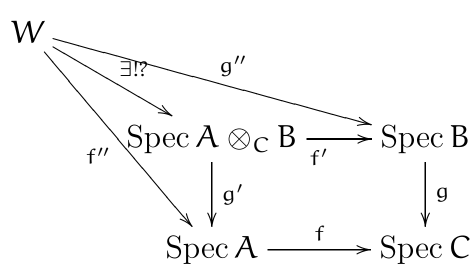
2.1. *Theorem (fibered products always exist). — Suppose $f : X \to Z$ and $g : Y \to Z$ are morphisms of schemes. Then the fibered product*



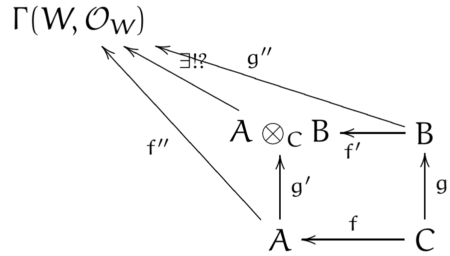
exists in the category of schemes.

We have an extended proof by universal property.

First, if X, Y, Z are affine schemes, say $X = \operatorname{Spec} A$, $Y = \operatorname{Spec} B$, $Z = \operatorname{Spec} C$, the fibered product exists, and is $\operatorname{Spec} A \otimes_C B$. Here's why. Suppose W is any scheme, along with morphisms $f'' : W \to X$ and $g'' : W \to Y$ such that $f \circ f'' = g \circ g''$ as morphisms $W \to Z$. We hope that there exists a unique $h : W \to \operatorname{Spec} A \otimes_C B$ such that $f'' = g' \circ h$ and $g'' = f' \circ h$.



But maps to affine schemes correspond precisely to maps of global sections in the other direction (class 19 exercise 0.1):



But this is precisely the universal property for tensor product! (The tensor product is the cofibered product in the category of rings.)

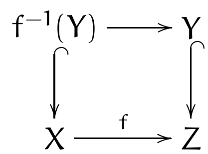
Thus indeed $\mathbb{A}^1 \times \mathbb{A}^1 \cong \mathbb{A}^2$, and more generally $(\mathbb{A}^1)^n \cong \mathbb{A}^n$.

Exercise. Show that the fibered product does not induce a bijection of points

$$\text{points}(\mathbb{A}^1_k) \times \text{points}(\mathbb{A}^1_k) \longrightarrow \text{points}(\mathbb{A}^2_k).$$

Thus products of schemes do something a little subtle on the level of sets.

Second, we note that the fibered product with open immersions always exists: if $Y \hookrightarrow Z$ an open immersion, then for any $f : X \to Z$, $X \times_Z Y$ is the open subset $f^{-1}(Y)$. (More precisely, this open subset satisfies the universal property.) We proved this in class 19 (exercise 1.2).



(An exercise to give you practice with this concept: show that the fibered product of two open immersions is their intersection.)

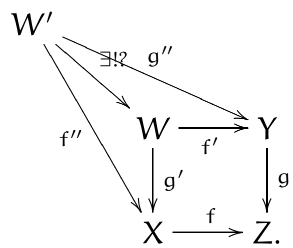
Hence the fibered product of a *quasiaffine* scheme (defined to be an open subscheme of an affine scheme) with an affine scheme over an affine scheme exists. *This isn't quite right; what we've shown, and what we'll use, is that the fibered product of a quasi-affine scheme with an affine scheme over an affine scheme Z exists so long as that quasi-affine scheme is an open subscheme of an affine scheme that also admits a map to Z extending the map from the quasiaffine. At some point I'll retype this to say this better. This sloppiness continues in later lectures, but the argument remains correct.*

Third, we show that $X \times_Z Y$ exists if Y and Z are affine and X is general. Before we show this, we remark that one special case of it is called "extension of scalars": if X is a k-scheme, and k' is a field extension (often k' is the algebraic closure of k), then $X \times_{\operatorname{Spec} k} \operatorname{Spec} k'$ (sometimes informally written $X \times_k k'$ or $X_{k'}$) is a k'-scheme. Often properties of X can be checked by verifying them instead on $X_{k'}$. This is the subject of *descent* — certain properties "descend" from $X_{k'}$ to X.

Let's verify this. It will follow from abstract nonsense and the gluing lemma. Recall the *gluing lemma* (a homework problem): assume we are given a bunch of schemes X_i indexed by some index set I, along with open subschemes $U_{ij} \subset X_i$ indexed by $I \times I$, and isomorphisms $f_{ij} : U_{ij} \xrightarrow{\sim} U_{ji}$, satisfying the cocycle condition: $f_{ij}(U_{ij} \cap U_{ik}) = U_{ji} \cap U_{jk}$, and $(f_{jk} \circ f_{ij})|_{U_{ij} \cap U_{ik}} = f_{ik}|_{U_{ij} \cap U_{ik}}$. Then they glue together to a unique scheme. (This was a homework problem long ago; I'll add a reference when I dig it up.)

We'll now apply this in our case. Cover X with affine open sets V_i. Let $V_{ij} = V_i \cap V_j$. Then for each of these, $X_i := V_i \times_Z Y$ exists, and each of them has open subsets $U_{ij} := V_{ij} \times_Z Y$, and isomorphisms satisfying the cocycle condition (because the V_i's and V_{ij}'s could be glued together via g_{ij} which satisfy the cocycle condition).

Call this glued-together scheme W. It comes with morphisms to X and Y (and their compositions to Z are the same). I claim that this satisfies the universal property for $X \times_Z Y$, basically because "morphisms glue" (yet another ancient exercise). Here's why. Suppose W' is any scheme, along with maps to X and Y that agree when they are composed to Z. We need to show that there is a unique morphism $W' \to W$ completing the diagram

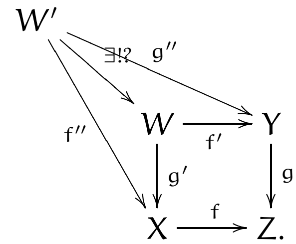


Now break W' up into open sets $W'_i = g''^{-1}(U_i)$. Then by the universal property for $V_i = U_i \times_Z Y$, there is a unique map $W'_i \to V_i$ (which we can interpret as $W'_i \to W$). (Thus we have already shown uniqueness of $W' \to W$.) These must agree on $W'_i \cap W'_j$, because there is only one map $W'_i \cap W'_j$ to W making the diagram commute (because of the second step — $(U_i \cap U_j) \times_Z Y$ exists). Thus all of these morphisms $W'_i \to W$ glue together; we have shown existence.

Fourth, we show that if Z is affine, and X and Y are arbitrary schemes, then $X \times_Z Y$ exists. We just repeat the process of the previous step, with the roles of X and Y repeated, using the fact that by the previous step, we can assume that the fibered product with an affine scheme with an arbitrary scheme over an affine scheme exists.

Fifth, we show that the fibered product of any two schemes over a *quasiaffine* scheme exists. Here is why: if $Z \hookrightarrow Z'$ is an open immersion into an affine scheme, then $X \times_Z Y = X \times_{Z'} Y$ are the same. (You can check this directly. But this is yet again an old exercise — problem set 1 problem A4 — following from the fact that $Z \hookrightarrow Z'$ is a monomorphism.)

Finally, we show that the fibered product of any scheme with any other scheme over any third scheme always exists. We do this in essentially the same way as the third step, using the gluing lemma and abstract nonsense. Say $f : X \to Z$, $g : Y \to Z$ are two morphisms of schemes. Cover Z with affine open subsets Z_i. Let $X_i = f^{-1}X_i$ and $Y_i = g^{-1}Y_i$. Define $Z_{ij} = Z_i \cap Z_j$, and X_{ij} and Y_{ij} analogously. Then $W_i := X_i \times_{Z_i} Y_i$ exists for all i, and has as open sets $W_{ij} := X_{ij} \times_{Z_{ij}} Y_{ij}$ along with gluing information satisfying the cocycle condition (arising from the gluing information for Z from the Z_i and Z_{ij}). Once again, we show that this satisfies the universal property. Suppose W' is any scheme, along with maps to X and Y that agree when they are composed to Z. We need to show that there is a unique morphism $W' \to W$ completing the diagram



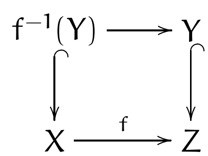
Now break W' up into open sets $W'_i = g'' \circ f^{-1}(Z_i)$. Then by the universal property for W_i, there is a unique map $W'_i \to W_i$ (which we can interpret as $W'_i \to W$). Thus we have already shown uniqueness of $W' \to W$. These must agree on $W'_i \cap W'_j$, because there is only one map $W'_i \cap W'_j$ to W making the diagram commute. Thus all of these morphisms $W'_i \to W$ glue together; we have shown existence. □

3. COMPUTING FIBERED PRODUCTS IN PRACTICE

There are four types of morphisms that it is particularly easy to take fibered products with, and all morphisms can be built from these four atomic components.

(1) *base change by open immersions*

We've already done the work for this one, and we used it above.

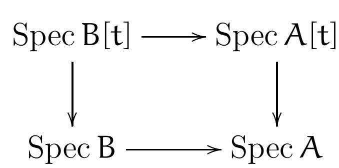


I'll describe the remaining three on the level of affine sets, because we obtain general fibered products by gluing.

(2) *adding an extra variable*

Exercise. Show that $B \otimes_A A[t] \cong B[t]$.

Hence the following is a fibered diagram.



(3) *base change by closed immersions*

If the right column is obtained by modding out by a certain ideal (i.e. if the morphism is a closed immersion, i.e. if the map of rings in the other direction is surjective), then the left column is obtained by modding out by the pulled back elements of that ideal. In other words, if $T \to R, S$ are two ring morphisms, and I is an ideal of R, and I^e is the extension of I to $R \otimes_T S$ (the elements $\sum_j i_j \otimes s_j$, where $i_j \in I$ and $s_j \in S$, then there is a natural isomorphism

$$R/I \otimes_T S \cong (R \otimes_T S)/I^e.$$

(This is precisely problem B3 on problem set 1.) Thus the natural morphism $R \otimes_T S \to R/I \otimes_T S$ is a surjection, and we have a base change diagram:

$$\begin{array}{ccc}
\operatorname{Spec}(R \otimes_T S)/I^e & \longrightarrow & \operatorname{Spec} R/I \\
\big\downarrow & & \big\downarrow \\
\operatorname{Spec} R \otimes_T S & \longrightarrow & \operatorname{Spec} R \\
\big\downarrow & & \big\downarrow \\
\operatorname{Spec} S & \longrightarrow & \operatorname{Spec} T
\end{array}$$

(where each rectangle is a fiber diagram).

Translation: the fibered product with a subscheme is the subscheme of the fibered product in the obvious way. We say that "closed immersions are preserved by base change".

(4) *base change by localization*

Exercise. Suppose $C \to B, A$ are two morphisms of rings. Suppose S is a multiplicative set of A. Then $(S \otimes 1)$ is a multiplicative set of $A \otimes_C B$. Show that there is a natural morphism $(S^{-1}A) \otimes_C B \cong (S \otimes 1)^{-1}(A \otimes_C B)$.

Hence we have a fiber diagram:

$$\begin{array}{ccc}
\operatorname{Spec}(S \otimes 1)^{-1}(A \otimes_C B) & \longrightarrow & \operatorname{Spec} S^{-1}A \\
\big\downarrow & & \big\downarrow \\
\operatorname{Spec} A \otimes_C B & \longrightarrow & \operatorname{Spec} A \\
\big\downarrow & & \big\downarrow \\
\operatorname{Spec} B & \longrightarrow & \operatorname{Spec} C
\end{array}$$

(where each rectangle is a fiber diagram).

Translation: the fibered product with a localization is the localization of the fibered product in the obvious way. We say that "localizations are preserved by base change". This is handy if the localization is of the form $A \hookrightarrow A_f$ (corresponding to taking distinguished open sets) or $A \hookrightarrow FF(A)$ (from A to the fraction field of A, corresponding to taking generic points), and various things in between.

These four tricks let you calculate lots of things in practice. For example,

$$\begin{aligned}&\operatorname{Spec} k[x_1,\dots,x_m]/(f_1(x_1,\dots,x_m),\dots,f_r(x_1,\dots,x_m))\otimes_k\\ &\operatorname{Spec} k[y_1,\dots,y_n]/(g_1(y_1,\dots,y_n),\dots,g_s(y_1,\dots,y_n))\\ \cong\ &\operatorname{Spec} k[x_1,\dots,x_m,y_1,\dots,y_n]/(f_1(x_1,\dots,x_m),\dots,f_r(x_1,\dots,x_m),\\ &g_1(y_1,\dots,y_n),\dots,g_s(y_1,\dots,y_n)).\end{aligned}$$

Here are many more examples.

4. Examples

One important example is of *fibers* of morphisms. Suppose $p \to Z$ is the inclusion of a point (not necessarily closed). Then if $g : Y \to Z$ is any morphism, the base change with $p \to Z$ is called the *fiber of* g *above* p or the *preimage of* p, and is denoted $g^{-1}(p)$. If Z is irreducible, the fiber above the generic point is called the *generic fiber*. In an affine open subscheme $\operatorname{Spec} A$ containing p, p corresponds to some prime ideal $\mathfrak{p}$, and the morphism corresponds to the ring map $A \to A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}$. this is the composition if localization and closed immersion, and thus can be computed by the tricks above.

Here is an interesting example, that we will consider multiple times during this course. Consider the projection of the parabola $y^2 = x$ to the x axis, corresponding to the map of rings $\mathbb{Q}[x] \to \mathbb{Q}[y]$, with $x \mapsto y^2$. (If $\mathbb{Q}$ alarms you, replace it with your favorite field and see what happens.)

Then the preimage of 1 is 2 points:

$$\begin{aligned}\operatorname{Spec}\mathbb{Q}[x,y]/(y^2-x)\otimes_{\mathbb{Q}}\operatorname{Spec}\mathbb{Q}[x]/(x-1) &\cong \operatorname{Spec}\mathbb{Q}[x,y]/(y^2-x,x-1)\\ &\cong \operatorname{Spec}\mathbb{Q}[y]/(y^2-1)\\ &\cong \operatorname{Spec}\mathbb{Q}[y]/(y-1)\coprod \operatorname{Spec}\mathbb{Q}[y]/(y+1).\end{aligned}$$

The preimage of 0 is 1 nonreduced point:

$$\operatorname{Spec}\mathbb{Q}[x,y]/(y^2-x,x) \cong \operatorname{Spec}\mathbb{Q}[y]/(y^2).$$

The preimage of -1 is 1 reduced point, but of "size 2 over the base field".

$$\operatorname{Spec}\mathbb{Q}[x,y]/(y^2-x,x+1) \cong \operatorname{Spec}\mathbb{Q}[y]/(y^2+1) \cong \operatorname{Spec}\mathbb{Q}[i].$$

The preimage of the generic fiber is again 1 reduced point, but of "size 2 over the residue field".

$$\operatorname{Spec}\mathbb{Q}[x,y]/(y^2-x)\otimes\mathbb{Q}(x) \cong \operatorname{Spec}\mathbb{Q}[y]\otimes\mathbb{Q}(y^2)$$

i.e. you take elements polynomials in y, and you are allowed to invert polynomials in y^2. A little thought shows you that you are then allowed to invert polynomials in y, as if $f(y)$ is any polynomial in y, then

$$\frac{1}{f(y)} = \frac{f(-y)}{f(y)f(-y)},$$

and the latter denominator is a polynomial in y^2. Thus

$$\operatorname{Spec} \mathbb{Q}[x, y]/(y^2 - x) \otimes \mathbb{Q}(x) \cong \mathbb{Q}(y)$$

which is a degree 2 field extension of $\mathbb{Q}(x)$.

For future reference notice the following interesting fact: in each case, the number of preimages can be interpreted as 2, where you count to two in several ways: you can count points; you can get non-reduced behavior; or you can have field extensions. This is going to be symptomatic of a very special and important kind of morphism (a finite flat morphism).

Here are some other examples.

4.1. *Exercise.* Prove that $\mathbb{A}^n_R \cong \mathbb{A}^n_{\mathbb{Z}} \times_{\operatorname{Spec} \mathbb{Z}} \operatorname{Spec} R$. Prove that $\mathbb{P}^n_R \cong \mathbb{P}^n_{\mathbb{Z}} \times_{\operatorname{Spec} \mathbb{Z}} \operatorname{Spec} R$.

4.2. *Exercise.* Show that for finite-type schemes over $\mathbb{C}$, the complex-valued points of the fibered product correspond to the fibered product of the complex-valued points. (You will just use the fact that $\mathbb{C}$ is algebraically closed.)

Here is a definition in common use. The terminology is a bit unfortunate, because it is a second (different) meaning of "points of a scheme". If T is a scheme, the T-*valued points of a scheme* X are defined to be the morphism $T \to X$. They are sometimes denoted $X(T)$. If R is a ring (most commonly in this context a field), the R-*valued points of a scheme* X are defined to be the morphism $\operatorname{Spec} R \to X$. They are sometimes denoted $X(R)$. For example, if k is an algebraically closed field, then the k-valued points of a finite type scheme are just the closed points; but in general, things can be weirder. (When we say "points of a scheme", and not T-valued points, we will always mean the usual meaning, not this meaning.)

Exercise. Describe a natural bijection $(X \times_Z Y)(T) \cong X(T) \times_{Z(T)} Y(T)$. (The right side is a fibered product of sets.) In other words, fibered products behaves well with respect to T-valued points. This is one of the motivations for this notion.

4.3. *Exercise.* Describe $\operatorname{Spec} \mathbb{C} \times_{\operatorname{Spec} \mathbb{R}} \operatorname{Spec} \mathbb{C}$. This small example is the first case of something incredibly important.

4.4. *Exercise.* Consider the morphism of schemes $X = \operatorname{Spec} k[t] \to Y = \operatorname{Spec} k[u]$ corresponding to $k[u] \to k[t]$, $t = u^2$. Show that $X \times_Y X$ has 2 irreducible components. Compare what is happening above the generic point of Y to the previous exercise.

4.5. *A little too vague to be an exercise.* More generally, suppose $K/\mathbb{Q}$ is a finite Galois field extension. Investigate the analogue of the previous two exercises. Try degree 2. Try degree 3.

4.6. *Hard but fascinating exercise for those familiar with the Galois group of $\overline{\mathbb{Q}}$ over $\mathbb{Q}$.* Show that the points of $\operatorname{Spec} \overline{\mathbb{Q}} \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}$ are in natural bijection with $\operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, and the Zariski topology on the former agrees with the profinite topology on the latter.

4.7. *Exercise (A weird scheme).* Show that $\operatorname{Spec} \mathbb{Q}(t) \otimes_{\mathbb{Q}} \mathbb{C}$ is an integral dimension one scheme, with closed points in natural correspondence with the transcendental complex numbers. (If the description $\operatorname{Spec} \mathbb{C}[t] \otimes_{\mathbb{Q}[t]} \mathbb{Q}(t)$ is more striking, you can use that instead.) This scheme doesn't come up in nature, but it is certainly neat!

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FOUNDATIONS OF ALGEBRAIC GEOMETRY CLASS 23

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CONTENTS

Last day: Morphisms to (quasi)projective schemes, and invertible sheaves; fibered products.

Today: Fibers of morphisms. Properties preserved by base change: open immersions, closed immersions, Segre embedding. Other schemes defined by universal property: reduction, normalization.

Last day, I showed you that fibered products exist, and I gave an argument that had fairly few moving parts: fibered products exist when the schemes in question are affine schemes; the universal property; and the fact that morphisms glue. I'll give you an exercise later today to give you a chance to make a similar argument, when I give the universal property for reducedness.

1. FIBERS OF MORPHISMS

We can informally interpret fibered product in the following geometric way. Suppose $Y \to Z$ is a morphism. We interpret this as a "family of schemes parametrized by a base scheme (or just plain *base*) Z." Then if we have another morphism $X \to Z$, we interpret the induced map $X \times_Z Y \to X$ as the "pulled back family". I drew a picture of this on the blackboard. I discussed the example: the family $y^2 z = x^3 + txz^2$ of cubics in $\mathbb{P}^2$ parametrized by the affine line, and what happens if you pull back to the affine plane via $t = uv$, to get the family $y^2 z = x^3 + uvxz^2$.

For this reason, fibered product is often called *base change* or *change of base* or *pullback*.

For instance, if X is a closed point of Z, then we will get the fiber over Z. As an example, consider the map of schemes $f : Y = \operatorname{Spec} \mathbb{Q}[t] \to Z = \operatorname{Spec} \mathbb{Q}[u]$ given by $u \mapsto t^2$ (or

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$u = t^2$). (I drew a picture on the blackboard. It looked like a parabola with horizontal axis of symmetry, projecting to the x-axis.) The fiber above $u = 1$ corresponds to the base change $X = \operatorname{Spec} \mathbb{Q}[u]/(u-1) \to \operatorname{Spec} \mathbb{Q}[u]$. Let's do the algebra: $X \times_Z Y = \operatorname{Spec} \mathbb{Q}[t, u]/(u-1, u-t^2) \cong \operatorname{Spec} \mathbb{Q}[t]/(t^2-1) \cong \operatorname{Spec} \mathbb{Q}[t]/(t-1) \times \mathbb{Q}[t]/(t+1)$. We see two reduced points (at "$u = 1, t = 1$ and $u = 1, t = -1$").

Next let's examine the fiber above $u = 0$. We get $\operatorname{Spec} \mathbb{Q}[t]/(t^2)$ — a point with non-reduced structure!

Finally, let's consider $u = -1$. We get $\operatorname{Spec} \mathbb{Q}[t]/(t^2 + 1)$. We get a single reduced point. The residue field $\mathbb{Q}(i)$ is a degree 2 field extension over $\mathbb{Q}$.

(Notice that in each case, we get something of "size two", informally speaking. One way of making this precise is that the rank of the sheaf $f_* \mathcal{O}_Y$ is rank 2 everywhere. In the first case, we see it as getting two different points. In the second, we get one point, with non-reduced behavior. In the last case, we get one point, of "size two". We will later see this "constant rank of $f_* \mathcal{O}_Y$" as symptomatic of the fact that this morphism is "particularly nice", i.e. finite and flat.)

We needn't look at fibers over just closed points; we can consider fibers over any points. More precisely, if p is a point of Z with residue field K, then we get a map $\operatorname{Spec} K \to Z$, and we can base change with respect to this morphism.

In the case of the generic point of $\operatorname{Spec} \mathbb{Q}[u]$ in the above example, we have $K = \mathbb{Q}(u)$, and $\mathbb{Q}[u] \to \mathbb{Q}(u)$ is the inclusion of the generic point. Let $X = \operatorname{Spec} \mathbb{Q}(u)$. Then you can verify that $X \times_Z Y = \operatorname{Spec} \mathbb{Q}[t, u]/(u - t^2) \otimes \mathbb{Q}(u) \cong \operatorname{Spec} \mathbb{Q}(t)$. We get the morphism $\mathbb{Q}(u) \to \mathbb{Q}(t)$ given by $u = t^2$ — a quadratic field extension.

Implicit here is a notion I should make explicit, about how you base change with respect to localization. Given $A \to B$, and a multiplicative set S of A, we have $(S^{-1}A) \otimes_A B \cong S^{-1}B$, where $S^{-1}B$ has the obvious interpretation. In other words,

$$\begin{array}{ccc} S^{-1}B & \longleftarrow & B \\ \uparrow & & \uparrow \\ S^{-1}A & \longleftarrow & A \end{array}$$

is "cofiber square" (or "pushout diagram").

1.1. *Remark: Geometric points.* We have already given two meanings for the "points of a scheme". We used one to define the notion of a scheme. Secondly, if T is a scheme, people sometimes say that $\operatorname{Hom}(T, X)$ are the "T-valued points of X". That's already confusing. But also, people say that the geometric points correspond to $\operatorname{Hom}(T, X)$ where T is the Spec of an algebraically closed field. Then for example the *geometric fibers* are the fibers over geometric points. In the example above, here is a geometric point: $\operatorname{Spec} \overline{\mathbb{Q}}[u]/(u - 1) \to \operatorname{Spec} \mathbb{Q}[u]$. And here is a geometric fiber: $\operatorname{Spec} \overline{\mathbb{Q}}[t]/(t^2-1)$. Notice that the geometric fiber above $u = -1$ also consists of two points, unlike the "usual" fiber.

(I should check: possibly the definition should just be for T the algebraic closure of the residue field of a not-necessarily-closed point.)

1.2. *Exercise for the arithmetically-minded.* Show that for the morphism $\operatorname{Spec}\mathbb{C} \to \operatorname{Spec}\mathbb{R}$, all geometric fibers consist of two reduced points. This exercise should be removed if I have the wrong definition of geometric point!

We will discuss more about geometric points and properties of geometric fibers shortly.

2. Properties preserved by base change

We now discuss a number of properties that behave well under base change.

We've already shown that the notion of "open immersion" is preserved by base change (problem 6 on problem set 9, see class 19). We did this by explicitly describing what the fibered product of an open immersion is: if $Y \hookrightarrow Z$ is an open immersion, and $f : X \to Z$ is any morphism, then we checked that the open subscheme $f^{-1}(Y)$ of X satisfies the universal property of fibered products.

2.1. *Important exercise (problem 8+ on the last problem set).* Show that the notion of "closed immersion" is preserved by base change. (This was stated in class 19.) Somewhat more precisely, given a fiber diagram

$$\begin{array}{ccc} W & \longrightarrow & X \\ \downarrow & & \downarrow \\ Y & \longrightarrow & Z \end{array}$$

where $Y \hookrightarrow Z$ is a closed immersion, then $W \hookrightarrow X$ is as well. (Hint: in the case of affine schemes, you have done this before in a different guise — see problem B3 on problem set 1!) In the course of the proof, you will show that W is cut out by the same equations in X as Y is in Z, or more precisely by pullback of those equations. Hence fibered products (over k) of schemes of finite type over k may be computed easily:

$$\operatorname{Spec} k[x_1,\dots,x_m]/(f_1(x_1,\dots,x_m),\dots,f_r(x_1,\dots,x_m)) \times_{\operatorname{Spec} k}$$

$$\operatorname{Spec} k[y_1,\dots,y_m]/(g_1(y_1,\dots,y_m),\dots,g_s(y_1,\dots,y_m))$$

$$\cong \operatorname{Spec} k[x_1,\dots,x_m,y_1,\dots,y_m]/(f_1(x_1,\dots,x_m),\dots,f_r(x_1,\dots,x_m),$$

$$g_1(y_1,\dots,y_m),\dots,g_s(y_1,\dots,y_m)).$$

We sometimes say that W is the *scheme-theoretic pullback of* Y, *scheme-theoretic inverse image*, or *inverse image scheme of* Y. The ideal sheaf of W is sometimes called the *inverse image (quasicoherent) ideal sheaf*.

Note for experts: It is not necessarily the quasicoherent pullback (f^*) of the ideal sheaf, as the following example shows. (Thanks Joe!)

$$\begin{array}{ccc} \operatorname{Spec} k[x]/(x) & \longrightarrow & \operatorname{Spec} k[x]/(x) \\ \downarrow & & \downarrow \\ \operatorname{Spec} k[x]/(x) & \longrightarrow & \operatorname{Spec} k[x] \end{array}$$

Instead, the correct thing to pullback (the thing that "pulls back well") is the surjection $\mathcal{O}_Z \to \mathcal{O}_Y \to 0$, which pulls back to $\mathcal{O}_X \to \mathcal{O}_W \to 0$. The key issue is that pullback of quasicoherent sheaves is right-exact, so we shouldn't expect the pullback of $0 \to \mathcal{I}_{Y/Z} \to \mathcal{O}_Z \to \mathcal{O}_Y \to 0$ to be exact, only right-exact. (Thus for example we get a natural map $f^*\mathcal{I}_{Y/Z} \to \mathcal{I}_{W/X}$.)

Similarly, other important properties are preserved by base change.

2.2. *Exercise.* Show that the notion of "morphism locally of finite type" is preserved by base change. Show that the notion of "affine morphism" is preserved by base change. Show that the notion of "finite morphism" is preserved by base change.

2.3. *Exercise.* Show that the notion of "quasicompact morphism" is preserved by base change.

2.4. *Exercise.* Show that the notion of "morphism of finite type" is preserved by base change.

2.5. *Exercise.* Show that the notion of "quasifinite morphism" (= finite type + finite fibers) is preserved by base change. (Note: the notion of "finite fibers" is not preserved by base change. $\operatorname{Spec} \overline{\mathbb{Q}} \to \operatorname{Spec} \mathbb{Q}$ has finite fibers, but $\operatorname{Spec} \overline{\mathbb{Q}} \otimes_{\mathbb{Q}} \overline{\mathbb{Q}} \to \operatorname{Spec} \overline{\mathbb{Q}}$ has one point for each element of $\operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$.)

2.6. *Exercise.* Show that surjectivity is preserved by base change (or fibered product). In other words, if $X \to Y$ is a surjective morphism, then for any $Z \to Y$, $X \times_Y Z \to Z$ is surjective. (You may end up using the fact that for any fields k_1 and k_2 containing k_3, $k_1 \otimes_{k_3} k_2$ is non-zero, and also the axiom of choice.)

2.7. *Exercise.* Show that the notion of "irreducible" is not necessarily preserved by base change. Show that the notion of "connected" is not necessarily preserved by base change. (Hint: $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$, $\mathbb{Q}[i] \otimes_{\mathbb{Q}} \mathbb{Q}[i]$.)

If X is a scheme over a field k, it is said to be *geometrically irreducible* if its base change to $\overline{k}$ (i.e. $X \times_{\operatorname{Spec} k} \operatorname{Spec} \overline{k}$) is irreducible. Similarly, it is *geometrically connected* if its base change to $\overline{k}$ (i.e. $X \times_{\operatorname{Spec} k} \operatorname{Spec} \overline{k}$) is connected. Similarly also for *geometrically reduced* and *geometrically integral*. We say that $f : X \to Y$ has *geometrically irreducible (resp. connected, reduced, integral) fibers* if the geometric fibers are geometrically irreducible (resp. connected, reduced, integral).

If you care about such notions, see Hartshorne Exercise II.3.15 for some facts (stated in a special case). In particular, to check geometric irreducibility, it suffices to check over *separably closed* (not necessarily algebraically closed) fields. To check geometric reducedness, it suffices to check over *perfect* fields.

2.8. *Exercise.* Show that $\operatorname{Spec} \mathbb{C}$ is not a geometrically irreducible $\mathbb{R}$-scheme. If $\operatorname{char} k = p$, show that $\operatorname{Spec} k(u)$ is not a geometrically reduced $\operatorname{Spec} k(u^p)$-scheme.

2.9. *Exercise.* Show that the notion of geometrically irreducible (resp. connected, reduced, integral) fibers behaves well with respect to base change.

On a related note:

2.10. *Exercise (less important).* Suppose that l/k is a finite field extension. Show that a k-scheme X is normal if and only if $X \times_{\operatorname{Spec} k} \operatorname{Spec} l$ is normal. Hence deduce that if k is any field, then $\operatorname{Spec} k[w, x, y, z]/(wz - xy)$ is normal. (I think this was promised earlier.) Hint: we showed earlier (Problem B4 on set 4) that $\operatorname{Spec} k[a, b, c, d]/(a^2 + b^2 + c^2 + d^2)$ is normal.

3. PRODUCTS OF PROJECTIVE SCHEMES: THE SEGRE EMBEDDING

I will next describe products of projective A-schemes over A. The case of greatest initial interest is if $A = k$. (A reminder of why we like projective schemes. (i) it is an easy way of getting interesting non-affine schemes. (ii) we get lots of schemes of classical interest. (iii) we have a hard time thinking of anything that isn't projective or an open subset of a projective. (iv) a k-scheme is a first approximation of what we mean by compact.)

In order to do this, I need only describe $\mathbb{P}^m_A \times_A \mathbb{P}^n_A$, because any projective scheme has a closed immersion in some $\mathbb{P}^m_A$, and closed immersions behave well under base change: so if $X \hookrightarrow \mathbb{P}^m_A$ and $Y \hookrightarrow \mathbb{P}^n_A$ are closed immersions, then $X \times_A Y \hookrightarrow \mathbb{P}^m_A \times_A \mathbb{P}^n_A$ is also a closed immersion, cut out by the equations of X and Y.

We'll describe $\mathbb{P}^m_A \times_A \mathbb{P}^n_A$, and see that it too is a projective A-scheme. Consider the map $\mathbb{P}^m_A \times_A \mathbb{P}^n_A \to \mathbb{P}^{mn+m+n}_A$ given by

$$([x_0; \dots; x_m], [y_0; \dots; y_n]) \to [z_{00}; z_{01}; \cdots; z_{ij}; \cdots; z_{mn}] = [x_0 y_0; x_0 y_1; \cdots; x_i y_j; \cdots x_m y_n].$$

First, you should verify that this is a well-defined morphism! On the open chart $U_i \times V_j$, this gives a map $(x_{0/i}, \dots, x_{m/i}, y_{0/j}, \dots, y_{n/j}) \mapsto [x_{0/i} y_{0/j}; \dots; x_{i/i} y_{j/j}; \dots; x_{m/i} y_{n/j}]$. Note that this gives an honest map to projective space — not all the entries on the right are zero, as one of the entries $(x_{i/i} y_{j/j})$ is 1.

(Aside: we now well know that a map to projective space corresponds to an invertible sheaf with a bunch of sections. The invertible sheaf on this case is $\pi_1^* \mathcal{O}_{\mathbb{P}^m_A}(1) \otimes \pi_2^* \mathcal{O}_{\mathbb{P}^n_A}(1)$, where π_i are the projections of the product onto the two factors. The notion $\boxtimes$ is often used for this notion, when you pull back sheaves from each factor of a product, and tensor. For example, this invertible sheaf could be written $\mathcal{O}(1) \boxtimes \mathcal{O}(1)$. People often write $\mathcal{O}(a) \boxtimes \mathcal{O}(b)$ for $\mathcal{O}(a, b)$.)

I claim this morphism is a closed immersion. (We are essentially using Exercise 3.2 in the class 21 notes, problem 40 in problem set 9. But don't waste your time by looking back at it.) Let's check this on the open set where $z_{ab} \neq 0$. Without loss of generality, I'll take $a = b = 0$, to make notation simpler. Then the preimage of this open set in $\mathbb{P}^m_A \times \mathbb{P}^n_A$ is the locus where $x_0 \neq 0$ and $y_0 \neq 0$, i.e. $U_0 \times V_0$, U_0 and V_0 are the usual distinguished open sets of $\mathbb{P}^m_A$ and $\mathbb{P}^n_A$ respectively. The coordinates here are $x_{1/0}, \ldots, x_{m/0}, y_{1/0}, \ldots, y_{n/0}$. Thus the map corresponds to $z_{ab/00} \to x_{a/0} y_{b/0}$, which clearly induces a surjection of rings

$$A[z_{00/00}, \ldots, z_{mn/00}] \to A[x_{1/0}, \ldots, x_{m/0}, y_{1/0}, \ldots, y_{n/0}].$$

(Recall that $z_{a0/00} \mapsto x_{a/0}$ and $z_{0b/00} \mapsto y_{b/0}$.)

Hence we are done! This map is called the *Segre morphism* or *Segre embedding*. If A is a field, the image is called the *Segre variety* — although we don't yet know what a variety is!

Here are some useful comments.

3.1. *Exercise.* Show that the Segre scheme (the image of the Segre morphism) is cut out by the equations corresponding to

$$\operatorname{rank} \begin{pmatrix} a_{00} & \cdots & a_{0n} \\ \vdots & \ddots & \vdots \\ a_{m0} & \cdots & a_{mn} \end{pmatrix} = 1,$$

i.e. that all 2×2 minors vanish. (Hint: suppose you have a polynomial in the a_{ij} that becomes zero upon the substitution $a_{ij} = x_i y_j$. Give a recipe for subtracting polynomials of the form monomial times 2×2 minor so that the end result is 0.)

3.2. *Example.* Let's consider the first non-trivial example, when $m = n = 1$. We get $\mathbb{P}^1 \times \mathbb{P}^1 \hookrightarrow \mathbb{P}^3$. We get a single equation

$$\operatorname{rank} \begin{pmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{pmatrix} = 1,$$

i.e. $a_{00}a_{11} - a_{01}a_{10} = 0$. We get our old friend, the quadric surface! Hence: the nonsingular quadric surface $wz - xy = 0$ is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. Note that we can reinterpret the rulings; I pointed this out on the model. Since (by diagonalizability of quadratics) all nonsingular quadratics over an algebraically closed field are isomorphic, we have that all nonsingular quadric surfaces over an algebraically closed field are isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$.

Note that this is not true even over a field that is not algebraically closed. For example, over $\mathbb{R}$, $w^2 + x^2 + y^2 + z^2 = 0$ is not isomorphic to $\mathbb{P}^1_{\mathbb{R}} \times_{\mathbb{R}} \mathbb{P}^1_{\mathbb{R}}$. Reason: the former has no real points, while the latter has lots of real points.

3.3. Let's return to the general Segre situation. We can describe the closed subscheme alternatively the Proj of the subring R of

$$A[x_0, \dots, x_m, y_0, \dots, y_n]$$

generated by monomials of equal degree in the x's and the y's. Using this, you can give a co-ordinate free description of this product (i.e. without using the co-ordinates x_i and y_j): $\mathbb{P}^m_A \times_A \mathbb{P}^n_A = \operatorname{Proj} R$ where

$$R = \oplus_{i=0}^{\infty} \operatorname{Sym}^i H^0(\mathbb{P}^m_A, \mathcal{O}(1)) \otimes \operatorname{Sym}^i H^0(\mathbb{P}^n_A, \mathcal{O}(1)).$$

Kirsten asks an interesting question: show that $\mathcal{O}(a, b)$ gives a closed immersion to projective space if $a, b > 0$.

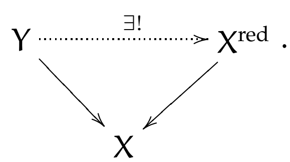
You may want to ponder how to think of products of three projective spaces.

4. Other schemes defined by universal property: reduction, normalization

I now want to define other schemes using universal properties, in ways that are vaguely analogous to fibered product.

As a warm-up, I'd like to revisit an earlier topic: reduction of a scheme. Recall that if X is a scheme, we defined a closed immersion $X^{\text{red}} \hookrightarrow X$. (See the comment just before §1.4 in class 19.) I'd like to revisit this.

4.1. *Potentially enlightening exercise.* Show that $X^{\text{red}} \to X$ satisfies the following universal property: any morphism from a reduced scheme Y to X factors uniquely through X^{red}.



You can use this as a definition for $X^{\text{red}} \to X$. Let me walk you through part of this. First, prove this for X affine. (Here you use the fact that we know that maps to an affine scheme correspond to a maps of global sections in the other direction.) Then use the universal property to show the result for quasiaffine X. Then use the universal property to show it in general. **Oops! I don't think I've defined quasiaffine before. It is any scheme that can be expressed as an open subset of an affine scheme. I should eventually put this definition earlier in the course notes, but may not get a chance to. It may appear in the class 22 notes, which are yet to be written up. The concept is reintroduced yet again in Exercise 4.4 below.**

4.2. Normalization.

I now want to tell you how to normalize a reduced Noetherian scheme. A normalization of a scheme X is a morphism $\nu : \tilde{X} \to X$ from a normal scheme, where ν induces a bijection of components of $\tilde{X}$ and X, and ν gives a birational morphism on each of the components; it will be nicer still, as it will satisfy a universal property. (I drew a picture of a normalization of a curve.) **Oops! I didn't define *birational* until class 27. Please just plow ahead! I may later patch this anachronism, but most likely I won't get the chance.**

I'll begin by dealing with the case where X is irreducible, and hence integral. (I'll then deal with the more general case, and also discuss normalization in a function field extension.)

In this case of X irreducible, the normalization satisfies dominant morphism from an irreducible normal scheme to X, then this morphism factors uniquely through ν:

$$\begin{array}{ccc} Y & \overset{\exists!}{\dashrightarrow} & \tilde{X} \\ & \searrow & \swarrow_{\nu} \\ & X & \end{array} .$$

Thus if it exists, then it is unique up to unique isomorphism. We now have to show that it exists, and we do this in the usual way. We deal first with the case where X is affine, say $X = \operatorname{Spec} R$, where R is an integral domain. Then let $\tilde{R}$ be the integral closure of R in its fraction field $\operatorname{Frac}(R)$.

4.3. *Exercise.* Show that $\nu : \operatorname{Spec} \tilde{R} \to \operatorname{Spec} R$ satisfies the universal property.

4.4. *Exercise.* Show that normalizations exist for any quasiaffine X (i.e. any X that can be expressed as an open subset of an affine scheme).

4.5. *Exercise.* Show that normalizations exist in general.

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FOUNDATIONS OF ALGEBRAIC GEOMETRY CLASS 24

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CONTENTS

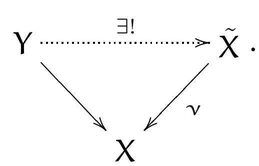
Last day: Fibers of morphisms. Properties preserved by base change: open immersions, closed immersions, Segre embedding. Other schemes defined by universal property: reduction, normalization.

Today: normalization (in a field extension), "sheaf Spec**", "sheaf** Proj**", projective morphism.**

1. NORMALIZATION, CONTINUED

Last day, I defined the normalization of a reduced scheme. I have an interesting question for experts: there is a reasonable extension to schemes in general; does anything go wrong? I haven't yet given this much thought, but it seems worth exploring.

I described normalization last day in the case when X is irreducible, and hence integral. In this case of X irreducible, the normalization satisfies the universal property, that if $Y \to X$ is any other dominant morphism from a normal scheme to X, then this morphism factors uniquely through ν:



Thus if it exists, then it is unique up to unique isomorphism. We then showed that it exists, using an argument we saw for the third time. (The first time was in the existence of the fibered product. The second was an argument for the existence of the reduction morphism.) The ring-theoretic case got us started: if $X = \operatorname{Spec} R$, then and $\tilde{R}$ is the integral closure of R in its fraction field $\operatorname{Frac}(R)$, then I gave as an exercise that $\nu : \operatorname{Spec} \tilde{R} \to \operatorname{Spec} R$ satisfies the universal property.

Date: Thursday, January 19, 2006. Last minor update June 28, 2007.

1.1. *Exercise.* Show that the normalization morphism is surjective. (Hint: Going-up!)

We now mention some bells and whistles. The following fact is handy.

1.2. *Theorem (finiteness of integral closure). — Suppose* A *is a domain,* $K = \mathrm{Frac}(A)$, L/K *is a finite field extension, and* B *is the integral closure of* A *in* L *("the integral closure of* A *in the field extension* L/K*", i.e. those elements of* L *integral over* A*).*
(a) if A *is integrally closed, then* B *is a finitely generated* A*-module.*
(b) if A *is a finitely generated* k*-algebra, then* B *(the integral closure of* A *in its fraction field) is a finitely generated* A*-module.*

I hope to type up a proof of these facts at some point to show you that they are not that bad. Much of part (a) was proved by Greg Brumfiel in 210B last year.

Warning: (b) does *not* hold for Noetherian A in general. I find this very alarming. I don't know an example offhand, but one is given in Eisenbud's book.

1.3. *Exercise.* Show that $\dim \tilde{X} = \dim X$ (hint: see our going-up discussion).

1.4. *Exercise.* Show that if X is an integral finite-type k-scheme, then its normalization $\nu : \tilde{X} \to X$ is a finite morphism.

1.5. *Exercise.* Explain how to generalize the notion of normalization to the case where X is a reduced Noetherian scheme (with possibly more than one component). This basically requires defining a universal property. I'm not sure what the "perfect" definition, but all reasonable universal properties should lead to the same space.

1.6. *Exercise.* Show that if X is an integral finite type k-scheme, then its non-normal points form a closed subset. (This is a bit trickier. Hint: consider where $\nu_* \mathcal{O}_{\tilde{X}}$ has rank 1.) I haven't thought through all the details recently, so I hope I've stated this correctly.

Here is an explicit example to think through some of these ideas.

1.7. Exercise. Suppose $X = \operatorname{Spec} \mathbb{Z}[15i]$. Describe the normalization $\tilde{X} \to X$. (Hint: it isn't hard to find an integral extension of $\mathbb{Z}[15i]$ that is integrally closed. By the above discussion, you've then found the normalization!) Over what points of X is the normalization not an isomorphism?

1.8. *Exercise.* (This is an important generalization!) Suppose X is an integral scheme. Define the *normalization of* X, $\nu : \tilde{X} \to X$, *in a given finite field extension of the function field of* X. Show that $\tilde{X}$ is normal. (This will be hard-wired into your definition.) Show that if either X is itself normal, or X is finite type over a field k, then the normalization in a finite field extension is a finite morphism.

Let's try this in a few cases.

1.9. Exercise. Suppose $X = \operatorname{Spec} \mathbb{Z}$ (with function field $\mathbb{Q}$). Find its integral closure in the field extension $\mathbb{Q}(i)$.

A finite extension K of $\mathbb{Q}$ is called a *number field*, and the integral closure of $\mathbb{Z}$ in K the *ring of integers of* K, denoted $\mathcal{O}_K$. (This notation is a little awkward given our other use of the symbol $\mathcal{O}$.) By the previous exercises, $\operatorname{Spec} \mathcal{O}_K$ is a Noetherian normal domain of dimension 1 (hence regular). This is called a *Dedekind domain*. We think of it as a smooth curve.

1.10. Exercise. (a) Suppose $X = \operatorname{Spec} k[x]$ (with function field $k(x)$). Find its integral closure in the field extension $k(y)$, where $y^2 = x^2 + x$. (Again we get a Dedekind domain.)
(b) Suppose $X = \mathbb{P}^1$, with distinguished open $\operatorname{Spec} k[x]$. Find its integral closure in the field extension $k(y)$, where $y^2 = x^2 + x$. (Part (a) involves computing the normalization over one affine open set; now figure out what happens over the "other".)

2. Sheaf Spec

Given an A-algebra, B, we can take its Spec to get an affine scheme over $\operatorname{Spec} A$: $\operatorname{Spec} B \to \operatorname{Spec} A$. I'll now give a universal property description of a globalization of that notation. We will take an arbitrary scheme X, and a quasicoherent sheaf of algebras $\mathcal{A}$ on it. We will define how to take Spec of this sheaf of algebras, and we will get a scheme $\underline{\operatorname{Spec}}\, \mathcal{A} \to X$ that is "affine over X", i.e. the structure morphism is an affine morphism.

We will do this as you might by now expect: for each affine on X, we use our affine construction, and show that everything glues together nicely. We do this instead by describing $\underline{\operatorname{Spec}}\, \mathcal{A} \to X$ in terms of a good universal property: given any morphism $\pi : Y \to X$ along with a morphism of $\mathcal{O}_X$-modules

$$\alpha : \mathcal{A} \to \pi_* \mathcal{O}_Y,$$

there is a unique map $Y \to \underline{\operatorname{Spec}}\, \mathcal{A}$ factoring π, i.e. so that the following diagram commutes,

$$\begin{array}{ccc} Y & \overset{\exists !}{\dashrightarrow} & \underline{\operatorname{Spec}}\, \mathcal{A} \\ & {\scriptstyle \pi} \searrow \quad \swarrow {\scriptstyle \beta} & \\ & X & \end{array}$$

and an isomorphism $\phi : \mathcal{A} \to \beta_* \mathcal{O}_{\underline{\operatorname{Spec}}\, \mathcal{A}}$ inducing α.

(For experts: we need $\mathcal{O}_X$-modules, and to leave our category of quasicoherent sheaves on X, because we only showed that the pushforward of quasicoherent sheaves are quasicoherent for certain morphisms, where the preimage of each affine was a finite union of affines, the pairwise intersection of which were also finite unions. This notion will soon be formalized as quasicompact and quasiseparated.)

At this point we're getting to be experts on this, so let's show that this $\underline{\mathrm{Spec}}\,\mathcal{A}$ exists. In the case where X is affine, we are done by our affine discussion. In the case where X is quasiaffine, we are done for the same reason as before. And finally, in the case where X is general, we are done once again!

In particular, note that $\underline{\mathrm{Spec}}\,\mathcal{A} \to X$ is an affine morphism.

2.1. *Exercise.* Show that if $f : Z \to X$ is an affine morphism, then we have a natural isomorphism $Z \cong \underline{\mathrm{Spec}}\, f_*\mathcal{O}_Z$ of X-schemes.

Hence we can recover any affine morphism in this way. More precisely, a morphism is affine if and only if it is of the form $\underline{\mathrm{Spec}}\,\mathcal{A} \to X$.

2.2. *Exercise ($\underline{\mathrm{Spec}}$ behaves well with respect to base change).* Suppose $f : Z \to X$ is any morphism, and $\mathcal{A}$ is a quasicoherent sheaf of algebras on X. Show that there is a natural isomorphism $Z \times_X \underline{\mathrm{Spec}}\,\mathcal{A} \cong \underline{\mathrm{Spec}}\, f^*\mathcal{A}$.

An important example of this $\underline{\mathrm{Spec}}$ construction is the total space of a finite rank locally free sheaf $\mathcal{F}$, which is a *vector bundle*. It is $\underline{\mathrm{Spec}}\,\mathrm{Sym}^*\,\mathcal{F}^\vee$.

2.3. *Exercise.* Show that this is a vector bundle, i.e. that given any point $p \in X$, there is a neighborhood $p \in U \subset X$ such that $\underline{\mathrm{Spec}}\,\mathrm{Sym}^*\,\mathcal{F}^\vee|_U \cong \mathbb{A}^n_U$. Show that $\mathcal{F}$ is isomorphic to the sheaf of sections of it.

As an easy example: if $\mathcal{F}$ is a *free* sheaf of rank n, then $\underline{\mathrm{Spec}}\,\mathrm{Sym}^*\,\mathcal{F}^\vee$ is called $\mathbb{A}^n_X$, generalizing our earlier notions of $\mathbb{A}^n_A$. As the notion of a free sheave behaves well with respect to base change, so does the notion of $\mathbb{A}^n_X$, i.e. given $X \to Y$, $\mathbb{A}^n_Y \times_Y X \cong \mathbb{A}^n_X$.

Here is one last fact that might come in handy.

2.4. *Exercise.* Suppose $f : \underline{\mathrm{Spec}}\,\mathcal{A} \to X$ is a morphism. Show that the category of quasicoherent sheaves on $\underline{\mathrm{Spec}}\,A$ is "essentially the same as" (=equivalent to) the category of quasicoherent sheaves on X with the structure of $\mathcal{A}$-modules (quasicoherent $\mathcal{A}$-modules on X).

The reason you could imagine caring is when X is quite simple, and $\underline{\mathrm{Spec}}\,\mathcal{A}$ is complicated. We'll use this before long when $X \cong \mathbb{P}^1$, and $\underline{\mathrm{Spec}}\,\mathcal{A}$ is a more complicated curve. (I drew a picture of this.)

3. Sheaf Proj

We'll now do a global (or "sheafy") version of Proj, which we'll denote $\underline{\mathrm{Proj}}$.

Suppose now that $\mathcal{S}_*$ is a quasicoherent sheaf of graded algebras of X. To be safe, let me assume that $\mathcal{S}_*$ is locally generated in degree 1 (i.e. there is a cover by small affine open sets, where for each affine open set, the corresponding algebra is generated in degree 1), and $\mathcal{S}_1$ is finite type. We will define $\underline{\operatorname{Proj}}\, \mathcal{S}_*$.

The essential ideal is that we do this affine by affine, and then glue the result together. But as before, this is tricky to do, but easier if you state the right universal property.

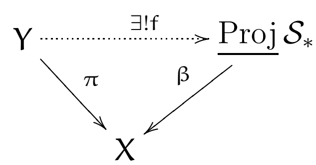
As a preliminary, let me re-examine our earlier theorem, that "Maps to $\mathbb{P}^n$ correspond to $n+1$ sections of an invertible sheaf, not all vanishing at any point (= generated by global sections), modulo sections of $\mathcal{O}_X^*$."

I will now describe this in a more "relative" setting, where relative means that we do this with morphisms of schemes. We begin with a relative notion of base-point free. Suppose $f : Y \to X$ is a morphism, and $\mathcal{L}$ is an invertible sheaf on Y. We say that $\mathcal{L}$ is *relatively base point free* if for every point $p \in X$, $q \in Y$, with $f(q) = p$, there is a neighborhood U for which there is a section of $\mathcal{L}$ over $f^{-1}(U)$ not vanishing at q. Similarly, we define *relatively generated by global sections* if there is a neighborhood U for which there are sections of $\mathcal{L}$ over $f^{-1}(U)$ generating *every* stalk of $f^{-1}(U)$. This is admittedly hideous terminology. (One can also define *relatively generated by global sections at a point* $p \in Y$. See class 16 where we defined these notions in a non-relative setting. In class 32, this will come up again.) More generally, we can define the notion of "relatively generated by global sections by a subsheaf of $f_*\mathcal{L}$".

Definition. $(\underline{\operatorname{Proj}}\, \mathcal{S}_*, \mathcal{O}_{\underline{\operatorname{Proj}}\, \mathcal{S}_*}(1)) \to X$ satisfies the following universal property. Given any other X-scheme Y with an invertible sheaf $\mathcal{L}$, and a map of graded $\mathcal{O}_X$-algebras

$$\alpha : \mathcal{S}_* \to \oplus_{n=0} \pi_* \mathcal{L}^{\otimes n},$$

such that $\mathcal{L}$ is relatively generated by the global sections of $\alpha(S_1)$, there is a unique factorization



and a canonical isomorphism $\mathcal{L} \cong f^*\mathcal{O}_{\underline{\operatorname{Proj}}\, \mathcal{S}_*}(1)$ and a morphism $\mathcal{S}_* \to \oplus_n \beta_* \mathcal{O}(n)$ inducing α.

In particular, $\underline{\operatorname{Proj}}\, \mathcal{S}_*$ comes with an invertible sheaf $\mathcal{O}_{\underline{\operatorname{Proj}}\, \mathcal{S}_*}(1)$, and this $\mathcal{O}(1)$ should be seen as part of the data.

This definition takes some getting used to.

But we prove this as usual!

We first deal with the case where X is affine, say $X = \operatorname{Spec} A$, $\mathcal{S}_* = \tilde{S}_*$. You won't be surprised to hear that in this case, $(\operatorname{Proj} S_*, \mathcal{O}(1))$ satisfies the universal property.

We outline why. Clearly, given a map $Y \to \operatorname{Proj} S_*$, we get a pullback map α. Conversely, given such a pullback map, we want to show that this induces a (unique) map $Y \to \operatorname{Proj} S_*$. Now because S_* is generated in degree 1, we have a closed immersion

$\operatorname{Proj} S_* \hookrightarrow \operatorname{Proj} \operatorname{Sym}^* S_1$. The map in degree 1, $S_1 \to \pi_* \mathcal{L}$, gives a map $Y \to \operatorname{Proj} \operatorname{Sym}^* S_1$ by our magic theorem "Maps to $\mathbb{P}^n$ correspond to $n+1$ sections of an invertible sheaf, not all vanishing at any point (= generated by global sections), modulo sections of $\mathcal{O}_X^*$."

3.1. Exercise. Complete this argument that if $X = \operatorname{Spec} A$, then $(\operatorname{Proj} \mathcal{S}_*, \mathcal{O}(1))$ satisfies the universal property.

3.2. Exercise. Show that $(\operatorname{Proj} \mathcal{S}_*, \mathcal{O}(1))$ exists in general, by following the analogous universal property argument: show that it exists for X quasiaffine, then in general.

3.3. *Exercise ($\underline{\mathrm{Proj}}$ behaves well with respect to base change).* Suppose $\mathcal{S}_*$ is a quasicoherent sheaf of graded algebras on X satisfying the required hypotheses above for $\underline{\mathrm{Proj}}\, \mathcal{S}_*$ to exist. Let $f : Y \to X$ be any morphism. Give a natural isomorphism

$$(\underline{\mathrm{Proj}}\, f^* \mathcal{S}_*, \mathcal{O}_{\underline{\mathrm{Proj}}\, f^* \mathcal{S}_*}(1)) \cong (Y \times_X \underline{\mathrm{Proj}}\, \mathcal{S}_*, g^* \mathcal{O}_{\underline{\mathrm{Proj}}\, \mathcal{S}_*}(1)) \cong$$

where g is the natural morphism in the base change diagram

$$\begin{array}{ccc} Y \times_X \underline{\mathrm{Proj}}\, \mathcal{S}_* & \xrightarrow{g} & \underline{\mathrm{Proj}}\, \mathcal{S}_* \\ \downarrow & & \downarrow \\ Y & \longrightarrow & X. \end{array}$$

3.4. *Definition.* If $\mathcal{F}$ is a finite rank locally free sheaf on X. Then $\underline{\mathrm{Proj}} \operatorname{Sym}^* \mathcal{F}$ is called its *projectivization*. If $\mathcal{F}$ is a free sheaf of rank $n+1$, then we define $\mathbb{P}^n_X := \underline{\mathrm{Proj}} \operatorname{Sym}^* \mathcal{F}$. (Then $\mathbb{P}^n_{\operatorname{Spec} A}$ agrees with our earlier definition of $\mathbb{P}^n_A$.) Clearly this notion behaves well with respect to base change.

This "relative $\mathcal{O}(1)$" we have constructed is a little subtle. Here are couple of exercises to give you practice with the concept.

3.5. *Exercise.* $\underline{\mathrm{Proj}}(\mathcal{S}_*[t]) \cong \underline{\mathrm{Spec}}\, \mathcal{S}_* \coprod \underline{\mathrm{Proj}}\, \mathcal{S}_*$, where $\underline{\mathrm{Spec}}\, \mathcal{S}_*$ is an open subscheme, and $\underline{\mathrm{Proj}}\, \mathcal{S}_*$ is a closed subscheme. Show that $\underline{\mathrm{Proj}}\, \mathcal{S}*$ is an effective Cartier divisor, corresponding to the invertible sheaf $\mathcal{O}_{\underline{\mathrm{Proj}}\, N}(1)$. (This is the generalization of the projective and affine cone. At some point I should give an explicit reference to our earlier exercise on this.)

3.6. *Exercise.* Suppose $\mathcal{L}$ is an invertible sheaf on X, and $\mathcal{S}_*$ is a quasicoherent sheaf of graded algebras on X satisfying the required hypotheses above for $\underline{\mathrm{Proj}}\, \mathcal{S}_*$ to exist. Define $\mathcal{S}'_* = \oplus_{n=0} \mathcal{S}_n \otimes \mathcal{L}_n$. Give a natural isomorphism of X-schemes

$$(\underline{\mathrm{Proj}}\, \mathcal{S}'_*, \mathcal{O}_{\underline{\mathrm{Proj}}\, \mathcal{S}'_*}(1)) \cong (\underline{\mathrm{Proj}}\, \mathcal{S}_*, \mathcal{O}_{\underline{\mathrm{Proj}}\, \mathcal{S}_*}(1) \otimes \pi^* \mathcal{L}),$$

where $\pi : \underline{\mathrm{Proj}}\, \mathcal{S}_* \to X$ is the structure morphism. In other words, informally speaking, the $\underline{\mathrm{Proj}}$ is the same, but the $\mathcal{O}(1)$ is twisted by $\mathcal{L}$.

3.7. Projective morphisms.

If you are tuning out because of these technicalities, please tune back in! I now want to define an essential notion.

Recall that we have recast affine morphisms in the following way: $X \to Y$ is an affine morphism if $X \cong \underline{\mathrm{Spec}}\, \mathcal{A}$ for some quasicoherent sheaf of algebras $\mathcal{A}$ on Y.

I will now *define* the notion of a projective morphism similarly.

3.8. *Definition.* A morphism $X \to Y$ is *projective* if there is an isomorphism

$$\begin{array}{ccc} X & \xrightarrow{\sim} & \underline{\mathrm{Proj}}\, \mathcal{S}_* \\ & \searrow \quad \swarrow & \\ & Y & \end{array}$$

for a quasicoherent sheaf of algebras $\mathcal{S}_*$ on Y satisfying the required hypothesis for $\underline{\mathrm{Proj}}$ to exist.

Two warnings! 1. Notice that I didn't say anything about the $\mathcal{O}(1)$, which is an important definition. The notion of affine morphism is affine-local on the target, but this notion is not affine-local on the target! (In nice circumstances it is, as we'll see later. We'll also see an example where this is not.) 2. Hartshorne gives a different definition; I'm following the more general definition of Grothendieck. But again, these definitions turn out to be the same in nice circumstances.

This is the "relative version" of $\mathrm{Proj}\, S_* \to \mathrm{Spec}\, A$.

3.9. *Exercise.* Show that closed immersions are projective morphisms. (Hint: Suppose the closed immersion $X \to Y$ corresponds to $\mathcal{O}_Y \to \mathcal{O}_X$. Consider $\mathcal{S}_0 = \mathcal{O}_X$, $\mathcal{S}_i = \mathcal{O}_Y$ for $i > 1$.)

3.10. *Exercise (suggested by Kirsten).* Suppose $f : X \hookrightarrow \mathbb{P}^n_S$ where S is some scheme. Show that the structure morphism $\pi : X \to S$ is a projective morphism as follows: let $\mathcal{L} = f^* \mathcal{O}_{\mathbb{P}^n_S}(1)$, and show that $X = \underline{\mathrm{Proj}}\, \pi_* \mathcal{L}^{\otimes n}$.

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FOUNDATIONS OF ALGEBRAIC GEOMETRY CLASS 25

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CONTENTS

Last day: Normalization (in a finite field extension), "sheaf Spec", "sheaf Proj", projective morphisms.

Today: separatedness, definition of variety.

0.1. Here is a notion I should have introduced earlier: *induced reduced subscheme structure*. Suppose X is a scheme, and S is a *closed subset* of X. Then there is a unique reduced closed subscheme Z of X "supported on S". More precisely, it can be defined by the following universal property: for any morphism from a *reduced* scheme Y to X, whose image lies in S (as a set), this morphism factors through Z uniquely. Over an affine $X = \mathrm{Spec}\, R$, we get $\mathrm{Spec}\, R/I(S)$. (Exercise: verify this.) For example, if S is the entire underlying set of X, we get X^{red}.

1. SEPARATED MORPHISMS

We will now describe a very useful notion, that of morphisms being *separated*. Separatedness is one of the definitions in algebraic geometry (like flatness) that seems initially unmotivated, but later turns out to be the answer to a large number of desiderata.

Here are some initial reasons. First, in some sense it is the analogue of Hausdorff. A better description is the following: if you take the definition I'm about to give you and apply it to the "usual" topology, you'll get a correct (if unusual) definition of Hausdorffness. The reason this doesn't give Hausdorffness in the category of schemes is because the topology on the product is not the product topology. (An earlier exercise was to show that $\mathbb{A}^2_k$ does not have the product topology on $\mathbb{A}^1_k \times_k \mathbb{A}^1_k$.) One benefit of this definition is that we will be finally ready to define a *variety*, in a way that corresponds to the classical definition.

Date: Tuesday, January 24, 2006. Small update June 20.

Second, a separated morphism has the property that the intersection of a two affine open sets is affine, which is precisely the odd hypothesis needed to make Cech cohomology work.

A third motivation is that nasty line with doubled origin, which is a counterexample to many statements one might hope are true. The line with double origin is not separated, and by adding a separatedness hypothesis, the desired statements turn out to be true.

A fourth motivation is to give a good foundation for the notion of rational maps, which we will discuss shortly.

A lesson arising from the construction is the importance of the diagonal morphism. More precisely given a morphism $X \to Y$, nice consequences can be leveraged from good behavior of the diagonal morphism $\delta : X \to X \times_Y X$, usually through fun diagram chases. This is a lesson that applies across many fields of mathematics. (Another nice gift the diagonal morphism: it will soon give us a good algebraic definition of differentials.)

1.1. *Proposition. — Let* $X \to Y$ *be a morphism of schemes. Then the diagonal morphism* $\delta : X \to X \times_Y X$ *is a locally closed immersion.*

This locally closed subscheme of $X \times_Y X$ (the diagonal) will be denoted Δ.

Proof. We will describe a union of open subsets of $X \times_Y X$ covering the image of X, such that the image of X is a closed immersion in this union.

1.2. Say Y is covered with affine opens V_i and X is covered with affine opens U_{ij}, with $\pi : U_{ij} \to V_i$. Then the diagonal is covered by $U_{ij} \times_{V_i} U_{ij}$. (Any point $p \in X$ lies in some U_{ij}; then $\delta(p) \in U_{ij} \times_{V_i} U_{ij}$.) Note that $\delta^{-1}(U_{ij} \times_{V_i} U_{ij}) = U_{ij}$: $U_{ij} \times_{V_i} U_{ij} \cong U_{ij} \times_Y U_{ij}$ because $V_i \hookrightarrow Y$ is a monomorphism. Then because open immersions behave well with respect to base change, we have the fiber diagram

$$\begin{array}{ccc} U_{ij} & \longrightarrow & X \\ \downarrow & & \downarrow \\ U_{ij} \times_Y X & \longrightarrow & X \times_Y X \end{array}$$

from which $\delta^{-1}(U_{ij} \times_Y X) = U_{ij}$. As $\delta^{-1}(U_{ij} \times_Y U_{ij})$ contains U_{ij}, we must have $\delta^{-1}(U_{ij} \times_Y U_{ij}) = U_{ij}$.

Finally, we'll check that $U_{ij} \to U_{ij} \times_{V_i} U_{ij}$ is a closed immersion. Say $V_i = \operatorname{Spec} S$ and $U_{ij} = \operatorname{Spec} R$. Then this corresponds to the natural ring map $R \times_S R \to R$, which is obviously surjective. □

(A picture is helpful here.)

Note that the open subsets we described may not cover $X \times_Y X$, so we have not shown that δ is a closed immersion.

1.3. *Definition.* A morphism $X \to Y$ is said to be **separated** if the diagonal morphism $\delta : X \to X \times_Y X$ is a closed immersion. If R is a ring, an R-scheme X is said to be *separated over* R if the structure morphism $X \to \operatorname{Spec} R$ is separated. When people say that a scheme (rather than a morphism) X is separated, they mean implicitly that some morphism is separated. For example, if they are talking about R-schemes, they mean that X is separated over R.

Thanks to Proposition 1.1, a morphism is separated if and only if the image of the diagonal morphism is closed.

1.4. *Important easy exercise.* Show that open immersions and closed immersions are separated. (Answer: Show that monomorphisms are separated. Open and closed immersions are monomorphisms, by earlier exercises. Alternatively, show this by hand.)

1.5. *Important easy exercise.* Show that every morphism of affine schemes is separated. (Hint: this was essentially done in Proposition 1.1.)

I'll now give you an example of something separated that is not affine. The following single calculation will eventually easily imply that all quasiprojective morphisms are separated.

1.6. *Proposition.* — $\mathbb{P}^n_{\mathbb{Z}} \to \operatorname{Spec} \mathbb{Z}$ *is separated.*

(The identical argument holds with $\mathbb{Z}$ replaced by any ring.)

Proof. We cover $\mathbb{P}^n_{\mathbb{Z}} \times_{\mathbb{Z}} \mathbb{P}^n_{\mathbb{Z}}$ with open sets of the form $U_i \times U_j$, where $U_0, \ldots, U_n$ form the "usual" affine open cover. The case $i = j$ was taken care of before, in the proof of Proposition 1.1. For $i \neq j$, we may take $i = 0$, $j = n$. Then

$$U_0 \times_{\mathbb{Z}} U_n \cong \operatorname{Spec} \mathbb{Z}[x_{1/0}, \ldots, x_{n/0}, y_{0/n}, \ldots, y_{n-1/n}],$$

and the image of the diagonal morphism meets this open set in the closed subscheme $y_{0/n}x_{n/0} = 1$, $x_{i/0} = x_{n/0}y_{i/n}$, $y_{j/n} = y_{0/n}x_{j/0}$. □

1.7. *Exercise.* Verify the last sentence of the proof. Note that you should check that the diagonal morphism restricted to this open set has source $U_0 \cap U_n$; see §1.2.

1.8. *Exercise.* Show that the line with doubled origin X is not separated, by verifying that the image of the diagonal morphism is not closed. (Another argument is given below, in Exercise 1.28.)

We finally define then notion of variety!

1.9. *Definition.* A **variety** over a field k is defined to be a reduced, separated scheme of finite type over k. We may use the language k-variety.

Example: a reduced finite type affine k-scheme is a variety. In other words, to check if $\operatorname{Spec} k[x_1, \ldots, x_n]/(f_1, \ldots, f_r)$ is a variety, you need only check reducedness.

Notational caution: In some sources (including, I think, Mumford), the additional condition of irreducibility is imposed. We will not do this. Also, it is often assumed that k is algebraically closed. We will not do this either.

Here is a very handy consequence of separatedness!

1.10. *Proposition.* — *Suppose $X \to \operatorname{Spec} R$ is a separated morphism to an affine scheme, and U and V are affine open sets of X. Then $U \cap V$ is an affine open subset of X.*

We'll prove this shortly.

Consequence: if $X = \operatorname{Spec} A$, then the intersection of any two affine opens is open (just take $R = \mathbb{Z}$ in the above proposition). This is certainly not an obvious fact! We know that the intersection of any two distinguished affine open sets is affine (from $D(f) \cap D(g) = D(fg)$), but we have very little handle on affine open sets in general.

Warning: this property does not characterize separatedness. For example, if $R = \operatorname{Spec} k$ and X is the line with doubled origin over k, then X also has this property. This will be generalized slightly in Exercise 1.31.

Proof. Note that $(U \times_{\operatorname{Spec} R} V) \cap \Delta = U \cap V$, where Δ is the diagonal. (This is clearest with a figure. See also §1.2.)

$U \times_{\operatorname{Spec} R} V$ is affine ($\operatorname{Spec} S \times_{\operatorname{Spec} R} \operatorname{Spec} T = \operatorname{Spec} S \otimes_R T$), and Δ is a closed subscheme of an affine scheme, and hence affine. □

1.11. Sample application: The graph morphism.

1.12. *Definition.* Suppose $f : X \to Y$ is a morphism of Z-schemes. The morphism $\Gamma : X \to X \times_Z Y$ given by $\Gamma = (\mathrm{id}, f)$ is called the **graph morphism**.

1.13. *Proposition.* — *Show that Γ is a locally closed immersion. Show that if Y is a separated Z-scheme (i.e. the structure morphism $Y \to Z$ is separated), then Γ is a closed immersion.*

This will be generalized in Exercise 1.29.

Proof by diagram.

$$\begin{array}{ccc} X & \longrightarrow & X \times_Z Y \\ \downarrow & & \downarrow \\ Y & \xrightarrow{\delta} & Y \times_Z Y \end{array}$$

□

1.14. Quasiseparated morphisms.

We now define a handy relative of separatedness, that is also given in terms of a property of the diagonal morphism, and has similar properties. The reason it is less famous is because it automatically holds for the sorts of schemes that people usually deal with. We say a morphism $f : X \to Y$ is **quasiseparated** if the diagonal morphism $\delta : X \to X \times_Y X$ is quasicompact. I'll give a more insightful translation shortly, in Exercise 1.15.

Most algebraic geometers will only see quasiseparated morphisms, so this may be considered a very weak assumption. Here are two large classes of morphisms that are quasiseparated. (a) As closed immersions are quasicompact (not hard), separated implies quasiseparated. (b) If X is a Noetherian scheme, then any morphism to another scheme is quasicompact (not hard; *Exercise*), so any $X \to Y$ is quasiseparated. Hence those working in the category of Noetherian schemes need never worry about this issue.

It is the following characterization which makes quasiseparatedness a useful hypothesis in proving theorems.

1.15. *Exercise.* Show that $f : X \to Y$ is quasiseparated if and only if for any affine open $\operatorname{Spec} R$ of Y, and two affine open subsets U and V of X mapping to $\operatorname{Spec} R$, $U \cap V$ is a *finite* union of affine open sets.

1.16. *Exercise.* Here is an example of a nonquasiseparated scheme. Let $X = \operatorname{Spec} k[x_1, x_2, \dots]$, and let U be $X - \mathfrak{m}$ where $\mathfrak{m}$ is the maximal ideal $(x_1, x_2, \dots)$. Take two copies of X, glued along U. Show that the result is not quasiseparated.

In particular, the condition of quasiseparatedness is often paired with quasicompactness in hypotheses of theorems. A morphism $f : X \to Y$ is quasicompact and quasiseparated if and only if the preimage of any affine open subset of Y is a *finite* union of affine open sets in X, whose pairwise intersections are all *also* finite unions of affine open sets.

This strong finiteness assumption can be very useful, as the following result shows:

1.17. *Proposition. — If $X \to Y$ is a quasicompact, quasiseparated morphism, and $\mathcal{F}$ is quasicoherent sheaf on X, show that $f_*\mathcal{F}$ is a quasicoherent sheaf on Y.*

Proof. The proof we gave earlier (Theorem 2.2 of Class 20) applies without change. We just didn't have the name "quasiseparated" to attach to these hypothesis. □

1.18. *Theorem. — Both separatedness and quasiseparatedness are preserved by base change.*

Proof. Suppose

$$\begin{array}{ccc} W & \longrightarrow & X \\ \downarrow & & \downarrow \\ Y & \longrightarrow & Z \end{array}$$

is a fiber square. We will show that if $Y \to Z$ is separated or quasiseparated, then so is $W \to X$. The reader should verify (using only category theory!) that

$$\begin{array}{ccc} W & \xrightarrow{\delta_W} & W \times_X W \\ \downarrow & & \downarrow \\ Y & \xrightarrow{\delta_Y} & Y \times_Z Y \end{array}$$

is a fiber diagram. As the property of being a closed immersion is preserved by base change (shown earlier when we showed many properties are well behaved under base change) , if δ_Y is a closed immersion, so is δ_X.

Quasiseparatedness follows in the identical manner, as quasicompactness is also preserved by base change. $\square$

1.19. *Proposition. — The condition of being separated is local on the target. Precisely, a morphism* $f : X \to Y$ *is separated if and only if for any cover of* Y *by open subsets* U_i, $f^{-1}(U_i) \to U_i$ *is separated for each* i.

Hence affine morphisms are separated, by Exercise 1.5. (Thus finite morphisms are separated.)

Proof. If $X \to Y$ is separated, then for any $U_i \hookrightarrow Y$, $f^{-1}(U_i) \to U_i$ is separated by Theorem 1.18. Conversely, to check if $\Delta \hookrightarrow X \times_Y X$ is a closed subset, it suffices to check this on an open cover. If $g : X \times_Y X \to Y$ is the natural morphism, our open cover U_i of Y induces an open cover $g^{-1}(U_i)$ of $X \times_Y X$. $\square$

1.20. *Exercise.* Prove that the condition of being quasiseparated is local on the target. (Hint: the condition of being quasicompact is local on the target by an earlier exercise; use a similar argument.)

1.21. *Proposition. — The condition of being separated is closed under composition. In other words, if* $f : X \to Y$ *is separated and* $g : Y \to Z$ *is separated, then* $g \circ f : X \to Z$ *is separated.*

Proof. This is a good excuse to show you a very useful fiber diagram:

$$\boxed{\begin{array}{ccc} U \times_X V & \longrightarrow & U \times_S V \\ \downarrow & & \downarrow \\ X & \longrightarrow & X \times_S X \end{array}}$$

We are given that $a : X \hookrightarrow X \times_Y X$ and $b : Y \to Y \times_Z Y$ are closed immersions, and we wish to show that $X \to X \times_Z X$ is a closed immersion. Consider the diagram

$$\begin{array}{ccccc} X & \xrightarrow{a} & X \times_Y X & \xrightarrow{c} & X \times_Z X \\ & & \downarrow & & \downarrow \\ & & Y & \xrightarrow{b} & Y \times_Z Y. \end{array}$$

The square on the right is a fiber diagram (see the very useful diagram above). As b is a closed immersion, c is too (closed immersions behave well under fiber diagrams). Thus $c \circ a$ is a closed immersion (the composition of two closed immersions is also a closed immersion). □

The identical argument (with "closed immersion" replaced by "quasicompact") shows that the condition of being quasiseparated is closed under composition.

1.22. *Proposition. — Any quasiprojective morphism is separated.*

As a corollary, any reduced quasiprojective k-scheme is a k-variety.

Proof. Open immersions are separated by Exercise 1.4. Hence by Proposition 1.21, it suffices to check that projective morphisms are separated. We can check that this locally on the target by Proposition 1.19, so it suffices to check that $f : X \to Z$ where f factors through $\mathbb{P}^n_Z$, and $X \hookrightarrow \mathbb{P}^n_Z$ is a closed immersion. But closed immersions are separated, so $X \hookrightarrow \mathbb{P}^n_Z$ is separated, so it suffices to check $\mathbb{P}^n_Z \to Z$ is separated. But this is obtained by base change from $\mathbb{P}^n_{\mathbb{Z}} \to \operatorname{Spec} \mathbb{Z}$, so we are done (as this latter morphism is separated by the previous proposition, and separatedness is preserved by base change by Proposition 1.18). □

1.23. *Proposition. — Suppose* $f : X \to Y$ *and* $f' : X' \to Y'$ *are separated morphisms of* S*-schemes. Then the product morphism* $f \times f' : X \times_S X' \to Y \times_S Y'$ *is separated.*

Proof. Consider the following diagram, and use the fact that separatedness is preserved under base change and composition.

$$\begin{array}{ccccccc} & X \times_S X' & \longrightarrow & X \times_S Y' & \longrightarrow & Y \times_S Y' & \\ \swarrow & & \swarrow & & \searrow & & \searrow \\ X' & \longrightarrow & Y' & & X & \longrightarrow & Y \end{array}$$

□

1.24. A very fun result.

We now come to a very useful, but bizarre-looking, result.

1.25. *Proposition. — Let* P *be a class of morphisms that is preserved by base change and composition. Suppose*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & {\scriptstyle h}\searrow \quad \swarrow{\scriptstyle g} & \\ & Z & \end{array}$$

is a commuting diagram of schemes.

(a) *Suppose that the diagonal morphism* $\delta_g : Y \to Y \times_Z Y$ *is in* P *and* $h : X \to Z$ *is in* P. *The* $f : X \to Y$ *is in* P.
(b) *In particular, if closed immersions are in* P, *then if* h *is in* P *and* g *is separated, then* f *is in* P.

I like this because when you plug in different P, you get very different-looking (and non-obvious) consequences.

Here are some examples.

Locally closed immersions are separated, so part (a) applies, and the first clause always applies. In other words, if you factor a locally closed immersion $X \to Z$ into $X \to Y \to Z$, then $X \to Y$ must be a locally closed immersion.

A morphism (over $\operatorname{Spec} k$) from a projective k-scheme to a separated k-scheme is always projective.

Possibilities for P in case (b) include: finite morphisms, morphisms of finite type, projective morphisms (needed exercise: closed immersions are projective), closed immersions, affine morphisms.

Proof. By the fibered square

$$\begin{array}{ccc} X & \xrightarrow{\Gamma} & X \times_Z Y \\ \downarrow & & \downarrow \\ Y & \xrightarrow{\delta_Y} & Y \times_Z Y \end{array}$$

we see that the graph morphism $\Gamma : X \to X \times_Z Y$ is in P (Definition 1.12), as P is closed under base change. By the fibered square

$$\begin{array}{ccc} X \times_Z Y & \xrightarrow{h'} & Y \\ \downarrow & & \downarrow{\scriptstyle g} \\ X & \xrightarrow{h} & Z \end{array}$$

the projection $h' : X \times_Z Y \to Y$ is in P as well. Thus $f = h' \circ \Gamma$ is in P □

1.26. *Exercise.* Show that a k-scheme is separated (over k) iff it is separated over $\mathbb{Z}$.

Here now are some fun and useful exercises.

1.27. *Useful exercise: The locus where two morphisms agree.* We can now make sense of the following statement. Suppose

$$f, g : \quad \begin{array}{ccc} X & \longrightarrow & Y \\ & \searrow \quad \swarrow & \\ & Z & \end{array}$$

are two morphisms over Z. Then the locus on X where f and g agree is a locally closed subscheme of X. If $Y \to Z$ is separated, then the locus is a closed subscheme of X. More precisely, define V to be the following fibered product:

$$\begin{array}{ccc} V & \longrightarrow & Y \\ \downarrow & & \downarrow \delta \\ X & \xrightarrow{(f,g)} & Y \times_Z Y. \end{array}$$

As δ is a locally closed immersion, $V \to X$ is too. Then if $h : W \to X$ is any scheme such that $g \circ h = f \circ h$, then h factors through V. (Put differently: we are describing $V \hookrightarrow X$ by way of a universal property. Taking this as the definition, it is not a priori clear that V is a locally closed subscheme of X, or even that it exists.) Now we come to the exercise: prove this (the sentence before the parentheses). (Hint: we get a map $g \circ h = f \circ h : W \to Y$. Use the definition of fibered product to get $W \to V$.)

1.28. *Exercise.* Show that the line with doubled origin X is not separated, by finding two morphisms $f_1, f_2 : W \to X$ whose domain of agreement is not a closed subscheme (cf. Proposition 1.1). (Another argument was given above, in Exercise 1.8.)

1.29. *Exercise.* Suppose $\pi : Y \to X$ is a morphism, and $s : X \to Y$ is a *section* of a morphism, i.e. $\pi \circ s$ is the identity on X. Show that s is a locally closed immersion. Show that if π is separated, then s is a closed immersion. (This generalizes Proposition 1.13.)

1.30. *Less important exercise.* Suppose P is a class of morphisms such that closed immersions are in P, and P is closed under fibered product and composition. Show that if $X \to Y$ is in P then $X^{red} \to Y^{red}$ is in P. (Two examples are the classes of separated morphisms and quasiseparated morphisms.) (Hint:

$$\begin{array}{ccccc} X^{red} & \longrightarrow & X \times_Y Y^{red} & \longrightarrow & Y^{red} \\ & \searrow & \downarrow & & \downarrow \\ & & X & \longrightarrow & Y \end{array}$$

)

1.31. *Exercise.* Suppose $\pi : X \to Y$ is a morphism or a ring R, Y is a separated R-scheme, U is an affine open subset of X, and V is an affine open subset of Y. Show that $U \cap \pi^{-1}V$ is an affine open subset of X. (Hint: this generalizes Proposition 1.9 of the Class 25 notes. Use Proposition 1.12 or 1.13.) This will be used in the proof of the Leray spectral sequence.

2. Valuative criteria for separatedness

Describe fact that some people love. It can be useful. I've never used it. But it gives good intuition.

It is possible to verify separatedness by checking only maps from valuations rings.

We begin with a valuative criterion that applies in a case that will suffice for the interests of most people, that of finite type morphisms of Noetherian schemes. We'll then give a more general version for more general readers.

2.1. *Theorem (Valuative criterion for separatedness for morphisms of finite type of Noetherian schemes). — Suppose* $f : X \to Y$ *is a morphism of finite type of Noetherian schemes. Then* f *is separated if and only if the following condition holds. For any discrete valuation ring* R *with function field* K*, and any diagram of the form*

(1)
$$\begin{array}{ccc} \operatorname{Spec} K & \longrightarrow & X \\ \big\downarrow & & \big\downarrow f \\ \operatorname{Spec} R & \longrightarrow & Y \end{array}$$

(where the vertical morphism on the left corresponds to the inclusion $R \hookrightarrow K$*), there is at most one morphism* $\operatorname{Spec} R \to X$ *such that the diagram*

(2)
$$\begin{array}{ccc} \operatorname{Spec} K & \longrightarrow & X \\ \big\downarrow & \nearrow & \big\downarrow f \\ \operatorname{Spec} R & \longrightarrow & Y \end{array}$$

commutes.

A useful thing to take away from this statement is the intuition behind it. We think of $\operatorname{Spec} R$ as a "germ of a curve", and $\operatorname{Spec} K$ as the "germ minus the origin". Then this says that if we have a map from a germ of a curve to Y, and have a lift of the map away from the origin to X, then there is at most one way to lift the map from the entire germ. (A picture is helpful here.)

For example, this captures the idea of what is wrong with the map of the line with the doubled origin over k: we take $\operatorname{Spec} R$ to be the germ of the affine line at the origin, and consider the map of the germ minus the origin to the line with doubled origin. Then we have two choices for how the map can extend over the origin.

2.2. *Exercise.* Make this precise: show that the line with the doubled origin fails the valuative criterion for separatedness.

Proof. (This proof is more telegraphic than I'd like. I may fill it out more later. Because we won't be using this result later in the course, you should feel free to skip it, but you may want to skim it.) One direction is fairly straightforward. Suppose $f : X \to Y$ is separated, and such a diagram (1) were given. suppose g_1 and g_2 were two morphisms

$\operatorname{Spec} R \to X$ making (2) commute. Then $g = (g_1, g_2) : \operatorname{Spec} R \to X \times_Y X$ is a morphism, with $g(\operatorname{Spec} K)$ contained in the diagonal. Hence as $\operatorname{Spec} K$ is dense in $\operatorname{Spec} R$, and g is continuous, $g(\operatorname{Spec} R)$ is contained in the closure of the diagonal. As the diagonal is closed (the separated hypotheses), $g(\operatorname{Spec} R)$ is also contained *set-theoretically* in the diagonal. As $\operatorname{Spec} R$ is reduced, g factors through the reduced induced subscheme structure (§0.1) of the diagonal. Hence g factors through the diagonal:

$$\operatorname{Spec} R \longrightarrow X \xrightarrow{\delta} X \times_Y X,$$

which means $g_1 = g_2$ by Exercise 1.27.

Suppose conversely that f is not separated, i.e. that the diagonal $\Delta \subset X \times_Y X$ is not closed. As $X \times_Y X$ is Noetherian (X is Noetherian, and $X \times_Y X \to X$ is finite type as it is obtained by base change from the finite type $X \to Y$) we have a well-defined notion of dimension of all irreducible closed subsets, and it is bounded. Let P be a point in $\overline{\Delta} - \Delta$ of largest dimension. Let Q be a point in Δ such that $P \in \overline{Q}$. (A picture is handy here.) Let Z be the scheme obtained by giving the reduced induced subscheme structure to $\overline{Q}$. Then P is a codimension 1 point on Z; let $R' = \mathcal{O}_{Z,P}$ be the local ring of Z at P. Then R' is a Noetherian local domain of dimension 1. Let R'' be the normalization of R. Choose any point P'' of $\operatorname{Spec} R''$ mapping to P; such a point exists because the normalization morphism $\operatorname{Spec} R \to \operatorname{Spec} R'$ is surjective (normalization is an integral extension, hence surjective by the Going-up theorem, lecture 21 theorem 1.5). Let R be the localization of R'' at P''. Then R is a normal Noetherian local domain of dimension 1, and hence a discrete valuation ring. Let K be its fraction field. Then $\operatorname{Spec} R \to X \times_Y X$ does not factor through the diagonal, but $\operatorname{Spec} K \to X \times_Y X$ does, and we are done. □

Here is a more general statement. I won't give a proof here, but I think the proof given in Hartshorne Theorem II.4.3 applies (even though the hypotheses are more restrictive).

2.3. *Theorem (Valuative criterion of separatedness). — Suppose* $f : X \to Y$ *is a quasicompact, quasiseparated morphism. Then* f *is separated if and only if the following condition holds. For any valuation ring* R *with function field* K*, and or any diagram of the form* (1)*, there is at most one morphism* $\operatorname{Spec} R \to X$ *such that the diagram* (2) *commutes.*

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FOUNDATIONS OF ALGEBRAIC GEOMETRY CLASS 26

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CONTENTS

Last day: separatedness, definition of variety.

Today: proper morphisms.

I said a little more about separatedness of moduli spaces, for those familiar such objects. Suppose we are interested in a moduli space of a certain kind of object. That means that there is a scheme M with a "universal family" of such objects over M, such that there is a bijection between families of such objects over an arbitrary scheme S, and morphisms $S \to B$. (One direction of this map is as follows: given a morphism $S \to B$, we get a family of objects over S by pulling back the universal family over B.) The separatedness of the moduli space (over the base field, for example, if there is one) can be interpreted as follows. Fix a valuation ring A (or even discrete valuation ring, if our moduli space of of finite type) with fraction field K. We interpret Spec intuitively as a germ of a curve, and we interpret $\operatorname{Spec} K$ as the germ minus the "origin" (an analogue of a small punctured disk). Then we have a family of objects over $\operatorname{Spec} K$ (or over the punctured disk), or equivalently a map $\operatorname{Spec} K \to M$, and the moduli space is separated if there is *at most one way* to fill in the family over the origin, i.e. a family over $\operatorname{Spec} A$.

1. PROPER MORPHISMS

I'll now tell you about a new property of morphisms, the notion of *properness*. You can think about this in several ways.

Recall that a map of topological spaces (also known as a continuous map!) is said to be proper if the preimage of compact sets is compact. Properness of morphisms is an analogous property. For example, proper varieties over $\mathbb{C}$ will be the same as compact in the "usual" topology.

Alternatively, we will see that projective morphisms are proper — this is the hardest thing we will prove — so you can see this as nice property satisfied by projective morphisms, and hence as a generalization of projective morphisms. Indeed, in some sense,

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essentially all interesting properties of projective morphisms that don't explicitly involve $\mathcal{O}(1)$ turn out to be properties of proper morphisms. The key tool in showing such results is Chow's Lemma, which I will state but not prove. Like separatedness, there is a valuative criterion for properness.

Definition. We say a map of topological spaces (i.e. a continuous map) $f : X \to Y$ is *closed* if for each closed subset $S \subset X$, $f(S)$ is also closed. (This is the definition used elsewhere in mathematics.) We say a morphism of schemes is closed if the underlying continuous map is closed. We say that a morphism of schemes $f : X \to Y$ is *universally closed* if for every morphism $g : Z \to Y$, the induced $Z \times_Y X \to Z$ is closed. In other words, a morphism is universally closed if it remains closed under any base change. (A note on terminology: if P is some property of schemes, then a morphism of schemes is said to be "universally P" if it remains P under any base change.)

A morphism $f : X \to Y$ *is* **proper** *if it is separated, finite type, and universally closed.*

As an example: we expect that $\mathbb{A}^1_{\mathbb{C}} \to \operatorname{Spec} \mathbb{C}$ is not proper, because the complex manifold corresponding to $\mathbb{A}^1_{\mathbb{C}}$ is not compact. However, note that this map is separated (it is a map of affine schemes), finite type, and closed. So the "universally" is what matters here. What's the base change that turns this into a non-closed map? Consider $\mathbb{A}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}} \to \mathbb{P}^1_{\mathbb{C}}$.

1.1. *Exercise.* Show that $\mathbb{A}^1_{\mathbb{C}} \to \operatorname{Spec} \mathbb{C}$ is not proper.

Here are some examples of proper maps.

1.2. Closed immersions are proper: they are clearly separated (as affine morphisms are separated). They are finite type. After base change, they remain closed immersions, and closed immersions are always closed.

More generally, finite morphisms are proper: they are separated (as affine), and finite type. The notion of "finite morphism" behaves well under base change, and we have checked that finite morphisms are always closed (I believe in class 21, using the Going-up theorem).

I mentioned that we are going to show that projective morphisms are proper. In fact, finite morphisms are projective (and closed immersions are finite), so the previous two facts will follow from our fancier fact. I should have explained earlier why finite morphisms are projective, but I'll do so now. Suppose $X \to Y$ is a finite morphism, i.e. $X = \underline{\operatorname{Spec}}\, \mathcal{A}$ where $\mathcal{A}$ is a finite type sheaf of algebras. I will now show that $X = \underline{\operatorname{Proj}}\, \mathcal{S}_*$, where $\mathcal{S}_*$ is a sheaf of graded algebras, satisfying all of our various conditions: $\mathcal{S}_0 = \mathcal{O}_Y$, $\mathcal{S}_*$ is "locally generated" by $\mathcal{S}_1$ as a $\mathcal{S}_0$-algebra (i.e. this is true over every open affine subset of Y). Given the statement, you might be able to guess what $\mathcal{S}_*$ should be. I must tell you what $\mathcal{S}_n$ is, and how to multiply. Take $\mathcal{S}_n = \mathcal{A}$ for $n > 0$, with the "obvious" map.

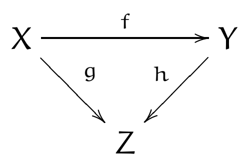
1.3. *Exercise.* Verify that $X = \underline{\operatorname{Proj}}\, \mathcal{S}_*$. What is $\mathcal{O}_{\underline{\operatorname{Proj}}\, \mathcal{S}_*}(1)$?

1.4. Properties of proper morphisms.

1.5. *Proposition. —*

(a) *The notion of "proper morphism" is stable under base change.*
(b) *The notion of "proper morphism" is local on the target (i.e.* $f : X \to Y$ *is proper if and only if for any affine open cover* $U_i \to Y$*,* $f^{-1}(U_i) \to U_i$ *is proper). Note that the "only if" direction follows from (a) — consider base change by* $U_i \hookrightarrow Y$*.*
(c) *The notion of "proper morphism" is closed under composition.*
(d) *The product of two proper morphisms is proper (i.e. if* $f : X \to Y$ *and* $g : X' \to Y'$ *are proper, where all morphisms are morphisms of* Z*-schemes) then* $f \times g : X \times_Z X' \to Y \times_Z Y'$ *is proper.*
(e) *Suppose*

(1)



is a commutative diagram, and g *is proper, and* h *is separated. Then* f *is proper.*
(f) *(I don't know if this is useful, but I may as well say it anyway.) Suppose* (1) *is a commutative diagram, and* f *is surjective,* g *is proper, and* h *is separated and finite type. Then* h *is proper.*

Proof. (a) We have already shown that the notions of separatedness and finite type are local on the target. The notion of closedness is local on the target, and hence so is the notion of universal closedness.

(b) The notions of separatedness, finite type, and universal closedness are all preserved by fiber product. (Notice that this is why universal closedness is better than closedness — it is automatically preserved by base change!)

(c) The notions of separatedness, finite type, and universal closedness are all preserved by composition.

(d) Both $X \times_Z Y \to X' \times_Z Y$ and $X' \times_Z Y \to X' \times_Z Y'$ are proper, because the notion is preserved by base change (part (b)). Then their composition is also proper (part (c)).

(e) Closed immersions are proper, so we invoke our magic and weird "property P fact" from last day.

(f) *Exercise.* □

We come to the hardest thing I will prove today.

1.6. *Theorem. — Projective morphisms are proper.*

It is not easy to come up with an example of a morphism that is proper but not projective! I'll give an simple example before long of a proper but not projective surface (over a field), once we have the notion of the fact that line bundles on nice families of curves have constant degree. Once we discuss blow-ups, I'll give Hironaka's example of a proper but not projective *nonsingular* threefold over $\mathbb{C}$.

I'll give part of the proof today, and the rest next day (because I thought I had a simplification that I realized this morning didn't work out).

Proof. Suppose $f : X \to Y$ is projective. Because the notion of properness is local on the base, we may assume that Y is affine, say $\operatorname{Spec} A$. Then $X \hookrightarrow \mathbb{P}^n_A$ for some n. As closed immersions are proper (§1.2), and the composition of two proper morphisms is proper, it suffices to prove that $\mathbb{P}^n_A \to \operatorname{Spec} A$ is proper. However, we have shown that projective morphisms are separated (last day), and finite type, so it suffices to show that $\mathbb{P}^n_A \to \operatorname{Spec} A$ is universally closed.

We will next show that it suffices to show that $\mathbb{P}^n_R \to \operatorname{Spec} R$ is closed for all rings R. Indeed, we need to show that given any base change $X \to \operatorname{Spec} A$, the resulting base changed morphisms $\mathbb{P}^n_X \to X$ is closed. But the notion of being "closed" is local on the base, so we can replace X by an affine cover.

Next day I will complete the proof by showing that $\mathbb{P}^n_A \to \operatorname{Spec} A$ is closed. This is sometimes called the fundamental theorem of elimination theory. Here are some examples to show you that this is a bit subtle.

First, let $A = k[a, b, c, \dots, i]$, and consider the closed subscheme of $\mathbb{P}^2_A$ (taken with coordinates x, y, z) corresponding to $ax + by + cz = 0$, $dx + ey + fz = 0$, $gx + hy + iz = 0$. Then we are looking for the locus in $\operatorname{Spec} A$ where these equations have a non-trivial solution. This indeed corresponds to a Zariski-closed set — where

$$\det \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = 0.$$

As a second example, let $A = k[a_0, a_1, \dots, a_m, b_0, b_1, \dots, b_n]$. Now consider the closed subscheme of $\mathbb{P}^1_A$ (taken with coordinates x and y) corresponding to $a_0 x^m + a_1 x^{m-1} y + \cdots + a_m y^m = 0$ and $b_0 x^n + b_1 x^{m-1} y + \cdots + b_n y^n = 0$. Then we are looking at the locus in $\operatorname{Spec} A$ where these two polynomials have a common root — this is known as the *resultant*. □

I'll end my discussion of properness with some results that I'll not prove and not use.

1.7. Miscellaneous facts.

Here are some enlightening facts.

(a) Proper and affine = finite. (b) Proper and quasifinite = finite.

(We'll show all three of this in the case of projective morphisms.)

As an application: quasifinite morphisms from proper schemes to separated schemes are finite. Here is why: suppose $X \to Y$ is a quasifinite morphism over Z, where X is proper over Z. Then by one of our weird "property P" facts (Proposition 1.24(b) in class 25), $X \to Y$ is proper. Hence by (b) above, it is finite.

Here is an explicit example: consider a morphism $\mathbb{P}^1 \to \mathbb{P}^1$ given by two distinct sections of $\mathcal{O}_{\mathbb{P}^1}(2)$. The fibers are finite, hence this is a finite morphism. (This could also be checked directly.)

Here is a third fact: If $\pi : X \to Y$ is proper, and $\mathcal{F}$ is a coherent sheaf on X, then $\pi_* \mathcal{F}$ is coherent.

In particular, if X is proper over k, $H^0(X, \mathcal{F})$ is finite-dimensional. (This is just the special case of the morphism $X \to k$.)

1.8. Valuative criterion.

There is a valuative criterion for properness too. I've never used it personally, but it *is* useful, both directly, and also philosophically. I'll make statements, and then discuss some philosophy.

1.9. *Theorem (Valuative criterion for properness for morphisms of finite type of Noetherian schemes).* — *Suppose* $f : X \to Y$ *is a morphism of finite type of locally Noetherian schemes. Then* f *is proper if and only if the following condition holds. For any discrete valuation ring* R *with function field* K*, and or any diagram of the form*

(2)
$$\begin{array}{ccc} \operatorname{Spec} K & \longrightarrow & X \\ \big\downarrow & & \big\downarrow f \\ \operatorname{Spec} R & \longrightarrow & Y \end{array}$$

(where the vertical morphism on the left corresponds to the inclusion $R \hookrightarrow K$*), there is* exactly *one morphism* $\operatorname{Spec} R \to X$ *such that the diagram*

(3)
$$\begin{array}{ccc} \operatorname{Spec} K & \longrightarrow & X \\ \big\downarrow & \nearrow & \big\downarrow f \\ \operatorname{Spec} R & \longrightarrow & Y \end{array}$$

commutes.

Recall that the valuative criterion for properness was the same, except that *exact* was replaced by *at most*.

In the case where Y is a field, you can think of this as saying that limits of one-parameters always exist, and are unique.

1.10. *Theorem (Valuative criterion of properness).* — *Suppose* $f : X \to Y$ *is a quasiseparated, finite type (hence quasicompact) morphism. Then* f *is proper if and only if the following condition* *holds. For any valuation ring* R *with function field* K*, and or any diagram of the form* (2)*, there is* exactly *one morphism* $\operatorname{Spec} R \to X$ *such that the diagram* (3) *commutes.*

Uses: (1) intuition. (2) moduli idea: exactly one way to fill it in (stable curves). (3) motivates the definition of properness for stacks.

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FOUNDATIONS OF ALGEBRAIC GEOMETRY CLASS 27

RAVI VAKIL

CONTENTS

Last day: proper morphisms.

Today: a little more propriety. Rational maps. Curves.

(These notes include some facts discussed in class 28, for the sake of continuity.)

1. PROPER MORPHISMS

Last day we mostly proved:

1.1. *Theorem. — Projective morphisms are proper.*

We had reduced it to the following fact:

1.2. *Proposition. —* $\pi : \mathbb{P}^n_A \to \operatorname{Spec} A$ *is a closed morphism.*

Proof. Suppose $Z \hookrightarrow \mathbb{P}^n_A$ is a closed *subset*. We wish to show that $\pi(Z)$ is closed.

Suppose $y \notin \pi(Z)$ is a *closed* point of $\operatorname{Spec} A$. We'll check that there is a distinguished open neighborhood $D(f)$ of y in $\operatorname{Spec} A$ such that $D(f)$ doesn't meet $\pi(Z)$. (If we could show this for *all* points of $\pi(Z)$, we would be done. But I prefer to concentrate on closed points for now.) Suppose y corresponds to the maximal ideal $\mathfrak{m}$ of A. We seek $f \in A - \mathfrak{m}$ such that $\pi^* f$ vanishes on Z.

A picture helps here, but I haven't put it in the notes.

Date: Tuesday, January 31, 2006. Updated March 8, 2006.

Let $U_0, \ldots, U_n$ be the usual affine open cover of $\mathbb{P}^n_A$. The closed subsets $\pi^{-1}y$ and Z do not intersect. On the affine open set U_i, we have two closed subsets that do not intersect, which means that the ideals corresponding to the two open sets generate the unit ideal, so in the ring of functions on U_i, we can write

$$1 = a_i + \sum m_{ij} g_{ij}$$

where $m_{ij} \in \mathfrak{m}$, and a_i vanishes on Z. Note that $a_i, g_{ij} \in A[x_{0/i}, x_{1/i}, \ldots, x_{n/i}]$. So by multiplying by a sufficiently high power x_i^n of x_i, we have an equality

$$x_i^N = a_i' + \sum m_{ij} g_{ij}'$$

on U_i, where both sides are expressions in $A[x_0, \ldots, x_n]$. We may take N large enough so that it works for all i. Thus for N' sufficiently large, we can write any monomial in $x_1, \ldots, x_n$ of degree N' as something vanishing on Z plus a linear combination of elements of $\mathfrak{m}$ times other polynomials. Hence if $S_* = A[x_0, \ldots, x_n]$,

$$S_{N'} = I(Z)_{N'} + \mathfrak{m} S_{N'}$$

where $I(Z)_*$ is the graded ideal of functions vanishing on Z. Hence by Nakayama's lemma, there exists $f \in A - \mathfrak{m}$ such that

$$f S_{N'} \subset I(Z)_{N'}.$$

Thus we have found our desired f!

We are now ready to tackle the proposition in general. Suppose $y \in \operatorname{Spec} A$ is no longer necessarily a closed point, and say $y = [\mathfrak{p}]$. Then we apply the same argument in $\operatorname{Spec} A_{\mathfrak{p}}$. We get $S_{N'} \otimes A_{\mathfrak{p}} = I(Z)_{N'} \otimes A_{\mathfrak{p}} + \mathfrak{m} S_{N'} \otimes A_{\mathfrak{p}}$, from which $g(S_{N'}/I(Z)_{N'}) \otimes A_{\mathfrak{p}} = 0$ for some $g \in A_{\mathfrak{p}} - \mathfrak{p} A_{\mathfrak{p}}$, from which $(S_{N'}/I(Z)_{N'}) \otimes A_{\mathfrak{p}} = 0$. Now $S_{N'}$ is a finitely generated A-module, so there is some $f \in R - \mathfrak{p}$ with $f S_N \subset I(Z)$ (if the module-generators of $S_{N'}$, and $f_1, \ldots, f_a$ are annihilate the generators respectively, then take $f = \prod f_i$), so once again we have found $D(f)$ containing $\mathfrak{p}$, with (the pullback of) f vanishing on Z. □

2. Scheme-theoretic closure, and scheme-theoretic image

Have I defined scheme-theoretic closure of a locally closed subscheme $W \hookrightarrow Y$? I think I have neglected to. It is the smallest closed subscheme of Y containing W. *Exercise.* Show that this notion is well-defined. More generally, if $f : W \to Y$ is any morphism, define the scheme-theoretic image as the smallest closed subscheme $Z \to Y$ so that f factors through $Z \hookrightarrow Y$. *Exercise.* Show that this is well-defined. (One possible hint: use a universal property argument.) If Y is affine, the ideal sheaf corresponds to the functions on Y that are zero when pulled back to Z. Show that the closed set underlying the image subscheme may be strictly larger than the closure of the set-theoretic image: consider $\coprod_{n \geq 0} \operatorname{Spec} k[t]/t^n \to \operatorname{Spec} k[t]$. (I suspect that such a pathology cannot occur for finite type morphisms of Noetherian schemes, but I haven't investigated.)

3. Rational maps

This is a very old topic, near the beginning of any discussion of varieties. It has appeared late for us because we have just learned about separatedness.

For this section, I will suppose that X and Y are integral and separated, although these notions are often useful in more general circumstances. The interested reader should consider the first the case where the schemes in question are reduced and separated (but not necessarily irreducible). Many notions can make sense in more generality (without reducedness hypotheses for example), but I'm not sure if there is a widely accepted definition.

A key example will be irreducible varieties, and the language of rational maps is most often used in this case.

A **rational map** $X \dashrightarrow Y$ is a morphism on a dense open set, with the equivalence relation: $(f : U \to Y) \sim (g : V \to Y)$ if there is a dense open set $Z \subset U \cap V$ such that $f|_Z = g|_Z$. (We will soon see that we can add: if $f|_{U\cap V} = g|_{U\cap V}$.)

An obvious example of a rational map is a morphism. Another example is a rational function, which is a rational map to $\mathbb{A}^1_{\mathbb{Z}}$ (*easy exercise*).

3.1. *Exercise.* Show that you can compose two rational maps $f : X \dashrightarrow Y$, $g : Y \dashrightarrow Z$ if f is dominant.

3.2. *Easy exercise.* Show that dominant rational maps give morphisms of function fields in the opposite direction. (This was problem 37 on problem set 9.)

It is not true that morphisms of function fields give dominant rational maps, or even rational maps. For example, $k[x]$ and $\operatorname{Spec} k(x)$ have the same function field ($k(x)$), but there is no rational map $\operatorname{Spec} k[x] \dashrightarrow k(x)$. Reason: that would correspond to a morphism from an open subset U of $\operatorname{Spec} k[x]$, say $k[x, 1/f(x)]$, to $k(x)$. But there is no map of rings $k(x) \to k[x, 1/f(x)]$ for any one $f(x)$.

However, this is true in the case of varieties (see Proposition 3.4 below).

A rational map $f : X \to Y$ is said to be *birational* if it is dominant, and there is another morphism (a "rational inverse") that is also dominant, such that $f \circ g$ is (in the same equivalence class as) the identity on Y, and $g \circ f$ is (in the same equivalence class as) the identity on X.

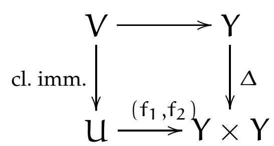
A morphism is **birational** if it is birational as a rational map. We say X and Y are *birational* to each other if there exists a birational map $X \dashrightarrow Y$. This is the same as our definition before. Birational maps induce isomorphisms of function fields.

3.3. *Important Theorem. — Two S-morphisms $f_1, f_2 : U \to Z$ from a reduced scheme to a separated S-scheme agreeing on a dense open subset of U are the same.*

Note that this generalizes the easy direction of the valuative criterion of separatedness (which is the special case where U is Spec of a discrete valuation ring — which consists of two points — and the dense open set is the generic point).

It is useful to see how this breaks down when we give up reducedness of the base or separatedness of the target. For the first, consider the two maps $\operatorname{Spec} k[x,y]/(x^2, xy) \to \operatorname{Spec} k[t]$, where we take f_1 given by $t \mapsto y$ and f_2 given by $t \mapsto y + x$; f_1 and f_2 agree on the distinguished open set $D(y)$. (A picture helps here!) For the second, consider the two maps from $\operatorname{Spec} k[t]$ to the line with the doubled origin, one of which maps to the "upper half", and one of which maps to the "lower half". these to morphisms agree on the dense open set $D(f)$.

Proof.



We have a closed subscheme of U containing the generic point. It must be all of U. □

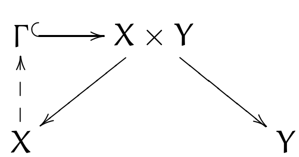
Consequence 1. Hence (as X is reduced and Y is separated) if we have two morphisms from open subsets of X to Y, say $f : U \to Y$ and $g : V \to Y$, and they agree on a dense open subset $Z \subset U \cap V$, then they necessarily agree on $U \cap V$.

Consequence 2. Also: a rational map has a largest *domain of definition* on which $f : U \dashrightarrow Y$ is a morphism, which is the union of all the domains of definition.

In particular, a rational function from a reduced scheme has a largest *domain of definition*.

We define the *graph* of a rational map $f : X \dashrightarrow Y$ as follows: let (U, f') be any representative of this rational map (so $f' : U \to Y$ is a morphism). Let Γ_f be the scheme-theoretic closure of $\Gamma_{f'} \hookrightarrow U \times Y \hookrightarrow X \times Y$, where the first map is a closed immersion, and the second is an open immersion. *Exercise.* Show that this is independent of the choice of U.

Here is a handy diagram involving the graph of a rational map:



(that "up arrow" should be dashed).

We now prove a Proposition promised earlier.

3.4. *Proposition. — Suppose* X, Y *are irreducible varieties, and we are given* $f^\# : FF(Y) \hookrightarrow FF(Y)$. *Then there exists a dominant rational map* $f : X \dashrightarrow Y$ *inducing* $f^\#$.

Proof. By replacing Y with an affine open set, we may assume Y is affine, say $Y = \operatorname{Spec} k[x_1, \ldots, x_n]/(f_1, \ldots, f_r)$. Then we have $x_1, \ldots, x_n \in K(X)$. Let U be an open subset of the domains of definition of these rational functions. Then we get a morphism $U \to \mathbb{A}^n_k$. But this morphism factors through $Y \subset \mathbb{A}^n$, as $x_1, \ldots, x_n$ satisfy all the relations $f_1, \ldots, f_r$. □

3.5. *Exercise.* Let K be a finitely generated field extension of transcendence degree m over k. Show there exists an irreducible k-variety W with function field K. (Hint: let $x_1, \ldots, x_n$ be generators for K over k. Consider the map $\operatorname{Spec} K \to \operatorname{Spec} k[t_1, \ldots, t_n]$ given by the ring map $t_i \mapsto x_i$. Take the scheme-theoretic closure of the image.)

3.6. *Proposition. — Suppose* X *and* Y *are integral and separated (our standard hypotheses from last day). Then* X *and* Y *are birational if and only if there is a dense=non-empty open subscheme* U *of* X *and a dense=non-empty open subscheme* V *of* Y *such that* $U \cong Y$.

This gives you a good idea of how to think of birational maps.

3.7. *Exercise.* Prove this. (Feel free to consult Iitaka or Hartshorne (Corollary I.4.5).)

4. Examples of rational maps

We now give a bunch of examples Here are some examples of rational maps, and birational maps. A recurring theme is that domains of definition of rational maps to projective schemes extend over nonsingular codimension one points. We'll make this precise when we discuss curves shortly.

(A picture is helpful here.) The first example is how you find a formula for Pythagorean triples. Suppose you are looking for rational points on the circle C given by $x^2 + y^2 = 1$. One rational point is $p = (1, 0)$. If q is another rational point, then pq is a line of rational (non-infinite) slope. This gives a rational map from the conic to $\mathbb{A}^1$. Conversely, given a line of slope m through p, where m is rational, we can recover q as follows: $y = m(x-1)$, $x^2 + y^2 = 1$. We substitute the first equation into the second, to get a quadratic equation in x. We know that we will have a solution $x = 1$ (because the line meets the circle at $(x, y) = (1, 0)$), so we expect to be able to factor this out, and find the other factor. This indeed works:

$$\begin{aligned} x^2 + (m(x-1))^2 &= 1 \\ (m^2+1)x^2 + (-2)x + (m^2-1) &= 0 \\ (x-1)((m^2+1)x - (m^2-1)) &= 0 \end{aligned}$$

The other solution is $x = (m^2-1)/(m^2+1)$, which gives $y = 2m/(m^2+1)$. Thus we get a birational map between the conic C and $\mathbb{A}^1$ with coordinate m, given by $f : (x, y) \mapsto y/(x-1)$ (which is defined for $x \neq 1$), and with inverse rational map given by $m \mapsto ((m^2-1)/(m^2+1), 2m/(m^2+1))$ (which is defined away from $m^2 + 1 = 0$).

We can extend this to a rational map $C \dashrightarrow \mathbb{P}^1$ via the inclusion $\mathbb{A}^1 \to \mathbb{P}^1$. Then f is given by $(x, y) \mapsto [y; x-1]$. (Remember that we give maps to projective space by giving sections of line bundles — in this case, we are using the structure sheaf.) We then have an interesting question: what is the domain of definition of f? It appears to be defined everywhere except for where $y = x - 1 = 0$, i.e. everywhere but p. But in fact it can be extended over p! Note that $(x, y) \mapsto [x+1; -y]$ (where $(x, y) \neq (-1, y)$) agrees with f on their common domains of definition, as $[x+1; -y] = [y; x-1]$. Hence this rational map can be extended farther than we at first thought. This will be a special case of a result we'll see later today.

(For the curious: we are working with schemes over $\mathbb{Q}$. But this works for any scheme over a field of characteristic not 2. What goes wrong in characteristic 2?)

4.1. *Exercise.* Use the above to find a "formula" for all Pythagorean triples.

4.2. *Exercise.* Show that the conic $x^2 + y^2 = z^2$ in $\mathbb{P}^2_k$ is isomorphic to $\mathbb{P}^1_k$ for any field k of characteristic not 2. (Presumably this is true for any ring in which 2 is invertible too...)

In fact, any conic in $\mathbb{P}^2_k$ with a k-valued point (i.e. a point with residue field k) is isomorphic to $\mathbb{P}^1_k$. (This hypothesis is certainly necessary, as $\mathbb{P}^1_k$ certainly has k-valued points. $x^2 + y^2 + z^2 = 0$ over $k = \mathbb{R}$ gives an example of a conic that is not isomorphic to $\mathbb{P}^1_k$.)

4.3. *Exercise.* Find all rational solutions to $y^2 = x^3 + x^2$, by finding a birational map to $\mathbb{A}^1$, mimicking what worked with the conic.

You will obtain a rational map to $\mathbb{P}^1$ that is not defined over the node $x = y = 0$, and *can't* be extended over this codimension 1 set. This is an example of the limits of our future result showing how to extend rational maps to projective space over codimension 1 sets: the codimension 1 sets have to be nonsingular. More on this soon!

4.4. *Exercise.* Use something similar to find a birational map from the quadric $Q = \{x^2 + y^2 = w^2 + z^2\}$ to $\mathbb{P}^2$. Use this to find all rational points on Q. (This illustrates a good way of solving Diophantine equations. You will find a dense open subset of Q that is isomorphic to a dense open subset of $\mathbb{P}^2$, where you can easily find all the rational points. There will be a closed subset of Q where the rational map is not defined, or not an isomorphism, but you can deal with this subset in an ad hoc fashion.)

4.5. *Exercise (a first view of a blow-up).* Let k be an algebraically closed field. (We make this hypothesis in order to not need any fancy facts on nonsingularity.) Consider the rational map $\mathbb{A}^2_k \dashrightarrow \mathbb{P}^1_k$ given by $(x, y) \mapsto [x; y]$. I think you have shown earlier that this rational map cannot be extended over the origin. Consider the graph of the birational map, which we denote $\mathrm{Bl}_{(0,0)} \mathbb{A}^2_k$. It is a subscheme of $\mathbb{A}^2_k \times \mathbb{P}^1_k$. Show that if the coordinates on $\mathbb{A}^2$ are x, y, and the coordinates on $\mathbb{P}^1$ are u, v, this subscheme is cut out in $\mathbb{A}^2 \times \mathbb{P}^1$ by the single equation $xv = yu$. Show that $\mathrm{Bl}_{(0,0)} \mathbb{A}^2_k$ is nonsingular. Describe the fiber of the morphism $\mathrm{Bl}_{(0,0)} \mathbb{A}^2_k \to \mathbb{P}^1_k$ over each closed point of $\mathbb{P}^1_k$. Describe the fiber of the morphism

$\text{Bl}_{(0,0)} \mathbb{A}^2_k \to \mathbb{A}^2_k$. Show that the fiber over $(0,0)$ is an effective Cartier divisor. It is called the *exceptional divisor*.

4.6. *Exercise (the Cremona transformation, a useful classical construction).* Consider the rational map $\mathbb{P}^2 \dashrightarrow \mathbb{P}^2$, given by $[x;y;z] \to [1/x;1/y;1/z]$. What is the the domain of definition? (It is bigger than the locus where $xyz \neq 0$!) You will observe that you can extend it over codimension 1 sets. This will again foreshadow a result we will soon prove.

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FOUNDATIONS OF ALGEBRAIC GEOMETRY CLASS 28

RAVI VAKIL

CONTENTS

Last day: More on properness. Rational maps.

Today: Curves.

(I also discussed rational maps a touch more, but I've included that in the class 27 notes for the sake of continuity.)

1. CURVES

Let's now use our technology to study something explicit! For our discussion here, we will temporarily define a *curve* to be an integral variety over k of dimension 1. (In particular, curves are reduced, irreducible, separated, and finite type over k.)

I gave an incomplete proof to the following proposition. Because I don't think I'll use it, I haven't tried to patch it. But if there is interest, I'll include the proof with the hole, in case one of you can figure out how to make it work. (We showed that each closed point gives a discrete valuation, and we showed that each discrete valuation gives a morphism from the Spec corresponding discrete valuation ring to the curve, but we didn't show that it was the local ring of the corresponding closed point. I would like to do this without invoking any algebra that we haven't yet proved.)

1.1. *Proposition. — Suppose C is a projective nonsingular curve. Then each closed point of C yields a discrete valuation ring, and hence a discrete valuation on $FF(C)$. This gives a bijection from closed points of C and discrete valuations on $FF(C)$.*

Thus a projective nonsingular curve is a convenient way of seeing all the discrete valuations at once, in a nice geometric package.

I had wanted to ask you the following exercise (for those with arithmetic proclivities), but I won't now: Suppose A is the ring of integers in a number field (i.e. the integral

Date: Thursday, February 2, 2006.

closure of $\mathbb{Z}$ in a finite field extension $K/\mathbb{Q}$ — $K = FF(A)$). Show that there is a natural bijection between discrete valuations on K are in bijection with the maximal ideals of A.

1.2. *Key Proposition. — Suppose C is a dimension 1 finite type k-scheme, and p is a nonsingular point of it. Suppose Y is a projective k-scheme. Then any morphism $C - p \to Y$ extends to $C \to Y$.*

Note: if such an extension exists, then it is unique: The non-reduced locus of C is a closed subset (we checked this earlier for any Noetherian scheme), not including p, so by replacing C by an open neighborhood of p that is reduced, we can use our recently-proved theorem that maps from reduced schemes to separated schemes are determined by their behavior on a dense open set (Important Theorem 3.3 in last day's notes).

I'd like to give two proofs, which are enlightening in different ways.

Proof 1. By restricting to an affine neighborhood of C, we can reduce to the case where C is affine.

We next reduce to the case where $Y = \mathbb{P}^n_k$. Here is how. Choose a closed immersion $Y \to \mathbb{P}^n_k$. If the result holds for $\mathbb{P}^n$, and we have a morphism $C \to \mathbb{P}^n$ with $C - p$ mapping to Y, then C must map to Y as well. Reason: we can reduce to the case where the source is an affine open subset, and the target is $\mathbb{A}^n_k \subset \mathbb{P}^n_k$ (and hence affine). Then the functions vanishing on $Y \cap \mathbb{A}^n_k$ pull back to functions that vanish at the generic point of C and hence vanish everywhere on C, i.e. C maps to Y.

Choose a uniformizer $t \in \mathfrak{m} - \mathfrak{m}^2$ in the local ring. By discarding the points of the vanishing set $V(t)$ aside from p, and taking an affine open subset of p in the remainder we reduce to the case where t cuts out precisely $\mathfrak{m}$ (i.e. $\mathfrak{m} = (y)$). Choose a dense open subset U of $C - p$ where the pullback of $\mathcal{O}(1)$ is trivial. Take an affine open neighborhood $\operatorname{Spec} A$ of p in $U \cup \{p\}$. Then the map $\operatorname{Spec} A - p \to \mathbb{P}^n$ corresponds to $n+1$ functions, say $f_0, \ldots, f_n \in A_{\mathfrak{m}}$, not all zero. Let m be the smallest valuation of all the f_i. Then $[t^{-m} f_0; \ldots; t^{-m} f_n]$ has all entries in A, and not all in the maximal ideal, and thus is defined at p as well. □

Proof 2. We extend the map $\operatorname{Spec} FF(C) \to Y$ to $\operatorname{Spec} \mathcal{O}_{C,p} \to Y$ as follows. Note that $\mathcal{O}_{C,p}$ is a discrete valuation ring. We show first that there is a morphism $\operatorname{Spec} \mathcal{O}_{C,p} \to \mathbb{P}^n$. The rational map can be described as $[a_0; a_1; \cdots; a_n]$ where $a_i \in \mathcal{O}_{C,p}$. Let m be the minimum valuation of the a_i, and let t be a uniformizer of $\mathcal{O}_{C,p}$ (an element of valuation 1). Then $[t^{-m} a_0; t^{-m} a_1; \ldots t^{-m} a_n]$ is another description of the morphism $\operatorname{Spec} FF(\mathcal{O}_{C,p}) \to \mathbb{P}^n$, and each of the entries lie in $\mathcal{O}_{C,p}$, and not all entries lie in $\mathfrak{m}$ (as one of the entries has valuation 0). This same expression gives a morphism $\operatorname{Spec} \mathcal{O}_{C,p} \to \mathbb{P}^n$.

Our intuition now is that we want to glue the maps $\operatorname{Spec} \mathcal{O}_{C,p} \to Y$ (which we picture as a map from a germ of a curve) and $C - p \to Y$ (which we picture as the rest of the curve). Let $\operatorname{Spec} R \subset Y$ be an affine open subset of Y containing the image of $\operatorname{Spec} \mathcal{O}_{C,p}$. Let $\operatorname{Spec} A \subset C$ be an affine open of C containing p, and such that the image of $\operatorname{Spec} A - p$ in Y lies in $\operatorname{Spec} R$, and such that p is cut out scheme-theoretically by a single equation (i.e.

there is an element $t \in A$ such that (t) is the maximal ideal corresponding to $\mathfrak{p}$. Then R and A are domains, and we have two maps $R \to A_{(t)}$ (corresponding to $\operatorname{Spec} \mathcal{O}_{C,\mathfrak{p}} \to \operatorname{Spec} R$) and $R \to A_t$ (corresponding to $\operatorname{Spec} A - \mathfrak{p} \to \operatorname{Spec} R$) that agree "at the generic point", i.e. that give the same map $R \to \mathrm{FF}(A)$. But $A_t \cap A_{(t)} = A$ in $\mathrm{FF}(A)$ (e.g. by Hartogs' theorem — elements of the fraction field of A that don't have any poles away from t, nor at t, must lie in A), so we indeed have a map $R \to A$ agreeing with both morphisms. □

1.3. *Exercise (Useful practice!).* Suppose X is a Noetherian k-scheme, and Z is an irreducible codimension 1 subvariety whose generic point is a nonsingular point of X (so the local ring $\mathcal{O}_{X,Z}$ is a discrete valuation ring). Suppose $X \dashrightarrow Y$ is a rational map to a projective k-scheme. Show that the domain of definition of the rational map includes a dense open subset of Z. In other words, rational maps from Noetherian k-schemes to projective k-schemes can be extended over nonsingular codimension 1 sets. We saw this principle in action with the Cremona transformation, in Class 27 Exercise 4.6. (By the easy direction of the valuative criterion of separatedness, or the theorem of uniqueness of extensions of maps from reduced schemes to separated schemes — Theorem 3.3 of Class 27 — this map is unique.)

1.4. *Theorem. — If C is a nonsingular curve, then there is some projective nonsingular curve C' and an open immersion $C \hookrightarrow C'$.*

This proof has a bit of a different flavor than proofs we've seen before. We'll use make particular use of the fact that one-dimensional Noetherian schemes have a boring topology.

Proof. Given a nonsingular curve C, take a non-empty=dense affine open set, and take any non-constant function f on that affine open set to get a rational map $C \dashrightarrow \mathbb{P}^1$ given by $[1; f]$. As a dense open set of a dimension 1 scheme consists of everything but a finite number of points, by Proposition 1.2, this extends to a morphism $C \to \mathbb{P}^1$.

We now take the normalization of $\mathbb{P}^1$ in the function field $\mathrm{FF}(C)$ of C (a finite extension of $\mathrm{FF}(\mathbb{P}^1)$), to obtain $C' \to \mathbb{P}^1$. Now C' is normal, hence nonsingular (as nonsingular = normal in dimension 1). By the finiteness of integral closure, $C' \to \mathbb{P}^1$ is a finite morphism. Moreover, finite morphisms are projective, so by considering the composition of projective morphisms $C' \to \mathbb{P}^1 \to \operatorname{Spec} k$, we see that C' is projective over k. Thus we have an isomorphism $\mathrm{FF}(C) \to \mathrm{FF}(C')$, hence a rational map $C \dashrightarrow C'$, which extends to a morphism $C \to C'$ by Key Proposition 1.2.

Finally, I claim that $C \to C'$ is an open immersion. If we can prove this, then we are done. I note first that this is an injection of sets:

- the generic point goes to the generic point
- the closed points of C correspond to distinct valuations on $\mathrm{FF}(C)$ (as C is separated, by the easy direction of the valuative criterion of separatedness)

Thus as sets, C is C' minus a finite number of points. As the topology on C and C' is the "cofinite topology" (i.e. the open sets include the empty set, plus everything minus a finite number of closed points), the map $C \to C'$ of topological spaces expresses C as a homeomorphism of C onto its image $\mathrm{im}(C)$. Let $f : C \to \mathrm{im}(C)$ be this morphism of schemes. Then the morphism $\mathcal{O}_{\mathrm{im}(C)} \to f_*\mathcal{O}_C$ can be interpreted as $\mathcal{O}_{\mathrm{im}(C)} \to \mathcal{O}_C$ (where we are identifying C and $\mathrm{im}(C)$ via the homeomorphism f). This morphism of sheaves is an isomorphism of stalks at all points $p \in \mathrm{im}(C)$ (it is the isomorphism the discrete valuation ring corresponding to $p \in C'$), and is hence an isomorphism. Thus $C \to \mathrm{im}(C)$ is an isomorphism of schemes, and thus $C \to C'$ is an open immersion. □

We now come to the big theorem of today (although the Key Proposition 1.2 above was also pretty big).

1.5. *Theorem. — The following categories are equivalent.*

(i) *nonsingular projective curves, and surjective morphisms.*
(ii) *nonsingular projective curves, and dominant morphisms.*
(iii) *nonsingular projective curves, and dominant rational maps*
(iv) *quasiprojective reduced curves, and dominant rational maps*
(v) *function fields of dimension* 1 *over* k*, and* k*-homomorphisms.*

(All morphisms and maps are assumed to be k-morphisms and k-rational maps, i.e. they are all over k. Remember that today we are working in the category of k-schemes.)

This has a lot of implications. For example, each quasiprojective reduced curve is isomorphic to precisely one projective nonsingular curve.

This leads to a motivating question that I mentioned informally last day (and that isn't in the notes). Suppose k is algebraically closed (such as $\mathbb{C}$). Is it true that all nonsingular projective curves are isomorphic to $\mathbb{P}^1_k$? Equivalently, are all quasiprojective reduced curves birational to $\mathbb{A}^1_k$? Equivalently, are all transcendence degree 1 extensions of k generated (as a field) by a single element? The answer (as most of you know) is *no*, but we can't yet see that.

1.6. *Exercise.* Show that all nonsingular proper curves are projective.

(We may eventually see that all reduced proper curves over k are projective, but I'm not sure; this will use the Riemann-Roch theorem, and I may just prove it for projective curves.)

Before we get to the proof, I want to mention a sticky point that came up in class. If $k = \mathbb{R}$, then we are allowing curves such as $\mathbb{P}^1_{\mathbb{C}}$ that "we don't want". One way of making this precise is noting that they are not geometrically irreducible (as $\mathbb{C}(t)_{\otimes_{\mathbb{R}}}\mathbb{C} \cong \mathbb{C}(t) \oplus \mathbb{C}(t)$). Another way is to note that this function field K does not satisfy $\overline{k} \cap K = k$ in $\overline{K}$. If this bothers you, then add it to each of the 5 categories. (For example, in (i)–(iii), we consider only nonsingular projective curves whose function field K satisfies $\overline{k} \cap K = k$ in $\overline{K}$.) If this doesn't bother you, please ignore this paragraph!

Proof. Any surjective morphism is a dominant morphism, and any dominant morphism is a dominant rational map, and each nonsingular projective curve is a quasiprojective curve, so we've shown (i) $\rightarrow$ (ii) $\rightarrow$ (iii) $\rightarrow$ (iv). To get from (iv) to (i), we first note that the nonsingular points on a quasiprojective reduced curve are dense. (One method, suggested by Joe: we know that normalization is an isomorphism away from a closed subset.) Given a dominant rational map between quasiprojective reduced curves $C \rightarrow C'$, we get a dominant rational map between their normalizations, which in turn gives a dominant rational map between their projective models $D \dashrightarrow D'$. The the dominant rational map is necessarily a morphism by Proposition 1.2, and then this morphism is necessarily projective and hence closed, and hence surjective (as the image contains the generic point of D', and hence its closure). Thus we have established (iv) $\rightarrow$ (i).

It remains to connect (i). Each dominant rational map of quasiprojective reduced curves indeed yields a map of function fields of dimension 1 (their fraction fields). Each function field of dimension 1 yields a reduced affine (hence quasiprojective) curve over k, and each map of two such yields a dominant rational map of the curves. $\square$

1.7. Degree of a morphism between projective nonsingular curves.

We conclude with a useful fact: Any non-constant morphism from one projective nonsingular curve to another has a well-behaved degree, in a sense that we will now make precise. We will also show that any non-constant finite morphism from one nonsingular curve to another has a well-behaved degree in the same sense.

Suppose $f : C \rightarrow C'$ is a surjective (or equivalently, dominant) map of nonsingular projective curves.

It is a finite morphism. Here is why. (If we had already proved that quasifinite projective or proper morphisms with finite fibers were finite, we would know this. Once we *do* know this, the contents of this section would extend to the case where C is not necessarily non-singular.) Let C'' be the normalization of C' in the function field of C. Then we have an isomorphism $FF(C) \cong FF(C'')$ which leads to birational maps $C \leftarrow\!\!\dashrightarrow C''$ which extend to morphisms as both C and C'' are nonsingular and projective. Thus this yields an isomorphism of C and C''. But $C'' \rightarrow C$ is a finite morphism by the finiteness of integral closure.

We can then use the following proposition, which applies in more general situations.

1.8. *Proposition. — Suppose that* $\pi : C \rightarrow C'$ *is a surjective finite morphism, where* C *is an integral curve, and* C' *is an integral nonsingular curve. Then* $\pi_* \mathcal{O}_C$ *is locally free of finite rank.*

As π is finite, $\pi_* \mathcal{O}_C$ is a finite type sheaf on $\mathcal{O}'_C$. In case you care, the hypothesis "integral" on C' is redundant.

Before proving the proposition. I want to remind you what this means. Suppose d is the rank of this allegedly locally free sheaf. Then the fiber over any point of C with residue field K is the Spec of an algebra of dimension d over K. This means that the number of points in the fiber, counted with appropriate multiplicity, is always d.

As a motivating example, consider the map $\mathbb{Q}[y] \to \mathbb{Q}[x]$ given by $x \mapsto y^2$. (We've seen this example before.) I picture this as the projection of the parabola $x = y^2$ to the x-axis. (i) The fiber over $x = 1$ is $\mathbb{Q}[y]/(y^2 - 1)$, so we get 2 points. (ii) The fiber over $x = 0$ is $\mathbb{Q}[y]/(y^2)$ — we get one point, with multiplicity 2, arising because of the nonreducedness. (iii) The fiber over $x = -1$ is $\mathbb{Q}[y]/(y^2 + 1) \cong \mathbb{Q}[i]$ — we get one point, with multiplicity 2, arising because of the field extension. (iv) Finally, the fiber over the generic point $\operatorname{Spec} \mathbb{Q}(x)$ is $\operatorname{Spec} \mathbb{Q}(y)$, which is one point, with multiplicity 2, arising again because of the field extension (as $\mathbb{Q}(y)/\mathbb{Q}(x)$ is a degree 2 extension). We thus see three sorts of behaviors (as (iii) and (iv) are the same behavior). Note that even if you only work with algebraically closed fields, you will still be forced to this third type of behavior, because residue fields at generic points tend not to be algebraically closed (witness case (iv) above).

Note that we need C' to be nonsingular for this to be true. (I gave a picture of the normalization of a nodal curve as an example. A picture would help here.)

We will see the proof next day.

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FOUNDATIONS OF ALGEBRAIC GEOMETRY CLASS 29

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CONTENTS

Last day: One last bit of rational maps. Curves.

Today: A bit more curves. Introduction to cohomology.

1. Scheme-theoretic closure, and scheme-theoretic image

I discussed the scheme-theoretic closure of a locally closed scheme, and more generally, the scheme-theoretic image of a morphism. I've moved this discussion into the class 27 notes.

2. Curves

Last day we proved a couple of important theorems:

2.1. *Key Proposition. — Suppose C is a dimension 1 finite type k-scheme, and p is a nonsingular point of it. Suppose Y is a projective k-scheme. Then any morphism $C - p \to Y$ extends to $C \to Y$.*

2.2. *Theorem. — If C is a nonsingular curve, then there is some projective nonsingular curve C' and an open immersion $C \hookrightarrow C'$.*

2.3. *Theorem. — The following categories are equivalent.*

Date: Tuesday, February 7, 2006. Updated June 25, 2007.

(i) *nonsingular projective curves, and surjective morphisms.*
(ii) *nonsingular projective curves, and dominant morphisms.*
(iii) *nonsingular projective curves, and dominant rational maps*
(iv) *quasiprojective reduced curves, and dominant rational maps*
(v) *fields of transcendence dimension* 1 *over* k*, and* k*-homomorphisms.*

We then discussed the degree of a morphism between projective nonsingular curves. In particular, we are in the midst of showing that any non-constant morphism from one projective nonsingular curve to another has a well-behaved degree. Suppose $f : C \to C'$ is a surjective (or equivalently, dominant) map of nonsingular projective curves. We showed that f is a finite morphism, by showing that f is the normalization of C' in the function field of C; hence the result follows by finiteness of integral closure.

2.4. *Proposition. — Suppose that* $\pi : C \to C'$ *is a surjective finite morphism, where* C *is an integral curve, and* C' *is an integral nonsingular curve. Then* $\pi_* \mathcal{O}_C$ *is locally free of finite rank.*

All we will really need is that C is reduced of pure dimension 1.

We are about to prove this.

Let's discuss again what this means. (I largely said this last day.) Suppose d is the rank of this allegedly locally free sheaf. Then the fiber over any point of C with residue field K is the Spec of an algebra of dimension d over K. This means that the number of points in the fiber, counted with appropriate multiplicity, is always d.

Proof. (For experts: we will later see that what matters here is that the morphism is finite and *flat*. But we don't yet know what flat is.)

The question is local on the target, so we may assume that C' is affine. Note that $\pi_* \mathcal{O}_C$ is torsion-free (as $\Gamma(C, \mathcal{O}_C)$ is an integral domain). Our plan is as follows: by an important exercise from last quarter (Exercise 5.2 of class 15; problem 10 on problem set 7), if the rank of the coherent sheaf $\pi_* \mathcal{O}_C$ is constant, then (as C' is reduced) $\pi_* \mathcal{O}_C$ is locally free. We'll show this by showing the rank at any closed point of C' is the same as the rank at the generic point.

The notion of "rank at a point" behaves well under base change, so we base change to the discrete valuation ring $\mathcal{O}_{C',p}$, where p is some closed point of C'. Then $\pi_* \mathcal{O}_C$ is a finitely generated module over a discrete valuation ring which is torsion-free. By the classification of finitely generated modules over a principal ideal domain, any finitely generate module over a principal ideal domain A is a direct sum of modules of the form $A/(d)$ for various $d \in A$. But if A is a discrete valuation ring, and $A/(d)$ is torsion-free, then $A/(d)$ is necessarily A (as for example all ideals of A are of the form 0 or a power of the maximal ideal). Thus we are done. □

Remark. If we are working with complex curves, this notion of degree is the same as the notion of the topological degree.

Suppose C is a projective curve, and $\mathcal{L}$ is an invertible sheaf. We will define $\deg \mathcal{L}$.

Let s be a non-zero rational section of $\mathcal{L}$. For any $p \in C$, recall the valuation of s at p ($\nu_p(s) \in \mathbb{Z}$). (Pick any local section t of $\mathcal{L}$ not vanishing at p. Then $s/t \in FF(C)$. $\nu_p(s) := \nu_p(s/t)$. We can show that this is well-defined.)

Define $\deg(\mathcal{L}, s)$ (where s is a non-zero rational section of $\mathcal{L}$) to be the number of zeros minus the number of poles, counted with appropriate multiplicity. (In other words, each point contributes the valuation at that point times the degree of the field extension.) We'll show that this is independent of s. (Note that we need the projective hypothesis: the sections x and 1 of the structure sheaf on $\mathbb{A}^1$ have different degrees.)

Notice that $\deg(\mathcal{L}, s)$ is additive under products: $\deg(\mathcal{L}, s)+\deg(\mathcal{M}, t) = \deg(\mathcal{L}\otimes\mathcal{M}, s\otimes t)$. Thus to show that $\deg(\mathcal{L}, s) = \deg(\mathcal{L}, t)$, we need to show that $\deg(\mathcal{O}_C, s/t) = 0$. Hence it suffices to show that $\deg(\mathcal{O}_C, u) = 0$ for a non-zero rational function u on C. Then u gives a rational map $C \dashrightarrow \mathbb{P}^1$. By our recent work (Proposition 2.1 above), this can be extended to a morphism $C \to \mathbb{P}^1$. The preimage of 0 is the number of 0's, and the preimage of ∞ is the number of ∞'s. But these are the same by our previous discussion of degree of a morphism! Finally, suppose $p \mapsto 0$. I claim that the valuation of u at p times the degree of the field extension is precisely the contribution of p to $u^{-1}(0)$. (A similar computation for ∞ will complete the proof of the desired result.) This is because the contribution of p to $u^{-1}(0)$ is precisely

$$\dim_k \mathcal{O}_{C,p}/(u) = \dim_k \mathcal{O}_{C,p}/\mathfrak{m}^{\nu_p(u)} = \nu_p(u) \dim_k \mathcal{O}_{C,p}/\mathfrak{m}.$$

□

We can define the degree of an invertible sheaf $\mathcal{L}$ on an integral *singular* projective curve C as follows: if $\nu : \tilde{C} \to C$ be the normalization, let $\deg_C \mathcal{L} := \deg_{\tilde{C}} \nu^*\mathcal{L}$. Notice that if s is a meromorphic section that has neither zeros nor poles at the singular points of C, then $\deg_C \mathcal{L}$ is still the number of zeros minus the number of poles (suitably counted), because the zeros and poles of $\nu^*\mathcal{L}$ are just the same as those of $\mathcal{L}$.

3.1. *Exercise.* Suppose $f : C \to C'$ is a degree d morphism of integral projective nonsingular curves, and $\mathcal{L}$ is an invertible sheaf on C'. Show that $\deg_C f^*\mathcal{L} = d \deg_{C'} \mathcal{L}$.

3.2. Degree of a Cartier divisor on a curve.

I said the following in class 30. (I've repeated this in the class 30 notes.)

Suppose D is an effective Cartier divisor on a projective curve, or a Cartier divisor on a projective nonsingular curve (over a field k). (I should really say: suppose D is a Cartier divisor on a projective curve, but I don't think I defined Cartier divisors in that generality.) Then define the *degree* of D (denoted $\deg D$) to be the degree of the corresponding invertible sheaf.

Exercise. If D is an effective Cartier divisor on a projective nonsingular curve, say $D = \sum n_i p_i$, prove that $\deg D = \sum n_i \deg p_i$, where $\deg p_i$ is the degree of the field extension of the residue field at p_i over k.

4. CECH COHOMOLOGY OF QUASICOHERENT SHEAVES

One idea behind the cohomology of quasicoherent sheaves is as follows. If $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$ is a short exact sequence of sheaves on X, we know that

$$0 \to \mathcal{F}(X) \to \mathcal{G}(X) \to \mathcal{H}(X).$$

In other words, $\Gamma(X, \cdot)$ is a left-exact functor. We dream that this is something called H^0, and that this sequence continues off to the right, giving a long exact sequence in cohomology. (In general, whenever we see a left-exact or right-exact functor, we should hope for this, and in most good cases our dreams are fulfilled. The machinery behind this is sometimes called *derived functor cohomology*, which we may discuss in the third quarter.)

We'll show that these cohomology groups exist. Before defining them explicitly, we first describe their important properties.

Suppose X is an R-scheme. Assume throughout that X is separated and quasicompact. Then for each quasicoherent sheaf $\mathcal{F}$ on X, we'll define R-modules $H^i(X, \mathcal{F})$. (In particular, if $R = k$, they are k-vector spaces.) First, $\boxed{H^0(X, \mathcal{F}) = \Gamma(X, \mathcal{F})}$. Each H^i will be a contravariant functor in the space X, and a $\boxed{\text{covariant functor in the sheaf } \mathcal{F}}$. The functor H^i behaves well under direct sums: $\boxed{H^i(X, \oplus_j \mathcal{F}_j) = \oplus_j H^i(X, \mathcal{F}_j)}$. (We will need infinite sums, not just finite sums.) If $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$ is a short exact sequence of quasicoherent sheaves on X, then we have a $\boxed{\text{long exact sequence}}$

$$\begin{aligned} 0 \to H^0(X, \mathcal{F}) \to H^0(X, \mathcal{G}) \to H^0(X, \mathcal{H}) \\ \to H^1(X, \mathcal{F}) \to H^1(X, \mathcal{G}) \to H^1(X, \mathcal{H}) \to \cdots. \end{aligned}$$

(The maps $H)(X, ?) \to H^i(X, ??)$ will be those coming from covariance; the *connecting homomorphisms* $H^i(x, \mathcal{H}) \to H^{i+1}(X, \mathcal{F})$ will have to be defined.) We'll see that if X can be covered by n affines, then $\boxed{H^i(X, \mathcal{F}) = 0}$ for $i \geq n$ for all $\mathcal{F}$, i. (In particular, all higher quasicoherent cohomology groups on affine schemes vanish.) If $X \hookrightarrow Y$ is a closed immersion, and $\mathcal{F}$ is a quasicoherent sheaf on X, then $\boxed{H^i(X, \mathcal{F}) = H^i(Y, f_* \mathcal{F})}$. (We'll care about this particularly in the case when $X \subset Y = \mathbb{P}^N_R$, which will let us reduce calculations on arbitrary projective R-schemes to calculations on $\mathbb{P}^n_R$.)

We will also identify the cohomology of all the invertible sheaves on $\mathbb{P}^n_R$:

4.1. *Proposition. —*

- $H^0(\mathbb{P}^n_R, \mathcal{O}_{\mathbb{P}^n_R}(m))$ *is a free* R*-module of rank* $\binom{n+m}{n}$ *if* $i = 0$ *and* $m \geq 0$*, and* 0 *otherwise.*
- $H^n(\mathbb{P}^n_R, \mathcal{O}_{\mathbb{P}^n_R}(m))$ *is a free* R*-module of rank* $\binom{-m-1}{-n-m-1}$ *if* $m \leq -n-1$*, and* 0 *otherwise.*
- $H^i(\mathbb{P}^n_R, \mathcal{O}_{\mathbb{P}^n_R}(m)) = 0$ *if* $0 < i < n$.

It is more helpful to say the following imprecise statement: $H^0(\mathbb{P}^n_R, \mathcal{O}_{\mathbb{P}^n_R}(m))$ should be interpreted as the homogeneous degree m polynomials in $x_0, \dots, x_n$ (with R-coefficients), and $H^n(\mathbb{P}^n_R, \mathcal{O}_{\mathbb{P}^n_R}(m))$ should be interpreted as the homogeneous degree m Laurent polynomials in $x_0, \dots, x_n$, where in each monomial, each x_i appears with degree at most -1.

We'll prove this next day.

Here are some features of this Proposition that I wish to point out, that will be the first appearances of things that we'll prove later.

- The cohomology of these bundles vanish above the dimension of the space *if* $R = k$; we'll generalize this for $\operatorname{Spec} R$, and even more, in before long.
- These cohomology groups are always finitely-generated R modules.
- The top cohomology group vanishes for $m > -n-1$. (This is a first appearance of "Kodaira vanishing".)
- The top cohomology group is "1-dimensional" for $m = -n-1$ if $R = k$. This is the first appearance of a dualizing sheaf.
- We have a natural duality $H^i(X, \mathcal{O}(m)) \times H^{n-i}(X, \mathcal{O}(-n-1-m)) \to H^n(X, \mathcal{O}(-n-1))$. This is the first appearance of Serre duality.

I'd like to use all these properties to prove things, so you'll see how handy they are. We'll worry later about defining cohomology, and proving these properties.

When we discussed global sections, we worked hard to show that for any coherent sheaf $\mathcal{F}$ on $\mathbb{P}^n_R$ we could find a surjection $\mathcal{O}(m)^{\oplus j} \to \mathcal{F}$, which yields the exact sequence

$$0 \to \mathcal{G} \to \mathcal{O}(m)^{\oplus j} \to \mathcal{F} \to 0 \tag{1}$$

for some coherent sheaf $\mathcal{G}$. We can use this to prove the following.

4.2. *Theorem. — (i) For any coherent sheaf $\mathcal{F}$ on a projective R-scheme where R is Noetherian, $h^i(X, \mathcal{F})$ is a finitely generated R-module. (ii) (Serre vanishing) Furthermore, for $m \gg 0$, $H^i(X, \mathcal{F}(m)) = 0$ for all i, even without Noetherian hypotheses.*

Proof. Because cohomology of a closed scheme can be computed on the ambient space, we may reduce to the case $X = \mathbb{P}^n_R$.

(i) Consider the long exact sequence:

$$0 \longrightarrow H^0(\mathbb{P}^n_R, \mathcal{G}) \longrightarrow H^0(\mathbb{P}^n_R, \mathcal{O}(m)^{\oplus j}) \longrightarrow H^0(\mathbb{P}^n_R, \mathcal{F}) \longrightarrow$$

$$H^1(\mathbb{P}^n_R, \mathcal{G}) \longrightarrow H^1(\mathbb{P}^n_R, \mathcal{O}(m)^{\oplus j}) \longrightarrow H^1(\mathbb{P}^n_R, \mathcal{F}) \longrightarrow \cdots$$

$$\cdots \longrightarrow H^{n-1}(\mathbb{P}^n_R, \mathcal{G}) \longrightarrow H^{n-1}(\mathbb{P}^n_R, \mathcal{O}(m)^{\oplus j}) \longrightarrow H^{n-1}(\mathbb{P}^n_R, \mathcal{F}) \longrightarrow$$

$$H^n(\mathbb{P}^n_R, \mathcal{G}) \longrightarrow H^n(\mathbb{P}^n_R, \mathcal{O}(m)^{\oplus j}) \longrightarrow H^n(\mathbb{P}^n_R, \mathcal{F}) \longrightarrow 0$$

The exact sequence ends here because $\mathbb{P}^n_R$ is covered by $n+1$ affines. Then $H^n(\mathbb{P}^n_R, \mathcal{O}(m)^{\oplus j})$ is finitely generated by Proposition 4.1, hence $H^n(\mathbb{P}^n_R, \mathcal{F})$ is finitely generated for all coherent sheaves $\mathcal{F}$. Hence in particular, $H^n(\mathbb{P}^n_R, \mathcal{G})$ is finitely generated. As $H^{n-1}(\mathbb{P}^n_R, \mathcal{O}(m)^{\oplus j})$ is finitely generated, and $H^n(\mathbb{P}^n_R, \mathcal{G})$ is too, we have that $H^{n-1}(\mathbb{P}^n_R, \mathcal{F})$ is finitely generated for all coherent sheaves $\mathcal{F}$. We continue inductively downwards.

(ii) Twist (4.1) by $\mathcal{O}(N)$ for $N \gg 0$. Then $H^n(\mathbb{P}^n_R, \mathcal{O}(m+N)^{\oplus j}) = 0$, so $H^n(\mathbb{P}^n_R, \mathcal{F}(N)) = 0$. Translation: for any coherent sheaf, its top cohomology vanishes once you twist by $\mathcal{O}(N)$ for N sufficiently large. Hence this is true for $\mathcal{G}$ as well. Hence from the long exact sequence, $H^{n-1}(\mathbb{P}^n_R, \mathcal{F}(N)) = 0$ for $N \gg 0$. As in (i), we induct downwards, until we get that $H^1(\mathbb{P}^n_R, \mathcal{F}(N)) = 0$. (The induction proceeds no further, as it is *not* true that $H^0(\mathbb{P}^n_R, \mathcal{O}(m+N)^{\oplus j}) = 0$ for large N — quite the opposite. □

Exercise for those who like working with non-Noetherian rings: Prove part (i) in the above result without the Noetherian hypotheses, assuming only that R is a coherent R-module (it is "coherent over itself"). (Hint: induct downwards as before. The order is as follows: $H^n(\mathbb{P}^n_R, \mathcal{F})$ finitely generated, $H^n(\mathbb{P}^n_R, \mathcal{G})$ finitely generated, $H^n(\mathbb{P}^n_R, \mathcal{F})$ coherent, $H^n(\mathbb{P}^n_R, \mathcal{G})$ coherent, $H^{n-1}(\mathbb{P}^n_R, \mathcal{F})$ finitely generated, $H^{n-1}(\mathbb{P}^n_R, \mathcal{G})$ finitely generated, etc.)

In particular, we have proved the following, that we would have cared about even before we knew about cohomology.

4.3. *Corollary. — Any projective k-scheme has a finite-dimensional space of global sections. More generally, if $\mathcal{F}$ is a coherent sheaf on a projective R-scheme, then $h^0(X, \mathcal{F})$ is a finitely generated R-module.*

This is true more generally for proper k-schemes, not just projective k-schemes, but I won't give the argument here.

Here is another a priori interesting consequence:

4.4. *Corollary. — If $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$ is an exact sequence of coherent sheaves on projective X with $\mathcal{F}$ coherent, then for $n \gg 0$, $0 \to H^0(X, \mathcal{F}(n)) \to H^0(X, \mathcal{G}(n)) \to H^0(X, \mathcal{H}(n)) \to 0$ is also exact.*

(Proof: for $n \gg 0$, $H^1(X, \mathcal{F}(n)) = 0$.)

This result can also be shown directly, without the use of cohomology.

5. Proving the things you need to know

As you read this, you should go back and check off all the facts, to make sure that I've shown all that I've promised.

5.1. Cech cohomology. Works nicely here. In general: take finer and finer covers. Here we take a single cover.

Suppose X is quasicompact and separated, e.g. X is quasiprojective over A. In particular, X may be covered by a finite number of affine open sets, and the intersection of any two affine open sets is also an affine open set; these are the properties we will use. Suppose $\mathcal{F}$ is a quasicoherent sheaf, and $\mathcal{U} = \{U_i\}_{i=1}^n$ is a *finite* set of affine open sets of X whose union is U. For $I \subset \{1, \dots, n\}$ define $U_i = \cap_{i \in I} U_i$. It is affine by the separated hypothesis. **Define** $H^i_{\mathcal{U}}(U, \mathcal{F})$ to be the ith cohomology group of the complex

$$0 \to \bigoplus_{\substack{|I|=1 \\ I \subset \{1,\dots,n\}}} \mathcal{F}(U_I) \to \cdots \to \bigoplus_{\substack{|I|=i \\ I \subset \{1,\dots,n\}}} \mathcal{F}(U_I) \to \bigoplus_{\substack{|I|=i+1 \\ I \subset \{1,\dots,n\}}} \mathcal{F}(U_I) \to \cdots. \tag{2}$$

Note that if X is an R-scheme, then $H^i_{\mathcal{U}}(X, \mathcal{F})$ is an R-module. Also $H^0_{\mathcal{U}}(X, \mathcal{F}) = \Gamma(X, \mathcal{F})$.

5.2. *Exercise.* Suppose $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ is a short exact sequence of sheaves on a topological space, and $\mathcal{U}$ is an open cover such that on any intersection the sections of $\mathcal{F}_2$ surject onto $\mathcal{F}_3$. Show that we get a long exact sequence of cohomology. (Note that this applies in our case!)

I ended by stating the following result, which we will prove next day.

5.3. *Theorem/Definition. — Recall that X is quasicompact and separated. $H^i_{\mathcal{U}}(U, \mathcal{F})$ is independent of the choice of (finite) cover $\{U_i\}$. More precisely,*

- (*) *for all k, for any two covers $\{U_i\} \subset \{V_i\}$ of size at most k, the maps $H^i_{\{V_i\}}(X, \mathcal{F}) \to H^i_{\{U_i\}}(X, \mathcal{F})$ induced by the natural maps of complex* (2) *are isomorphisms.*

Define the Cech cohomology group $H^i(X, \mathcal{F})$ to be this group.

I needn't have stated in terms of some k; I've stated it in this way so I can prove it by induction.

(For experts: we'll get natural quasiisomorphisms of Cech complexes for various $\mathcal{U}$.)

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FOUNDATIONS OF ALGEBRAIC GEOMETRY CLASS 30

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CONTENTS

Last day: More curves. Cohomology take 1.

Today: Cohomology continued. Hilbert functions and Hilbert polynomials.

1. LEFT-OVER: DEGREE OF A CARTIER DIVISOR ON A PROJECTIVE CURVE

As always, there is something small that I should have said last day. Suppose D is an effective Cartier divisor on a projective curve, or a Cartier divisor on a projective non-singular curve (over a field k). (I should really say: suppose D is a Cartier divisor on a projective curve, but I don't think I defined Cartier divisors in that generality.) Then define the *degree* of D (denoted $\deg D$) to be the degree of the corresponding invertible sheaf.

Exercise. If D is an effective Cartier divisor on a projective nonsingular curve, say $D = \sum n_i p_i$, prove that $\deg D = \sum n_i \deg p_i$, where $\deg p_i$ is the degree of the field extension of the residue field at p_i over k.

(This is also now in the class 29 notes, where it belongs.)

2. COHOMOLOGY CONTINUED

Last day, I gave you lots of facts that we wanted cohomology to satisfy. Suppose X is a separated and quasicompact R-scheme. In particular, X can be covered by a finite number of affine open sets, and the intersection of any two affine open sets is another affine open

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set. We are going to define $H^i(X, \mathcal{F})$ for any quasicoherent sheaf $\mathcal{F}$ on X, that satisfies the following properties.

- $H^0(X, \mathcal{F}) = \Gamma(X, \mathcal{F})$
- H^i is a contravariant functor in X and a covariant functor in $\mathcal{F}$.
- $H^i(X, \oplus_j \mathcal{F}_j) = \oplus_j H^i(X, \mathcal{F}_j)$: cohomology commutes with arbitrary direct sums.
- long exact sequences
- $H^i(\operatorname{Spec} R, \mathcal{F}) = 0$.
- If $X \hookrightarrow Y$ is a closed immersion, and $\mathcal{F}$ is a quasicoherent sheaf on X, then $H^i(X, \mathcal{F}) = H^i(Y, f_*\mathcal{F})$.
- $H^i(\mathbb{P}^n_R, \mathcal{O}_{\mathbb{P}^n_R}(r))$ is something nice (we described it in a statement last day that we will prove today)

Last day, we defined these cohomology groups given the additional data of an affine open cover $\mathcal{U}$; I used the notation $H^i_{\mathcal{U}}(X, \mathcal{F})$. We'll start today by showing that this is independent of $\mathcal{U}$.

2.1. *Theorem/Definition. — Recall that* X *is quasicompact and separated.* $H^i_{\mathcal{U}}(U, \mathcal{F})$ *is independent of the choice of (finite) cover* $\{U_i\}$. *More precisely,*

(*) *for all* k, *for any two covers* $\{U_i\} \subset \{V_i\}$ *of size at most* k, *the maps* $H^i_{\{V_i\}}(X, \mathcal{F}) \to H^i_{\{U_i\}}(X, \mathcal{F})$ *induced by the natural maps of complex* (1) *are isomorphisms.*

Define the Cech cohomology group $H^i(X, \mathcal{F})$ *to be this group.*

$$0 \to \bigoplus_{\substack{|I|=1 \\ I \subset \{1,\ldots,n\}}} \mathcal{F}(U_I) \to \cdots \to \bigoplus_{\substack{|I|=i \\ I \subset \{1,\ldots,n\}}} \mathcal{F}(U_I) \to \bigoplus_{\substack{|I|=i+1 \\ I \subset \{1,\ldots,n\}}} \mathcal{F}(U_I) \to \cdots . \tag{1}$$

I needn't have stated in terms of some k; I've stated it in this way so I can prove it by induction.

(For experts: we'll get natural quasiisomorphisms of Cech complexes for various $\mathcal{U}$.)

Proof. We prove this by induction on k. The base case is trivial. We need only prove the result for $\{U_i\}_{i=1}^n \subset \{U_i\}_{i=0}^n$, where the case $k = n$ is known. Consider the exact sequence of complexes

$$
\begin{array}{ccccccc}
 & 0 & & 0 & & 0 & \\
 & \downarrow & & \downarrow & & \downarrow & \\
\cdots \longrightarrow & \bigoplus_{\substack{|I|=i-1\\ 0\in I\subset\{0,\dots,n\}}} \mathcal{F}(U_I) & \longrightarrow & \bigoplus_{\substack{|I|=i\\ 0\in I\subset\{0,\dots,n\}}} \mathcal{F}(U_I) & \longrightarrow & \bigoplus_{\substack{|I|=i+1\\ 0\in I\subset\{0,\dots,n\}}} \mathcal{F}(U_I) & \longrightarrow \cdots \\
 & \downarrow & & \downarrow & & \downarrow & \\
\cdots \longrightarrow & \bigoplus_{\substack{|I|=i-1\\ I\subset\{0,\dots,n\}}} \mathcal{F}(U_I) & \longrightarrow & \bigoplus_{\substack{|I|=i\\ I\subset\{0,\dots,n\}}} \mathcal{F}(U_I) & \longrightarrow & \bigoplus_{\substack{|I|=i+1\\ I\subset\{0,\dots,n\}}} \mathcal{F}(U_I) & \longrightarrow \cdots \\
 & \downarrow & & \downarrow & & \downarrow & \\
\cdots \longrightarrow & \bigoplus_{\substack{|I|=i-1\\ I\subset\{1,\dots,n\}}} \mathcal{F}(U_I) & \longrightarrow & \bigoplus_{\substack{|I|=i\\ I\subset\{1,\dots,n\}}} \mathcal{F}(U_I) & \longrightarrow & \bigoplus_{\substack{|I|=i+1\\ I\subset\{1,\dots,n\}}} \mathcal{F}(U_I) & \longrightarrow \cdots \\
 & \downarrow & & \downarrow & & \downarrow & \\
 & 0 & & 0 & & 0 &
\end{array}
$$

We get a long exact sequence of cohomology from this. Thus by Exercise 5.2 of last day, we wish to show that the top row is exact. But the ith cohomology of the top row is precisely $H^i_{\{U_i\cap U_0\}_{i>0}}(U_i, \mathcal{F})$ except at step 0, where we get 0 (because the complex starts off $0 \to \mathcal{F}(U_0) \to \oplus_{j=1}^n \mathcal{F}(U_0 \cap U_j)$). So we just need to show that higher Cech groups of affine schemes are 0. Hence we are done by the following result. □

2.2. *Theorem. — The higher Cech cohomology $H^i_{\mathcal{U}}(X, \mathcal{F})$ of an affine R-scheme X vanishes (for any affine cover $\mathcal{U}$, $i > 0$, and quasicoherent $\mathcal{F}$).*

Serre describes this as a partition of unity argument.

A spectral sequence argument can make quick work of this, but I'd like to avoid introducing spectral sequences until I have to.

Proof. We want to show that the "extended" complex (where you tack on global sections to the front) has no cohomology, i.e. that

$$0 \to \mathcal{F}(X) \to \oplus_{|I|=1}\mathcal{F}(U_I) \to \oplus_{|I|=2}\mathcal{F}(U_I) \to \cdots \qquad (2)$$

is exact. We do this with a trick.

Suppose first that some U_i (say U_0) is X. Then the complex can be described as the middle row of the following short exact sequence of complexes

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & 0 & \longrightarrow & \oplus_{|I|=1,0\in I}\mathcal{F}(U_I) & \longrightarrow & \oplus_{|I|=2,0\in I}\mathcal{F}(U_I) & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{F}(X) & \longrightarrow & \oplus_{|I|=1}\mathcal{F}(U_I) & \longrightarrow & \oplus_{|I|=2}\mathcal{F}(U_I) & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{F}(X) & \longrightarrow & \oplus_{|I|=1,0\notin I}\mathcal{F}(U_I) & \longrightarrow & \oplus_{|I|=2,0\notin I}\mathcal{F}(U_I) & \longrightarrow & \cdots
\end{array}
$$

The top row is the same as the bottom row, slid over by 1. The corresponding long exact sequence of cohomology shows that the central row has vanishing cohomology. (Topological experts will recognize a *mapping cone* in the above construction.)

We next prove the general case by sleight of hand. Say $X = \operatorname{Spec} S$. We wish to show that the complex of R-modules (2) is exact. It is also a complex of S-modules, so we wish to show that the complex of S-modules (2) is exact. To show that it is exact, it suffices to show that for a cover of $\operatorname{Spec} S$ by distinguished opens $D(f_i)$ ($1 \leq i \leq s$) (i.e. $(f_1, \ldots, f_s) = 1$ in S) the complex is exact. (Translation: exactness of a sequence of sheaves may be checked locally.) We choose a cover so that each $D(f_i)$ is contained in some $U_j = \operatorname{Spec} R_j$. Consider the complex localized at f_i. As

$$\Gamma(\operatorname{Spec} R, \mathcal{F})_f = \Gamma(\operatorname{Spec}(R_j)_f, \mathcal{F})$$

(as this is one of the definitions of a quasicoherent sheaf), as $U_j \cap D(f_i) = D(f_i)$, we are in the situation where one of the U_i's is X, so we are done. □

2.3. *Exercise.* Suppose $V \subset U$ are open subsets of X. Show that we have restriction morphisms $H^i(U, \mathcal{F}) \to H^i(V, \mathcal{F})$ (if U and V are quasicompact, and U hence V is separated). Show that restrictions commute. Hence if X is a Noetherian space, $H^i(;\mathcal{F})$ this is a contravariant functor from the category $\mathrm{Top}(X)$ to abelian groups. (For experts: this means that it is a presheaf. But this is not a good way to think about it, as its sheafification is 0, as it vanishes on the affine base.) The same argument will show more generally that for any map $f : X \to Y$, there exist natural maps $H^i(X, \mathcal{F}) \to H^i(X, f^*\mathcal{F})$; I should have asked this instead.

2.4. *Exercise.* Show that if $\mathcal{F} \to \mathcal{G}$ is a morphism of quasicoherent sheaves on separated and quasicompact X then we have natural maps $H^i(X, \mathcal{F}) \to H^i(X, \mathcal{G})$. Hence $H^i(X, \cdot)$ is a covariant functor from quasicoherent sheaves on X to abelian groups (or even R-modules).

In particular, we get the following facts.

1. If $X \hookrightarrow Y$ is a closed subscheme then $H^i(X, \mathcal{F}) = H^i(Y, f_*\mathcal{F})$, as promised at start of our discussion on cohomology.

2. Also, if X can be covered by n affine open sets, then $H^i(X, \mathcal{F}) = 0$ for all quasicoherent $\mathcal{F}$, and $i \geq n$. In particular, $H^i(\operatorname{Spec} R, \mathcal{F}) = 0$ for $i > 0$.

3. Cohomology behaves well for arbitrary direct sums of quasicoherent sheaves.

2.5. Dimensional vanishing for projective k-schemes.

2.6. *Theorem. — Suppose X is a projective k-scheme, and $\mathcal{F}$ is a quasicoherent sheaf on X. Then $H^i(X, \mathcal{F}) = 0$ for $i > \dim X$.*

In other words, cohomology vanishes above the dimension of X. We will later show that this is true when X is a *quasiprojective* k-scheme.

Proof. Suppose $X \hookrightarrow \mathbb{P}^N$, and let $n = \dim X$. We show that X may be covered by n affine open sets. Long ago, we had an exercise saying that we could find n Cartier divisors on $\mathbb{P}^N$ such that their complements $U_0, \dots, U_n$ covered X. (We did this as follows. Lemma: Suppose $Y \hookrightarrow \mathbb{P}^N$ is a projective scheme. Then Y is Noetherian, and hence has a finite number of components. We can find a hypersurface H containing none of their associated points. Then H contains no component of Y, the dimension of $H \cap Y$ is strictly smaller than Y, and if $\dim Y = 0$, then $H \cap Y = \emptyset$.) Then U_i is affine, so $U_i \cap X$ is affine, and thus we have covered X with n affine open sets. □

Remark. We actually *need* n affine open sets to cover X, but I don't see an easy way to prove it. One way of proving it is by showing that the complement of an affine set is always pure codimension 1.

3. Cohomology of line bundles on projective space

I'll now pay off that last IOU.

3.1. *Proposition. —*

- $H^0(\mathbb{P}^n_R, \mathcal{O}_{\mathbb{P}^n_R}(m))$ *is a free R-module of rank $\binom{n+m}{n}$ if $i = 0$ and $m \geq 0$, and 0 otherwise.*
- $H^n(\mathbb{P}^n_R, \mathcal{O}_{\mathbb{P}^n_R}(m))$ *is a free R-module of rank $\binom{-m-1}{-n-m-1}$ if $m \leq -n-1$, and 0 otherwise.*
- $H^i(\mathbb{P}^n_R, \mathcal{O}_{\mathbb{P}^n_R}(m)) = 0$ *if $0 < i < n$.*

It is more helpful to say the following imprecise statement: $H^0(\mathbb{P}^n_R, \mathcal{O}_{\mathbb{P}^n_R}(m))$ should be interpreted as the homogeneous degree m polynomials in $x_0, \dots, x_n$ (with R-coefficients), and $H^n(\mathbb{P}^n_R, \mathcal{O}_{\mathbb{P}^n_R}(m))$ should be interpreted as the homogeneous degree m Laurent polynomials in $x_0, \dots, x_n$, where in each monomial, each x_i appears with degree at most -1.

Proof. The H^0 statement was an (important) exercise last quarter.

Rather than consider $\mathcal{O}(m)$ for various m, we consider them all at once, by considering $\mathcal{F} = \oplus_m \mathcal{O}(m)$.

Of course we take the standard cover $U_0 = D(x_0), \ldots, U_n = D(x_n)$ of $\mathbb{P}^n_R$. Notice that if $I \subset \{1, \ldots, n\}$, then $\mathcal{F}(U_I)$ corresponds to the Laurent monomials where each x_i for $i \notin I$ appears with non-negative degree.

We consider the H^n statement. $H^n(\mathbb{P}^n_R, \mathcal{F})$ is the cokernel of the following surjection

$$\oplus_{i=0}^n \mathcal{F}(U_{\{1,\ldots,n\}-\{i\}}) \to \mathcal{F}_{U_{\{1,\ldots,n\}}}$$

i.e.

$$\oplus_{i=0}^n R[x_0, \ldots, x_n, x_0^{-1}, \ldots, \widehat{x_i^{-1}}, \ldots x_n^{-1}] \to R[x_0, \ldots, x_n, x_0^{-1}, \ldots, x_n^{-1}].$$

This cokernel is precisely as described.

We last consider the H^i statement ($0 < i < n$). We prove this by induction on n. The cases $n = 0$ and 1 are trivial. Consider the exact sequence of quasicoherent sheaves:

$$0 \longrightarrow \mathcal{F} \xrightarrow{\times x_n} \mathcal{F} \longrightarrow \mathcal{F}' \longrightarrow 0$$

where $\mathcal{F}'$ is analogous sheaf on the hyperplane $x_n = 0$ (isomorphic to $\mathbb{P}^{n-1}_R$). (This exact sequence is just the direct sum over all m of the exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n_R}(m-1) \xrightarrow{\times x_n} \mathcal{O}_{\mathbb{P}^n_R}(m) \longrightarrow \mathcal{O}_{\mathbb{P}^{n-1}_R}(m) \longrightarrow 0 ,$$

which in turn is obtained by twisting the closed subscheme exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n_R}(m-1) \xrightarrow{\times x_n} \mathcal{O}_{\mathbb{P}^n_R}(m) \longrightarrow \mathcal{O}_{\mathbb{P}^{n-1}_R}(m) \longrightarrow 0$$

by $\mathcal{O}_{\mathbb{P}^n_R}(m)$.)

The long exact sequence in cohomology gives us:

$$0 \longrightarrow H^0(\mathbb{P}^n_R, \mathcal{F}) \xrightarrow{\times x_n} H^0(\mathbb{P}^n_R, \mathcal{F}) \longrightarrow H^0(\mathbb{P}^{n-1}_R, \mathcal{F}') \quad .$$

$$\longrightarrow H^1(\mathbb{P}^n_R, \mathcal{F}) \xrightarrow{\times x_n} H^1(\mathbb{P}^n_R, \mathcal{F}) \longrightarrow H^1(\mathbb{P}^{n-1}_R, \mathcal{F}')$$

$$\cdots \longrightarrow H^{n-1}(\mathbb{P}^n_R, \mathcal{F}) \xrightarrow{\times x_n} H^{n-1}(\mathbb{P}^n_R, \mathcal{F}) \longrightarrow H^{n-1}(\mathbb{P}^{n-1}_R, \mathcal{F}')$$

$$\longrightarrow H^n(\mathbb{P}^n_R, \mathcal{F}) \xrightarrow{\times x_n} H^n(\mathbb{P}^n_R, \mathcal{F}) \longrightarrow 0$$

We will now show that this gives an isomorphism

$$\boxed{\times x_n : H^i(\mathbb{P}^n_R, \mathcal{F}) \to H^i(\mathbb{P}^n_R, \mathcal{F})} \tag{3}$$

for $0 < i < n$. The inductive hypothesis gives us this except for $i = 1$ and $i = n-1$, where we have to pay a bit more attention. For the first, note that $H^0(\mathbb{P}^n_R, \mathcal{F}) \longrightarrow H^0(\mathbb{P}^{n-1}_R, \mathcal{F}')$ is surjective: this map corresponds to taking the set of all polynomials in $x_0, \ldots, x_n$, and setting $x_n = 0$. The last is slightly more subtle: $H^{n-1}(\mathbb{P}^{n-1}_R, \mathcal{F}') \to H^n(\mathbb{P}^n_R, \mathcal{F})$ is injective, and corresponds to taking a Laurent polynomial in $x_0, \dots, x_{n-1}$ (where in each monomial, each x_i appears with degree at most -1) and multiplying by x_n^{-1}, which indeed describes the kernel of $H^n(\mathbb{P}^n_R, \mathcal{F}) \xrightarrow{\times x_n} H^n(\mathbb{P}^n_R, \mathcal{F})$. (This is a worthwhile calculation! See the exercise after the end of this proof.) We have thus established (3) above.

We will now show that the localization $H^i(\mathbb{P}^n_R, \mathcal{F})_{x_n} = 0$. (Here's what we mean by localization. Notice $H^i(\mathbb{P}^n_R, \mathcal{F})$ is naturally a module over $R[x_0, \dots, x_n]$ — we know how to multiply by elements of R, and by (3) we know how to multiply by x_i. Then we localize this at x_n to get an $R[x_0, \dots, x_n]_{x_n}$-module.) This means that each element $\alpha \in H^i(\mathbb{P}^n_R, \mathcal{F})$ is killed by some power of x_i. But by (3), this means that $\alpha = 0$, concluding the proof of the theorem.

Consider the Cech complex computing $H^i(\mathbb{P}^n_R, \mathcal{F})$. Localize it at x_n. Localization and cohomology commute (basically because localization commutes with operations of taking quotients, images, etc.), so the cohomology of the new complex is $H^i(\mathbb{P}^n_R, \mathcal{F})_{x_n}$. But this complex computes the cohomology of $\mathcal{F}_{x_n}$ on the affine scheme U_n, and the higher cohomology of *any* quasicoherent sheaf on an affine scheme vanishes (by Theorem 2.2 which we've just proved — in fact we used the same trick there), so $H^i(\mathbb{P}^n_R, \mathcal{F})_{x_n} = 0$ as desired. $\square$

3.2. *Exercise.* Verify that $H^{n-1}(\mathbb{P}^{n-1}_R, \mathcal{F}') \to H^n(\mathbb{P}^n_R, \mathcal{F})$ is injective (likely by verifying that it is the map on Laurent monomials we claimed above).

4. Application of cohomology: Hilbert polynomials and Hilbert functions; degrees

We've already seen some powerful uses of this machinery, to prove things about spaces of global sections, and to prove Serre vanishing. We'll now see some classical constructions come out very quickly and cheaply.

In this section, we will work over a field k. Define $h^i(X, \mathcal{F}) := \dim_k H^i(X, \mathcal{F})$.

Suppose $\mathcal{F}$ is a coherent sheaf on a projective k-scheme X. Define the *Euler characteristic*

$$\chi(X, \mathcal{F}) = \sum_{i=0}^{\dim X} (-1)^i h^i(X, \mathcal{F}).$$

We will see repeatedly here and later that while Euler characteristics behave better than individual cohomology groups. As one sign, notice that for fixed n, and $m \geq 0$,

$$h^0(\mathbb{P}^n_k, \mathcal{O}(m)) = \binom{n+m}{m} = \frac{(m+1)(m+2)\cdots(m+n)}{n!}.$$

Notice that the expression on the right is a polynomial in m of degree n. (For later reference, I want to point out that the leading term is $m^n/n!$.) But it is not true that

$$h^0(\mathbb{P}^n_k, \mathcal{O}(m)) = \frac{(m+1)(m+2)\cdots(m+n)}{n!}$$

for *all* m — it breaks down for $m \leq -n-1$. Still, you can check that

$$\chi(\mathbb{P}^n_k, \mathcal{O}(m)) = \frac{(m+1)(m+2)\cdots(m+n)}{n!}.$$

So one lesson is this: if one cohomology group (usual the top or bottom) behaves well in a certain range, and then messes up, likely it is because (i) it is actually the Euler characteristic which is behaving well *always*, and (ii) the other cohomology groups vanish in that range.

In fact, we will see that it is often hard to calculate cohomology groups (even h^0), but it is often easier calculating Euler characteristics. So one important way of getting a hold of cohomology groups is by computing the Euler characteristics, and then showing that all the *other* cohomology groups vanish. Hence the ubiquity and importance of *vanishing theorems*. (A vanishing theorem usually states that a certain cohomology group vanishes under certain conditions.)

The following exercise already shows that Euler characteristic behaves well.

4.1. *Exercise.* Show that Euler characteristic is additive in exact sequences. In other words, if $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$ is an exact sequence of coherent sheaves on X, then $\chi(X, \mathcal{G}) = \chi(X, \mathcal{F}) + \chi(X, \mathcal{H})$. (Hint: consider the long exact sequence in cohomology.) More generally, if

$$0 \to \mathcal{F}_1 \to \cdots \to \mathcal{F}_n \to 0$$

is an exact sequence of sheaves, show that

$$\sum_{i=1}^{n} (-1)^i \chi(X, \mathcal{F}_i) = 0.$$

4.2. *Exercise.* Prove the *Riemann-Roch theorem* for line bundles on a nonsingular projective curve C over k: suppose $\mathcal{L}$ is an invertible sheaf on C. Show that $\chi(\mathcal{L}) = \deg \mathcal{L} + \chi(C, \mathcal{O}_C)$. (Possible hint: Write $\mathcal{L}$ as the difference of two effective Cartier divisors, $\mathcal{L} \cong \mathcal{O}(Z - P)$ ("zeros" minus "poles"). Describe two exact sequences $0 \to \mathcal{O}_C(-P) \to \mathcal{O}_C \to \mathcal{O}_P \to 0$ and $0 \to \mathcal{L}(-Z) \to \mathcal{L} \to \mathcal{O}_Z \otimes \mathcal{L} \to 0$, where $\mathcal{L}(-Z) \cong \mathcal{O}_C(P)$.)

If $\mathcal{F}$ is a coherent sheaf on X, define the *Hilbert function of* $\mathcal{F}$:

$$h_{\mathcal{F}}(n) := h^0(X, \mathcal{F}(n)).$$

The *Hilbert function of* X is the Hilbert function of the structure sheaf. The ancients were aware that the Hilbert function is "eventually polynomial", i.e. for large enough n, it agrees with some polynomial, called the *Hilbert polynomial* (and denoted $p_{\mathcal{F}}(n)$ or $p_X(n)$). In modern language, we expect that this is because the Euler characteristic should be a polynomial, and that for $n \gg 0$, the higher cohomology vanishes. This is indeed the case, as we now verify.

I ended by stating the following, which we will prove next day.

4.3. *Claim.* — *For* $n \gg 0$, $h^0(X, \mathcal{F}(n))$ *is a polynomial of degree equal to the dimension of the support of* $\mathcal{F}$. *In particular,* $h^0(X, \mathcal{O}_X(n))$ *is "eventually polynomial" with degree* = $\dim X$.

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FOUNDATIONS OF ALGEBRAIC GEOMETRY CLASS 31

RAVI VAKIL

CONTENTS

Last day: Cohomology continued. Beginning of Hilbert functions

Today: Hilbert polynomials and Hilbert functions. Higher direct image sheaves.

1. APPLICATION OF COHOMOLOGY: HILBERT POLYNOMIALS AND HILBERT FUNCTIONS; DEGREES

We're in the process of seeing applications of cohomology. In this section, we will work over a field k. We defined $h^i(X, \mathcal{F}) := \dim_k H^i(X, \mathcal{F})$. If $\mathcal{F}$ is a coherent sheaf on a projective k-scheme X, we defined the *Euler characteristic*

$$\chi(X, \mathcal{F}) = \sum_{i=0}^{\dim X} (-1)^i h^i(X, \mathcal{F}).$$

We will see repeatedly here and later that Euler characteristics behave better than individual cohomology groups.

If $\mathcal{F}$ is a coherent sheaf on X, define the *Hilbert function of* $\mathcal{F}$:

$$h_{\mathcal{F}}(m) := h^0(X, \mathcal{F}(m)).$$

The *Hilbert function of* X is the Hilbert function of the structure sheaf $\mathcal{O}_X$. The ancients were aware that the Hilbert function is "eventually polynomial", i.e. for large enough n, it agrees with some polynomial, called the *Hilbert polynomial* (and denoted $p_{\mathcal{F}}(m)$ or $p_X(m)$). In modern language, we expect that this is because the Euler characteristic should be a polynomial, and that for $m \gg 0$, the higher cohomology vanishes. This is indeed the case, as we now verify.

Date: Tuesday, February 14, 2006. Tiny update April 9.

1.1. *Theorem. — If $\mathcal{F}$ is a coherent sheaf on a projective k-scheme $X \hookrightarrow \mathbb{P}^n_k$, for $m \gg 0$, $h^0(X, \mathcal{F}(m))$ is a polynomial of degree equal to the dimension of the support of $\mathcal{F}$. In particular, for $m \gg 0$, $h^0(X, \mathcal{O}_X(m))$ is polynomial with degree* = $\dim X$.

(Here $\mathcal{O}_X(m)$ is the restriction or pullback of $\mathcal{O}_{\mathbb{P}^n_k}(1)$.)

I realize now that I will use the notion of associated primes of a *module*, not just of a ring. I think I only discussed associated primes of a ring last quarter, because I had hoped not to need this slightly more general case. Now I really don't need it, and if you want to ignore this issue, you can just prove the second half of the theorem, which is all we will use anyway. But the argument carries through with no change, so please follow along if you can.

Proof. For $m \gg 0$, $h^i(X, \mathcal{F}(m)) = 0$ by Serre vanishing (class 29 Theorem 4.2(ii)), so instead we will prove that for *all* m, $\chi(X, \mathcal{F}(m))$ is a polynomial of degree equal to the dimension of the support of $\mathcal{F}$. Define $p_{\mathcal{F}}(m) = \chi(X, \mathcal{F}(m))$; we'll show that $p_{\mathcal{F}}(m)$ is a polynomial of the desired degree.

Our approach will be a little weird. We'll have two steps, and they will be very similar. If you can streamline, please let me know.

Step 1. We first show that for all n, if $\mathcal{F}$ is scheme-theoretically supported a linear subspace of dimension k (i.e. $\mathcal{F}$ is the pushforward of a coherent sheaf on some linear subspace of dimension k), then $p_{\mathcal{F}}(m)$ is a polynomial of degree at most k. (In particular, for any coherent $\mathcal{F}$, $p_{\mathcal{F}}(m)$ is a polynomial of degree at most n.)

We prove this by induction on the dimension of the support. I'll leave the base case $k = 0$ (or better yet, $k = -1$) to you (*exercise*). Suppose now that X is supported in a linear space Λ of dimension k, and we know the result for all $k' < k$. Then let $x = 0$ be a hyperplane not containing Λ, so $\Lambda' = \dim(x = 0) \cap \Lambda = k - 1$. Then we have an exact sequence

(1)
$$0 \longrightarrow \mathcal{K} \longrightarrow \mathcal{F} \xrightarrow{\times x} \mathcal{F}(1) \longrightarrow \mathcal{K}' \longrightarrow 0$$

where $\mathcal{K}$ (resp. $\mathcal{K}'$) is the kernel (resp. cokernel) of the map $\times x$. Notice that $\mathcal{K}$ and $\mathcal{K}'$ are both supported on Λ'. (This corresponds to an algebraic fact: over an affine open $\operatorname{Spec} A$, the exact sequence is

$$0 \longrightarrow K \longrightarrow M \xrightarrow{\times x} M \longrightarrow K' \longrightarrow 0$$

and both $K = \ker(\times x) = (0 : x)$ and $K' \cong M/xM$ are (A/x)-modules.) Twist (1) by $\mathcal{O}(m)$ and take Euler-characteristics to obtain $p_{\mathcal{F}}(m+1) - p_{\mathcal{F}}(m) = p_{\mathcal{K}'}(m) - p_{\mathcal{K}}(m)$. By the inductive hypothesis, the right side of this equation is a polynomial of degree at most $k - 1$. Hence (by an easy induction) $p(m)$ is a polynomial of degree at most k.

Step 2. We'll now show that the degree of this polynomial is precisely $\dim \operatorname{Supp} \mathcal{F}$. As $\mathcal{F}$ is a coherent sheaf on a Noetherian scheme, it has a finite number of associated points, so we can find a hypersurface $H = (f = 0)$ not containing any of the associated points. (This is that problem from last quarter that we have been repeatedly using recently: problem 24(c) on set 5, which was exercise 1.19 in the class 11 notes.) In particular, $\dim H \cap \operatorname{Supp} \mathcal{F}$

is strictly less than $\dim \operatorname{Supp} \mathcal{F}$, and in fact one less by Krull's Principal Ideal Theorem. Let $d = \deg f$. Then I claim that $\times f : \mathcal{F}(-d) \to \mathcal{F}$ is an inclusion. Indeed, on any affine open set, the map is of the form $\times \overline{f} : M \to M$ (where $\overline{f}$ is the restriction of f to this open set), and the fact that $f = 0$ contains no associated points *means* that this is an injection of modules. (Remember that those ring elements annihilating elements of M are precisely the associated primes, and $\overline{f}$ is contained in none of them.) Then we have

$$0 \to \mathcal{F}(-d) \to \mathcal{F} \to \mathcal{K}' \to 0.$$

Twisting by $\mathcal{O}(m)$ yields

$$0 \to \mathcal{F}(m-d) \to \mathcal{F}(m) \to \mathcal{K}'(m) \to 0.$$

Taking Euler characteristics gives $p_{\mathcal{F}}(m) - p_{\mathcal{F}}(m-d) = p_{\mathcal{K}'}(m)$. Now by step 1, we know that $p_{\mathcal{F}}(m)$ is a polynomial. Also, by our inductive hypothesis, and Exercise 1.2 below, the right side is a polynomial of degree of precisely $\dim \operatorname{Supp} \mathcal{F} - 1$. Hence $p(m)$ is a polynomial of degree $\dim \operatorname{Supp} \mathcal{F}$. □

1.2. *Exercise.* Consider the short exact sequence of A-modules $0 \longrightarrow M \xrightarrow{\times f} M \longrightarrow K' \longrightarrow 0$. Show that $\operatorname{Supp} K' = \operatorname{Supp}(M) \cap \operatorname{Supp}(A/f)$.

Notice that we needed the first part of the proof to ensure that $p_{\mathcal{F}}(m)$ is in fact a polynomial; otherwise, the second part would just show that $p_{\mathcal{F}}(m)$ is just a polynomial when m is fixed modulo d.

(For experts: here is a different way to avoid having two similar steps. If k is an infinite field, e.g. if it were algebraically closed, then we could find a hypersurface as in step 2 of degree 1, using that problem from last quarter mentioned in the proof. So what to do if k is not infinite? Note that if you have a complex of k-vector spaces, and you take its cohomology, and then tensor with $\overline{k}$, you get the same thing as if you tensor first, and then take the cohomology. By this trick, we can assume that k is algebraically closed. In fancy language: we have taken a *faithfully flat* base extension. I won't define what this means here; it will turn up early in the third quarter.)

Example 1. $p_{\mathbb{P}^n}(m) = \binom{m+n}{n}$, where we interpret this as the polynomial $(m+1)\cdots(m+n)/n!$.

Example 2. Suppose H is a degree d hypersurface in $\mathbb{P}^n$. Then from

$$0 \to \mathcal{O}_{\mathbb{P}^n}(-d) \to \mathcal{O}_{\mathbb{P}^n} \to \mathcal{O}_H \to 0,$$

we have

$$p_H(m) = p_{\mathbb{P}^n}(m) - p_{\mathbb{P}^n}(m-d) = \binom{m+n}{n} - \binom{m+n-d}{n}.$$

1.3. *Exercise.* Show that the twisted cubic (in $\mathbb{P}^3$) has Hilbert polynomial $3m+1$.

1.4. *Exercise.* Find the Hilbert polynomial for the dth Veronese embedding of $\mathbb{P}^n$ (i.e. the closed immersion of $\mathbb{P}^n$ in a bigger projective space by way of the line bundle $\mathcal{O}(d)$).

From the Hilbert polynomial, we can extract many invariants, of which two are particularly important. The first is the *degree*. Classically, the degree of a complex projective variety of dimension n was defined as follows. We slice the variety with n generally chosen hyperplane. Then the intersection will be a finite number of points. The degree is this number of points. Of course, this requires showing all sorts of things. Instead, we will define the *degree of a projective k-scheme of dimension* n to be leading coefficient of the Hilbert polynomial (the coefficient of m^n) times $n!$.

For example, the degree of $\mathbb{P}^n$ in itself is 1. The degree of the twisted cubic is 3.

1.5. *Exercise.* Show that the degree of a degree d hypersurface is d (preventing a notational crisis).

1.6. *Exercise.* Suppose a curve C is embedded in projective space via an invertible sheaf of degree d. (In other words, this line bundle determines a closed immersion.) Show that the degree of C under this embedding is d (preventing another notational crisis). (Hint: Riemann-Roch.)

1.7. *Exercise.* Find the degree of the dth Veronese embedding of $\mathbb{P}^n$.

1.8. *Exercise (Bezout's theorem).* Suppose X is a projective scheme of dimension at least 1, and H is a degree d hypersurface not containing any associated points of X. (For example, if X is a projective variety, then we are just requiring H not to contain any irreducible components of X.) Show that $\deg H \cap X = d \deg X$.

This is a very handy theorem! For example: if two projective plane curves of degree m and degree n share no irreducible components, then they intersect in mn points, counted with appropriate multiplicity. The notion of multiplicity of intersection is just the degree of the intersection as a k-scheme.

We trot out a useful example for a third time: let $k = \mathbb{Q}$, and consider the parabola $x = y^2$. We intersect it with the four usual suspects: $x = 1$, $x = 0$, $x = -1$, and $x = 2$, and see that we get 2 each time (counted with the same convention as with the last time we saw this example).

If we intersect it with $y = 2$, we only get one point — but that's of course because this isn't a projective curve, and we really should be doing this intersection on $\mathbb{P}^2_k$ — and in this case, the conic meets the line in two points, one of which is "at ∞".

1.9. *Exercise.* Determine the degree of the d-fold Veronese embedding of $\mathbb{P}^n$ in a different way as follows. Let $v_d : \mathbb{P}^n \to \mathbb{P}^N$ be the Veronese embedding. To find the degree of the image, we intersect it with n hyperplanes in $\mathbb{P}^N$ (scheme-theoretically), and find the number of intersection points (counted with multiplicity). But the pullback of a hyperplane in $\mathbb{P}^N$ to $\mathbb{P}^n$ is a degree d hypersurface. Perform this intersection in $\mathbb{P}^n$, and use Bezout's theorem. (If already you know the answer by the earlier exercise on the degree of the Veronese embedding, this will be easier.)

There is another nice bit of information residing in the Hilbert polynomial. Notice that $p_X(0) = \chi(X, \mathcal{O}_X)$, which is an *intrinsic* invariant of the scheme X, which does not depend on the projective embedding.

Imagine how amazing this must have seemed to the ancients: they defined the Hilbert function by counting how many "functions of various degrees" there are; then they noticed that when the degree gets large, it agrees with a polynomial; and then when they plugged 0 into the polynomial — extrapolating backwards, to where the Hilbert function and Hilbert polynomials didn't agree — they found a magic invariant!

And now I can give you a nonsingular curve over an algebraically closed field that is not $\mathbb{P}^1$! Note that the Hilbert polynomial of $\mathbb{P}^1$ is $(m+1)/1 = m+1$, so $\chi(\mathcal{O}_{\mathbb{P}^1}) = 1$. Suppose C is a degree d curve in $\mathbb{P}^2$. Then the Hilbert polynomial of C is

$$p_{\mathbb{P}^2}(m) - p_{\mathbb{P}^2}(m-d) = (m+1)(m+2)/2 - (m-d+1)(m-d+2)/2.$$

Plugging in $m = 0$ gives us $-(d^2 - 3d)/2$. Thus when $d > 2$, we have a curve that cannot be isomorphic to $\mathbb{P}^1$! (I think I gave you an earlier exercise that there is a *nonsingular* degree d curve. Note however that the calculation above didn't use nonsingularity.)

Now from $0 \to \mathcal{O}_{\mathbb{P}^2}(-d) \to \mathcal{O}_{\mathbb{P}^2} \to \mathcal{O}_C \to 0$, using $h^1(\mathcal{O}_{\mathbb{P}^2}(d)) = 0$, we have that $h^0(C, \mathcal{O}_C) = 1$. As $h^0 - h^1 = \chi$, we have

$$h^1(C, \mathcal{O}_C) = (d-1)(d-2)/2.$$

Motivated by geometry, we define the *arithmetic genus* of a scheme X as $1 - \chi(X, \mathcal{O}_X)$. This is sometimes denoted $p_a(X)$. In the case of nonsingular complex curves, this corresponds to the topological genus. For irreducible reduced curves (or more generally, curves with $h^0(X, \mathcal{O}_X) \cong k$), $p_a(X) = h^1(X, \mathcal{O}_X)$. (In higher dimension, this is a less natural notion.)

We thus now have examples of curves of genus $0, 1, 3, 6, 10, \dots$ (corresponding to degree 1 or $2, 3, 4, 5, \dots$).

This begs some questions, such as: are there curves of other genera? (Yes.) Are there other genus 1 curves? (Not if k is algebraically closed, but yes otherwise.) Do we have all the curves of genus 3? (Almost all, but not quite all.) Do we have all the curves of genus 6? (We're missing most of them.)

Caution: The Euler characteristic of the structure sheaf doesn't distinguish between isomorphism classes of nonsingular projective schemes over algebraically closed fields — for example, $\mathbb{P}^1 \times \mathbb{P}^1$ and $\mathbb{P}^2$ both have Euler characteristic 1, but are not isomorphic (as for example $\operatorname{Pic} \mathbb{P}^2 \cong \mathbb{Z}$ while $\operatorname{Pic} \mathbb{P}^1 \times \mathbb{P}^1 \cong \mathbb{Z} \oplus \mathbb{Z}$).

Important Remark. We can restate the Riemann-Roch formula as:

$$h^0(C, \mathcal{L}) - h^1(C, \mathcal{L}) = \deg \mathcal{L} - p_a + 1.$$

This is the most common formulation of the Riemann-Roch formula.

1.10. *Complete intersections.* We define a *complete intersection* in $\mathbb{P}^n$ as follows. $\mathbb{P}^n$ is a complete intersection in itself. A closed subscheme $X_r \hookrightarrow \mathbb{P}^n$ of dimension r (with $r < n$) is a complete intersection if there is a complete intersection X_{r+1}, and X_r is a Cartier divisor in class $\mathcal{O}_{X_{r+1}}(d)$.

Exercise. Show that if X is a complete intersection of dimension r in $\mathbb{P}^n$, then $H^i(X, \mathcal{O}_X(m)) = 0$ for all $0 < i < r$ and all m. Show that if $r > 0$, then $H^0(\mathbb{P}^n, \mathcal{O}(m)) \to H^0(X, \mathcal{O}(m))$ is surjective.

Now in my definition, X_r is the zero-divisor of a section of $\mathcal{O}_{X_{r+1}}(m)$ for some m. But this section is the restriction of a section of $\mathcal{O}(m)$ on $\mathbb{P}^n$. Hence X_r is the scheme-theoretic intersection of X_{r+1} with a hypersurface. Thus inductively we can show that X_r is the scheme-theoretic intersection of $n - r$ hypersurfaces. (By Bezout's theorem, $\deg X_r$ is the product of the degree of the defining hypersurfaces.)

Exercise. Show that complete intersections of positive dimension are connected. (Hint: show $h^0(X, \mathcal{O}_X) = 1$.)

Exercise. Find the genus of the intersection of 2 quadrics in $\mathbb{P}^3$. (We get curves of more genera by generalizing this!)

Exercise. Show that the rational normal curve of degree d in $\mathbb{P}^d$ is *not* a complete intersection if $d > 2$.

Exercise. Show that the union of 2 distinct planes in $\mathbb{P}^4$ is not a complete intersection. (This is the first appearance of another universal counterexample!) Hint: it is connected, but you can slice with another plane and get something not connected.

This is another important scheme in algebraic geometry that is an example of many sorts of behavior. We will see more of it later!

2. Higher direct image sheaves

I'll now introduce a notion generalizing these Cech cohomology groups. Cohomology groups were defined for $X \to \operatorname{Spec} A$ where the structure morphism is quasicompact and separated; for any quasicoherent $\mathcal{F}$ on X, we defined $H^i(X, \mathcal{F})$.

We'll now do something similar for quasicompact and separated morphisms $\pi : X \to Y$: for any quasicoherent $\mathcal{F}$ on X, we'll define $R^i\pi_*\mathcal{F}$, a quasicoherent sheaf on Y.

We have many motivations for doing this. In no particular order:

(1) It "globalizes" what we were doing anywhere.
(2) If $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$ is a short exact sequence of quasicoherent sheaves on X, then we know that $0 \to \pi_*\mathcal{F} \to \pi_*\mathcal{G} \to \pi_*\mathcal{H}$ is exact, and higher pushforwards will extend this to a long exact sequence.

(3) We'll later see that this will show how cohomology groups vary in families, especially in "nice" situations. Intuitively, if we have a nice family of varieties, and a family of sheaves on them, we could hope that the cohomology varies nicely in families, and in fact in "nice" situations, this is true. (As always, "nice" usually means "flat", whatever that means.)

There will be no extra work involved for us.

Suppose $\pi : X \to Y$, and $\mathcal{F}$ is a quasicoherent sheaf on X. For each $\operatorname{Spec} A \subset Y$, we have A-modules $H^i(\pi^{-1}(\operatorname{Spec} A), \mathcal{F})$. We will show that these patch together to form a quasicoherent sheaf. We need check only one fact: that this behaves well with respect to taking distinguished opens. In other words, we must check that for each $f \in A$, the natural map $H^i(\pi^{-1}(\operatorname{Spec} A), \mathcal{F}) \to H^i(\pi^{-1}(\operatorname{Spec} A), \mathcal{F})_f$ (induced by the map of spaces in the opposite direction — H^i is contravariant in the space) is precisely the localization $\otimes_A A_f$. But this can be verified easily: let $\{U_i\}$ be an affine cover of $\pi^{-1}(\operatorname{Spec} A)$. We can compute $H^i(\pi^{-1}(\operatorname{Spec} A), \mathcal{F})$ using the Cech complex. But this induces a cover $\operatorname{Spec} A_f$ in a natural way: If $U_i = \operatorname{Spec} A_i$ is an affine open for $\operatorname{Spec} A$, we define $U_i' = \operatorname{Spec}(A_i)_f$. The resulting Cech complex for $\operatorname{Spec} A_f$ is the localization of the Cech complex for $\operatorname{Spec} A$. As taking cohomology of a complex commutes with localization, we have defined a quasicoherent sheaf on Y by one of our definitions of quasicoherent sheaves.

2.1. (Something important happened in that last sentence — localization commuting with taking cohomology. If you want practice with this notion, here is an *exercise:* suppose $C^0 \to C^1 \to C^2$ is a complex in an abelian category, and F is an exact functor to another abelian category. Show that F applied to the cohomology of this complex is naturally isomorphic to the cohomology of F of this complex. Translation: taking cohomology commutes with exact functors. In the particular case of this construction, the exact functor in equation is the localization functor $\otimes_A A_f$ from A-modules to A_f-modules. I'll discuss this a bit more at the start of the class 32 notes.)

Define the i**th higher direct image sheaf** or the i**th (higher) pushforward sheaf** to be this quasicoherent sheaf.

2.2. *Theorem. —*

(a) *$R^0\pi_*\mathcal{F}$ is canonically isomorphic to $\pi_*\mathcal{F}$.*
(b) *$R^i\pi_*$ is a covariant functor from the category of quasicoherent sheaves on X to the category of quasicoherent sheaves on Y, and a contravariant functor in Y-schemes X.*
(c) *A short exact sequence $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$ of sheaves on X induces a long exact sequence*

$$0 \longrightarrow R^0\pi_*\mathcal{F} \longrightarrow R^0\pi_*\mathcal{G} \longrightarrow R^0\pi_*\mathcal{H} \longrightarrow$$

$$R^1\pi_*\mathcal{F} \longrightarrow R^1\pi_*\mathcal{G} \longrightarrow R^1\pi_*\mathcal{H} \longrightarrow \cdots$$

of sheaves on Y. (This is often called the corresponding **long exact sequence of higher pushforward sheaves***.)*

(d) *(projective pushforwards of coherent are coherent) If π is a projective morphism and $\mathcal{O}_Y$ is coherent on Y (this hypothesis is automatic for Y locally Noetherian), and $\mathcal{F}$ is a coherent sheaf on X, then for all i, $R^i\pi_*\mathcal{F}$ is a coherent sheaf on Y.*

Proof. Because it suffices to check each of these results on affine opens, they all follow from the analogous statements in Cech cohomology. □

The following result is handy (and essentially immediate from our definition).

2.3. *Exercise.* Show that if π is affine, then for $i > 0$, $R^i\pi_*\mathcal{F} = 0$. Moreover, show that if Y is quasicompact and quasiseparated then the natural morphism $H^i(X, \mathcal{F}) \to H^i(Y, f_*\mathcal{F})$ is an isomorphism. (A special case of the first sentence is a special case we showed earlier, when π is a closed immersion. Hint: use any affine cover on Y, which will induce an affine cover of X.)

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FOUNDATIONS OF ALGEBRAIC GEOMETRY CLASS 32

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CONTENTS

Last day: Hilbert polynomials and Hilbert functions. Higher direct image sheaves.

Today: Applications of higher pushforwards; crash course in spectral sequences; towards the Leray spectral sequence.

1. A USEFUL ALGEBRAIC FACT

I'd like to start with an algebra exercise that is very useful.

1.1. *Exercise (Important algebra exercise).* Suppose $M_1 \xrightarrow{\alpha} M_2 \xrightarrow{\beta} M_3$ is a complex of A-modules (i.e. $\beta \circ \alpha = 0$), and N is an A-module. (a) Describe a natural homomorphism of the cohomology of the complex, tensored with N, with the cohomology of the complex you get when you tensor with N, $H(M_*) \otimes_A B \to H(M_* \otimes_A N)$, i.e.

$$\left(\frac{\ker \beta}{\operatorname{im} \alpha}\right) \otimes_A N \to \frac{\ker(\beta \otimes N)}{\operatorname{im}(\alpha \otimes N)}.$$

I always forget which way this map is supposed to go.
(b) If N is *flat*, i.e. $\otimes N$ is an exact functor, show that the morphism defined above is an isomorphism. (Hint: This is actually a categorical question: if M_* is an exact sequence in an abelian category, and F is a right-exact functor, then (a) there is a natural morphism $FH(M_*) \to H(FM_*)$, and (b) if F is an exact functor, this morphism is an isomorphism.)

Example: localization is exact, so $S^{-1}A$ is a *flat* A-algebra for all multiplicative sets S. In particular, A_f is a flat A-algebra. We used (b) implicitly last day, when I said that given a quasicompact, separated morphism $\pi : X \to Y$, and an affine open subset $\operatorname{Spec} A$ of Y, and a distinguished affine open $\operatorname{Spec} A_f$ of that, the cohomology of any Cech complex computing the cohomology $\pi^{-1}(\operatorname{Spec} A)$, tensored with A_f, would be naturally isomorphic to the cohomology of the complex you get when you tensor with A_f.

Date: Thursday, February 16, 2006. Updated June 26.

Here is another example.

1.2. *Exercise (Higher pushforwards and base change).* (a) Suppose $f : Z \to Y$ is any morphism, and $\pi : X \to Y$ as usual is quasicompact and separated. Suppose $\mathcal{F}$ is a quasicoherent sheaf on X. Let

$$\begin{array}{ccc} W & \xrightarrow{f'} & X \\ \downarrow{\scriptstyle \pi'} & & \downarrow{\scriptstyle \pi} \\ Z & \xrightarrow{f} & Y \end{array}$$

is a fiber diagram. Describe a natural morphism $f^*(R^i\pi_*\mathcal{F}) \to R^i\pi'_*(f')^*\mathcal{F}$.
(b) If $f : Z \to Y$ is an affine morphism, and for a cover $\operatorname{Spec} A_i$ of Y, where $f^{-1}(\operatorname{Spec} A_i) = \operatorname{Spec} B_i$, B_i is a *flat* A-algebra, show that the natural morphism of (a) is an isomorphism. (You can likely generalize this immediately, but this will lead us into the concept of flat morphisms, and we'll hold off discussing this notion for a while.)

A useful special case if the following. If f is a closed immersion of a closed point in Y, the right side is the cohomology of the fiber, and the left side is the fiber of the cohomology. In other words, the fiber of the higher pushforward maps naturally to the cohomology of the fiber. We'll later see that in good situations this is an isomorphism, and thus the higher direct image sheaf indeed "patches together" the cohomology on fibers.

Here is one more consequence of our algebraic fact.

1.3. *Exercise (projection formula).* Suppose $\pi : X \to Y$ is quasicompact and separated, and $\mathcal{E}, \mathcal{F}$ are quasicoherent sheaves on X and Y respectively. (a) Describe a natural morphism

$$(R^i\pi_*\mathcal{E}) \otimes \mathcal{F} \to R^i\pi_*(\mathcal{E} \otimes \pi^*\mathcal{F}).$$

(b) If $\mathcal{F}$ is locally free, show that this natural morphism is an isomorphism.

Here is another consequence, that I stated in class 33. (It is still also in the class 33 notes.)

Exercise. Suppose that X is a quasicompact separated k-scheme, where k is a field. Suppose $\mathcal{F}$ is a quasicoherent sheaf on X. Let $X_{\overline{k}} = X \times_{\operatorname{Spec} k} \operatorname{Spec} \overline{k}$, and $f : X_{\overline{k}} \to X$ the projection. Describe a natural isomorphism $H^i(X, \mathcal{F}) \otimes_k \overline{k} \to H^i(X_{\overline{k}}, f^*\mathcal{F})$. Recall that a k-scheme X is *geometrically integral* if $X_{\overline{k}}$ is integral. Show that if X is geometrically integral and projective, then $H^0(X, \mathcal{O}_X) \cong k$. (This is a clue that $\mathbb{P}^1_{\mathbb{C}}$ is not a geometrically integral $\mathbb{R}$-scheme.)

2. Fun applications of the higher pushforward

Last day we proved that if $\pi : X \to Y$ is a projective morphism, and $\mathcal{F}$ is a coherent sheaf on X, then $\pi_*\mathcal{F}$ is coherent (under a technical assumption: if either Y and hence X are Noetherian; or more generally if $\mathcal{O}_Y$ is a coherent sheaf).

As a nice immediate consequence is the following. Finite morphisms are affine (from the definition) and projective (an earlier exercise); the converse also holds.

2.1. *Corollary. — If $\pi : X \to Y$ is projective and affine and $\mathcal{O}_Y$ is coherent, then π is finite.*

In fact, more generally, if π is universally closed and affine, then π is finite. We won't use this, so I won't explain why, but you can read about it in Atiyah-Macdonald, Exercise 5.35.

Proof. By the theorem from last day, $\pi_* \mathcal{O}_X$ is coherent and hence finitely generated. □

Here is another handy theorem.

2.2. *Theorem (relative dimensional vanishing). — If $f : X \to Y$ is a projective morphism and $\mathcal{O}_Y$ is coherent, then the higher pushforwards vanish in degree higher than the maximum dimension of the fibers.*

This is false without the projective hypothesis. Here is an example of why.

Exercise. Consider the open immersion $\pi : \mathbb{A}^n - 0 \to \mathbb{A}^n$. By direct calculation, show that $R^{n-1} f_* \mathcal{O}_{\mathbb{A}^n - 0} \neq 0$.

Proof. Let m be the maximum dimension of all the fibers.

The question is local on Y, so we'll show that the result holds near a point p of Y. We may assume that Y is affine, and hence that $X \hookrightarrow \mathbb{P}^n_Y$.

Let k be the residue field at p. Then $f^{-1}(p)$ is a projective k-scheme of dimension at most m. Thus we can find affine open sets $D(f_1), \dots, D(f_{m+1})$ that cover $f^{-1}(p)$. In other words, the intersection of $V(f_i)$ does not intersect $f^{-1}(p)$.

If $Y = \operatorname{Spec} A$ and $p = [\mathfrak{p}]$ (so $k = A_{\mathfrak{p}} / \mathfrak{p} A_{\mathfrak{p}}$), then arbitrarily lift each f_i from an element of $k[x_0, \dots, x_n]$ to an element f_i' of $A_{\mathfrak{p}}[x_0, \dots, x_n]$. Let F be the product of the denominators of the f_i'; note that $F \notin \mathfrak{p}$, i.e. $p = [\mathfrak{p}] \in D(F)$. Then $f_i' \in A_F[x_0, \dots, x_n]$. The intersection of their zero loci $\cap V(f_i') \subset \mathbb{P}^n_{A_F}$ is a closed subscheme of $\mathbb{P}^n_{A_F}$. Intersect it with X to get another closed subscheme of $\mathbb{P}^n_{A_F}$. Take its image under f; as projective morphisms are closed, we get a closed subset of $D(F) = \operatorname{Spec} A_F$. But this closed subset does not include p; hence we can find an affine neighborhood $\operatorname{Spec} B$ of p in Y missing the image. But if f_i'' are the restrictions of f_i' to $B[x_0, \dots, x_n]$, then $D(f_i'')$ cover $f^{-1}(\operatorname{Spec} B)$; in other words, over $f^{-1}(\operatorname{Spec} B)$ is covered by $m+1$ affine open sets, so by the affine-cover vanishing theorem, its cohomology vanishes in degree at least $m+1$. But the higher-direct image sheaf is computed using these cohomology groups, hence the higher direct image sheaf $R^i f_* \mathcal{F}$ vanishes on $\operatorname{Spec} B$ too. □

2.3. *Important Exercise.* Use a similar argument to prove *semicontinuity of fiber dimension of projective morphisms*: suppose $\pi : X \to Y$ is a projective morphism where $\mathcal{O}_Y$ is coherent. Show that $\{y \in Y : \dim f^{-1}(y) > k\}$ is a Zariski-closed subset. In other words, the dimension of the fiber "jumps over Zariski-closed subsets". (You can interpret the case $k = -1$ as the fact that projective morphisms are closed.) This exercise is rather important for having a sense of how projective morphisms behave! Presumably the result is true more generally for proper morphisms.

Here is another handy theorem, that is proved by a similar argument. We know that finite morphisms are projective, and have finite fibers. Here is the converse.

2.4. *Theorem (projective + finite fibers = finite). — Suppose $\pi : X \to Y$ is such that $\mathcal{O}_Y$ is coherent. Then π is projective and finite fibers if and only if it is finite. Equivalently, π is projective and quasifinite if and only it is finite.*

(Recall that quasifinite = finite fibers + finite type. But projective includes finite type.)

It is true more generally that proper + quasifinite = finite. (We may see that later.)

Proof. We show it is finite near a point $y \in Y$. Fix an affine open neighborhood $\operatorname{Spec} A$ of y in Y. Pick a hypersurface H in $\mathbb{P}^n_A$ missing the preimage of y, so $H \cap X$ is closed. (You can take this as a hint for Exercise 2.3!) Let $H' = \pi_*(H \cap X)$, which is closed, and doesn't contain y. Let $U = \operatorname{Spec} R - H'$, which is an open set containing y. Then above U, π is projective and affine, so we are done by the previous Corollary 2.1. $\square$

Here is one last potentially useful fact. (To be honest, I'm not sure if we'll use it in this course.)

2.5. *Exercise.* Suppose $f : X \to Y$ is a projective morphism, with $\mathcal{O}(1)$ on X. Suppose Y is quasicompact and $\mathcal{O}_Y$ is coherent. Let $\mathcal{F}$ be coherent on X. Show that

(a) $f^* f_* \mathcal{F}(n) \to \mathcal{F}(n)$ is surjective for $n \gg 0$. (First show that there is a natural map for any n! Hint: by adjointness of f_* with f_*.) Translation: for $n \gg 0$, $\mathcal{F}(n)$ is relatively generated by global sections.
(b) For $i > 0$ and $n \gg 0$, $R^i f_* \mathcal{F}(n) = 0$.

3. Toward the Leray spectral sequence: Crash course in spectral sequences

My goal now is to tell you enough about spectral sequences that you'll have a good handle on how to use them in practice, and why you shouldn't be frightened when they come up in a seminar. There will be some key points that I will not prove; it would be good, once in your life, to see a proof of these facts, or even better, to prove it yourself. Then in good conscience you'll know how the machine works, and you can close the hood once and for all and just happily drive the powerful machine.

My philosophy will be to tell you just about a stripped down version of spectral sequences, which frankly is what is used most of the time. You can always gussy it up later on. But it will be enough to give a quick proof of the Leray spectral sequence.

A good reference as always is Weibel. I learned it from Lang's *Algebra*. I don't necessarily endorse that, but at least his exposition is just a few pages long.

Let's get down to business.

For me, a *double complex* (in an abelian category) will be a bunch of objects $A^{p,q}$ ($p, q \in \mathbb{Z}$), which are zero unless $p, q \geq 0$, and morphisms $d^{p,q} : A^{p,q} \to A^{p+1,q}$ and $\delta^{p,q} : A^{p,q} \to A^{p,q+1}$ (we will always write these as d and δ and ignore the subscripts) satisfying $d^2 = 0$ and $\delta^2 = 0$, and one more condition: either $d\delta = \delta d$ ("all the squares commute") or $d\delta + \delta d = 0$ (they all "anticommute"). Both come up, and you can switch from one to the other by replacing $\delta^{p,q}$ with $(-1)^p \delta^{p,q}$. So I'll hereafter presume that all the squares anticommute, but that you know how to turn the commuting case into this one.

Also, there are variations on this definition, where for example the vertical arrows go downwards, or some different subset of the $A^{p,q}$ are required to be zero, but I'll leave these straightforward variations to you.

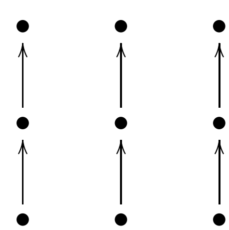
From the double complex (with the anticommuting convention), we construct a corresponding (single) complex A^* with $A^k = \oplus_i A^{i,k-i}$, with $D = d + \delta$. Note that $D^2 = (d+\delta)^2 = d^2 + (d\delta + \delta d) + \delta^2 = 0$, so A^* is indeed a complex. (Be sure you see how to interpret this in $A^{*,*}$!)

The *cohomology* of the single complex is sometimes called the *hypercohomology* of the double complex.

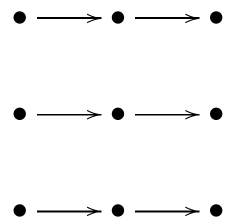
Our motivating goal will be to find the hypercohomology of the double complex. (You'll see later that we'll have other real goals, and that this is a red herring.)

Then here is recipe for computing (information) about the cohomology. We create a countable sequence of tables as follows. Table 0, denoted $E_0^{p,q}$, is defined as follows: $E_0^{p,q} = A^{p,q}$.

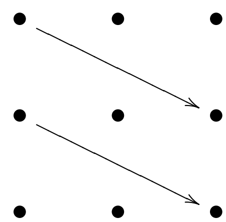
We then look just at the vertical arrows (the δ-arrows).



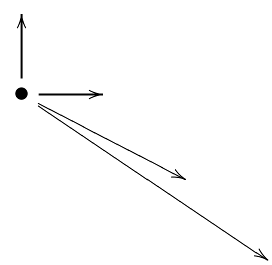
The columns are complexes, so we take cohomology of these vertical complexes, resulting in a new table, $E_1^{p,q}$. Then there are natural morphisms from each entry of the new table to the entry on the right. (This needs to be checked!)



The composition of two of these morphisms is again zero, so again we have complexes. We take cohomology of these as well, resulting in a new table, $E_2^{p,q}$. It turns out that there are natural morphisms from each entry to the entry two to the right and one below, and that the composition of these two is 0.



This can go on until the cows come home. The order of the morphisms is shown pictorially below.



(Notice that the map always is "degree 1" in the grading of the single complex.)

Now if you follow any entry in our original table, eventually the arrow into it will come from outside of the first quadrant, and the arrow out of it will go to outside the first quadrant, so after a certain stage the complex will look like $0 \to E_?^{p,q} \to 0$. Then after that stage, the (p,q)-entry will never change. We define $E_\infty^{p,q}$ to be the table whose (p,q)th entry is this object. We say that $E_k^{p,q}$ *converges to* $E_\infty^{p,q}$.

Then it is a fact (or even a theorem) that there is a filtration of $H^k(A^*)$ by More precisely you can filter $H^k(A^*)$ with $k+1$ objects whose successive quotients are $E_\infty^{i,k-i}$, where the sub-object is $E_\infty^{k,0}$, and the quotient $H^k(A^*)$ by the next biggest object is $E_\infty^{0,k}$. I hope that is clear; please let me know if I can say it better! The following may help:

	$E_\infty^{0,k}$		$E_\infty^{1,k-1}$		$E_\infty^{k-1,1}$		$E_\infty^{k,0}$	
$H^k(A^*)$	$\supset$	$?$	$\supset$	$? \supset \cdots \supset ?$	$\supset$	$?$	$\supset$	0

(I always forget which way the quotients are supposed to go. One way of remembering it is by having some idea of how the result is proved. The picture here is that the double complex is filtered by subcomplexes $\oplus_{p \geq k, q \geq 0} A^{p,q}$, and the first term corresponding by taking the cohomology of the subquotients of this filtration. Then the "biggest quotient" corresponds to the left column, which remains true at the level of cohomology. If this doesn't help you, just ignore this parenthetical comment. If you have a better way of remembering this, even a mnemonic trick, please let me know!)

The sequence $E_k^{p,q}$ is called a *spectral sequence*, and we say that it *abuts* to $H^*(A^*)$. We often say that $E_2^{p,q}$ (or any other term) abuts to H.

Unfortunately, you only get partial information about $H^*(A^*)$. But there are some cases where you get more information: if all $E_\infty^{i,k-i}$ are zero, or if all but one of them are zero; or if we are in the category of vector spaces over a field k, and are interested only in the dimension of $H^*(A^*)$.

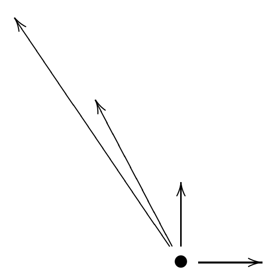
Also, in good circumstances, E_2 (or some other low term) already equals E_∞.

3.1. *Exercise.* Show that $H^0(A^*) = E_\infty^{0,0} = E_2^{0,0}$ and

$$0 \to E_2^{1,0} \to H^1(A^*) \to E_2^{0,1} \to E_2^{2,0} \to H^2(A^*).$$

3.2. *Exercise.* Suppose we are working in the category of vector spaces over a field k, and $\oplus_{p,q} E_2^{p,q}$ is a finite-dimensional vector space. Show that $\chi(H^*(A^*))$ is well-defined, and equals $\sum_{p,q}(-1)^{p+q} E_2^{p,q}$. (It will sometimes happen that $\oplus E_0^{p,q}$ will be an infinite-dimensional vector space, but that $E_2^{p,q}$ will be finite-dimensional!)

Eric pointed out that I was being a moron, and I could just as well have done everything in the opposite direction, i.e. reversing the roles of horizontal and vertical morphisms. Then the sequences of arrows giving the spectral sequence would look like this:



Then we would again get pieces of a filtration of $H^*(A^*)$ (where we have to be a bit careful with the order with which $E_\infty^{p,q}$ corresponds to the subquotients — it in the opposite order to the previous case).

I tried unsuccessfully to convince that Eric that I am *not* a moron, and that this was my secret plan all along. Both algorithms compute the same thing, and usually we don't care about the final answer — we often care about the answer we get in one way, and we get at it by doing the spectral sequence in the *other* way.

Now we're ready to try this out, and see how to use it in practice.

The moral of these examples is what follows: in the past, you've had prove various facts involving various sorts of diagrams, which involved chasing elements all around. Now, you'll just plug them into a spectral sequence, and let the spectral sequence machinery do your chasing for you.

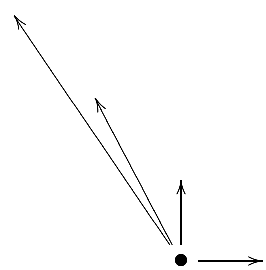
Example: Proving the snake lemma. Consider the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & D & \longrightarrow & E & \longrightarrow & F & \longrightarrow & 0 \\ & & \uparrow{\scriptstyle \alpha} & & \uparrow{\scriptstyle \beta} & & \uparrow{\scriptstyle \gamma} & & \\ 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \end{array}$$

where the rows are exact and the squares commute. (Normally the snake lemma is described with the vertical arrows pointing downwards, but I want to fit this into my spectral sequence conventions.) We wish to show that there is an exact sequence

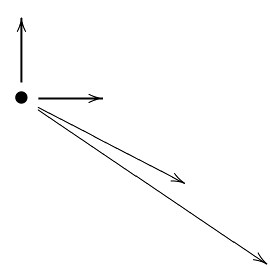
$$0 \to \ker \alpha \to \ker \beta \to \ker \gamma \to \operatorname{im} \alpha \to \operatorname{im} \beta \to \operatorname{im} \gamma \to 0. \tag{1}$$

We plug this into our spectral sequence machinery. We first compute the hypercohomology by taking the rightward morphisms first, i.e. using the order



Then because the rows are exact, $E_1^{p,q} = 0$, so the spectral sequence has already converged: $E_\infty^{p,q} = 0$.

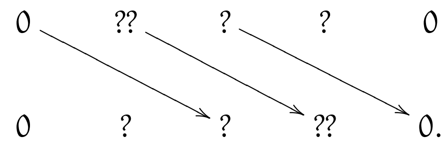
We next compute this "0" in another way, by computing the spectral sequence starting in the other direction.



Then $E_1^{*,*}$ (with its arrows) is:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \operatorname{im} \alpha & \longrightarrow & \operatorname{im} \beta & \longrightarrow & \operatorname{im} \gamma & \longrightarrow & 0 \\ & & & & & & & & \\ 0 & \longrightarrow & \ker \alpha & \longrightarrow & \ker \beta & \longrightarrow & \ker \gamma & \longrightarrow & 0. \end{array}$$

Then we compute $E_2^{*,*}$ and find:



Then we see that after E_2, all the terms will stabilize except for the double question marks; and after E_3, even these two will stabilize. But in the end our complex must be the 0 complex. This means that in E_2, all the entries must be zero, except for the two double question marks; and these two must be the same. This means that $0 \to \ker \alpha \to \ker \beta \to \ker \gamma$ and $\operatorname{im} \alpha \to \operatorname{im} \beta \to \operatorname{im} \gamma \to 0$ are both exact (that comes from the vanishing of the single-question-marks), and

$$\operatorname{coker}(\ker \beta \to \ker \gamma) \cong \ker(\operatorname{im} \alpha \to \operatorname{im} \beta)$$

is an isomorphism (that comes from the equality of the double-question-marks). Taken together, we have proved the snake lemma (1)!

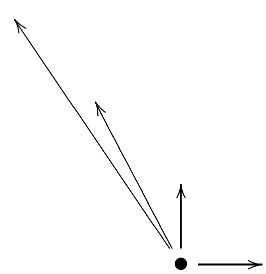
Example: the Five Lemma. Suppose

(2)
$$\begin{array}{ccccccccc} F & \longrightarrow & G & \longrightarrow & H & \longrightarrow & I & \longrightarrow & J \\ {\scriptstyle\alpha}\uparrow & & {\scriptstyle\beta}\uparrow & & {\scriptstyle\gamma}\uparrow & & {\scriptstyle\delta}\uparrow & & {\scriptstyle\epsilon}\uparrow \\ A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & D & \longrightarrow & E \end{array}$$

where the rows are exact and the squares commute.

Suppose α, β, δ, ϵ are isomorphisms. We'll show that γ is an isomorphism.

We first compute the cohomology of the total complex by starting with the rightward arrows:

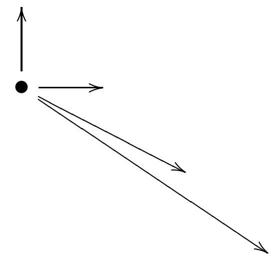


(I chose this because I see that we will get lots of zeros.) Then E_1 looks like this:

$$\begin{array}{ccccc} ? & 0 & 0 & 0 & ? \\ \uparrow & \uparrow & \uparrow & \uparrow & \uparrow \\ ? & 0 & 0 & 0 & ? \end{array}$$

Then E_2 looks similar, and the sequence will converge by E_2 (as we'll never get any arrows between two non-zero entries in a table thereafter). We can't conclude that the cohomology of the total complex vanishes, but we can note that it vanishes in all but four degrees — and most important, in the two degrees corresponding to the entries C and H (the source and target of γ).

We next compute this in the other direction:



Then E_1 looks like this:

$$0 \longrightarrow 0 \longrightarrow ? \longrightarrow 0 \longrightarrow 0$$

$$0 \longrightarrow 0 \longrightarrow ? \longrightarrow 0 \longrightarrow 0$$

and the spectral sequence converges at this step. We wish to show that those two ?'s are zero. But they are precisely the cohomology groups of the total complex that we just showed *were* zero — so we're done!

Exercise. By looking at this proof, prove a subtler version of the five lemma, where one of the isomorphisms can instead just be required to be an injection, and another can instead just be required to be a surjection. (I'm deliberately not telling you which ones, so you can see how the spectral sequence is telling you how to improve the result.) I've heard this called the "subtle five lemma", but I like calling it the $4\frac{1}{2}$-lemma.

Exercise. If β and δ (in (2)) are injective, and α is surjective, show that γ is injective. State the dual statement. (The proof of the dual statement will be essentially the same.)

Exercise. Use spectral sequences to show that a short exact sequence of complexes gives a long exact sequence in cohomology.

3.3. *Exercise.* Suppose $\mu : A^* \to B^*$ is a morphism of complexes. Suppose C^* is the single complex associated to the double complex $A^* \to B^*$. (C^* is called the *mapping cone* of μ.) Show that there is a long exact sequence of complexes:

$$\cdots \to H^{i-1}(C^*) \to H^i(A^*) \to H^i(B^*) \to H^i(C^*) \to H^{i+1}(A^*) \to \cdots .$$

(There is a slight notational ambiguity here; depending on how you index your double complex, your long exact sequence might look slightly different.) In particular, people often use the fact μ induces an isomorphism on cohomology if and only if the mapping cone is exact.

(Does anyone else have some classical important fact that would be useful practice for people learning spectral sequences?)

Next day, I'll state and prove the Leray spectral sequence in algebraic geometry.

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FOUNDATIONS OF ALGEBRAIC GEOMETRY CLASS 33

RAVI VAKIL

CONTENTS

Last day: Applications of higher pushforwards; crash course in spectral sequences.

Today: The Leray spectral sequence. Beginning fun with curves: the Riemann-Hurwitz formula.

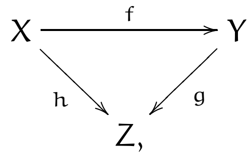
Before I start, here is one small comment I should have made earlier. In the notation $R^j f_* \mathcal{F}$ for higher pushforward sheaves, the "R" stands for "right derived functor", and "corresponds" to the fact that we get a long exact sequence in cohomology extending to the right (from the 0th terms). More generally, next quarter we will see that in good circumstances, if we have a left-exact functor, there may be a long exact sequence going off to the right, in terms of right derived functors. Similarly, if we have a right-exact functor (e.g. if M is an A-module, then $\otimes_A M$ is a right-exact functor from the category of A-modules to itself), there may be a long exact sequence going off to the left, in terms of left derived functors.

Here is another exercise that I should have asked earlier. I have also now included it in the class 32 notes (in section 1).

Exercise. Suppose that X is a quasicompact separated k-scheme, where k is a field. Suppose $\mathcal{F}$ is a quasicoherent sheaf on X. Let $X_{\overline{k}} = X \times_{\operatorname{Spec} k} \operatorname{Spec} \overline{k}$, and $f : X_{\overline{k}} \to X$ the projection. Describe a natural isomorphism $H^i(X, \mathcal{F}) \otimes_k \overline{k} \to H^i(X_{\overline{k}}, f^* \mathcal{F})$. Recall that a k-scheme X is *geometrically integral* if $X_{\overline{k}}$ is integral. Show that if X is geometrically integral and projective, then $H^0(X, \mathcal{O}_X) \cong k$. (This is a clue that $\mathbb{P}^1_{\mathbb{C}}$ is not a geometrically integral $\mathbb{R}$-scheme.)

Date: Tuesday, February 21, 2006. Updated June 26.

Suppose



with f and g (and hence h) quasicompact and separated. Suppose $\mathcal{F}$ is a quasicoherent sheaf on X. The Leray spectral sequence lets us find out about the higher pushforwards of h in terms of the higher pushforwards under g of the higher pushforwards under f.

1.1. *Theorem (Leray spectral sequence). — There is a spectral sequence whose* $E_2^{p,q}$*-term is* $R^j g_*(R^i f_* \mathcal{F})$*, abutting to* $R^{i+j} h_* \mathcal{F}$*.*

An important special case is if $Z = \operatorname{Spec} k$, or Z is some other base ring. Then this gives us handle on the cohomology of $\mathcal{F}$ on X in terms of the cohomology of its higher pushforwards to Y.

Proof. We assume Z is an affine ring, say $\operatorname{Spec} A$. Our construction will be "natural" and will hence glue. (At worst, we you can check that it behaves well under localization.)

Fix a finite affine cover of X, U_i. Fix a finite affine cover of Y, V_j. Create a double complex

$$E_0^{a,b} = \oplus_{|I|=a+1,|J|=b+1} \mathcal{F}(U_I \cap \pi^{-1} V_J)$$

for $a, b \geq 0$, with obvious Cech differential maps. By exercise 15 on problem set 11 (class 25, exercise 1.31), $U_I \cap \pi^{-1} V_J$ is affine (for all I, J).

Let's choose the filtration that corresponds to first taking the arrow in the vertical (V) direction. For each I, we'll get a Cech covering of U_I. The Cech cohomology of an affine is trivial except for H^0, so the E_1 term will be 0 except when $j = 0$. There, we'll get $\oplus \mathcal{F}(U_I)$. Then the E_2 term will be $E_2^{p,q} = H^p(X, \mathcal{F}) = \Gamma(Z, R^p h_* \mathcal{F})$ if $q = 0$ and 0 otherwise, and it will converge there.

Let's next choose the filtration that corresponds to first taking the arrow in the horizontal (U) direction. For each V_J, we will get a Cech covering of $\pi^{-1} V_J$. The entries of E_1 will thus be $\oplus_J H^i(f^{-1} V_j, \mathcal{F}) = \oplus_j \Gamma(V_j, R^i \pi_* \mathcal{F})$. Thus E_2 will be as advertised in the statement of Leray. $\square$

Here are some useful examples.

Consider $h^i(\mathbb{P}_k^m \times_k \mathbb{P}_k^n, \mathcal{O}_{\mathbb{P}_k^m \times_k \mathbb{P}_k^n})$. We get 0 unless $i = 0$, in which case we get 1. (The same argument shows that $h^i(\mathbb{P}_A^m \times_A \mathbb{P}_A^n, \mathcal{O}_{\mathbb{P}_A^m \times_A \mathbb{P}_A^n}) \cong A$ if $i = 0$, and 0 otherwise.) You should make this precise:

Exercise. Suppose Y is any scheme, and $\pi : \mathbb{P}^n_Y \to Y$ is the trivial projective bundle over Y. Show that $\pi_* \mathcal{O}_{\mathbb{P}^n_Y} \cong \mathcal{O}_Y$. More generally, show that $R^j \pi_* \mathcal{O}(m)$ is a finite rank free sheaf on Y, and is 0 if $j \neq 0, n$. Find the rank otherwise.

More generally, let's consider $H^i(\mathbb{P}^m_k \times_k \mathbb{P}^n_k, \mathcal{O}(a, b))$. I claim that for each (a, b) at most one cohomology group is non-trivial, and it will be $i = 0$ if $a, b \geq 0$; $i = m + n$ if $a \leq -m - 1$, $b \leq -n - 1$; $i = m$ if $a \geq 0$, $b \leq -n - 1$, and $i = n$ if $a \leq -m - 1$, $b = 0$. I attempted to show this to you in a special case, in the hope that you would see how the argument goes. I tried to show that $h^i(\mathbb{P}^2_k \times_k \mathbb{P}^1_k, \mathcal{O}(-4, 1))$ is 6 if $i = 2$ and 0 otherwise. The following exercise will help you see if you understood this.

Exercise. Let A be any ring. Suppose a is a negative integer and b is a positive integer. Show that $H^i(\mathbb{P}^m_A \times_A \mathbb{P}^n_A, \mathcal{O}(a, b))$ is 0 unless $i = m$, in which case it is a free A-module. Find the rank of this free A-module. (Hint: Use the previous exercise, and the projection formula, which was Exercise 1.3 of class 32, and exercise 17 of problem set 14.)

We can now find curves of any (non-negative) genus, over any algebraically closed field. An integral projective nonsingular curve over k is *hyperelliptic* if admits a finite degree 2 morphism (or "cover") of $\mathbb{P}^1$.

1.2. *Exercise.* (a) Find the genus of a curve in class $(2, n)$ on $\mathbb{P}^1_k \times_k \mathbb{P}^1_k$. (A curve in class $(2, n)$ is any effective Cartier divisor corresponding to invertible sheaf $\mathcal{O}(2, n)$. Equivalently, it is a curve whose ideal sheaf is isomorphic to $\mathcal{O}(-2, -n)$. Equivalently, it is a curve cut out by a non-zero form of bidegree $(2, n)$.)
(b) Suppose for convenience that k is algebraically closed of characteristic not 2. Show that there exists an integral nonsingular curve in class $(2, n)$ on $\mathbb{P}^1_k \times \mathbb{P}^1_k$ for each $n > 0$.

1.3. *Exercise.* Suppose X and Y are projective k-schemes, and $\mathcal{F}$ and $\mathcal{G}$ are coherent sheaves on X and Y respectively. Recall that if $\pi_1 : X \times Y \to X$ and $\pi_2 : X \times Y \to Y$ are the two projections, then $\mathcal{F} \boxtimes \mathcal{G} := \pi_1^* \mathcal{F} \otimes \pi_2^* \mathcal{G}$. Prove the following, adding additional hypotheses if you find them necessary.
(a) Show that $H^0(X \times Y, \mathcal{F} \boxtimes \mathcal{G}) = H^0(X, \mathcal{F}) \otimes H^0(Y, \mathcal{G})$.
(b) Show that $H^{\dim X + \dim Y}(X \times Y, \mathcal{F} \boxtimes \mathcal{G}) = H^{\dim X}(X, \mathcal{F}) \otimes_k H^{\dim Y}(Y, \mathcal{G})$.
(c) Show that $\chi(X \times Y, \mathcal{F} \boxtimes \mathcal{G}) = \chi(X, \mathcal{F}) \chi(Y, \mathcal{G})$.

I suspect that this Leray spectral sequence converges in this case at E^2, so that $h^n(X \times Y, \mathcal{F} \boxtimes \mathcal{G}) = \sum_{i+j=n} h^i(X, \mathcal{F}) h^j(Y, \mathcal{G})$. Or if this is false, I'd like to see a counterexample. It might even be true that

$$H^n(X \times Y, \mathcal{F} \boxtimes \mathcal{G}) = \oplus_{i+j=n} H^i(X, \mathcal{F}) \otimes H^j(Y, \mathcal{G}).$$

2. Fun with Curves

We already know enough to study curves in a great deal of detail, so this seems like a good way to end this quarter. We get much more mileage if we have a few facts involving differentials, so I'll introduce these facts, and take them as a black box. The actual black boxes we'll need are quite small, but I want to tell you some of the background behind them.

For this topic, we will assume that all curves are projective geometrically integral nonsingular curves over a field k. We will sometimes add the hypothesis that k is algebraically closed.

Most people are happy with working over algebraically closed fields, and all of you should ignore the adverb "geometrically" in the previous paragraph. For those interested in non-algebraically closed fields, an example of a curve that is integral but not geometrically integral is $\mathbb{P}^1_{\mathbb{C}}$ over $\mathbb{R}$. Upon base change to the algebraic closure $\mathbb{C}$ of $\mathbb{R}$, this curve has two components.

2.1. **Differentials on curves.** There is a sheaf of differentials on a curve C, denoted Ω_C, which is an invertible sheaf. (In general, nonsingular k-varieties of dimension d will have a sheaf of differentials over k that will be locally free of rank k. And differentials will be defined in vastly more generality.) We will soon see that this invertible sheaf has degree equal to twice the genus minus 2: $\boxed{\deg \Omega_C = 2g_c - 2}$. For example, if $C = \mathbb{P}^1$, then $\Omega_C \cong \mathcal{O}(-2)$.

Differentials pull back: any surjective morphism of curves $f : C \to C'$ induces a natural map $f^*\Omega_{C'} \to \Omega_C$.

2.2. **The Riemann-Hurwitz formula.** Whenever we invoke this formula (in this section), we will assume that k is algebraically closed and characteristic 0. These conditions aren't necessary, but save us some extra hypotheses. Suppose $f : C \to C'$ is a dominant morphism. Then it turns out $f^*\Omega_{C'} \hookrightarrow \Omega_C$ is an inclusion of invertible sheaves. (This is a case when inclusions of invertible sheaves does not mean what people normally mean by inclusion of line bundles, which are always isomorphisms.) Its cokernel is supported in dimension 0:

$$0 \to f^*\Omega_{C'} \to \Omega_C \to [\text{dimension } 0] \to 0.$$

The divisor R corresponding to those points (with multiplicity), is called the *ramification divisor*.

We can study this in local coordinates. We don't have the technology to describe this precisely yet, but you might still find this believable. If the map at $q \in C'$ looks like $u \mapsto u^n = t$, then $dt \mapsto d(u^n) = nu^{n-1}du$, so dt when pulled back vanishes to order $n-1$. Thus branching of this sort $u \mapsto u^n$ contributes $n-1$ to the ramification divisor. (More correctly, we should look at the map of Spec's of discrete valuation rings, and then u is a uniformizer for the stalk at q, and t is a uniformizer for the stalk at $f(q)$, and t is actually a unit times u^n. But the same argument works.)

Now in a recent exercise on pullbacks of invertible sheaves under maps of curves, we know that a degree of the pullback of an invertible sheaf is the degree of the map times the degree of the original invertible sheaf. Thus if d is the degree of the cover, $\deg \Omega_C =$

$d \deg \Omega_{C'} + \deg R$. Conclusion: if $C \to C'$ is a degree d cover of curves, then

$$\boxed{2g_c - 2 = d(2g_{C'} - 2) + \deg R}$$

Here are some applications.

Example. When I drew a sample branched cover of one complex curve by another, I showed a genus 2 curve covering a genus 3 curve. Show that this is impossible. (Hint: $\deg R \geq 0$.)

Example: Hyperelliptic curves. Hyperelliptic curves are curves that are double covers of $\mathbb{P}^1_k$. If they are genus g, then they are branched over $2g + 2$ points, as each ramification can happen to order only 1. (Caution: we are in characteristic 0!) You may already have heard about genus 1 complex curves double covering $\mathbb{P}^1$, branched over 4 points.

Application 1. First of all, the degree of R is even: any cover of a curve must be branched over an even number of points (counted with multiplicity).

Application 2. The only connected unbranched cover of $\mathbb{P}^1_k$ is the isomorphism. Reason: if $\deg R = 0$, then we have $2 - 2g_C = 2d$ with $d \geq 1$ and $g_c \geq 0$, from which $d = 1$ and $g_C = 0$.

Application 3: Luroth's theorem. Suppose $g(C) = 0$. Then from the Riemann-Hurwitz formula, $g(C') = 0$. (Otherwise, if $g_{C'}$ were at least 1, then the right side of the Riemann-Hurwitz formula would be non-negative, and thus couldn't be -2, which is the left side. This has a non-obvious algebraic consequence, by our identification of covers of curves with field extensions (class 28 Theorem 1.5). Hence all subfields of $k(x)$ containing k are of the form $k(y)$ where $y = f(x)$. (Here we have the hypothesis where k is algebraically closed. We'll patch that later.) Kirsten said that an algebraic proof was given in Math 210.

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FOUNDATIONS OF ALGEBRAIC GEOMETRY CLASS 34

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CONTENTS

Last day: The Leray spectral sequence. Beginning fun with curves: Ω_C, and the Riemann-Hurwitz formula.

Today: More fun with curves: Serre duality, criterion for closed immersion, series of useful remarks, curves of genus 0 and 2

1. LAST DAY

Last day we began to talk about curves over a field k. Our standing assumptions will be that a curve C is projective, geometrically integral and nonsingular over a field k.

(People happy to work over algebraically closed fields can continue to ignore the adverb "geometrically".)

I'm in the process of telling you a few facts that we will prove next quarter. We will use these facts to prove lots of things about curves.

Last day I defined Ω_C, sheaf of differentials on C. I really should have called it $\boxed{\Omega_{C/k}}$, to make clear that this sheaf on C depends on the structure morphism $C \to k$. I stated that $\Omega_{C/k}$ is an invertible sheaf, and told you that we will soon see that has degree $\boxed{\deg \Omega_C = 2g_c - 2}$. I stated that differentials pullback under covers $f : C \to C'$ (i.e. that there is a morphism $f^*\Omega_{C'/k} \to \Omega_{C/k}$), and if we are in characteristic 0, then this yields an

Date: Thursday, February 23, 2006. Minor correction June 25, 2007.

inclusion of invertible sheaves, which yields $0 \to f^*\Omega_{C'} \to \Omega_C \to R \to 0$, where R corresponds to the *ramification divisor* on C, which keeps track of the branching of $C \to C'$. From this I claimed that we will deduce the *Riemann-Hurwitz formula*

$$\boxed{2g_C - 2 = d(2g_{C'} - 2) + \deg R}$$

1.1. **Serre duality.** (We are not requiring k to be algebraically closed.) In general, nonsingular varieties will have a special invertible sheaf $\mathcal{K}_X$ which is the determinant of Ω_X. This invertible sheaf is called the *canonical bundle*, and will later be defined in much greater generality. In our case, $X = C$ is a curve, so $\mathcal{K}_C = \Omega_C$, and from here on in, we'll use $\mathcal{K}_C$ instead of Ω_C. The reason it is called the dualizing sheaf is because it arises in Serre duality. Serre duality states that $H^1(C, \mathcal{K}) \cong k$, or more precisely that there is a *trace morphism* $\boxed{H^1(C, \mathcal{K}) \to k}$ that is an isomorphism. (Example: if $C = \mathbb{P}^1$, then we indeed have $h^1(\mathbb{P}^1, \mathcal{O}(-2)) = 1$.)

Further, for any coherent sheaf $\mathcal{F}$, the natural map

$$\boxed{H^0(C, \mathcal{F}) \otimes_k H^1(C, \mathcal{K} \otimes \mathcal{F}^\vee) \to H^1(C, \mathcal{K})}$$

is a perfect pairing. Thus in particular, $h^0(C, \mathcal{F}) = h^1(C, \mathcal{K} \otimes \mathcal{F}^\vee)$. Recall we defined the arithmetic genus of a curve to be $h^1(C, \mathcal{O}_C)$. Then $h^0(C, \mathcal{K}) = g$ as well.

Recall that Riemann-Roch for a invertible sheaf $\mathcal{L}$ states that

$$h^0(C, \mathcal{L}) - h^1(C, \mathcal{L}) = \deg \mathcal{L} - g + 1.$$

Applying this to $\mathcal{L} = \mathcal{K}$, we get

$$\deg \mathcal{K} = h^0(C, \mathcal{K}) - h^1(C, \mathcal{K}) + g - 1 = h^1(C, \mathcal{O}) - h^0(C, \mathcal{O}) + g - 1 = g - 1 + g - 1 = 2g - 2$$

as promised earlier.

1.2. **A criterion for when a morphism is a closed immersion.** We'll also need a criterion for when something is a closed immersion. To help set it up, let's observe some facts about closed immersions. Suppose $f : X \to Y$ is a closed immersion. Then f is projective, and it is injective on points. This is not enough to ensure that it is a closed immersion, as the example of the normalization of the cusp shows (Figure 1). Another example is the Frobenius morphism from $\mathbb{A}^1$ to $\mathbb{A}^1$, given by $k[t] \to k[u]$, $u \to t^p$, where k has characteristic p.

The additional information you need is that the tangent map is an isomorphism at all closed points. (Exercise: show this is false in those two examples.)

1.3. *Theorem. — Suppose k is an algebraically closed field, and $f : X \to Y$ is a projective morphism of finite-type k-schemes that is injective on closed points and injective on tangent vectors of closed points. Then f is a closed immersion.*

The example of $\operatorname{Spec} \mathbb{C} \to \operatorname{Spec} \mathbb{R}$ shows that we need the hypothesis that k is algebraically closed.

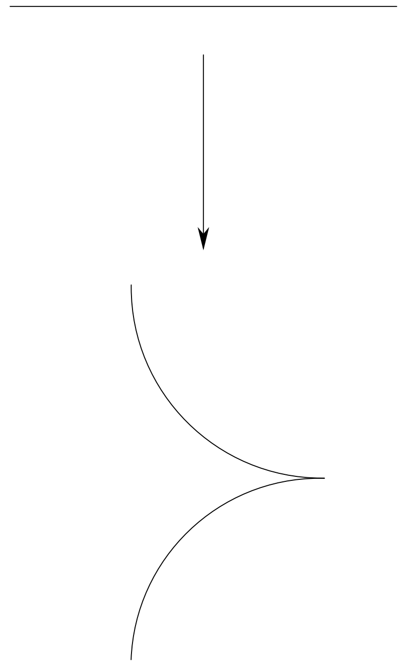

FIGURE 1. Projective morphisms that are injective on points need not be closed immersions

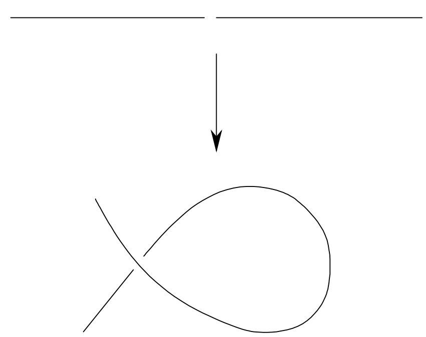

FIGURE 2. We need the projective hypothesis in Theorem 1.3

We need the hypothesis of projective morphism, as shown by the following example (which was described at the blackboard, see Figure 2). We map $\mathbb{A}^1$ to the plane, so that its image is a curve with one node. We then consider the morphism we get by discarding one of the preimages of the node. Then this morphism is an injection on points, and is also injective on tangent vectors, but it is not a closed immersion. (In the world of differential geometry, this fails to be an embedding because the map doesn't give a homeomorphism onto its image.)

Suppose $f(p) = q$, where p and q are closed points. We will use the hypothesis that X and Y are k-schemes where k is algebraically closed at only one point of the argument: that the map induces an isomorphism of residue fields at p and q.

(For those of you who are allergic to algebraically closed fields: still pay attention, as we'll use this to prove things about curves over k where k is *not* necessarily algebraically closed.)

This is the hardest result of today. We will kill the problem in old-school French style: death by a thousand cuts.

Proof. We may assume that Y is affine, say $\operatorname{Spec} B$.

I next claim that f has finite fibers, not just finite fibers above closed points: the fiber dimension for projective morphisms is upper-semicontinuous (Class 32 Exercise 2.3), so the locus where the fiber dimension is at least 1 is a closed subset, so if it is non-empty, it must contain a closed point of Y. Thus the fiber over any point is a dimension 0 finite type scheme over that point, hence a finite set.

Hence f is a projective morphism with finite fibers, thus affine, and even finite (Class 32 Corollary 2.4).

Thus X is affine too, say $\operatorname{Spec} A$, and f corresponds to a ring morphism $B \to A$. We wish to show that this is a surjection of rings, or (equivalently) of B-modules. We will show that for any maximal ideal $\mathfrak{n}$ of B, $B_\mathfrak{n} \to A_\mathfrak{n}$ is a surjection of $B_\mathfrak{n}$-modules. (This will show that $B \to A$ is a surjection. Here is why: if K is the cokernel, so $B \to A \to K \to 0$, then we wish to show that $K = 0$. Now A is a finitely generated B-module, so K is as well, being a homomorphic image of A. Thus $\operatorname{Supp} K$ is a closed set. If $K \neq 0$, then $\operatorname{Supp} K$ is non-empty, and hence contains a closed point $[\mathfrak{n}]$. Then $K_\mathfrak{n} \neq 0$, so from the exact sequence $B_\mathfrak{n} \to \mathbb{A}_\mathfrak{n} \to K_\mathfrak{n} \to 0$, $B_\mathfrak{n} \to A_\mathfrak{n}$ is not a surjection.)

If $A_\mathfrak{n} = 0$, then clearly $B_\mathfrak{n}$ surjects onto $A_\mathfrak{n}$, so assume otherwise. I claim that $A_\mathfrak{n} = A \otimes_B B_\mathfrak{n}$ is a local ring. Proof: $\operatorname{Spec} A_\mathfrak{n} \to \operatorname{Spec} B_\mathfrak{n}$ is a finite morphism (as it is obtained by base change from $\operatorname{Spec} A \to \operatorname{Spec} B$), so we can use the going-up theorem. $A_\mathfrak{n} \neq 0$, so $A_\mathfrak{n}$ *has* a prime ideal. Any point p of $\operatorname{Spec} A_\mathfrak{n}$ maps to some point of $\operatorname{Spec} B_\mathfrak{n}$, which has $[\mathfrak{n}]$ in its closure. Thus there is a point q in the closure of p that maps to $[\mathfrak{n}]$. But there is only one point of $\operatorname{Spec} A_\mathfrak{n}$ mapping to $[\mathfrak{n}]$, which we denote $[\mathfrak{m}]$. Thus we have shown that $\mathfrak{m}$ contains all other prime ideals of $\operatorname{Spec} A_\mathfrak{n}$, so $A_\mathfrak{n}$ is a local ring.

Injectivity of tangent vectors *means* surjectivity of cotangent vectors, i.e. $\mathfrak{n}/\mathfrak{n}^2 \to \mathfrak{m}/\mathfrak{m}^2$ is a surjection, i.e. $\mathfrak{n} \to \mathfrak{m}/\mathfrak{m}^2$ is a surjection. Claim: $\mathfrak{n}A_\mathfrak{n} = \mathfrak{m}A_\mathfrak{n}$. Reason: By Nakayama's lemma for the local ring $A_\mathfrak{n}$ and the $A_\mathfrak{n}$-module $\mathfrak{m}A_\mathfrak{n}$, we conclude that $\mathfrak{n}A_\mathfrak{n} = \mathfrak{m}A_\mathfrak{n}$.

Next apply Nakayama's Lemma to the $B_\mathfrak{n}$-module $A_\mathfrak{n}$. The element $1 \in A_\mathfrak{n}$ gives a generator for $A_\mathfrak{n}/\mathfrak{n}A_\mathfrak{n} = A_\mathfrak{n}/\mathfrak{m}A_\mathfrak{n}$, which equals $B_\mathfrak{n}/\mathfrak{n}B_\mathfrak{n}$ (as both equal k), so we conclude that 1 also generates $A_\mathfrak{n}$ as a $B_\mathfrak{n}$-module as desired. □

1.4. *Exercise.* Use this to show that the dth Veronese morphism from $\mathbb{P}^n_k$, corresponding to the complete linear series (see Class 22) $|\mathcal{O}_{\mathbb{P}^n_k}(d)|$, is a closed immersion. Do the same for the Segre morphism from $\mathbb{P}^m_k \times_{\operatorname{Spec} k} \mathbb{P}^n_k$. (This is just for practice for using this criterion.

This is a weaker result than we had before; we've earlier checked this over an arbitrary base ring, and we are now checking it only over algebraically closed fields.)

2. A SERIES OF USEFUL REMARKS

Suppose now that $\mathcal{L}$ is an invertible sheaf on a curve C (which as always in this discussion is projective, geometrically integral and nonsingular, over a field k which is not necessarily algebraically closed). I'll give a series of small useful remarks that we will soon use to great effect.

2.1. $h^0(C, \mathcal{L}) = 0$ if $\deg \mathcal{L} < 0$. Reason: if there is a non-zero section, then the degree of $\mathcal{L}$ can be interpreted as the number of zeros minus the number of poles. But there are no poles, so this would have to be non-negative. A slight refinement gives:

2.2. $h^0(C, \mathcal{L}) = 0$ or 1 if $\deg \mathcal{L} = 0$. This is because if there is a section, then the degree of $\mathcal{L}$ is the number of zeros minus the number of poles. Then as there are no poles, there can be no zeros. Thus the section (call it s) vanishes nowhere, and gives a trivialization for the invertible sheaf. (Recall how this works: we have a natural bijection for any open set $\Gamma(U, \mathcal{L}) \leftrightarrow \Gamma(U, \mathcal{O}_U)$, where the map from left to right is $s' \mapsto s'/s$, and the map from right to left is $f \mapsto sf$.) Thus if there is a section, $\mathcal{L} \cong \mathcal{O}$. But we've already checked that for a geometrically integral and nonsingular curve C, $h^0(C, \mathcal{L}) = 1$.

2.3. Suppose p is any closed point of degree 1. (In other words, the residue field of p is k.) Then $h^0(C, \mathcal{L}) - h^0(C, \mathcal{L}(-p)) = 0$ or 1. Reason: consider $0 \to \mathcal{O}_C(-p) \to \mathcal{O}_C \to \mathcal{O}_p \to 0$, tensor with $\mathcal{L}$ (this is exact as $\mathcal{L}$ is locally free) to get

$$0 \to \mathcal{L}(-p) \to \mathcal{L} \to \mathcal{L}|_p \to 0.$$

Then $h^0(C, \mathcal{L}|_p) = 1$, so as the long exact sequence of cohomology starts off

$$0 \to H^0(C, \mathcal{L}(-p)) \to H^0(C, \mathcal{L}) \to H^0(C, \mathcal{L}|_p),$$

we are done.

2.4. Suppose for this remark that k is algebraically closed. (In particular, *all* closed points have degree 1 over k.) Then if $h^0(C, \mathcal{L}) - h^0(C, \mathcal{L}(-p)) = 1$ for *all* closed points p, then $\mathcal{L}$ is base-point-free, and hence induces a morphism from C to projective space. (Note that $\mathcal{L}$ has a finite-dimensional vector space of sections: all cohomology groups of all coherent sheaves on a projective k-scheme are finite-dimensional.) Reason: given any p, our equality shows that there exists a section of $\mathcal{L}$ that does not vanish at p.

2.5. Next, suppose p and q are distinct points of degree 1. Then $h^0(C, \mathcal{L}) - h^0(C, \mathcal{L}(-p - q)) = 0, 1$, or 2 (by repeating the argument of 2.3 twice). If $h^0(C, \mathcal{L}) - h^0(C, \mathcal{L}(-p-q)) = 2$, then necessarily

$$h^0(C, \mathcal{L}) = h^0(C, \mathcal{L}(-p)) + 1 = h^0(C, \mathcal{L}(-q)) + 1 = h^0(C, \mathcal{L}(-p - q)) + 2. \quad (1)$$

I claim that the linear system $\mathcal{L}$ separates points p and q, by which I mean that the corresponding map f to projective space satisfies $f(p) \neq f(q)$. Reason: there is a hyperplane of projective space passing through p but not passing through q, or equivalently, there is a section of $\mathcal{L}$ vanishing at p but not vanishing at q. This is because of the last equality in (1).

2.6. By the same argument as above, if p is a point of degree 1, then $h^0(C, \mathcal{L}) - h^0(C, \mathcal{L}(-2p)) = 0, 1$, or 2. I claim that if this is 2, then map corresponds to $\mathcal{L}$ (which is already seen to be base-point-free from the above) separates the tangent vectors at p. To show this, I need to show that the cotangent map is *surjective*. To show surjectivity onto a one-dimensional vector space, I just need to show that the map is non-zero. So I need to give a function on the target vanishing at the image of p that pulls back to a function that vanishes at p to order 1 but not 2. In other words, I want a section of $\mathcal{L}$ vanishing at p to order 1 but not 2. But that is the content of the statement $h^0(C, \mathcal{L}(-p)) - h^0(C, \mathcal{L}(-2p)) = 1$.

2.7. Combining some of our previous comments: suppose C is a curve over an *algebraically closed* field k, and $\mathcal{L}$ is an invertible sheaf such that for *all* closed points p and q, *not necessarily distinct*, $h^0(C, \mathcal{L}) - h^0(C, \mathcal{L}(-p-q)) = 2$, then $\mathcal{L}$ gives a closed immersion into projective space, as it separates points and tangent vectors, by Theorem 1.3.

2.8. We now bring in Serre duality. I claim that $\deg \mathcal{L} > 2g-2$ implies

$$\boxed{h^0(C, \mathcal{L}) = \deg \mathcal{L} - g - 1.}$$

This is important — remember this! Reason: $h^1(C, \mathcal{L}) = h^0(C, \mathcal{K} \otimes \mathcal{L}^\vee)$; but $\mathcal{K} \otimes \mathcal{L}^\vee$ has negative degree (as $\mathcal{K}$ has degree $2g-2$), and thus this invertible sheaf has no sections. Thus Riemann-Roch gives us the desired result.

Exercise. Suppose $\mathcal{L}$ is a degree $2g-2$ invertible sheaf. Show that it has $g-1$ or g sections, and it has g sections if and only if $\mathcal{L} \cong \mathcal{K}$.

2.9. We now come to our most important conclusion. Thus if k is algebraically closed, then $\deg \mathcal{L} \geq 2g$ implies that $\mathcal{L}$ is basepoint free (and hence determines a morphism to projective space). Also, $\deg \mathcal{L} \geq 2g+1$ implies that this is in fact a closed immersion. Remember this! [k need not be algebraically closed.]

2.10. I now claim (for the people who like fields that are not algebraically closed) that *the previous remark holds true even if* k *is not algebraically closed.* Here is why: suppose C is our curve, and $C_{\overline{k}} := C \otimes_k \overline{k}$ is the base change to the algebraic closure (which we are assuming is connected and nonsingular), with $\pi : C_{\overline{k}} \to C$ (which is an affine morphism, as it is obtained by base change from the affine morphism $\operatorname{Spec} \overline{k} \to \operatorname{Spec} k$). Then $H^0(C, \mathcal{L}) \otimes_k \overline{k} \cong H^0(C_{\overline{k}}, \pi^* \mathcal{L})$ for reasons I explained last day (see the first exercise on the class 33 notes, and also on problem set 15).

$$\begin{array}{ccc} C_{\overline{k}} & \xrightarrow{\pi} & C \\ \downarrow & & \downarrow \\ \operatorname{Spec} \overline{k} & \longrightarrow & \operatorname{Spec} k \end{array}$$

Let $s_0, \dots, s_n$ be a basis for the k-vector space $H^0(C, \mathcal{L})$; they give a basis for the $\overline{k}$-vector space $H^0(C_{\overline{k}}, \pi^*\mathcal{L})$. If $\mathcal{L}$ has degree at least $2g$, then these sections have no common zeros on $C_{\overline{k}}$; but this means that they have no common zeros on C. If $\mathcal{L}$ has degree at least $2g+1$, then these sections give a closed immersion $C_{\overline{k}} \hookrightarrow \mathbb{P}^n_{\overline{k}}$. Then I claim that $f : C \to \mathbb{P}^n_k$ (given by the same sections) is also a closed immersion. Reason: we can check this on each affine open subset $U = \operatorname{Spec} A \subset \mathbb{P}^n_k$. Now f has finite fibers, and is projective, hence is a finite morphism (and in particular affine). Let $\operatorname{Spec} B = f^{-1}(U)$. We wonder if $A \to B$ is a surjection of rings. But we know that this is true upon base changing by $\overline{k}$: $A \otimes_k \overline{k} \to B \otimes_k \overline{k}$ is surjective. So we are done.

We're now ready to take these facts and go to the races.

3. Genus 0

3.1. *Claim. — Suppose C is genus 0, and C has a k-valued point. Then $C \cong \mathbb{P}^1_k$.*

Of course C automatically has a k-point if k is algebraically closed. Thus we see that all genus 0 (integral, nonsingular) curves over an algebraically closed field are isomorphic to $\mathbb{P}^1$.

If k is not algebraically closed, then C needn't have a k-valued point: witness $x^2 + y^2 + z^2 = 0$ in $\mathbb{P}^2_{\mathbb{R}}$. We have already observed that this curve is *not* isomorphic to $\mathbb{P}^1_{\mathbb{R}}$, because it doesn't have an $\mathbb{R}$-valued point.

Proof. Let p be the point, and consider $\mathcal{L} = \mathcal{O}(p)$. Then $\deg \mathcal{L} = 1$, so we can apply what we know above: first of all, $h^0(C, \mathcal{L}) = 2$, and second of all, these two sections give a closed immersion in to $\mathbb{P}^1_k$. But the only closed immersion of a curve into $\mathbb{P}^1_k$ is the isomorphism! □

As a fun bonus, we see that the weird real curve $x^2 + y^2 + z^2 = 0$ in $\mathbb{P}^2_{\mathbb{R}}$ has no *divisors* of degree 1 over $\mathbb{R}$; otherwise, we could just apply the above argument to the corresponding line bundle.

Our weird curve shows us that over a non-algebraically closed field, there can be genus 0 curves that are not isomorphic to $\mathbb{P}^1_k$. The next result lets us get our hands on them as well.

3.2. *Claim. — All genus 0 curves can be described as conics in $\mathbb{P}^2_k$.*

Proof. Any genus 0 curve has a degree -2 line bundle — the canonical bundle $\mathcal{K}$. Thus any genus 0 curve has a degree 2 line bundle: $\mathcal{L} = \mathcal{K}^\vee$. We apply our machinery to this bundle: $h^0(C, \mathcal{L}) = 3 \geq 2g + 1$, so this line bundle gives a closed immersion into $\mathbb{P}^2$. [This proof is not complete if $k = \overline{k}$, as the criterion we are using requires this hypothesis. Exercise: Use §2.10 to give a complete proof.] □

3.3. *Exercise.* Suppose C is a genus 0 curve (projective, geometrically integral and nonsingular). Show that C has a point of degree at most 2.

We will use the following result later.

3.4. *Claim. — Suppose* C *is not isomorphic to* $\mathbb{P}^1_k$ *(with no restrictions on the genus of* C*), and* $\mathcal{L}$ *is an invertible sheaf of degree* 1*. Then* $h^0(C, \mathcal{L}) < 2$*.*

Proof. Otherwise, let s_1 and s_2 be two (independent) sections. As the divisor of zeros of s_i is the degree of $\mathcal{L}$, each vanishes at a single point p_i (to order 1). But $p_1 \neq p_2$ (or else s_1/s_2 has no poles or zeros, i.e. is a constant function, i.e. s_1 and s_2 are dependent). Thus we get a map $C \to \mathbb{P}^1$ which is basepoint free. This is a finite degree 1 map of nonsingular curves, which induces a degree 1 extension of function fields, i.e. an isomorphism of function fields, which means that the curves are isomorphic. But we assumed that C is not isomorphic to $\mathbb{P}^1_k$. □

4. Genus ≥ 2

It might make most sense to jump to genus 1 at this point, but the theory of elliptic curves is especially rich and beautiful, so I'll leave it for the end.

In general, the curves have quite different behaviors (topologically, arithmetically, geometrically) depending on whether $g = 0$, $g = 1$, or $g > 2$. This trichotomy extends to varieties of higher dimension. I gave a very brief discussion of this trichotomy for curves. For example, arithmetically, genus 0 curves can have lots and lots of points, genus 1 curves can have lots of points, and by Faltings' Theorem (Mordell's Conjecture) any curve of genus at least 2 has at most finitely many points. (Thus we knew before Wiles that $x^n + y^n = z^n$ in $\mathbb{P}^2$ has at most finitely many solutions for $n \geq 4$, as such curves have genus $\binom{n-1}{2} > 1$.) Geometrically, Riemann surfaces of genus 0 are positively curved, Riemann surfaces of genus 1 are flat, and Riemann surfaces of genus 1 are negatively curved. We will soon see that curves of genus at least 2 have finite automorphism groups, while curves of genus 1 have some automorphisms (a one-dimensional family), and (we've seen earlier) curves of genus 1 (over an algebraically closed field) have a three-dimensional automorphism group.

4.1. **Genus 2.** Fix a curve C of genus 2. Then $\mathcal{K}$ is degree 2, and has 2 sections. I claim that $\mathcal{K}$ is base-point-free. Otherwise, if p is a base point, then $\mathcal{K}(-p)$ is a degree 1 invertible sheaf with 2 sections, and we just showed (Claim 3.4) that this is impossible. Thus we have a double cover of $\mathbb{P}^1$. Conversely, any double cover $C \to \mathbb{P}^1$ arises from a degree 2 invertible sheaf with at least 2 sections, so by one of our useful facts, if $g(C) = 2$, this invertible sheaf must be the canonical bundle (as the only degree 2 invertible sheaf on a genus 2 curve with at least 2 sections is $\mathcal{K}_C$). Hence we have a natural bijection between genus 2 curves and genus 2 double covers of $\mathbb{P}^1$.

We now specialize to the case where $k = \overline{k}$, and the characteristic of k is 0. (All we will need, once we actually prove the Riemann-Hurwitz formula, is that the characteristic be distinct from 2.) Then the Riemann-Hurwitz formula shows that the cover is branched over 6 points. We will see next day that a double cover is determined by its branch points. Hence genus 2 curves are in bijection with unordered sextuples of points on $\mathbb{P}^1$. There is thus a 3-dimensional family of genus 2 curves — we have found them all!

(This is still a little imprecise; we would like to say that the moduli space of genus 2 curves is of dimension 3, but we haven't defined what we mean by moduli space!)

More generally, we may see next week (admittedly informally) that if $g > 1$, the curves of genus g "form a family" of dimension $3g - 3$. (If we knew the meaning of "moduli space", we would say that the dimension of the moduli space of genus g curves $\mathcal{M}_g$ is $3g - 3$.) What goes wrong in genus 0 and 1? The following table (as yet unproved by us!) might help.

genus	dimension of family of curves	dimension of automorphism group of curve
0	0	3
1	1	1
2	3	0
3	6	0
4	9	0
5	12	0
$\vdots$	$\vdots$	$\vdots$

You can probably see the pattern. This is a little like the behavior of the Hilbert function: the dimension of the moduli space is "eventually polynomial", so there is something that is better-behaved that is an alternating sum, and once the genus is sufficiently high, the "error term" becomes zero. The interesting question then becomes: why is the "right" notion the second column of the table minus the third? (In fact the second column is $h^1(C, T_C)$, where T_C is the tangent bundle — not yet defined — and the third column is $h^0(C, T_C)$. All other cohomology groups of the tangent bundle vanish by dimensional vanishing.)

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FOUNDATIONS OF ALGEBRAIC GEOMETRY CLASS 34 CRIB SHEET

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This is a summary of useful facts we proved or assumed. We will use them in the next two classes.

All curves C are projective, and geometrically integral and nonsingular over a field k.

There is an invertible sheaf (rank bundle) $\mathcal{K}$, called the *dualizing sheaf*; it is also the sheaf of differentials (in this guise it is called $\Omega_{C/k}$), and the cotangent bundle. $\deg \mathcal{K} = 2g-2$.

The *Riemann-Hurwitz formula* is $2g_C - 2 = d(2g_{C'} - 2) + \deg R$, where R is the *ramification divisor*.

Serre duality. There is an isomorphism $H^0(C, \mathcal{K}) \xrightarrow{\sim} k$ For any coherent sheaf $\mathcal{F}$, the natural map

$$\boxed{H^0(C,\mathcal{F}) \otimes_k H^1(C, \mathcal{K} \otimes \mathcal{F}^\vee) \to H^0(C, \mathcal{K})}$$

is a perfect pairing, so in particular, $h^0(C,\mathcal{F}) = h^1(C, \mathcal{K} \otimes \mathcal{F}^\vee)$. (As $g := h^1(C, \mathcal{O}_C)$, we get $h^0(C, \mathcal{K}) = g$ as well.) Hence Riemann-Roch now states:

$$h^0(C,\mathcal{L}) - h^1(C,\mathcal{L}) = \deg \mathcal{L} - g + 1.$$

Applying this to $\mathcal{L} = \mathcal{K}$, we get $\deg \mathcal{K} = 2g - 2$ (promised earlier).

Suppose now that $\mathcal{L}$ is an invertible sheaf on C.

0.1. $h^0(C,\mathcal{L}) = 0$ if $\deg \mathcal{L} < 0$. $h^0(C,\mathcal{L}) = 0$ or 1 if $\deg \mathcal{L} = 0$.

0.2. Suppose p is any closed point of degree 1. (In other words, the residue field of p is k.) Then $h^0(C,\mathcal{L}) - h^0(C,\mathcal{L}(-p)) = 0$ or 1.

0.3. Suppose for this remark that k is algebraically closed. (In particular, *all* closed points have degree 1 over k.) Then if $h^0(C,\mathcal{L}) - h^0(C,\mathcal{L}(-p)) = 1$ for *all* closed points p, then $\mathcal{L}$ is base-point-free, and hence induces a morphism from C to projective space.

0.4. Suppose p and q are distinct points of degree 1. Then $h^0(C,\mathcal{L}) - h^0(C,\mathcal{L}(-p-q)) = 0$, 1, or 2. If $h^0(C,\mathcal{L}) - h^0(C,\mathcal{L}(-p-q)) = 2$, then $\mathcal{L}$ separates points p and q, by which I mean that the corresponding map f to projective space satisfies $f(p) \neq f(q)$.

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0.5. If p is a point of degree 1, then $h^0(C, \mathcal{L}) - h^0(C, \mathcal{L}(-2p)) = 0$, 1, or 2. If it is 2, then the map corresponding to $\mathcal{L}$ separates the tangent vectors at p.

0.6. Combining some of our previous comments: suppose C is a curve over an *algebraically closed* field k, and $\mathcal{L}$ is an invertible sheaf such that for *all* closed points p and q, *not necessarily distinct*, $h^0(C, \mathcal{L}) - h^0(C, \mathcal{L}(-p-q)) = 2$, then $\mathcal{L}$ gives a closed immersion into projective space.

0.7. We now bring in Serre duality. $\deg \mathcal{L} > 2g - 2$ implies

$$\boxed{h^0(C, \mathcal{L}) = \deg \mathcal{L} - g - 1.}$$

If $\mathcal{L}$ is a degree $2g-2$ invertible sheaf, then $\mathcal{L}$ has $g-1$ or g sections, and it has g sections if and only if $\mathcal{L} \cong \mathcal{K}$.

0.8. *Our most important conclusion.* $\deg \mathcal{L} \geq 2g$ implies that $\mathcal{L}$ is basepoint free (and hence determines a morphism to projective space). Also, $\deg \mathcal{L} \geq 2g+1$ implies that this is in fact a closed immersion. Remember this!

0.9. Suppose C is not isomorphic to $\mathbb{P}^1_k$ (with no restrictions on the genus of C), and $\mathcal{L}$ is an invertible sheaf of degree 1. Then $h^0(C, \mathcal{L}) < 2$.

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FOUNDATIONS OF ALGEBRAIC GEOMETRY CLASS 35

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CONTENTS

Last day: More fun with curves: Serre duality, criterion for closed immersion, a series of useful remarks, curves of genus 0 and 2.

Today: hyperelliptic curves; curves of genus at least 2; elliptic curves take 1.

Last day we started studying curves in detail, using things we'd proved. Today, we'll continue to use these things. (See the "Class 34 crib sheet" for a reminder of what we know.)

1. HYPERELLIPTIC CURVES

As usual, we begin by working over an arbitrary field k, and specializing only when we need to. A curve C of genus at least 2 is *hyperelliptic* if it admits a degree 2 cover of $\mathbb{P}^1$. This map is often called the *hyperelliptic map*.

Equivalently, C is hyperelliptic if it admits a degree 2 invertible sheaf $\mathcal{L}$ with $h^0(C, \mathcal{L}) = 2$.

1.1. *Exercise..* Verify that these notions are the same. Possibly in the course of doing this, verify that if C is a curve, and $\mathcal{L}$ has a degree 2 invertible sheaf with at least 2 (linearly independent) sections, then $\mathcal{L}$ has precisely two sections, and that this $\mathcal{L}$ is base-point free and gives a hyperelliptic map.

The degree 2 map $C \to \mathbb{P}^1$ gives a degree 2 extension of function fields $\mathrm{FF}(C)$ over $\mathrm{FF}(\mathbb{P}^1) \cong k(t)$. If the characteristic is not 2, this extension is necessarily Galois, and the induced involution on C is called the *hyperelliptic involution*.

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1.2. *Proposition. — If $\mathcal{L}$ corresponds to a hyperelliptic cover $C \to \mathbb{P}^1$, then $\mathcal{L}^{\otimes(g-1)} \cong \mathcal{K}_C$.*

Proof. Compose the hyperelliptic map with the $(g-1)$th Veronese map:

$$C \xrightarrow{\mathcal{L}} \mathbb{P}^1 \xrightarrow{\mathcal{O}_{\mathbb{P}^1}(g-1)} \mathbb{P}^{g-1}.$$

The composition corresponds to $\mathcal{L}^{\otimes(g-1)}$. This invertible sheaf has degree $2g-2$, and the image is nondegenerate in $\mathbb{P}^{g-1}$, and hence has at least g sections. But one of our useful facts (and indeed an exercise) was that the only invertible sheaf of degree $2g-2$ with (at least) g sections is the canonical sheaf. $\square$

1.3. *Proposition. — If a curve (of genus at least 2) is hyperelliptic, then it is hyperelliptic in "only one way". In other words, it admits only one double cover of $\mathbb{P}^1$.*

Proof. If C is hyperelliptic, then we can recover the hyperelliptic map by considering the canonical map: it is a double cover of a degree $g-1$ rational normal curve (by the previous Proposition), and this double cover is the hyperelliptic cover (also by the proof of the previous Proposition). $\square$

Next, we invoke the Riemann-Hurwitz formula. We assume the $\operatorname{char} k = 0$, and $k = \overline{k}$, so we can invoke this black box. However, when we actually discuss differentials, and prove the Riemann-Hurwitz formula, we will see that we can just require $\operatorname{char} k \neq 2$ (and $k = \overline{k}$).

The Riemann-Hurwitz formula implies that hyperelliptic covers have precisely $2g+2$ (distinct) branch points. We will see in a moment that the branch points determine the curve (Claim 1.4).

Assuming this, we see that hyperelliptic curves of genus g correspond to precisely $2g+2$ points on $\mathbb{P}^1$ modulo S_{2g+2}, and modulo automorphisms of $\mathbb{P}^1$. Thus "the space of hyperelliptic curves" has dimension

$$2g + 2 - \dim \operatorname{Aut} \mathbb{P}^1 = 2g - 1.$$

(As usual, this is not a well-defined statement, because as yet we don't know what we mean by "the space of hyperelliptic curves". For now, take it as a plausibility statement.) If we believe that the curves of genus g form a family of dimension $3g-3$, we have shown that "most curves are not hyperelliptic" if $g > 2$ (or on a milder note, there exists a hyperelliptic curve of each genus $g > 2$).

1.4. *Claim. — Assume $\operatorname{char} k \neq 2$ and $k = \overline{k}$. Given n distinct points on $\mathbb{P}^1$, there is precisely one cover branched at precisely these points if n is even, and none if n is odd.*

In particular, the branch points determine the hyperelliptic curve. (We also used this fact when discussing genus 2 curves last day.)

Proof. Suppose we have a double cover of $\mathbb{A}^1$, $C \to \mathbb{A}^1$, where x is the coordinate on $\mathbb{A}^1$. This induces a quadratic field extension K over $k(x)$. As $\operatorname{char} k \neq 2$, this extension is Galois. Let σ be the hyperelliptic involution. Let y be an element of K such that $\sigma(y) = -y$, so 1 and y form a basis for K over the field $k(x)$ (and are eigenvectors of σ). Now $y^2 \in k(x)$, so we can replace y by an appropriate $k(x)$-multiple so that y^2 is a polynomial, with no repeated factors, and monic. (This is where we use the hypothesis that k is algebraically closed, to get leading coefficient 1.) Thus $y^2 = x^n + a_{n-1}x^{n-1} + \cdots + a_0$. The branch points correspond to those values of x for which there is exactly one value of y, i.e. the roots of the polynomial. As we have no double roots, the curve is nonsingular. Let this cover be $C' \to \mathbb{A}^1$. Both C and C' are normalizations of $\mathbb{A}^1$ in this field extension, and are thus isomorphic. Thus every double cover can be written in this way, and in particular, if the branch points are $r_1, \dots, r_n$, the cover is $y^2 = (x - r_1) \cdots (x - r_n)$.

We now consider the situation over $\mathbb{P}^1$. A double cover can't be branched over an odd number of points by the Riemann-Hurwitz formula. Given an even number of points r_1, …, r_n in $\mathbb{P}^1$, choose an open subset $\mathbb{A}^1$ containing all n points. Construct the double cover of $\mathbb{A}^1$ as explained in the previous paragraph: $y^2 = (x - r_1) \cdots (x - r_n)$. Then take the normalization of $\mathbb{P}^1$ in this field extension. Over the open $\mathbb{A}^1$, we recover this cover. We just need to make sure we haven't accidentally acquired a branch point at the missing point $\infty = \mathbb{P}^1 - \mathbb{A}^1$. But the total number of branch points is even, and we already have an even number of points, so there is no branching at ∞. □

Remark. If k is not algebraically closed (but of characteristic not 2), the above argument shows that if we have a double cover of $\mathbb{A}^1$, then it is of the form $y^2 = af(x)$, where f is monic, and $a \in k^*/(k^*)^2$. So (assuming the field doesn't contain all squares) a double cover does *not* determine the same curve. Moreover, see that this failure is classified by $k^*/(k^*)^2$. Thus we have lots of curves that are not isomorphic over k, but become isomorphic over $\overline{k}$. These are often called *twists* of each other.

(In particular, even though haven't talked about elliptic curves yet, we definitely have two elliptic curves over $\mathbb{Q}$ with the same j-invariant, that are not isomorphic.)

2. Curves of genus 3

Suppose C is a curve of genus 3. Then $\mathcal{K}$ has degree $2g - 2 = 4$, and has $g = 3$ sections.

2.1. *Claim. — $\mathcal{K}$ is base-point-free, and hence gives a map to $\mathbb{P}^2$.*

Proof. We check base-point-freeness by working over the algebraic closure $\overline{k}$. For any point p, by Riemann-Roch,

$$h^0(C, \mathcal{K}(-p)) - h^0(C, \mathcal{O}(p)) = \deg(\mathcal{K}(-p)) - g + 1 = 3 - 3 + 1 = 1.$$

But $h^0(C, \mathcal{O}(p)) = 0$ by one of our useful facts, so

$$h^0(C, \mathcal{K}(-p)) = 2 = h^0(C, \mathcal{K}) - 1.$$

Thus p is not a base-point of $\mathcal{K}$, so $\mathcal{K}$ is base-point-free. □

The next natural question is: Is this a closed immersion? Again, we can check over algebraic closure. We use our "closed immersion test" (again, see our useful facts). If it *isn't* a closed immersion, then we can find two points p and q (possibly identical) such that

$$h^0(C, \mathcal{K}) - h^0(C, \mathcal{K}(-p-q)) = 2,$$

i.e. $h^0(C, \mathcal{K}(-p-q)) = 2$. But by Serre duality, this means that $h^0(C, \mathcal{O}(p+q)) = 2$. We have found a degree 2 divisor with 2 sections, so C is hyperelliptic. (Indeed, I could have skipped that sentence, and made this observation about $\mathcal{K}(-p-q)$, but I've done it this way in order to generalize to higher genus.) Conversely, if C is hyperelliptic, then we already know that $\mathcal{K}$ gives a double cover of a nonsingular conic in $\mathbb{P}^2$ (also known as a rational normal curve of degree 2).

Thus we conclude that if C is not hyperelliptic, then the canonical map describes C as a degree 4 curve in $\mathbb{P}^2$.

Conversely, any quartic plane curve is canonically embedded. Reason: the curve has genus 3 (we can compute this — see our discussion of Hilbert functions), and is mapped by an invertible sheaf of degree 4 with 3 sections. Once again, we use the useful fact saying that the only invertible sheaf of degree $2g-2$ with g sections is $\mathcal{K}$.

Exercise. Show that the nonhyperelliptic curves of genus 3 form a family of dimension 6. (Hint: Count the dimension of the family of nonsingular quartics, and quotient by $\operatorname{Aut} \mathbb{P}^2 = PGL(3)$.)

The genus 3 curves thus seem to come in two families: the hyperelliptic curves (a family of dimension 5), and the nonhyperelliptic curves (a family of dimension 6). This is misleading — they actually come in a single family of dimension 6.

In fact, hyperelliptic curves are naturally limits of nonhyperelliptic curves. We can write down an explicit family. (This next paragraph will necessarily require some handwaving, as it involves topics we haven't seen yet.) Suppose we have a hyperelliptic curve branched over $2g+2=8$ points of $\mathbb{P}^1$. Choose an isomorphism of $\mathbb{P}^1$ with a conic in $\mathbb{P}^2$. There is a nonsingular quartic meeting the conic at precisely those 8 points. (This requires Bertini's theorem, so I'll skip that argument.) Then if f is the equation of the conic, and g is the equation of the quartic, then $f^2 + t^2 g$ is a family of quartics that are nonsingular for most t (nonsingular is an open condition as we will see). The $t=0$ case is a double conic. Then it is a fact that if you normalize the family, the central fiber (above $t=0$) turns into our hyperelliptic curve. Thus we have expressed our hyperelliptic curve as a limit of nonhyperelliptic curves.

3. Genus at least 3

We begin with two exercises in general genus, and then go back to genus 4.

Exercise Suppose C is a genus g curve. Show that if C is not hyperelliptic, then the canonical bundle gives a closed immersion $C \hookrightarrow \mathbb{P}^{g-1}$. (In the hyperelliptic case, we have already seen that the canonical bundle gives us a double cover of a rational normal curve.) Hint: follow the genus 3 case. Such a curve is called a *canonical curve*.

Exercise. Suppose C is a curve of genus $g > 1$, over a field k that is not algebraically closed. Show that C has a closed point of degree at most $2g-2$ over the base field. (For comparison: if $g = 1$, there is no such bound!)

We next consider nonhyperelliptic curves C of genus 4. Note that $\deg \mathcal{K} = 6$ and $h^0(C, \mathcal{K}) = 4$, so the canonical map expresses C as a sextic curve in $\mathbb{P}^3$. We shall see that all such C are complete intersections of quadric surfaces and cubic surfaces, and vice versa.

By Riemann-Roch, $\mathcal{K}^{\otimes 2}$ has $\deg \mathcal{K}^{\otimes 2} - g + 1 = 12 - 4 + 1 = 9$ sections. That's one less than $\dim \operatorname{Sym}^2 \Gamma(C, \mathcal{K}) = \binom{4+1}{2}$. Thus there is at least one quadric in $\mathbb{P}^3$ that vanishes on our curve C. Translation: C lies on at least on quadric Q. Now quadrics are either double planes, or the union of two planes, or cones, or nonsingular quadrics. (They corresponds to quadric forms of rank 1, 2, 3, and 4 respectively.) Note that C can't lie in a plane, so Q must be a cone or nonsingular. In particular, Q is irreducible.

Now C can't lie on *two* (distinct) such quadrics, say Q and Q'. Otherwise, as Q and Q' have no common components (they are irreducible and not the same!), $Q \cap Q'$ is a curve (not necessarily reduced or irreducible). By Bezout's theorem, it is a curve of degree 4. Thus our curve C, being of degree 6, cannot be contained in $Q \cap Q'$.

We next consider cubics. By Riemann-Roch, $\mathcal{K}^{\otimes 3}$ has $\deg \mathcal{K} \otimes 3 - g + 1 = 18 - 4 + 1 = 15$ sections. Now $\dim \operatorname{Sym}^3 \Gamma(C, \mathcal{K})$ has dimension $\binom{4+2}{3} = 20$. Thus C lies on at least a 5-dimensional vector space of cubics. Admittedly 4 of them come from multiplying the quadric Q by a linear form $(?w + ?x + ?y + ?z)$. But hence there is still one cubic K whose underlying form is not divisible by the quadric form Q (i.e. K doesn't contain Q.) Then K and Q share no component, so $K \cap Q$ is a complete intersection. By Bezout's theorem, we obtain a curve of degree 6. Our curve C has degree 6. This suggests that $C = K \cap Q$. In fact, $K \cap Q$ and C have the same Hilbert polynomial, and $C \subset K \cap Q$. Hence $C = K \cap Q$ by the following exercise.

Exercise. Suppose $X \subset Y \subset \mathbb{P}^n$ are a sequence of closed subschemes, where X and Y have the same Hilbert polynomial. Show that $X = Y$. Hint: consider the exact sequence

$$0 \to \mathcal{I}_{X/Y} \to \mathcal{O}_Y \to \mathcal{O}_X \to 0.$$

Show that if the Hilbert polynomial of $\mathcal{I}_{X/Y}$ is 0, then $\mathcal{I}_{X/Y}$ must be the 0 sheaf.

We now consider the converse, and who that any nonsingular complete intersection C of a quadric surface with a cubic surface is a canonically embedded genus 4 curve. It is not hard to check that it has genus 3 (again, using our exercises involving Hilbert functions). *Exercise.* Show that $\mathcal{O}_C(1)$ has 4 sections. (Translation: C doesn't lie in a hyperplane.) Hint: long exact sequences! Again, the only degree $2g-2$ invertible sheaf with g sections is the canonical sheaf, so $\mathcal{O}_C(1) \cong \mathcal{K}_C$, and C is indeed canonically embedded.

Exercise. Conclude that nonhyperelliptic curves of genus 4 "form a family of dimension $9 = 3g - 3$". (Again, this isn't a mathematically well-formed question. So just give a plausibility argument.)

On to genus 5!

Exercise. Suppose C is a nonhyperelliptic genus 5 curve. The canonical curve is degree 8 in $\mathbb{P}^4$. Show that it lies on a three-dimensional vector space of quadrics (i.e. it lies on 3 independent quadrics). Show that a nonsingular complete intersection of 3 quadrics is a canonical genus 5 curve.

In fact a canonical genus 5 is always a complete intersection of 3 quadrics.

Exercise. Show that the complete intersections of 3 quadrics in $\mathbb{P}^4$ form a family of dimension $12 = 3g - 3$.

This suggests that the nonhyperelliptic curves of genus 5 form a dimension 12 family.

So we've managed to understand curves of genus up to 5 (starting with 3) by thinking of canonical curves as complete intersections. Sadly our luck has run out.

Exercise. Show that if $C \subset \mathbb{P}^{g-1}$ is a canonical curve of genus $g \geq 6$, then C is *not* a complete intersection. (Hint: Bezout.)

4. Genus 1

Finally, we come to the very rich case of curves of genus 1.

Note that $\mathcal{K}$ is an invertible sheaf of degree $2g - 2 = 0$ with $g = 1$ section. But the only degree 0 invertible sheaf with a section is the trivial sheaf, so we conclude that $\mathcal{K} \cong \mathcal{O}$.

Next, note that if $\deg \mathcal{L} > 0$, then Riemann-Roch and Serre duality gives

$$h^0(C, \mathcal{L}) = h^0(C, \mathcal{L}) - h^0(C, \mathcal{K} \otimes \mathcal{L}^\vee) = h^0(C, \mathcal{L}) - h^0(C, \mathcal{L}^\vee) = \deg \mathcal{L}$$

as an invertible sheaf $\mathcal{L}^\vee$ of negative degree necessarily has no sections.

An *elliptic curve* is a genus 1 curve E with a choice of k-valued point p. (Note: it is *not* the same as a genus 1 curve — some genus 1 curves have no k-valued points. However, if $k = \overline{k}$, then any closed point is k-valued; but still, the choice of a closed point should always be considered part of the definition of an elliptic curve.)

Note that $\mathcal{O}_E(2p)$ has 2 sections, so the argument given in the hyperelliptic section shows that E admits a double cover of $\mathbb{P}^1$. One of the branch points is $2p$: one of the sections of $\mathcal{O}_E(2p)$ vanishes to p of order 2, so there is a point of $\mathbb{P}^1$ consists of p (with multiplicity 2). Assume now that $k = \overline{k}$, so we can use the Riemann-Hurwitz formula. Then the Riemann-Hurwitz formula shows that E has 4 branch points (p and three others). Conversely, given 4 points in $\mathbb{P}^1$, we get a map ($y^2 = \cdots$). This determines C (as shown in the hyperelliptic section). Thus elliptic curves correspond to 4 points in $\mathbb{P}^1$, where one is marked p, up to automorphisms of $\mathbb{P}^1$. (Equivalently, by placing p at ∞, elliptic curves correspond to 3 points in $\mathbb{A}^1$, up to affine maps $x \mapsto ax + b$.)

If the three other points are temporarily labeled q_1, q_2, q_3, there is a unique automorphism of $\mathbb{P}^1$ taking p, q_1, q_2 to $(\infty, 0, 1)$ respectively (as $\operatorname{Aut} \mathbb{P}^1$ is three-transitive). Suppose that q_3 is taken to some number λ under this map. Notice that $\lambda \neq 0, 1, \infty$.

- If we had instead sent p, q_2, q_1 to $(\infty, 0, 1)$, then q_3 would have been sent to $1 - \lambda$.
- If we had instead sent p, q_1, q_3 to $(\infty, 0, 1)$, then q_2 would have been sent to $1/\lambda$.
- If we had instead sent p, q_3, q_1 to $(\infty, 0, 1)$, then q_2 would have been sent to $1 - 1/\lambda = (\lambda - 1)/\lambda$.
- If we had instead sent p, q_2, q_3 to $(\infty, 0, 1)$, then q_2 would have been sent to $1/(1 - \lambda)$.
- If we had instead sent p, q_3, q_2 to $(\infty, 0, 1)$, then q_2 would have been sent to $1 - 1/(1 - \lambda) = \lambda/(\lambda - 1)$.

Thus these six values (in bijection with S_3) yield the same elliptic curve, and this elliptic curve will (upon choosing an ordering of the other 3 branch points) yield one of these six values.

Thus the elliptic curves over k corresponds to k-valued points of $\mathbb{P}^1 - \{0, 1, \lambda\}$, modulo the action of S_3 on λ given above. Consider the subfield of $k(\lambda)$ fixed by S_3. By Luroth's theorem, it must be of the form $k(j)$ for some $j \in k(\lambda)$. Note that λ should satisfy a sextic polynomial over $k(\lambda)$, as for each j-invariant, there are six values of λ in general.

At this point I should just give you j:

$$j = 2^8 \frac{(\lambda^2 - \lambda + 1)^3}{\lambda^2 (\lambda - 1)^2}.$$

But this begs the question: where did this formula come from? How did someone think of it?

Far better is to guess what j is. We want to come up with some $j(\lambda)$ such that $j(\lambda) = j(1/\lambda) = \cdots$. Hence we want some expression in λ that is invariant under this S_3-action. A silly choice would be the product of the six numbers $\lambda(1/\lambda) \cdots$ as this is 1.

A better idea is to add them all together. Unfortunately, if you do this, you'll get 3. (Here is one reason to realize this can't work: if you look at the sum, you'll realize that you'll get something of the form "degree at most 3" divided by "degree at most 2" (before cancellation). Then if $j' = p(\lambda)/q(\lambda)$, then λ satisfies (at most) a cubic over j. But we said that λ should satisfy a sextic over j'. The only way we avoid a contradiction is if $j' \in k$.

Our next attempt is to add up the six squares. When you do this by hand (it isn't hard), you get

$$j'' = \frac{2\lambda^6 - 6\lambda^5 + 9\lambda^4 - 8\lambda^3 + 9\lambda^2 - 6\lambda + 2}{\lambda^2 (\lambda - 1)^2}.$$

This works just fine: $k(j) \cong k(j'')$. If you really want to make sure that I'm not deceiving you, you can check (again by hand) that

$$2j/2^8 = \frac{2\lambda^6 - 6\lambda^5 + 12\lambda^4 - 14\lambda^3 + 12\lambda^2 - 6\lambda + 2}{\lambda^2(\lambda - 1)^2}.$$

The difference is 3.

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FOUNDATIONS OF ALGEBRAIC GEOMETRY CLASS 36

RAVI VAKIL

CONTENTS

Last day: More fun with curves: hyperelliptic curves; curves of genus at least 2; elliptic curves take 1.

Today: elliptic curves; the Picard variety; "the moduli space of curves has dimension $3g-3$."

This is the last class of the quarter! We'll finish off using what we know (and a little of what we'll know soon) to learn a great deal about curves.

There will be one more homework out early next week, due Thursday of the week after, covering this week's notes. We may well have a question-and-answer question on the last morning of class.

Once again, I'm going to use those important facts that we proved a couple of days ago, so I'll refer you to the class 34 crib sheet.

Let me first give you an exercise I should have given you last day.

Exercise. (a) Suppose C is a projective curve. Show that $C-p$ is affine. (Hint: show that $n \gg 0$, $\mathcal{O}(np)$ gives an embedding of C into some projective space $\mathbb{P}^m$, and that there is some hyperplane H meeting C precisely at p. Then $C-p$ is a closed subscheme of $\mathbb{P}^n - H$.) (b) If C is a geometrically integral nonsingular curve over a field k (i.e. all of our standing assumptions, minus projectivity), show that it is projective or affine.

Date: Thursday, March 2, 2006. Minor update January 30, 2007.

We're in the process of studying elliptic curves, i.e. curves E (projective, geometrically integral and nonsingular, over a field k) of genus 1, with a choice of a k-valued point p. (It is typical to use the letter E for the curve rather than C.)

So far we have seen that they admit double covers of $\mathbb{P}^1$, and that if $k = \overline{k}$, then the elliptic curves are classified by the j-invariant. The double cover corresponded to the invertible sheaf $\mathcal{O}_E(2p)$. We'll now consider $\mathcal{O}_E(np)$ for larger n.

1.1. **Degree 3.** Consider the degree 3 invertible sheaf $\mathcal{O}_E(3p)$. We consult our useful facts. By Riemann-Roch, $h^0(E, \mathcal{O}_E(3p)) = \deg(3p) - g + 1 = 3$. As $\deg E > 2g$, this gives a closed immersion. Thus we have a closed immersion $E \hookrightarrow \mathbb{P}^2_k$ as a cubic curve. Moreover, there is a line in $\mathbb{P}^2_k$ meeting E at point p with multiplicity 3. (Remark: a line in the plane meeting a smooth curve with multiplicity at least 2 is said to be a *tangent line*. A line in the plane meeting a smooth curve with multiplicity at least 3 is said to be a *flex line*.)

We can choose projective coordinates on $\mathbb{P}^2_k$ so that p maps to $[0; 1; 0]$, and the flex line is the line at infinity $z = 0$. Then the cubic is of the following form:

$$\begin{array}{ccccccc} ?x^3 & + & 0x^2y & + & 0xy^2 & + & 0y^3 \\ + & ?x^2z & + & ?xyz & + & ?y^2z & \\ & + & ?xz^2 & + & ?yz^2 & & \\ & & + & ?z^3 & & & = 0 \end{array}$$

The co-efficient of x is not 0 (or else this cubic is divisible by z). We can scale x so that the coefficient of x^3 is 1. The coefficient of y^2z is not 0 either (or else this cubic is singular at $x = z = 0$). As k is algebraically closed, we can scale y so that the coefficient of y^2z is 1. (More precisely, we are changing variables, say $y' = ay$ for some $a \in k$.) If the characteristic of k is not 2, then we can then replace y by $y + ?x + ?z$ so that the coefficients of xyz and yz^2 are 0, and if the characteristic of k is not 3, we can replace x by $x + ?z$ so that the coefficient of x^2z is also 0. In conclusion, if k is algebraically closed of characteristic not 2 or 3, we can write our elliptic curve in the form

$$y^2z = x^3 + ax^2z + bz^3.$$

This is called *Weierstrass normal form.* (If only some of the "bonus hypotheses" $k = \overline{k}$, $\operatorname{char} k \neq 2, 3$ is true, then we can perform only some of the reductions of course.)

Notice that we see the hyperelliptic description of the curve (by setting $z = 1$, or more precisely, by working in the distinguished open set $z \neq 0$ and using inhomogeneous coordinates). In particular, we can compute the j-invariant.

Here is the geometric explanation of why the double cover description is visible in the cubic description.

I drew a picture of the projective plane, showing the cubic, and where it met the z-axis (the line at infinity) — where the z-axis and x-axis meet — it has a flex there. I drew the lines through that point — vertical lines. Equivalently, you're just taking 2 of the 3 sections: x and z. These are two sections of $\mathcal{O}(3p)$, but they have a common zero — a base point at p. So you really get two sections of $\mathcal{O}(2p)$.

Exercise. Show that $\mathcal{O}(4p)$ embeds E in $\mathbb{P}^3$ as the complete intersection of two quadrics.

1.2. The group law.

1.3. *Theorem. — The closed points of* E *are in natural bijection with* $\operatorname{Pic}^0(E)$*, via* $x \leftrightarrow x - p$*. In particular, as* $\operatorname{Pic}^0(E)$ *is a group, we have endowed the closed points of* E *with a group structure.*

For those of you familiar with the complex analytic picture, this isn't surprising: E is isomorphic to the complex numbers modulo a lattice: $E \cong \mathbb{C}/\Lambda$.

This is currently just a bijection of sets. Given that E has a much richer structure (it has a generic point, and the structure of a variety), this is a sign that there should be a way of defining some *scheme* $\operatorname{Pic}^0(E)$, and that this should be an isomorphism of schemes.

Proof. For injectivity: $\mathcal{O}(x - p) \cong \mathcal{O}(y - p)$ implies $\mathcal{O}(x - y) \cong \mathcal{O}$. But as E is not genus 0, this is possible only if $x = y$.

For surjectivity: any degree 1 invertible sheaf has a section, so if $\mathcal{L}$ is any degree 0 invertible sheaf, then $\mathcal{O}(\mathcal{L}(p)) \cong \mathcal{O}(x)$ for some x. □

Note that more naturally, $\operatorname{Pic}^1(E)$ is in bijection with the points of E (without any choice of point p).

From now on, we will conflate closed points of E with degree 0 invertible sheaves on E.

Remark. The 2-torsion points in the group are the branch points in the double cover! Reason: q is a 2-torsion point if and only if $2q \sim 2p$ if and only if there is a section of $\mathcal{O}(2p)$ vanishing at q to order 2. (This is characteristic-independent.) Now assume that the characteristic is 0. (In fact, we'll only be using the fact that the characteristic is not 2.) By the Riemann-Hurwitz formula, there are 3 non-trivial torsion points. (Again, given the complex picture $E \cong \mathbb{C}/\Lambda$, this isn't surprising.)

Follow-up remark. An elliptic curve with *full level* n*-structure* is an elliptic curve with an isomorphism of its n-torsion points with $(\mathbb{Z}/n)^2$. (This notion will have problems if n is divisible by $\operatorname{char} k$.) Thus an elliptic curve with *full level 2 structure* is the same thing as an elliptic curve with an ordering of the three other branch points in its degree 2 cover description. Thus (if $k = \overline{k}$) these objects are parametrized by the λ-line (see the discussion last day).

Follow-up to the follow-up. There is a notion of moduli spaces of elliptic curves with full level n structure. Such moduli spaces are smooth curves (where this is interpreted appropriately), and have smooth compactifications. A *weight* k *level* n *modular form* is a section of $\mathcal{K}^{\otimes k}$ where $\mathcal{K}$ is the canonical sheaf of this "modular curve".

But let's get back down to earth.

1.4. *Proposition. — There is a morphism of varieties* $E \to E$ *sending a (degree* 1*) point to its inverse.*

In other words, the "inverse map" in the group law actually arises from a morphism of schemes — it isn't just a set map. This is another clue that $\text{Pic}^0(E)$ really wants to be a scheme.

Proof. It is the hyperelliptic involution $y \mapsto -y$! Here is why: if q and r are "hyperelliptic conjugates", then $q + r \sim 2p = 0$. □

We can describe addition in the group law using the cubic description. (Here a picture is absolutely essential, and at some later date, I hope to add it.) To find the sum of q and r on the cubic, we draw the line through q and r, and call the third point it meets s. Then we draw the line between p and s, and call the third point it meets t. Then $q + r = t$. Here's why: $q + r + s = p + s + t$ gives $(q - p) + (r - p) = (s - p)$.

(When the group law is often defined on the cubic, this is how it is done. Then you have to show that this is indeed a group law, and in particular that it is associative. We don't need to do this — $\text{Pic}^0 E$ is a group, so it is automatically associative.)

Note that this description works in all characteristics; we haven't required the cubic to be in Weierstrass normal form.

1.5. *Proposition. — There is a morphism of varieties* $E \times E \to E$ *that on degree* 1 *points sends* (q, r) *to* $q + r$.

"Proof". We just have to write down formulas for the construction on the cubic. This is no fun, so I just want to convince you that it can be done, rather than writing down anything explicit. They key idea is to define another map $E \times E \to E$, where if the input is (a, b), the output is the third point where the cubic meets the line, with the natural extension if the line doesn't meet the curve at three distinct points. Then we can use this to construct addition on the cubic. □

Aside: Discussion on group varieties and group schemes.

A *group variety* X over k is something that can be defined as follows: We are given an element $e \in X(k)$ (a k-valued point of X), and maps $i : X \to X$, $m : X \times X \to X$. They satisfy the hypotheses you'd expect from the definition of a group.

(i) associativity:

$$\begin{array}{ccc} X \times X \times X & \xrightarrow{(m,id)} & X \times X \\ \downarrow{\scriptstyle (id,m)} & & \downarrow{\scriptstyle m} \\ X \times X & \xrightarrow{m} & X \end{array}$$

commutes.

(ii) $X \xrightarrow{e,id} X \times X \xrightarrow{m} X$ and $X \xrightarrow{id,e} X \times X \xrightarrow{m} X$ are both the identity.

(iii) $X \xrightarrow{i,id} X \times X \xrightarrow{m} X$ and $X \xrightarrow{id,i} X \times X \xrightarrow{m} X$ are both e.

More generally, a *group scheme over a base* B is a scheme $X \to B$, with a section $e : B \to X$, and B-morphisms $i : X \to X$, $m : X \times_B X \to X$, satisfying the three axioms above.

More generally still, a *group object in a category* $\mathcal{C}$ is the above data (in a category $\mathcal{C}$), satisfying the same axioms. The e map is from the final object in the category to the group object.

You can check that a group object in the category of sets is in fact the same thing as a group. (This is symptomatic of how you take some notion and make it categorical. You write down its axioms in a categorical way, and if all goes well, if you specialize to the category of sets, you get your original notion. You can apply this to the notion of "rings" in an exercise below.)

1.6. *The functorial description.* It is often cleaner to describe this in a functorial way. Notice that if X is a group object in a category $\mathcal{C}$, then for any other element of the category, the set $\operatorname{Hom}(Y, X)$ is a group. Moreover, given any $Y_1 \to Y_2$, the induced map $\operatorname{Hom}(Y_2, X) \to \operatorname{Hom}(Y_1, X)$ is group homomorphism.

We can instead define a group object in a category to be an object X, along with morphisms $m : X \times X \to X$, $i : X \to X$, and e : final object $\to X$, such that these induce a natural group structure on $\operatorname{Hom}(Y, X)$ for each Y in the category, such that the forgetful maps are group homomorphisms. This is much cleaner!

Exercise. Verify that the axiomatic definition and the functorial definition are the same.

Exercise. Show that (E, p) is a group scheme. (Caution! we've stated that only the closed points form a group — the group Pic^0. So there is something to show here. The main idea is that with varieties, lots of things can be checked on closed points. First assume that $k = \overline{k}$, so the closed points are dimension 1 points. Then the associativity diagram is commutative on closed points; argue that it is hence commutative. Ditto for the other categorical requirements. Finally, deal with the case where k is not algebraically closed, by working over the algebraic closure.)

We've seen examples of group schemes before. For example, $\mathbb{A}^1_k$ is a group scheme under addition. $\mathbb{G}_m = \operatorname{Spec} k[t, t^{-1}]$ is a group scheme.

Easy exercise. Show that $\mathbb{A}^1_k$ is a group scheme under addition, and $\mathbb{G}_m$ is a group scheme under multiplication. You'll see that the functorial description trumps the axiomatic description here! (Recall that $\mathrm{Hom}(X, \mathbb{A}^1_k)$ is canonically $\Gamma(X, \mathcal{O}_X)$, and $\mathrm{Hom}(X, \mathbb{G}_m)$ is canonically $\Gamma(X, \mathcal{O}_X)^*$.)

Exercise. Define the group scheme $GL(n)$ over the integers.

Exercise. Define μ_n to be the kernel of the map of group schemes $\mathbb{G}_m \to \mathbb{G}_m$ that is "taking nth powers". In the case where n is a prime p, which is also $\mathrm{char}\, k$, describe μ_p. (I.e. how many points? How "big" = degree over k?)

Exercise. Define a *ring scheme*. Show that $\mathbb{A}^1_k$ is a ring scheme.

1.7. *Hopf algebras.* Here is a notion that we'll certainly not use, but it is easy enough to define now. Suppose $G = \mathrm{Spec}\, A$ is an affine group scheme, i.e. a group scheme that is an affine scheme. The categorical definition of group scheme can be restated in terms of the ring A. Then these axioms define a *Hopf algebra*. For example, we have a "comultiplication map" $A \to A \otimes A$. *Exercise.* As $\mathbb{A}^1_k$ is a group scheme, $k[t]$ has a Hopf algebra structure. Describe the comultiplication map $k[t] \to k[t] \otimes_k k[t]$.

2. Fun counterexamples using elliptic curves

We have a morphism $(\times n) : E \to E$ that is "multiplication by n", which sends p to np. If $n = 0$, this has degree 0. If $n = 1$, it has degree 1. Given the complex picture of a torus, you might not be surprised that the degree of $\times n$ is n^2. If $n = 2$, we have almost shown that it has degree 4, as we have checked that there are precisely 4 points q such that $2p = 2q$. All that really shows is that the degree is at least 4.

2.1. *Proposition. — For each $n > 0$, the "multiplication by n" map has positive degree. In other words, there are only a finite number of n torsion points.*

Proof. We prove the result by induction; it is true for $n = 1$ and $n = 2$.

If n is odd, then assume otherwise that $nq = 0$ for all closed points q. Let r be a non-trivial 2-torsion point, so $2r = 0$. But $nr = 0$ as well, so $r = (n-2[n/2])r = 0$, contradicting $r \neq 0$.

If n is even, then $[\times n] = [\times 2] \circ [\times (n/2)]$, and by our inductive hypothesis both $[\times 2]$ and $[\times (n/2)]$ have positive degree. □

In particular, the total number of torsion points on E is countable, so if k is an uncountable field, then E has an uncountable number of closed points (consider an open subset of the curve as $y^2 = x^3 + ax + b$; there are uncountably many choices for x, and each of them has 1 or 2 choices for y).

Thus *almost all* points on E are non-torsion. I'll use this to show you some pathologies.

An example of an affine open set that is not distinguished. I can give you an affine scheme X and an affine open subset Y that is not distinguished in X. Let $X = E - p$, which is affine (easy, or see Exercise).

Let q be another point on E so that $q-p$ is non-torsion. Then $E-p-q$ is affine (Exercise). Assume that it is distinguished. Then there is a function f on $E - p$ that vanishes on q (to some positive order d). Thus f is a rational function on E that vanishes at q to order d, and (as the total number of zeros minus poles of f is 0) has a pole at p of order d. But then $d(p-q) = 0$ in $\operatorname{Pic}^0 E$, contradicting our assumption that $p - q$ is non-torsion.

An Example of a scheme that is locally factorial at a point p, but such that no affine open neighborhood of p has ring that is a Unique Factorization Domain.

Consider $p \in E$. Then an open neighborhood of E is of the form $E - q_1 - \cdots - q_n$. I claim that its Picard group is nontrivial. Recall the exact sequence:

$$\mathbb{Z}^n \xrightarrow{(a_1,\dots,a_n)\mapsto a_1 q_1 + \cdots + a_n q_n} \operatorname{Pic} E \longrightarrow \operatorname{Pic}(E - q_1 - \cdots - q_n) \longrightarrow 0 .$$

But the group on the left is countable, and the group in the middle is uncountable, so the group on the right is non-zero.

Example of variety with non-finitely-generated space of global sections.

This is related to Hilbert's fourteenth problem, although I won't say how.

Before we begin we have a preliminary exercise.

Exercise. Suppose X is a scheme, and L is the total space of a line bundle corresponding to invertible sheaf $\mathcal{L}$, so $L = \operatorname{Spec} \oplus_{n \geq 0} (\mathcal{L}^\vee)^{\otimes n}$. Show that $H^0(L, \mathcal{O}_L) = \oplus H^0(X, (\mathcal{L}^\vee)^{\otimes n})$.

Let E be an elliptic curve over some ground field k, N a degree 0 non-torsion invertible sheaf on E, and P a positive-degree invertible sheaf on E. Then $H^0(E, N^m \otimes P^n)$ is nonzero if and only if either (i) $n > 0$, or (ii) $m = n = 0$ (in which case the sections are elements of k). Thus the ring $R = \oplus_{m,n \geq 0} H^0(E, N^m \otimes P^n)$ is not finitely generated.

Now let X be the total space of the vector bundle $N \oplus P$ over E. Then the ring of global sections of X is R.

3. More serious stuff

I'll conclude the quarter by showing the following.

- If C has genus g, then "$\operatorname{Pic}^0(C)$ has dimension g".
- "The moduli space of curves of genus g "is dimension $3g - 3$."

We'll work over an algebraically closed field k. We haven't yet made the above notions precise, so what follows are just plausibility arguments. (It is worth trying to think of a way of making these notions precise! There are several ways of doing this usefully.)

3.1. The Picard group has dimension g: "$\dim \operatorname{Pic}^0 C = g$". There are quotes around this equation because so far, $\operatorname{Pic}^0 C$ is simply a set, so this will just be a plausibility argument. Let p be any (closed, necessarily degree 1) point of C. Then twisting by p gives an isomorphism of $\operatorname{Pic}^d C$ and $\operatorname{Pic}^{d+1} C$, via $\mathcal{L} \leftrightarrow \mathcal{L}(p)$. Thus we'll consider $\operatorname{Pic}^d C$, where $d \gg 0$ (in fact $d > \deg \mathcal{K} = 2g-2$ will suffice). Say $\dim \operatorname{Pic}^d C = h$. We ask: how many degree d *effective divisors* are there (i.e. what is the dimension of this family)? The answer is clearly d, and C^d surjects onto this set (and is usually $d!$-to-1).

But we can count effective divisors in a different way. There is an h-dimensional family of line bundles by hypothesis, and each one of these has a $(d-g+1)$-dimensional family of non-zero sections, each of which gives a divisor of zeros. But two sections yield the same divisor if one is a multiple of the other. Hence we get: $h+(d-g+1)-1 = h+d-g$.

Thus $d = h+d-g$, from which $h = g$ as desired.

Note that we get a bit more: if we believe that Pic^d has an algebraic structure, we have a fibration $(C^d)/S_d \to \operatorname{Pic}^d$, where the fibers are isomorphic to $\mathbb{P}^{d-g}$. In particular, Pic^d is reduced, and irreducible.

3.2. The moduli space of genus g curves has dimension $3g-3$. Let $\mathcal{M}_g$ be the set of nonsingular genus g curves, and pretend that we can give it a variety structure. Say $\mathcal{M}_g$ has dimension p. By our useful Riemann-Roch facts, if $d \gg 0$, and D is a divisor of degree d, then $h^0(C, \mathcal{O}(D)) = d-g+1$. If we take two general sections s, t of the line bundle $\mathcal{O}(D)$, we get a map to $\mathbb{P}^1$, and this map is degree d. Conversely, any degree d cover $f : C \to \mathbb{P}^1$ arises from two linearly independent sections of a degree d line bundle. Recall that (s,t) gives the same map to $\mathbb{P}^1$ as (s',t') if and only (s,t) is a scalar multiple of (s',t'). Hence the number of maps to $\mathbb{P}^1$ arising from a fixed curve C and a fixed line bundle $\mathcal{L}$ correspond to the choices of two sections $(2(d-g+1))$, minus 1 to forget the scalar multiple, for a total of $2d-2g+1$. If we let the the line bundle vary, the number of maps from a fixed curve is $2d-2g+1+\dim \operatorname{Pic}^d(C) = 2d-g+1$. If we let the curve also vary, we see that the number of degree d genus g covers of $\mathbb{P}^1$ is $\boxed{p+2d-g+1}$.

But we can also count this number using the Riemann-Hurwitz formula. I'll need one believable fact: there are a finite number of degree d covers with a given set of branch points. (In the complex case, this is believable for the following reason. If $C \to \mathbb{P}^1$ is a branched cover of $\mathbb{P}^1$, branched over $p_1, \dots, p_r$, then by discarding the branch points and their preimages, we have an unbranched cover $C' \to \mathbb{P}^1 - \{p_1, \dots, p_r\}$. Then you can check that (i) the original map $C \to \mathbb{P}^1$ is determined by this map (because C is the normalization of $\mathbb{P}^1$ in this function field extension $FF(C')/FF(\mathbb{P}^1)$), and (ii) there are a finite number of such covers (corresponding to the monodromy data around these r points; we have r elements of S_d once we take branch cuts). This last step is where the characteristic 0 hypothesis is necessary.)

By the Riemann-Hurwitz formula, for a fixed g and d, the total amount of branching is $2g+2d-2$ (including multiplicity). Thus if the branching happens at no more than $2g+2d-2$ points, and if we have the simplest possible branching at $2g+2d-2$ points, the covering curve is genus g. Thus

$$p + 2d - g + 1 = 2g + 2d - 2,$$

from which $\boxed{p = 3g - 3}$.

Thus there is a $3g-3$-dimensional family of genus g curves! (By showing that the space of branched covers is reduced and irreducible, we could again "show" that the moduli space is reduced and irreducible.)

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FOUNDATIONS OF ALGEBRAIC GEOMETRY CLASS 37

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CONTENTS

Welcome back to the third quarter! The theme for this quarter, insofar as there is one, will be "useful ideas to know". We'll start with differentials for the first three lectures.

I prefer to start any topic with a number of examples, but in this case I'm going to spend a fair amount of time discussing technicalities, and then get to a number of examples. Here is the main message I want you to get. Differentials are an intuitive geometric notion, and we're going to figure out the right description of them algebraically. I find the algebraic manifestation a little non-intuitive, so I always like to tie it to the geometry. So please don't tune out of the statements. Also, I want you to notice that although the algebraic statements are odd, none of the proofs are hard or long.

This topic could have been done as soon as we knew about morphisms and quasicoherent sheaves.

1. MOTIVATION AND GAME PLAN

Suppose X is a "smooth" k-variety. We hope to define a tangent bundle. We'll see that the right way to do this will easily apply in much more general circumstances.

• We'll see that cotangent is more "natural" for schemes than tangent bundle. This is similar to the fact that the Zariski *cotangent* space is more natural than the *tangent space* (i.e. if A is a ring and $\mathfrak{m}$ is a maximal ideal, then $\mathfrak{m}/\mathfrak{m}^2$ is "more natural" than $(\mathfrak{m}/\mathfrak{m}^2)^\vee$. So we'll define the cotangent sheaf first.

• Our construction will work for general X, even if X is not "smooth" (or even at all nice, e.g. finite type). The cotangent sheaf won't be locally free, but it will still be a quasicoherent sheaf.

• Better yet, this construction will work "relatively". For *any* $X \to Y$, we'll define $\Omega_{X/Y}$, a quasicoherent sheaf on X, the sheaf of *relative differentials*. This will specialize to the earlier

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case by taking $Y = \operatorname{Spec} k$. The idea is that this glues together the cotangent sheaves of the fibers of the family. (I drew an intuitive picture in the "smooth" case. I introduced the phrase "vertical (co)tangent vectors".)

2. The affine case: three definitions

We'll first study the affine case. Suppose A is a B-algebra, so we have a morphism of rings $\phi : B \to A$ and a morphism of schemes $\operatorname{Spec} A \to \operatorname{Spec} B$. I will define an A-module $\Omega_{A/B}$ in three ways. This is called the *module of relative differentials* or the *module of Kähler differentials*. The module of differentials will be defined to be this module, as well as a map $d : A \to \Omega_{A/B}$ satisfying three properties.
(i) additivity. $da + da' = d(a + a')$
(ii) Leibniz. $d(aa') = a\, da' + a' da$
(iii) triviality on pullbacks. $db = 0$ for $b \in \phi(B)$.

As motivation, think of the case $B = k$. So for example, $da^n = na^{n-1} da$, and more generally, if f is a polynomial in one variable, $df(a) = f'(a)\, da$ (where f' is defined formally: if $f = \sum c_i x^i$ then $f' = \sum c_i i x^{i-1}$).

I'll give you three definitions of this sheaf in the affine case (i.e. this module). The first is a concrete hands-on definition. The second is by universal property. And the third will globalize well, and will allow us to define $\Omega_{X/Y}$ conveniently in general.

The first two definitions are analogous to what we have seen for tensor product. Recall that there are two common definitions of $\otimes$. The first is in terms of formal symbols satisfying some rules. This is handy for showing certain things, e.g. if $M \to M'$ is surjective, then so is $M \otimes N \to M' \otimes N$. The second is by universal property.

2.1. First definition of differentials: explicit description. We define $\Omega_{A/B}$ to be finite A-linear combinations of symbols "da" for $a \in A$, subject to the three rules (i)–(iii) above. For example, take $A = k[x, y]$, $B = k$. Then a sample differential is $3x^2\, dy + 4dx \in \Omega_{A/B}$. We have identities such as $d(3xy^2) = 3y^2\, dx + 6xy\, dy$.

Key fact. Note that if A is generated over B (as an algebra) by $x_i \in A$ (where i lies in some index set, possibly infinite), subject to some relations r_j (where j lies in some index set, and each is a polynomial in some finite number of the x_i), then the A-module $\Omega_{A/B}$ is generated by the dx_i, subject to the relations (i)—(iii) and $dr_j = 0$. In short, we needn't take every single element of A; we can take a generating set. And we needn't take every single relation among these generating elements; we can take generators of the relations.

2.2. *Exercise.* Verify the above key fact.

In particular:

2.3. *Proposition. — If* A *is a finitely generated* B*-algebra, then* $\Omega_{A/B}$ *is a finite type (i.e. finitely generated)* A*-module. If* A *is a finitely presented* B*-algebra, then* $\Omega_{A/B}$ *is a finitely presented* A*-module.*

("Finitely presented" algebra means finite number of generators (=finite type) and finite number of relations. If A is Noetherian, then the two hypotheses are the same, so most of you will not care.)

Let's now see some examples. Among these examples are three particularly important kinds of ring maps that we often consider: adding free variables; localizing; and taking quotients. If we know how to deal with these, we know (at least in theory) how to deal with any ring map.

2.4. Example: taking a quotient. If $A = B/I$, then $\Omega_{A/B} = 0$ basically immediately: $da = 0$ for all $a \in A$, as each such a is the image of an element of B. This should be believable; in this case, there are no "vertical tangent vectors".

2.5. Example: adding variables. If $A = B[x_1, \dots, x_n]$, then $\Omega_{A/B} = A dx_1 \oplus \cdots \oplus A dx_n$. (Note that this argument applies even if we add an arbitrarily infinite number of indeterminates.) The intuitive geometry behind this makes the answer very reasonable. The cotangent bundle should indeed be trivial of rank n.

2.6. Example: two variables and one relation. If $B = \mathbb{C}$, and $A = \mathbb{C}[x, y]/(y^2 - x^3)$, then $\Omega_{A/B} = \mathbb{C}\ dx \oplus \mathbb{C}\ dy/(2y\ dy - 3x^2\ dx)$.

2.7. Example: localization. If S is a multiplicative set of B, and $A = S^{-1}B$, then $\Omega_{A/B} = 0$. Reason: Note that the quotient rule holds. (If $b = as$, then $db = a\ ds + s\ da$, which can be rearranged to give $da = (s\ db - b\ ds)/s^2$.) Thus if $a = b/s$, then $da = (s\ db - b\ ds)/s^2 = 0$. (If $A = B_f$ for example, this is intuitively believable; then $\operatorname{Spec} A$ is an open subset of $\operatorname{Spec} B$, so there should be no "vertical cotangent vectors".)

2.8. Exercise: localization (stronger form). If S is a multiplicative set of A, show that there is a natural isomorphism $\Omega_{S^{-1}A/B} \cong S^{-1}\Omega_{A/B}$. (Again, this should be believable from the intuitive picture of "vertical cotangent vectors".) If T is a multiplicative set of B, show that there is a natural isomorphism $\Omega_{S^{-1}A/T^{-1}B} \cong S^{-1}\Omega_{A/B}$ where S is the multiplicative set of A that is the image of the multiplicative set $T \subset B$.

2.9. *Exercise.* (a) *(pullback of differentials)* If

$$\begin{array}{ccc} A' & \longleftarrow & A \\ \uparrow & & \uparrow \\ B' & \longleftarrow & B \end{array}$$

is a commutative diagram, show that there is a natural homomorphism of A'-modules $A' \otimes_A \Omega_{A/B} \to \Omega_{A'/B'}$. An important special case is $B = B'$.

(b) *(differentials behave well with respect to base extension, affine case)* If furthermore the above diagram is a tensor diagram (i.e. $A' \cong B' \otimes_B A$) then show that $A' \otimes_A \Omega_{A/B} \to \Omega_{A'/B'}$ is an isomorphism.

2.10. *Exercise.* Suppose k is a field, and K is a separable algebraic extension of k. Show that $\Omega_{K/k} = 0$.

2.11. *Exercise (Jacobian description of $\Omega_{A/B}$).* — Suppose $A = B[x_1, \dots, x_n]/(f_1, \dots, f_r)$. Then $\Omega_{A/B} = \{\oplus_i B dx_i\}/\{df_j = 0\}$ maybe interpreted as the cokernel of the Jacobian matrix $J : A^{\oplus r} \to A^{\oplus n}$.

I now want to tell you two handy (geometrically motivated) exact sequences. The arguments are a bit tricky. They are useful, but a little less useful than the foundation facts above.

2.12. *Theorem (the relative cotangent sequence, affine version). — Suppose* $C \to B \to A$ *are ring homomorphisms. Then there is a natural exact sequence of* A*-modules*

$$A \otimes_B \Omega_{B/C} \to \Omega_{A/C} \to \Omega_{A/B} \to 0.$$

Before proving this, I drew a picture motivating the statement. I drew pictures of two maps of schemes, $\operatorname{Spec} A \xrightarrow{f} \operatorname{Spec} B \xrightarrow{g} \operatorname{Spec} C$, where $\operatorname{Spec} C$ was a point, $\operatorname{Spec} B$ was $\mathbb{A}^1$ (or a "smooth curve"), and $\operatorname{Spec} A$ was $\mathbb{A}^2$ (or a "smooth surface"). The tangent space to a point upstairs has a subspace that is the tangent space to the vertical fiber. The cokernel is the pullback of the tangent space to the image point in $\operatorname{Spec} B$. Thus we have an exact sequence $0 \to T_{\operatorname{Spec} A/\operatorname{Spec} B} \to T_{\operatorname{Spec} A/\operatorname{Spec} C} \to f^* T_{\operatorname{Spec} B/\operatorname{Spec} C} \to 0$. We want the corresponding sequence of cotangent vectors, so we dualize. We end up with precisely the statement of the Theorem, except we also have left-exactness. This discrepancy is because the statement of the theorem is more general; we'll see that in the "smooth" case, we'll indeed have left-exactness.

Proof. (Before we start, note that surjectivity is clear, from $da \mapsto da$. The composition over the middle term is clearly 0: $db \to db \to 0$.) We wish to identify $\Omega_{A/B}$ as the cokernel of $A \otimes_B \Omega_{B/C} \to \Omega_{A/C}$. Now $\Omega_{A/B}$ is exactly the same as $\Omega_{A/C}$, except we have extra relations: $db = 0$ for $b \in B$. These are precisely the images of $1 \otimes db$ on the left. $\square$

2.13. *Theorem (Conormal exact sequence, affine version). — Suppose* B *is a* C*-algebra,* I *is an ideal of* B*, and* $A = B/I$*. Then there is a natural exact sequence of* A*-modules*

$$I/I^2 \xrightarrow{\delta: i \mapsto 1 \otimes di} A \otimes_B \Omega_{B/C} \xrightarrow{a \otimes db \mapsto a\, db} \Omega_{A/C} \longrightarrow 0.$$

Before getting to the proof, some discussion is necessary. (The discussion is trickier than the proof itself!)

The map δ is a bit subtle, so I'll get into its details before discussing the geometry. For any $i \in I$, $\delta i = 1 \otimes di$. Note first that this is well-defined: If $i, i' \in I$, $i \equiv i' \pmod{I^2}$, say $i - i' = i''i'''$ where $i'', i''' \in I$, then $\delta i - \delta i' = 1 \otimes (i''\, di''' + i'''\, di'') \in I\Omega_{B/C}$ is 0 in $A \otimes_B \Omega_{B/C} = (B/I) \otimes_B \Omega_{B/C}$. Next note that I/I^2 indeed is an $A = (B/I)$-module. Finally, note that the map $I/I^2 \to A \otimes_B \Omega_{B/C}$ is indeed a homomorphism of A-modules: If $a \in A$, $b \in I$, then $ab \mapsto 1 \otimes d(ab) = 1 \otimes (a\, db + b\, da) = 1 \otimes (a\, db) = a(1 \otimes db)$.

Having dispatched that formalism, let me get back to the geometry. I drew a picture where $\operatorname{Spec} C$ is a point, $\operatorname{Spec} B$ is a plane, and $\operatorname{Spec} A$ is something smooth in it. Let j be the inclusion. Then we have $0 \to T_{\operatorname{Spec} A/\operatorname{Spec} C} \to j^* T_{\operatorname{Spec} B/\operatorname{Spec} C} \to N_{\operatorname{Spec} B/\operatorname{Spec} C} \to 0$. Dualizing it, we get $0 \to N^\vee_{A/B} \to A \otimes \Omega_{B/C} \to \Omega_{A/C} \to 0$. This exact sequence reminds me of several things above and beyond the theorem. First of all, I/I^2 will later be the conormal bundle — hence the name of the theorem. Second, in good circumstances, the conormal exact sequence of Theorem 2.13 will be injective on the left.

2.14. *Aside: Why should I/I^2 be the conormal bundle?.* We'll *define* I/I^2 to be the conormal bundle later, so I'll try to give you an idea as to why this is reasonable. You believe now that $\mathfrak{m}/\mathfrak{m}^2$ should be the cotangent space to a point in $\mathbb{A}^n$. In other words, $(x_1, \dots, x_n)/(x_1, \dots, x_n)^2$ is the cotangent space to $\vec{0}$ in $\mathbb{A}^n$. Translation: it is the conormal space to the point $\vec{0} \in \mathbb{A}^n$. Then you might believe that in $\mathbb{A}^{n+m}$, $(x_1, \dots, x_n)/(x_1, \dots, x_n)^2$ is the conormal bundle to the coordinate n-plane $\mathbb{A}^m \subset \mathbb{A}^{n+m}$.

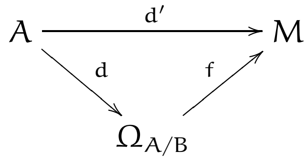
Let's finally prove the conormal exact sequence.

Proof of the conormal exact sequence (affine version) 2.13. We need to identify the cokernel of $\delta : I/I^2 \to A \otimes_B \Omega_{B/C}$ with $\Omega_{A/C}$. Consider $A \otimes_B \Omega_{B/C}$. As an A-module, it is generated by db ($b \in B$), subject to three relations: $dc = 0$ for $c \in \phi(C)$ (where $\phi : C \to B$ describes B as a C-algebra), additivity, and the Leibniz rule. Given any relation *in* B, d of that relation is 0.

Now $\Omega_{A/C}$ is defined similarly, except there are more relations *in* A; these are precisely the elements of $i \in B$. Thus we obtain $\Omega_{A/C}$ by starting out with $A \otimes_B \Omega_{B/C}$, and adding the additional relations di where $i \in I$. But this is precisely the image of δ! $\square$

2.15. Second definition: universal property. Here is a second definition that we'll use at least once, and is certainly important philosophically. Suppose A is a B-algebra, and M is a A-module. An *B-linear derivation of A into M* is a map $d : A \to M$ of B-modules (*not necessarily A-modules*) satisfying the Leibniz rule: $d(fg) = f\, dg + g\, df$. As an example, suppose $B = k$, and $A = k[x]$, and $M = A$. Then an example of a k-linear derivation is d/dx. As a second example, if $B = k$, $A = k[x]$, and $M = k$. Then an example of a k-linear derivation is $d/dx|_0$.

Then $d : A \to \Omega_{A/B}$ is defined by the following universal property: any other B-linear derivation $d' : A \to M$ factors uniquely through d:



Here f is a map of A-modules. (Note again that d and d' are not! They are only B-linear.) By universal property nonsense, if it exists, it is unique up to unique isomorphism. The candidate I described earlier clearly satisfies this universal property (in particular, it is a derivation!), hence this is it. [Thus Ω is the "unversal derivation". I should rewrite this paragraph at some point.]

The next result will give you more evidence that this deserves to be called the (relative) cotangent bundle.

2.16. *Proposition.* *Suppose* B *is a* k*-algebra, with residue field* k*. Then the natural map* $\delta : \mathfrak{m}/\mathfrak{m}^2 \to \Omega_{B/k} \otimes_B k$ *is an isomorphism.*

I skipped this proof in class, but promised it in the notes.

Proof. By the conormal exact sequence 2.13 with $I = \mathfrak{m}$ and $A = C = k$, δ is a surjection (as $\Omega_{k/k} = 0$), so we need to show that it is injection, or equivalently that $\mathrm{Hom}_k(\Omega_{B/k} \otimes_B k, k) \to \mathrm{Hom}_k(\mathfrak{m}/\mathfrak{m}^2, k)$ is a surjection. But any element on the right is indeed a derivation from B to k (an earlier exercise from back in the dark ages on the Zariski tangent space), which is precisely an element of $\mathrm{Hom}_B(\Omega_{B/k}, k)$ (by the universal property of $\Omega_{B/k}$), which is canonically isomorphic to $\mathrm{Hom}_k(\Omega_{B/k} \otimes_B k, k)$ as desired. □

Remark. As a corollary, this (in combination with the Jacobian exercise 2.11 above) gives a second proof of an exercise from the first quarter, showing the Jacobian criterion for nonsingular varieties over an algebraically closed field.

Aside. If you wish, you can use the universal property to show that $\Omega_{A/B}$ behaves well with respect to localization. For example, if S is a multiplicative set of A, then there is a natural isomorphism $\Omega_{S^{-1}A/B} \cong S^{-1}\Omega_{A/B}$. This can be used to give a different solution to Exercise 2.8. It can also be used to give a second definition of $\Omega_{X/Y}$ for a morphism of schemes $X \to Y$ (different from the one given below): we define it as a quasicoherent sheaf, by describing how it behaves on affine open sets, and showing that it behaves well with respect to distinguished localization.

Next day, I'll give a third definition which will globalize well, and we'll see that we already understand differentials for morphisms of schemes.

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FOUNDATIONS OF ALGEBRAIC GEOMETRY CLASS 38

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CONTENTS

Last day I introduced differentials on affine schemes, for a morphism $B \to A$. The differential was an A-module, as well as a homomorphism of B-modules, $d : A \to \Omega_{A/B}$. The A-module $\Omega_{A/B}$ is generated by da, and d satisfies 3 rules: additivity, Leibniz rule, and $db = 0$ (or $d1 = 0$). It satisfies a universal property: any derivation $A \to M$ uniquely factors through an A-module homomorphism $\Omega_{A/B} \to M$.

1. A THIRD DEFINITION OF Ω, SUITABLE FOR EASY GLOBALIZATION

1.1. Third definition. We now want to globalize this definition for an arbitrary morphism of schemes $f : X \to Y$. We could do this "affine by affine"; we just need to make sure that the above notion behaves well with respect to "change of affine sets". Thus a relative differential on X would be the data of, for every affine $U \subset X$, a differential of the form $\sum a_i \; db_i$, and on the intersection of two affine open sets $U \cap U'$, with representatives $\sum a_i \, db_i$ on U and $\sum a'_i \, db'_i$ on the second, an equality on the overlap. Instead, we'll take a different tack. We'll get what intuitively seems to be a very weird definition! I'll give the definition, then give you some intuition, and then get back to the definition.

Suppose $f : X \to Y$ be any morphism of schemes. Recall that $\delta : X \to X \times_Y X$ is a locally closed immersion (Class 9, p. 5). Thus there is an open subscheme $U \subset X \times_Y X$ for which $\delta : X \to U$ is a closed immersion, cut out by a quasicoherent sheaf of ideals $\mathcal{I}$. Then $\mathcal{I}/\mathcal{I}^2$ is a quasicoherent sheaf naturally supported on X (affine-locally this is the statement that I/I^2 is naturally an A/I-module). We call this the *conormal sheaf to* X (or somewhat more precisely,to the locally closed immersion). (For the motivation for this name, see last day's notes.) We denote it by $\mathcal{N}^{\vee}_{X/X \times_Y X}$. Then we will *define* $\Omega_{X/Y}$ as this conormal sheaf.

(Small technical point for pedants: what does $\mathcal{I}^2$ mean? In general, if $\mathcal{I}$ and $\mathcal{J}$ are quasicoherent ideal sheaves on a scheme Z, what does $\mathcal{I}\mathcal{J}$ mean? Of course it means that on each affine, we take the product of the two corresponding ideals. To make sure this

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is well-defined, we need only check that if A is a ring, and $f \in A$, and $I, J \subset A$ are two ideals, then $(IJ)_f = I_f J_f$ in A_f.)

Brief aside on (co)normal sheaves to locally closed immersions. For any locally closed immersion $W \to Z$, we can define the *conormal sheaf* $\mathcal{N}^\vee_{W/Z}$, a quasicoherent sheaf on W, similarly, and the *normal sheaf* as its dual $\mathcal{N}_{W/Z} := \underline{\mathrm{Hom}}(\mathcal{N}^\vee, \mathcal{O}_W)$. This is somewhat imperfect notation, as it suggests that the dual of $\mathcal{N}$ is always $\mathcal{N}^\vee$. This is not always true, as for A-modules, the natural morphism from a module to its double-dual is not always an isomorphism. (Modules for which this is true are called *reflexive*, but we won't use this notion.)

1.2. *Exercise: normal bundles to effective Cartier divisors.* Suppose $D \subset X$ is an effective Cartier divisor. Show that the conormal sheaf $\mathcal{N}^\vee_{D/X}$ is $\mathcal{O}(-D)|_D$ (and in particular is an invertible sheaf), and hence that the normal sheaf is $\mathcal{O}(D)|_D$. It may be surprising that the normal sheaf should be locally free if $X \cong \mathbb{A}^2$ and D is the union of the two axes (and more generally if X is nonsingular but D is singular), because you may be used to thinking that the normal bundle is isomorphic to a "tubular neighborhood".

Let's get back to talking about differentials.

We now define the d operator $d : \mathcal{O}_X \to \Omega_{X/Y}$. Let $\pi_1, \pi_2 : X \times_Y X \to X$ be the two projections. Then define $d : \mathcal{O}_X \to \Omega_{X/Y}$ on the open set U as follows: $df = \pi_2^* f - \pi_1^* f$. (*Warning:* this is not a morphism of quasicoherent sheaves, although it *is* $\mathcal{O}_Y$-linear.) We'll soon see that this is indeed a derivation, and at the same time see that our new notion of differentials agrees with our old definition on affine open sets, and hence globalizes the definition.

Before we do, let me try to convince you that this is a reasonable definition to make. (This paragraph is informal, and is in no way mathematically rigorous.) Say for example that Y is a point, and X is something smooth. Then the tangent space to $X \times X$ is $T_X \oplus T_X$: $T_{X \times X} = T_X \oplus T_X$. Restrict this to the diagonal Δ, and look at the normal bundle exact sequence:

$$0 \to T_\Delta \to T_{X \times X}|_\Delta \to N_{\Delta, X} \to 0.$$

Now the left morphism sends v to (v, v), so the cokernel can be interpreted as $(v, -v)$. Thus $N_{\Delta,X}$ is isomorphic to T_X. Thus we can turn this on its head: we know how to find the normal bundle (or more precisely the conormal sheaf), and we can use this to define the tangent bundle (or more precisely the cotangent sheaf). (Experts may want to ponder the above paragraph when Y is more general, but where $X \to Y$ is "nice". You may wish to think in the category of manifolds, and let $X \to Y$ be a submersion.)

Let's now see how this works for the special case $\operatorname{Spec} A \to \operatorname{Spec} B$. Then the diagonal $\operatorname{Spec} A \hookrightarrow \operatorname{Spec} A \otimes_B A$ corresponds to the ideal I of $A \otimes_B A$ that is the cokernel of the ring map

$$\sum x_i \otimes y_i \to \sum x_i y_i.$$

The derivation is $d : A \to A \otimes_B A$, $a \mapsto da := 1 \otimes a - a \otimes 1$ (taken modulo I^2). (I shouldn't really call this "d" until I've verified that it agrees with our earlier definition, but bear with me.)

Let's check that this satisfies the 3 conditions, i.e. that it is a derivation. Two are immediate: it is linear, vanishes on elements of b. Let's check the Leibniz rule:

$$\begin{aligned} d(aa') - a\,da' - a'\,da &= 1 \otimes aa' - aa' \otimes 1 - a \otimes a' + aa' \otimes 1 - a' \otimes a + a'a \otimes 1 \\ &= -a \otimes a' - a' \otimes a + a'a \otimes 1 + 1 \otimes aa' \\ &= (1 \otimes a - a \otimes 1)(1 \otimes a' - a' \otimes 1) \\ &\in I^2. \end{aligned}$$

Thus by the universal property of $\Omega_{A/B}$, we get a natural morphism $\Omega_{A/B} \to I/I^2$ of A-modules.

1.3. *Theorem. — The natural morphism $f : \Omega_{A/B} \to I/I^2$ induced by the universal property of $\Omega_{A/B}$ is an isomorphism.*

Proof. We'll show this as follows. (i) We'll show that f is surjective, and (ii) we will describe $g : I/I^2 \to \Omega_{A/B}$ such that $g \circ f : \Omega_{A/B} \to \Omega_{A/B}$ is the identity. Both of these steps will be very short. Then we'll be done, as to show $f \circ g$ is the identity, we need only show (by surjectivity of g) that $(f \circ g)(f(a)) = f(a)$, which is true (by (ii) $g \circ f = id$).

(i) For surjectivity, we wish to show that I is generated (modulo I^2) by $a \otimes 1 - 1 \otimes a$ as a runs over the elements of A. This has a one sentence explanation: If $\sum x_i \otimes y_i \in I$, i.e. $\sum x_i y_i = 0$ in A, then $\sum_i x_i \otimes y_i = \sum_i x_i(1 \otimes y_i - y_i \otimes 1)$.

(ii) Define $g : I/I^2 \to \Omega_{A/B}$ by $x \otimes y \mapsto x\,dy$. We need to check that this is well-defined, i.e. that elements of I^2 are sent to 0, i.e. we need that

$$\left(\sum x_i \otimes y_i\right)\left(\sum x'_j \otimes y'_j\right) = \sum_{i,j} x_i x'_j \otimes y_i y'_j \mapsto 0$$

where $\sum_i x_i y_i = \sum x'_j y'_j = 0$. But by the Leibniz rule,

$$\begin{aligned} \sum_{i,j} x_i x'_j\, d(y_i y'_j) &= \sum_{i,j} x_i x'_j y_i\, dy'_j + \sum_{i,j} x_i x'_j y'_j\, dy_i \\ &= \left(\sum_i x_i y_i\right)\left(\sum_j x'_j\, dy'_j\right) + \left(\sum_i x_i\, dy_i\right)\left(\sum_j x'_j y'_j\right) \\ &= 0. \end{aligned}$$

Then $f \circ g$ is indeed the identity, as

$$da \xrightarrow{g} 1 \otimes a - a \otimes 1 \xrightarrow{f} 1\,da - a\,d1 = da$$

as desired. □

We can now use our understanding of how Ω works on affine open sets to state some global results.

1.4. *Exercise.* Suppose $f : X \to Y$ is locally of finite type, and X is locally Noetherian. Show that $\Omega_{X/Y}$ is a coherent sheaf on X.

The relative cotangent exact sequence and the conormal exact sequence for schemes now directly follow.

1.5. *Theorem. — (Relative cotangent exact sequence) Suppose* $X \xrightarrow{f} Y \xrightarrow{g} Z$ *be morphisms of schemes. Then there is an exact sequence of quasicoherent sheaves on* X

$$f^*\Omega_{Y/Z} \to \Omega_{X/Z} \to \Omega_{X/Y} \to 0.$$

(Conormal exact sequence) Suppose $f : X \to Y$ *morphism of schemes,* $Z \hookrightarrow X$ *closed subscheme of* X*, with ideal sheaf* $\mathcal{I}$*. Then there is an exact sequence of sheaves on* Z*:*

$$\mathcal{I}/\mathcal{I}^2 \xrightarrow{\delta} \Omega_{X/Y} \otimes \mathcal{O}_Z \longrightarrow \Omega_{Z/Y} \longrightarrow 0.$$

Similarly, the sheaf of relative differentials pull back, and behave well under base change.

1.6. *Theorem (pullback of differentials). — (a) If*

$$\begin{array}{ccc} X' & \xrightarrow{g} & X \\ \downarrow & & \downarrow \\ Y' & \longrightarrow & Y \end{array}$$

is a commutative diagram of schemes, there is a natural homomorphism of quasicoherent sheaves on X' $g^*\Omega_{X/Y} \to \Omega_{X'/Y'}$*. An important special case is* $Y = Y'$*.*
(b) (Ω behaves well under base change) If furthermore the above diagram is a tensor diagram (i.e. $X' \cong X \otimes_Y Y'$*) then* $g^*\Omega_{X/Y} \to \Omega_{X'/Y'}$ *is an isomorphism.*

This follows immediately from an Exercise in last day's notes. Part (a) implicitly came up in our earlier discussion of the Riemann-Hurwitz formula.

As a particular case of part (b), the fiber of the sheaf of relative differentials is indeed the sheaf of differentials of the fiber. *Thus this notion indeed glues together the differentials on each fiber.*

2. Examples

2.1. The projective line. As an important first example, let's consider $\mathbb{P}^1_k$, with the usual projective coordinates x_0 and x_1. As usual, the first patch corresponds to $x_0 \neq 0$, and is of the form $\operatorname{Spec} k[x_{1/0}]$ where $x_{1/0} = x_1/x_0$. The second patch corresponds to $x_1 \neq 0$, and is of the form $\operatorname{Spec} k[x_{0/1}]$ where $x_{0/1} = x_0/x_1$.

Both patches are isomorphic to $\mathbb{A}^1_k$, and $\Omega_{\mathbb{A}^1_k} = \mathcal{O}_{\mathbb{A}^1_k}$. (More precisely, $\Omega_{k[x]/k} = k[x]\ dx$.) Thus $\Omega_{\mathbb{P}^1_k}$ is an invertible sheaf (a line bundle). Now we have classified the invertible sheaves on $\mathbb{P}^1_k$ — they are each of the form $\mathcal{O}(m)$. So which invertible sheaf is $\Omega_{\mathbb{P}^1/k}$?

Let's take a section, $dx_{1/0}$ on the first patch. It has no zeros or poles there, so let's check what happens on the other patch. As $x_{1/0} = 1/x_{0/1}$, we have $dx_{1/0} = -(1/x_{0/1}^2)\ dx_{0/1}$. Thus this section has a double pole where $x_{0/1} = 0$. Hence $\Omega_{\mathbb{P}^1_k/k} \cong \mathcal{O}(-2)$.

Note that the above argument did not depend on k being a field, and indeed we could replace k with any ring A (or indeed with any base scheme).

2.2. A plane curve. Consider next the plane curve $y^2 = x^3 - x$ in $\mathbb{A}^2_k$, where the characteristic of k is not 2. Then the differentials are generated by dx and dy, subject to the constraint that

$$2y\ dy = (3x^2 - 1)\ dx.$$

Thus in the locus where $y \neq 0$, dx is a generator (as dy can be expressed in terms of dx). Similarly, in the locus where $3x^2 - 1 \neq 0$, dy is a generator. These two loci cover the entire curve, as solving $y = 0$ gives $x^3 - x = 0$, i.e. $x = 0$ or ± 1, and in each of these cases $3x^2 - 1 \neq 0$.

Now consider the differential dx. Where does it vanish? Answer: precisely where $y = 0$. You should find this believable from the picture (which I gave in class).

2.3. Exercise: differentials on hyperelliptic curves. Consider the double cover $f : C \to \mathbb{P}^1_k$ branched over $2g+2$ distinct points. (We saw earlier that this curve has genus g.) Then $\Omega_{C/k}$ is again an invertible sheaf. What is its degree? (Hint: let x be a coordinate on one of the coordinate patches of $\mathbb{P}^1_k$. Consider $f^* dx$ on C, and count poles and zeros.) In class I gave a sketch showing that you should expect the answer to be $2g - 2$.

2.4. Exercise: differentials on nonsingular plane curves. Suppose C is a nonsingular plane curve of degree d in $\mathbb{P}^2_k$, where k is algebraically closed. By considering coordinate patches, find the degree of $\Omega_{C/k}$. Make any reasonable simplifying assumption (so that you believe that your result still holds for "most" curves).

Because Ω behaves well under pullback, note that the assumption that k is algebraically closed may be quickly excised:

2.5. *Exercise.* Suppose that C is a nonsingular projective curve over k such that $\Omega_{C/k}$ is an invertible sheaf. (We'll see that for nonsingular curves, the sheaf of differentials is always locally free. But we don't yet know that.) Let $C_{\overline{k}} = C \times_{\operatorname{Spec} k} \operatorname{Spec} \overline{k}$. Show that $\Omega_{C_{\overline{k}}/\overline{k}}$ is locally free, and that

$$\deg \Omega_{C_{\overline{k}}/\overline{k}} = \deg \Omega_{C/k}.$$

2.6. Projective space. We next examine the differentials of projective space $\mathbb{P}^n_k$. As projective space is covered by affine open sets of the form $\mathbb{A}^n$, on which the differential form a rank n locally free sheaf, $\Omega_{\mathbb{P}^n_k/k}$ is also a rank n locally free sheaf.

2.7. *Theorem (the Euler exact sequence). — The sheaf of differentials $\Omega_{\mathbb{P}^n_k/k}$ satisfies the following exact sequence*

$$0 \to \Omega_{\mathbb{P}^n_A} \to \mathcal{O}_{\mathbb{P}^n_A}(-1)^{\oplus(n+1)} \to \mathcal{O}_{\mathbb{P}^n_A} \to 0.$$

This is handy, because you can get a hold of Ω in a concrete way. Next day I will give an explicit example, to give you some practice.

I discussed some philosophy behind this theorem. Next day, I'll give a proof, and repeat the philosophy.

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FOUNDATIONS OF ALGEBRAIC GEOMETRY CLASSES 39 AND 40

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CONTENTS

These are notes from both class 39 and class 40.

Today: the Euler exact sequence. Discussion of nonsingular varieties over algebraically closed fields: Bertini's theorem, the Riemann-Hurwitz formula, and the (co)normal exact sequence for nonsingular subvarieties of nonsingular varieties.

We have now established the general theory of differentials, and we are now going to apply it.

1. PROJECTIVE SPACE AND THE EULER EXACT SEQUENCE

We next examine the differentials of projective space $\mathbb{P}^n_k$, or more generally $\mathbb{P}^n_A$ where A is an arbitrary ring. As projective space is covered by affine open sets, on which the differentials form a rank n locally free sheaf, $\Omega_{\mathbb{P}^n_A/A}$ is also a rank n locally free sheaf.

1.1. *Important Theorem (the Euler exact sequence). — The sheaf of differentials $\Omega_{\mathbb{P}^n_A/A}$ satisfies the following exact sequence*

$$0 \to \Omega_{\mathbb{P}^n_A} \to \mathcal{O}(-1)^{\oplus(n+1)} \to \mathcal{O}_{\mathbb{P}^n_A} \to 0.$$

This is handy, because you can get a hold of Ω in a concrete way. Here is an explicit example, to give you practice.

1.2. *Exercise.* Show that $H^1(\mathbb{P}^n_A, T_{\mathbb{P}^n_A}) = 0$. (This later turns out to be an important calculation for the following reason. If X is a nonsingular variety, $H^1(X, T_X)$ parametrizes deformations of the variety. Thus projective space can't deform, and is "rigid".)

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Let's prove the Euler exact sequence. I find this an amazing fact, and while I can prove it, I don't understand in my bones why this is true. Maybe someone can give me some enlightenment.

Proof. (What's really going on in this proof is that we consider those differentials on $\mathbb{A}_A^{n+1} \setminus \{0\}$ that are pullbacks of differentials on $\mathbb{P}_A^n$.)

I'll describe a map $\mathcal{O}(-1)^{\oplus(n+1)} \to \mathcal{O}$, and later identify the kernel with $\Omega_{X/Y}$. The map is given by

$$(s_0, s_1, \dots, s_n) \mapsto x_0 s_0 + x_1 s_1 + \cdots + x_n s_n.$$

Note that this is a degree 1 map.

Now I have to identify the kernel of this map with differentials, and I can do this on each open set (so long as I do it in a way that works simultaneously for each open set). So let's consider the open set U_0, where $x_0 \neq 0$, and we have coordinates $x_{j/0} = x_j/x_0$ $(1 \leq j \leq n)$. Given a differential

$$f_1(x_{1/0}, \dots, x_{n/0})\ dx_{1/0} + \cdots + f_n(x_{1/0}, \dots, x_{n/0})\ dx_{n/0}$$

we must produce $n+1$ sections of $\mathcal{O}(-1)$. As motivation, let me just look at the first term, and pretend that the projective coordinates are actual coordinates.

$$\begin{aligned} f_1\ dx_{1/0} &= f_1\ d(x_1/x_0) \\ &= f_1 \frac{x_0\ dx_1 - x_1\ dx_0}{x_0^2} \\ &= -\frac{x_1}{x_0^2} f_1\ dx_0 + \frac{f_1}{x_0}\ dx_1 \end{aligned}$$

Note that x_0 times the "coefficient of dx_0" plus x_1 times the "coefficient of dx_1" is 0, and also both coefficients are of homogeneous degree -1. Motivated by this, we take:

$$f_1\ dx_{1/0} + \cdots + f_n\ dx_{n/0} \mapsto \left(-\frac{x_1}{x_0^2} f_1 - \cdots - \frac{x_n}{x_0^2} f_n, \frac{f_1}{x_0}, \frac{f_2}{x_0}, \cdots, \frac{f_n}{x_0} \right) \tag{1}$$

Note that over U_0, this indeed gives an injection of $\Omega_{\mathbb{P}_A^n}$ to $\mathcal{O}(-1)^{\oplus(n+1)}$ that surjects onto the kernel of $\mathcal{O}(-1)^{\oplus(n+1)} \to \mathcal{O}_X$ (if $(g_0, \dots, g_n)$ is in the kernel, take $f_i = x_0 g_i$ for $i > 0$).

Let's make sure this construction, applied to two different coordinate patches (say U_0 and U_1) gives the same answer. (This verification is best ignored on a first reading.) Note that

$$\begin{aligned} f_1\ dx_{1/0} + f_2\ dx_{2/0} + \cdots &= f_1\ d\frac{1}{x_{0/1}} + f_2\ d\frac{x_{2/1}}{x_{0/1}} + \cdots \\ &= -\frac{f_1}{x_{0/1}^2} dx_{0/1} + \frac{f_2}{x_{0/1}} dx_{2/1} - \frac{f_2 x_{2/1}}{x_{0/1}^2} dx_{0/1} + \cdots \\ &= -\frac{f_1 + f_2 x_{2/1} + \cdots}{x_{0/1}^2} dx_{0/1} + \frac{f_2 x_1}{x_0} dx_{2/1} + \cdots . \end{aligned}$$

Under this map, the $dx_{2/1}$ term goes to the second factor (where the factors are indexed 0 through n) in $\mathcal{O}(-1)^{\oplus(n+1)}$, and yields f_2/x_0 as desired (and similarly for $dx_{j/1}$ for $j > 2$).

Also, the $dx_{0/1}$ term goes to the "zero" factor, and yields

$$(\sum_{j=1}^{n} f_i(x_i/x_1)/(x_0/x_1)^2)/x_1 = f_i x_i/x_0^2$$

as desired. Finally, the "first" factor must be correct because the sum over i of x_i times the ith factor is 0. □

Generalizations of the Euler exact sequence are quite useful. We won't use them later this year, so I'll state them without proof. Note that the argument applies without change if $\operatorname{Spec} A$ is replaced by an arbitrary base scheme. The Euler exact sequence further generalizes in a number of ways. As a first step, suppose V is a rank $n+1$ locally free sheaf (or vector bundle) on a scheme X. Then $\Omega_{\mathbb{P}V/X}$ sits in an Euler exact sequence:

$$0 \to \Omega_{\mathbb{P}V/X} \to \mathcal{O}(-1) \otimes V^\vee \to \mathcal{O}_X \to 0$$

If $\pi : \mathbb{P}V \to X$, the map $\mathcal{O}(-1) \otimes V^\vee \to \mathcal{O}_X$ is induced by $V^\vee \otimes \pi_* \mathcal{O}(1) \cong (V^\vee \otimes V) \otimes \mathcal{O}_X \to \mathcal{O}_X$, where $V^\vee \otimes V \to A$ is the trace map.

For another generalization, fix a base field, and let $G(m, n+1)$ be the space of vector spaces of dimension m in an $(n+1)$-dimensional vector space V. (This is called the *Grassmannian*. We have not shown that this is actually a variety in any natural way, but it is. The case $m = 1$ is $\mathbb{P}^n$.) Then over $G(m, n+1)$ we have a short exact sequence of locally free sheaves

$$0 \to S \to V \otimes \mathcal{O}_{G(m,n+1)} \to Q \to 0$$

where $V \otimes \mathcal{O}_{G(m,n+1)}$ is a trivial bundle, and S is the "universal subbundle" (such that over a point $[V' \subset V]$ of the Grassmannian $G(m, n+1)$, $S|_{[V' \subset V]}$ is V if you can see what that means). Then

$$\Omega_{G(m,n+1)/k} \cong \underline{\operatorname{Hom}}(Q, S). \tag{2}$$

1.3. *Exercise.* In the case of projective space, $m = 1$, $S = \mathcal{O}(-1)$. Verify (2) in this case.

This Grassmannian fact generalizes further to Grassmannian bundles.

2. Varieties over algebraically closed fields

We'll now discuss differentials in the case of interest to most people: varieties over algebraically closed fields. I'd like to begin with a couple of remarks.

2.1. *Remark: nonsingularity may be checked at closed points.* Recall from the first quarter a deep fact about regular local rings that we haven't proved: Any localization of a regular local ring at a prime is again regular local ring. (For a reference, see Matsumura's *Commutative Algebra*, p. 139.) I'm going to continue to use this without proof. It is possible I'll write up a proof later. But in any case, if this bothers you, you could re-define nonsingularity of locally finite type schemes over fields to be what other people call "nonsingularity at closed points", and the results of this section will hold.

2.2. *Remark for non-algebraically closed people.* Even if you are interested in non-algebraically closed fields, this section should still be of interest to you. In particular, if X is a variety over a field k, and $X_{\overline{k}} = X \times_{\operatorname{Spec} k} \operatorname{Spec} \overline{k}$, then $X_{\overline{k}}$ nonsingular implies that X is nonsingular. (You may wish to prove this yourself. By Remark 2.1, it suffices to check at closed points.) *Possible exercise.* In fact if k is separably closed, then $X_{\overline{k}}$ is nonsingular if and only if X is nonsingular, but this is a little bit harder.

Suppose for the rest of this section that X is a pure n-dimensional locally finite type scheme over an algebraically closed field k (e.g. a k-variety).

2.3. *Proposition. — $\Omega_{X/k}$ is locally free of rank n if and only if X is nonsingular.*

Proof. By Remark 2.1, it suffices to prove that $\Omega_{X/k}$ is locally free of rank n if and only if the closed points of X is nonsingular. Now $\Omega_{X/k}$ is locally free of rank n if and only if its fibers at all the closed points are rank n (recall that fibers jump in closed subsets). As the fiber of the cotangent sheaf is canonically isomorphic to the Zariski tangent space at closed points (done earlier), the Zariski tangent space at every closed point must have dimension n, i.e. the closed points are all nonsingular. □

Using this Proposition, we can get a new result using a neat trick.

2.4. *Theorem. — If X is integral, there is an dense open subset U of X which is nonsingular.*

Proof. The $n = 0$ case is immediate, so we assume $n > 0$.

We will show that the rank at the generic point is n. Then by uppersemicontinuity of the rank of a coherent sheaf (done earlier), it must be n in an open neighborhood of the generic point, and we are done by Proposition 2.3.

We thus have to check that if K is the fraction field of a dimension n integral finite-type k-scheme, i.e. if K is a transcendence degree n extension of k, then $\Omega_{K/k}$ is an n-dimensional vector space. But any transcendence degree $n > 1$ extension is separably generated: we can find n algebraically independent elements of K over k, say $x_1, \dots, x_n$, such that $K/k(x_1, \dots, x_n)$ is separable. (This is a fact about transcendence theory.) Then $\Omega_{K/k}$ is generated by $dx_1, \dots, dx_n$ (as $dx_1, \dots, dx_n$ generate $\Omega_{k(x_1,\dots,x_n)/k}$, and any element of K is separable over $k(x_1, \dots, x_n)$ — this is summarized most compactly using the affine form of the relative cotangent sequence). □

2.5. *Bertini's Theorem. — Suppose X is a nonsingular closed subvariety of $\mathbb{P}^n_k$ (where the standing hypothesis for this section, that k is algebraically closed, holds). Then there is an open subset of hyperplanes H of $\mathbb{P}^n_k$ such that H doesn't contain any component of X, and the scheme $H \cap X$ is a nonsingular variety. More precisely, this is an open subset of the dual projective space $\mathbb{P}^{n\vee}_k$. In particular, there exists a hyperplane H in $\mathbb{P}^n_k$ not containing any component of X such that the scheme $H \cap X$ is also a nonsingular variety.*

(We've already shown in our section on cohomology that if X is connected, then $H \cap X$ is connected.)

We may have used this before to show the existence of nonsingular curves of any genus, for example, although I don't think we did. (We discussed Bertini in class 35, p. 4.)

Note that this implies that a general degree $d > 0$ hypersurface in $\mathbb{P}^n_k$ also intersects X in a nonsingular subvariety of codimension 1 in X: replace $X \hookrightarrow \mathbb{P}^n$ with the composition $X \hookrightarrow \mathbb{P}^n \hookrightarrow \mathbb{P}^N$ where the latter morphism is the dth Veronese map.

Proof. In order to keep the language of the proof as clean as possible, I'll assume X is irreducible, but essentially the same proof applies in general.

The central idea of the proof is quite naive and straightforward. We'll describe the hyperplanes that are "bad", and show that they form a closed subset of dimension at most $n-1$ of $\mathbb{P}^{n\vee}_k$, and hence that the complement is a dense open subset. More precisely, we will define a projective variety $Y \subset X \times \mathbb{P}^{n\vee}_k$ that will be:

$$Y = \{(p \in X, H \subset \mathbb{P}^n_k) : p \in H, p \text{ is a singular point of } H \cap X, \text{ or } X \subset H\}$$

We will see that $\dim Y \leq n-1$. Thus the image of Y in $\mathbb{P}^{n\vee}_k$ will be a closed subset (the image of a closed subset by a projective hence closed morphism!), of dimension of $n-1$, and its complement is open.

We'll show that Y has dimension $n-1$ as follows. Consider the map $Y \to X$, sending (p, H) to p. Then a little thought will convince you that there is a $(n - \dim X - 1)$-dimensional family of hyperplanes through $p \in X$ such that $X \cap H$ is singular at p, or such that X is contained in H. (Those two conditions can be summarized quickly as: H contains the "first-order formal neighborhood of p in X", $\operatorname{Spec} \mathcal{O}_{X,p}/\mathfrak{m}^2$ where $\mathfrak{m}$ is the maximal ideal of $\mathcal{O}_{X,p}$.) Hence we expect Y to be a projective bundle, whose fibers are dimension $n - \dim X - 1$, and hence that Y has dimension at most $\dim X + (n - \dim X - 1) = n - 1$. In fact this is the case, but we'll show a little less (e.g. we won't show that $Y \to X$ is a projective bundle) because we don't need to prove this full statement to complete our proof of Bertini's theorem.

Let's put this strategy into action. We first define Y more precisely, in terms of equations on $\mathbb{P}^n \times \mathbb{P}^{n\vee}$, where the coordinates on $\mathbb{P}^n$ are $x_0, \dots x_n$, and the dual coordinates on $\mathbb{P}^{n\vee}$ are $a_0, \dots, a_n$. Suppose X is cut out by $f_1, \dots, f_r$. (We will soon verify that this definition of Y is independent of these equations.) Then we take these equations as some of the defining equations of Y. (So far we have defined the subscheme $X \times \mathbb{P}^{n\vee}$.) We also add the equation $a_0 x_0 + \cdots + a_n x_n = 0$. (So far we have described the subscheme of $\mathbb{P}^n \times \mathbb{P}^{n\vee}$ corresponding to points (p, H) where $p \in X$ and $p \in H$.) Note that the Jacobian matrix

$$\begin{pmatrix} \frac{\partial f_1}{\partial x_1}(p) & \cdots & \frac{\partial f_r}{\partial x_1}(p) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_1}{\partial x_n}(p) & \cdots & \frac{\partial f_r}{\partial x_n}(p) \end{pmatrix}$$

has corank equal to $\dim X$ at all closed points of X — this is precisely the Jacobian condition for nonsingularity (class 12, p. 3, 1.6). (Although we won't use this fact, in fact it has that corank $\dim X$ everywhere on X. Reason: the locus where the corank jumps is a closed locus, as this is described by equations, namely determinants of minors. Thus as the corank is constant at all closed points, it is constant everywhere.) We then require that the Jacobian matrix with a new row $(a_0, \cdots, a_n)$ has corank $\geq \dim X$ (hence $= \dim X$). This is cut out by equations (determinants of minors). By the Jacobian description of the Zariski tangent space, this condition encodes the requirement that the Zariski tangent space of $H \cap X$ at p has dimension precisely $\dim X$, which is $\dim H \cap X + 1$ (i.e. $H \cap X$ is singular at p) if H does not contain X, or if H contains X. This is precisely the notion that we hoped to capture.

Before getting on with our proof, let's do an example to convince ourselves that this algebra is describing the geometry we desire. Consider the plane conic $x_0^2 - x_1^2 - x_2^2 = 0$ over a field of characteristic not 2, which I picture as the circle $x^2 + y^2 = 1$ from the real picture in the chart U_0. (At this point I drew a picture.) Consider the point $(1, 1, 0)$, corresponding to $(1, 0)$ on the circle. We expect the tangent line in the affine plane to be $x = 1$, which should correspond to $x_0 - x_1 = 0$. Let's see what the algebra gives us. The Jacobian matrix is $\begin{pmatrix} 2x_0 & -2x_1 & -2x_2 \end{pmatrix} = \begin{pmatrix} 2 & -2 & 0 \end{pmatrix}$, which indeed has rank 1 as expected. Our recipe asks that the matrix $\begin{pmatrix} 2 & -2 & 0 \\ a_0 & a_1 & a_2 \end{pmatrix}$ have rank 1, which means that $(a_0, a_1, a_2) = (a_0, -a_0, 0)$, and also that $a_0x_0 + a_1x_1 + a_2x_2 = 0$, which is precisely what we wanted!

Returning to our construction, we can see that the Y just described is independent of the choice of $f_1, \dots, f_r$ (although we won't need this fact).

Here's why. It suffices to show that if we add in a redundant equation (some homogeneous f_0 that is a $k[x_0, \dots, x_n]$-linear combination of the f_i), we get the same Y (as then if we had a completely different set of f's, we could add them in one at a time, and then remove the old f's one at a time). If we add in a redundant equation, then that row in the Jacobian matrix will be a $k[x_0, \dots, x_n]$-linear combination of other rows, and thus the rank remains unchanged. (There is a slight issue I am glossing over here — f_0 may vanish on Y despite not being a linear combination of $f_1, \dots, f_n$.)

We'll next show that $\dim Y = n - 1$. For each $p \in X$, let Z_p be the locus of hyperplanes containing p, such that $H \cap X$ is singular at p, or else contains all of X; what is the dimension of Z_p? (For those who have heard of these words: what is the dimension of the locus of hyperplanes containing a first-order formal neighborhood of p in X?) Suppose $\dim X = d$. Then this should impose $d + 1$ conditions on hyperplanes. This means that it is a codimension $d + 1$, or dimension $n - d - 1$, projective space. Thus we should expect $Y \to X$ to be a projective bundle of relative dimension $n - d - 1$ over a variety of dimension d, and hence that $\dim Y = n - 1$. For convenience, I'll verify a little less: that $\dim Y \leq n - 1$.

Suppose Y has dimension N. Let $H_1, \dots, H_d$ be general hyperplanes such that $H_1 \cap \cdots \cap H_d \cap X$ is a finite set of points (this was an exercise from long ago, class 31, ex. 1.5, p. 4). Then if $\pi : Y \to X$ is the projection to X, then (using Krull's Principal Ideal Theorem)

$$n - d - 1 = \dim Y \cap \pi^* H_1 \cap \cdots \cap \pi^* H_d \geq \dim Y - d$$

from which $\dim Y \leq n - 1$. □

2.6. *Exercise-.* Show that Bertini's theorem still holds even if X is singular in dimension 0. (This isn't that important.)

2.7. *Remark..* The image in $\mathbb{P}^n$ tends to be a divisor. This is classically called the *dual variety*. The following exercise will give you some sense of it.

2.8. *Exercise.* Suppose $C \subset \mathbb{P}^2$ is a nonsingular conic over a field of characteristic not 2. Show that the dual variety is also a conic. (More precisely, suppose C is cut out by $f(x_0, x_1, x_2) = 0$. Show that $\{(a_0, a_1, a_2) : a_0x_0 + a_1x_1 + a_2x_2 = 0\}$ is cut out by a quadratic equation.) Thus for example, through a general point in the plane, there are two tangents to C. (The points on a line in the dual plane corresponds to those lines through a point of the original plane.)

We'll soon find the degree of the dual to a degree d curve (after we discuss the Riemann-Hurwitz formula), at least modulo some assumptions.

2.9. The Riemann-Hurwitz formula.

We're now ready to discuss and prove the Riemann-Hurwitz formula. We continue to work over an algebraically closed field k. Everything below can be mildly modified to work for a perfect field, e.g. any field of characteristic 0, and I'll describe this at the end of the discussion (Remark 2.17).

Definition (separable morphisms). A finite morphism between integral schemes $f : X \to Y$ is said to be *separable* if it is dominant, and the induced extension of function fields $FF(X)/FF(Y)$ is a separable extension. (Similarly, a generically finite morphism is *generically separable* if it is dominant, and the induced extension of function fields is a separable extension. We may not use this notion.) Note that this comes for free in characteristic 0.

2.10. *Proposition. — If $f : X \to Y$ is a finite separable morphism of nonsingular integral curves, then we have an exact sequence*

$$0 \to f^*\Omega_{Y/k} \to \Omega_{X/k} \to \Omega_{X/Y} \to 0.$$

Proof. We have right-exactness by the relative cotangent sequence, so we need to check only that $\phi : f^*\Omega_{Y/k} \to \Omega_{X/k}$ is injective. Now $\Omega_{Y/k}$ is an invertible sheaf on Y, so $f^*\Omega_{Y/k}$ is an invertible sheaf on X. Thus it has no torsion subsheaf, so we need only check that ϕ is an inclusion at the generic point. We thus tensor with $\mathcal{O}_\eta$ where η is the generic point of X. This is an exact functor (it is localization), and $\mathcal{O}_\eta \otimes \Omega_{X/Y} = 0$ (as $FF(X)/FF(Y)$ is a separable by hypothesis, and Ω for separable field extensions is 0 by Ex. 2.10, class 37, which was also Ex. 4, problem set 17). Also, $\mathcal{O}_\eta \otimes f^*\Omega_{Y/k}$ and $\mathcal{O}_\eta \otimes \Omega_{X/k}$ are both one-dimensional $\mathcal{O}_\eta$-vector spaces (they are the stalks of invertible sheaves at the generic point). Thus by considering

$$\mathcal{O}_\eta \otimes f^*\Omega_{Y/k} \to \mathcal{O}_\eta \otimes \Omega_{X/k} \to \mathcal{O}_\eta \otimes \Omega_{X/Y} \to 0$$

(which is

$$\mathcal{O}_\eta \to \mathcal{O}_\eta \to 0 \to 0)$$

we see that $\mathcal{O}_\eta \otimes f^*\Omega_{Y/k} \to \mathcal{O}_\eta \otimes \Omega_{X/k}$ is injective, and thus that $f^*\Omega_{Y/k} \to \Omega_{X/k}$ is injective. □

2.11. It is worth noting what goes wrong for non-separable morphisms. For example, suppose k is a field of characteristic p, consider the map $f : \mathbb{A}^1_k = \operatorname{Spec} k[t] \to \mathbb{A}^1_k = \operatorname{Spec} k[u]$ given by $u = t^p$. Then $\Omega_{\mathbb{A}^1_k/k}$ is the trivial invertible sheaf generated by dt. As another (similar but different) example, if $K = k(x)$ and $K' = K(x^p)$, then the inclusion $K' \hookrightarrow K$ induces $f : \operatorname{Spec} K[t] \to \operatorname{Spec} K'[t]$. Once again, Ω_f is an invertible sheaf, generated by dx (which in this case is pulled back from $\Omega_{K/K'}$ on $\operatorname{Spec} K$). In both of these cases, we have maps from one affine line to another, and there are vertical tangent vectors.

2.12. The sheaf $\Omega_{X/Y}$ on the right side of Proposition 2.10 is a coherent sheaf not supported at the generic point. Hence it is supported at a finite number of points. These are called the *ramification points* (and the images downstairs are called the *branch points*). I drew a picture here.

Let's check out what happens at closed points. We have two discrete valuation rings, say $\operatorname{Spec} A \to \operatorname{Spec} B$. I've assumed that we are working over an algebraically closed field k, so this morphism $B \to A$ induces an isomorphism of residue fields (with k). Suppose their uniformizers are s and t respectively, with $t \mapsto us^n$ where u is a unit of A. Then

$$dt = d(us^n) = uns^{n-1}\, ds + s^n\, du.$$

This vanishes to order at least $n-1$, and precisely $n-1$ if n doesn't divide the characteristic. The former case is called *tame* ramification, and the latter is called *wild* ramification. We call this order the *ramification order* at this point of X.

Define the *ramification divisor* on X as the sum of all points with their corresponding ramifications (only finitely many of which are non-zero). The image of this divisor on Y is called the *branch divisor*.

2.13. *Straightforward exercise: interpreting the ramification divisor in terms of number of preimages.* Suppose all the ramification above $y \in Y$ is tame. Show that the degree of the branch divisor at y is $\deg(f : X \to Y) - \#f^{-1}(y)$. Thus the multiplicity of the branch divisor counts the extent to which the number of preimages is less than the degree.

2.14. *Proposition. — Suppose R is the ramification divisor of $f : X \to Y$. Then $\Omega_X(-R) \cong f^*\Omega_Y$.*

Note that we are making no assumption that X or Y is projective.

Proof. This says that we can interpret the invertible sheaf $f^*\Omega_Y$ over an open set of X as those differentials on X vanishing along the ramification divisor. But that is the content of Proposition 2.10. □

Then the Riemann-Hurwitz formula follows!

2.15. *Theorem (Riemann-Hurwitz). — Suppose* $f : X \to Y$ *is a finite separable morphism of curves. Let* $n = \deg f$. *Then* $2g(X) - 2 = n(2g(Y) - 2) + \deg R$.

Note that we now need the projective hypotheses in order to take degrees of invertible sheaves.

Proof. This follows by taking the degree of both sides of Proposition 2.14 (and using the fact that the pullback of a degree d line bundle by a finite degree n morphism is dn, which was an earlier exercise, Ex. 3.1, class 29, p. 3, or Ex. 2, problem set 13). □

2.16. *Exercise: degree of dual curves.* Describe the degree of the dual to a nonsingular degree d plane curve C as follows. Pick a general point $p \in \mathbb{P}^2$. Find the number of tangents to C through p, by noting that projection from p gives a degree d map to $\mathbb{P}^1$ (why?) by a curve of known genus (you've calculated this before), and that ramification of this cover of $\mathbb{P}^1$ corresponds to a tangents through p. (Feel free to make assumptions, e.g. that for a general p this branched cover has the simplest possible branching — this should be a back-of-an-envelope calculation.)

2.17. *Remark: Riemann-Hurwitz over perfect fields.* This discussion can be extended to work when the base field is not algebraically closed; perfect will suffice. The place we assumed that the base field was algebraically closed was after we reduced to understanding the ramification of the morphism of the spectrum of one discrete valuation ring over our base field k to the spectrum of another, and we assumed that this map induced an isomorphism of residue fields. In general, it can be a finite extension. Let's analyze this case explicitly. Consider a map $\operatorname{Spec} A \to \operatorname{Spec} B$ of spectra of discrete valuation rings, corresponding to a ring extension $B \to A$. Let s be the uniformizer of A, and t the uniformizer of B. Let $\mathfrak{m}$ be the maximal ideal of A, and $\mathfrak{n}$ the maximal ideal of A. Then as $A/\mathfrak{m}$ is a finite extension of $B/\mathfrak{n}$, it is generated over $B/\mathfrak{n}$ by a single element (we're invoking here the theorem of the primitive element, and we use the "perfect" assumption here). Let s' be any lift of this element of $A/\mathfrak{m}$ to A. Then A is generated over B by s and s', so $\Omega_{A/B}$ is generated by ds and ds'. The contribution of ds is as described above. You can show that $ds' = 0$. Thus all calculations above carry without change, except for the following.

(i) We have to compute the degree of the ramification divisor appropriately: we need to include as a factor the degree of the field extension of the residue field of the point on the *source* (over k).

(ii) Exercise 2.13 doesn't work, but can be patched by replacing $\#f^{-1}(y)$ with the number of *geometric* preimages.

As an example of what happens differently in (ii), consider the degree 2 finite morphism $X = \operatorname{Spec} \mathbb{Z}[i] \to Y = \operatorname{Spec} \mathbb{Z}$. We can compute $\Omega_{\mathbb{Z}[i]/\mathbb{Z}}$ directly, as $\mathbb{Z}[i] \cong \mathbb{Z}[x]/(x^2+1)$: $\Omega_{\mathbb{Z}[i]/\mathbb{Z}} \cong \mathbb{Z}[i]dx/(2dx)$. In other words, it is supported at the prime $(1+i)$ (the unique prime above $[(2)] \in \operatorname{Spec} \mathbb{Z}$). However, the number of preimages of points in $\operatorname{Spec} \mathbb{Z}$ is not always 2 away from the point $[(2)]$; half the time (including, for example, over $[(3)]$) there is one point, but the field extension is separable.

2.18. *Exercise (aside): Artin-Schreier covers.* In characteristic 0, the only connected unbranched cover of $\mathbb{A}^1$ is the isomorphism $\mathbb{A}^1 \xrightarrow{\sim} \mathbb{A}^1$; that was an earlier example/exercise, when we discussed Riemann-Hurwitz the first time. In positive characteristic, this needn't be true, because of wild ramification. Show that the morphism corresponding to $k[x] \to k[x,y]/(y^p - x^p - y)$ is such a map. (Once the theory of the algebraic fundamental group is developed, this translates to: "$\mathbb{A}^1$ is not simply connected in characteristic p.")

2.19. The conormal exact sequence for nonsingular varieties.

Recall the conormal exact sequence. Suppose $f : X \to Y$ morphism of schemes, $Z \hookrightarrow X$ closed subscheme of X, with ideal sheaf $\mathcal{I}$. Then there is an exact sequence of sheaves on Z:

$$\mathcal{I}/\mathcal{I}^2 \xrightarrow{\delta} \Omega_{X/Y} \otimes \mathcal{O}_Z \longrightarrow \Omega_{Z/Y} \longrightarrow 0.$$

I promised you that in good situations this is exact on the left as well, as our geometric intuition predicts. Now let $Z = \operatorname{Spec} k$ (where $k = \overline{k}$), and Y a nonsingular k-variety, and $X \subset Y$ an irreducible closed subscheme cut out by the quasicoherent sheaf of ideals $\mathcal{I} \subset \mathcal{O}_Y$.

2.20. *Theorem (conormal exact sequence for nonsingular varieties). — X is nonsingular if and only if (i) $\Omega_{X/k}$ is locally free, and (ii) the conormal exact sequence is exact on the left also:*

$$0 \longrightarrow \mathcal{I}/\mathcal{I}^2 \xrightarrow{\delta} \Omega_{X/Y} \otimes \mathcal{O}_Z \longrightarrow \Omega_{Z/Y} \longrightarrow 0.$$

Moreover, if Y is nonsingular, then $\mathcal{I}$ is locally generated by $\operatorname{codim}(X,Y)$ elements, and $\mathcal{I}/\mathcal{I}^2$ is a locally free of rank $\operatorname{codim}(X,Y)$.

This latter condition is the definition of something being a *local complete intersection* in a nonsingular scheme.

You can read a proof of this in Hartshorne II.8.17. I'm not going to present it in class, as we'll never use it. The only case I've ever seen used is the implication that if X is nonsingular, then (i) and (ii) hold; and we've already checked (i). This implication (that in the case of a nonsingular subvariety of a nonsingular variety, the conormal and hence normal exact sequence is exact) is very useful for relating the differentials on a nonsingular subvariety to the normal bundle.

The real content is that in the case of a nonsingular subvariety of a nonsingular variety, the conormal exact sequence is exact on the left as well, and in this nice case we have a short exact sequence of locally free sheaves (vector bundles). By dualizing, i.e. applying $\underline{\operatorname{Hom}}(\cdot, \mathcal{O}_X)$, we obtain the *normal exact sequence*

$$0 \to T_{X/k} \to T_{Y/k} \to \mathcal{N}_{X/Y} \to 0$$

which is very handy. Note that dualizing an exact sequence will give you a left-exact sequence in general, but dualizing an exact sequence of locally free sheaves will always be locally free. (In fact, all you need is that the third term is locally free. I could make this an exercise; it may also follow if I define Ext soon after defining Tor, as an exercise.)

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FOUNDATIONS OF ALGEBRAIC GEOMETRY CLASSES 41 AND 42

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CONTENTS

Today: Flatness; Tor**; ideal-theoretic characterization of flatness; for coherent modules over a Noetherian local ring flat = free, flatness over a nonsingular curve.**

1. Introduction to flatness

We come next to the important concept of flatness. This topic is also not a hard topic, and we could have dealt with it as soon as we had discussed quasicoherent sheaves and morphisms. But it is an intuitively unexpected one, and the algebra and geometry are not obviously connected, so we've left it for relatively late. It is answer to many of your geometric prayers, but you just haven't realized it yet.

The notion of flatness apparently was first defined in Serre's landmark "GAGA" paper.

Here are some of the reasons it is a good concept. We would like to make sense of the notion of "fibration" in the algebraic category (i.e. in algebraic geometry, as opposed to differential geometry), and it turns out that flatness is essential to this definition. It turns out that flat is the right algebraic version of a "nice" or "continuous" family, and this notion is more general than you might think. For example, the double cover $\mathbb{A}^1 \to \mathbb{A}^1$ over an algebraically closed field given by $y \mapsto x^2$ is a flat family, which we interpret as two points coming together to a fat point. The fact that the degree of this map always is 2 is a symptom of how this family is well-behaved. Another key example is that of a family of smooth curves degenerating to a nodal curve, that I sketched on the board in class. One can prove things about smooth curves by first proving them about the nodal curve, and then showing that the result behaves well in flat families. In general, we'll see that certain things behave well in nice families, such as cohomology groups (and even better Euler characteristics) of fibers.

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There is a second flavor of prayer that is answered. It would be very nice if tensor product (of quasicoherent sheaves, or of modules over a ring) were an exact functor, and certain statements of results and proofs we have seen would be cleaner if this were true. Those modules for which tensoring is always exact are flat (this will be the definition!), and hence for flat modules (or quasicoherent sheaves, or soon, morphisms) we'll be able to get some very useful statements. A flip side of that is that exact sequences of *flat* modules remain exact when tensored with *any* other module.

In this section, we'll discuss flat morphisms. When introducing a new notion, I prefer to start with a number of geometric examples, and figure out the algebra on the fly. In this case, because there is enough algebra, I'll instead discuss the algebra at some length and then later explain why you care geometrically. This will require more patience than usual on your part.

2. Algebraic definition and easy facts

Many facts about flatness are easy or immediate, and a few are tricky. I'm going to try to make clear which is which, to help you remember the easy facts and the idea of proof for the harder facts.

The definition of a *flat A-module* is very simple. Recall that if

$$0 \to N' \to N \to N'' \to 0 \qquad (1)$$

is a short exact sequence of A-modules, and M is another A-module, then

$$M \otimes_A N' \to M \otimes_A N \to M \otimes_A N'' \to 0$$

is exact. In other words, $M \otimes_A$ is a right-exact functor. We say that M *is a flat* A*-module* if $M \otimes_A$ is an exact functor, i.e. if for all exact sequences (1),

$$0 \to M \otimes_A N' \to M \otimes_A N \to M \otimes_A N'' \to 0$$

is exact as well.

Exercise. If $N' \to N \to N''$ is exact and M is a flat A-module, show that $M \otimes_A N' \to M \otimes_A N \to M \otimes_A N''$ is exact. Hence *any* exact sequence of A-modules remains exact upon tensoring with M. (We've seen things like this before, so this should be fairly straightforward.)

We say that *a ring homomorphism* $B \to A$ *is flat* if A is flat as a B-module. (We don't care about the algebra structure of A.)

Here are two key examples of flat ring homomorphisms:

(i) free modules A-modules are clearly flat.
(ii) Localizations are flat: Suppose S is a multiplicative subset of B. Then $B \to S^{-1}B$ is a flat ring morphism.

Exercise. Verify (ii). We have used this before: localization is an exact functor.

Here is a useful way of recognizing when a module is *not* flat. Flat modules are torsion-free. More precisely, if x is a non-zero-divisor of A, and M is a flat A-module, then $M \xrightarrow{\times x} M$ is injective. Reason: apply the exact functor $M \otimes_A$ to the exact sequence $0 \longrightarrow A \xrightarrow{\times x} A$.

We make some quick but important observations:

2.1. *Proposition (flatness is a stalk/prime-local property). — An A-module M is flat if and only if $M_{\mathfrak{p}}$ is a flat $A_{\mathfrak{p}}$-module for all primes $\mathfrak{p}$.*

Proof. Suppose first that M is flat. Given any exact sequence of $A_{\mathfrak{p}}$-modules (1),

$$0 \to M \otimes_A N' \to M \otimes_A N \to M \otimes_A N'' \to 0$$

is exact too. But $M \otimes_A N$ is canonically isomorphic to $M \otimes_{A_{\mathfrak{p}}} N_{\mathfrak{p}}$ (exercise: show this!), so $M_{\mathfrak{p}}$ is a flat $A_{\mathfrak{p}}$-module.

Suppose next that M is *not* flat. Then there is some short exact sequence (1) that upon tensoring with M becomes

$$0 \to K \to M \otimes_A N' \to M \otimes_A N \to M \otimes_A N'' \to 0 \tag{2}$$

where $K \neq 0$ is the kernel of $M \otimes_A N' \to M \otimes_A N$. Then as $K \neq 0$, K has non-empty support, so there is some prime $\mathfrak{p}$ such that $K_{\mathfrak{p}} \neq 0$. Then

$$0 \to N'_{\mathfrak{p}} \to N_{\mathfrak{p}} \to N''_{\mathfrak{p}} \to 0 \tag{3}$$

is a short exact sequence of $A_{\mathfrak{p}}$-modules (recall that localization is exact — see (ii) before the statement of the Proposition), but is no longer exact upon tensoring (over $A_{\mathfrak{p}}$) with $M_{\mathfrak{p}}$ (as

$$0 \to K_{\mathfrak{p}} \to M_{\mathfrak{p}} \otimes_{A_{\mathfrak{p}}} N'_{\mathfrak{p}} \to M_{\mathfrak{p}} \otimes_{A_{\mathfrak{p}}} N_{\mathfrak{p}} \to M_{\mathfrak{p}} \otimes_{A_{\mathfrak{p}}} N''_{\mathfrak{p}} \to 0 \tag{4}$$

is exact). (Here we use that localization commutes with tensor product.) □

2.2. *Proposition (flatness is preserved by change of base ring). — If M flat A-module, $A \to B$ is a homomorphism, then $M \otimes_A B$ is a flat B-module.*

Proof. Exercise. □

2.3. *Proposition (transitivity of flatness). — If B is a flat A-algebra, and M is B-flat, then it is also A-flat.*

Proof. Exercise. (Hint: consider the natural isomorphism $(M \otimes_A B) \otimes_B \cdot \cong M \otimes_B (B \otimes_A \cdot)$.) □

The extension of this notion to schemes is straightforward.

2.4. *Definition: flat quasicoherent sheaf.* We say that a quasicoherent sheaf $\mathcal{F}$ on a scheme X is flat (over X) if for all $x \in X$, $\mathcal{F}_x$ is a flat $\mathcal{O}_{X,x}$-module. In light of Proposition 2.1, we can check this notion on affine open cover of X.

2.5. *Definition: flat morphism.* Similarly, we say that a morphism of schemes $\pi : X \to Y$ is flat if for all $x \in X$, $\mathcal{O}_{X,x}$ is a flat $\mathcal{O}_{Y,\pi(x)}$-module. Again, we can check locally, on maps of affine schemes.

We can combine these two definitions into a single definition.

2.6. *Definition: flat quasicoherent sheaf over some base.* Suppose $\pi : X \to Y$ is a morphism of schemes, and $\mathcal{F}$ is a quasicoherent sheaf on X. We say that $\mathcal{F}$ *is flat over* Y if for all $x \in X$, $\mathcal{F}_x$ is a flat $\mathcal{O}_{Y,\pi(x)}$-module.

Definitions 2.4 and 2.5 correspond to the cases $X = Y$ and $\mathcal{F} = \mathcal{O}_X$ respectively.

This definition can be extended without change to the category of ringed spaces, but we won't need this.

All of the Propositions above carry over naturally. For example, flatness is preserved by base change. (More explicitly: suppose $\pi : X \to Y$ is a morphism, and $\mathcal{F}$ is a quasicoherent sheaf on X, flat over Y. If $Y' \to Y$ is any morphism, and $p : X \times_Y Y' \to X$ is the projection, then $p^*\mathcal{F}$ is flat over Y'.) Also, flatness is transitive. (More explicitly: suppose $\pi : X \to Y$ and $\mathcal{F}$ is a quasicoherent sheaf on X, flat over Y. Suppose also that $\psi : Y \to Z$ is a flat morphism. Then $\mathcal{F}$ is flat over Z.)

We also have other statements easily. For example: open immersions are flat.

2.7. *Exercise.* If X is a scheme, and η is the generic point for an irreducible component, show that the natural morphism $\operatorname{Spec} \mathcal{O}_{X,\eta} \to X$ is flat. (Hint: localization is flat.)

We earlier proved the following important fact, although we did not have the language of flatness at the time.

2.8. *Theorem (cohomology commutes with flat base change). — Suppose*

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g} & Y \end{array}$$

is a fiber diagram, and f *(and thus* f'*) is quasicompact and separated (so higher pushforwards exist). Suppose also that* g *is flat, and* $\mathcal{F}$ *is a quasicoherent sheaf on* X*. Then the natural morphisms* $g^* R^i f_* \mathcal{F} \to R^i f'_*(g^*\mathcal{F})$ *are isomorphisms.*

A special case that is often useful is the case where Y' is the generic point of a component of Y. In other words, in light of Exercise 2.7, the stalk of the higher pushforward of

$\mathcal{F}$ at the generic point is the cohomology of $\mathcal{F}$ on the fiber over the generic point. This is a first example of something important: understanding cohomology of (quasicoherent sheaves on) fibers in terms of higher pushforwards. (We would certainly hope that higher pushforwards would tell us something about higher cohomology of fibers, but this is certainly not a priori clear!)

(I might dig up the lecture reference later, but I'll tell you now where proved it: where we described this natural morphism, I had a comment that if we had exactness of tensor product, them morphisms would be an isomorphism.)

We will spend the rest of our discussion on flatness as follows. First, we will ask ourselves: what are the flat modules over particularly nice rings? More generally, how can you check for flatness? And how should you picture it geometrically? We will then prove additional facts about flatness, and using flatness, answering the essential question: "why do we care?"

2.9. *Faithful flatness.* The notion of *faithful flatness* is handy, although we probably won't use it. We say that an extension of rings $B \to A$ is *faithfully flat* if for every A-module M, M is A-flat if and only if $M \otimes_A B$ is B-flat. We say that a morphism of schemes $X \to Y$ is *faithfully flat* if it is flat and surjective. These notions are the "same", as shown by the following exercise.

Exercise. Show that $B \to A$ is faithfully flat if and only if $\operatorname{Spec} A \to \operatorname{Spec} B$ is faithfully flat.

3. The "Tor" functors, and a "cohomological" criterion for flatness

In order to prove more facts about flatness, it is handy to have the notion of Tor. (Tor is short for "torsion". The reason for this name is that the 0th and/or 1st Tor-group measures common torsion in abelian groups (aka $\mathbb{Z}$-modules).) If you have never seen this notion before, you may want to just remember its properties, which are natural. But I'd like to prove everything anyway — it is surprisingly easy.

The idea behind Tor is as follows. Whenever we see a right-exact functor, we always hope that it is the end of a long-exact sequence. Informally, given a short exact sequence (1), we are hoping to see a long exact sequence

$$\cdots \longrightarrow \operatorname{Tor}_i^A(M, N') \longrightarrow \operatorname{Tor}_i^A(M, N) \longrightarrow \operatorname{Tor}_i^A(M, N'') \longrightarrow \cdots \tag{5}$$

$$\longrightarrow \operatorname{Tor}_1^A(M, N') \longrightarrow \operatorname{Tor}_1^A(M, N) \longrightarrow \operatorname{Tor}_1^A(M, N'')$$

$$\longrightarrow M \otimes_A N' \longrightarrow M \otimes_A N \longrightarrow M \otimes_A N'' \longrightarrow 0.$$

More precisely, we are hoping for *covariant functors* $\operatorname{Tor}_i^A(\cdot, N)$ from A-modules to A-modules (giving 2/3 of the morphisms in that long exact sequence), with $\operatorname{Tor}_0^A(M, N) \equiv$

$M \otimes_A N$, and natural δ morphisms $\mathrm{Tor}_{i+1}^A(M, N'') \to \mathrm{Tor}_i^A(M, N')$ for every short exact sequence (1) giving the long exact sequence. (In case you care, "natural" means: given a morphism of short exact sequences, the natural square you would write down involving the δ-morphism must commute. I'm not going to state this explicitly.)

It turns out to be not too hard to make this work, and this will later motivate derived functors. I'll now define $\mathrm{Tor}_i^A(M, N)$. Take any resolution $\mathcal{R}$ of N by free modules:

$$\cdots \longrightarrow A^{\oplus n_2} \longrightarrow A^{\oplus n_1} \longrightarrow A^{\oplus n_0} \longrightarrow N \to 0.$$

More precisely, build this resolution from right to left. Start by choosing generators of N as an A-module, giving us $A^{\oplus n_0} \to N \to 0$. Then choose generators of the kernel, and so on. Note that we are not requiring the n_i to be finite, although if N is a finitely-generated module and A is Noetherian (or more generally if N is coherent and A is coherent over itself), we can choose the n_i to be finite. Truncate the resolution, by stripping off the last term. Then tensor with M (which may lose exactness!). Let $\mathrm{Tor}_A^i(M, N)_{\mathcal{R}}$ be the homology of this complex at the ith stage ($i \geq 0$). The subscript $\mathcal{R}$ reminds us that our construction depends on the resolution, although we will soon see that it is independent of the resolution.

We make some quick observations.

- $\mathrm{Tor}_0^A(M, N)_{\mathcal{R}} \cong M \otimes_A N$ (and this isomorphism is canonical). Reason: as tensoring is right exact, and $A^{\oplus n_1} \to A^{\oplus n_0} \to N \to 0$ is exact, we have that $M^{\oplus n_1} \to M^{\oplus n_0} \to M \otimes_A N \to 0$ is exact, and hence that the homology of the truncated complex $M^{\oplus n_1} \to M^{\oplus n_0} \to 0$ is $M \otimes_A N$.
- If M is flat, then $\mathrm{Tor}_i^A(M, N)_{\mathcal{R}} = 0$ for all i.

Now given two modules N and N' and resolutions $\mathcal{R}$ and $\mathcal{R}'$ of N and N', we can "lift" any morphism $N \to N'$ to a morphism of the two resolutions:

$$\begin{array}{ccccccccccccc}
\cdots & \longrightarrow & A^{\oplus n_i} & \longrightarrow & \cdots & \longrightarrow & A^{\oplus n_1} & \longrightarrow & A^{\oplus n_0} & \longrightarrow & N & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & A^{\oplus n_i'} & \longrightarrow & \cdots & \longrightarrow & A^{\oplus n_1'} & \longrightarrow & A^{\oplus n_0'} & \longrightarrow & N' & \longrightarrow & 0
\end{array}$$

Denote the choice of lifts by $\mathcal{R} \to \mathcal{R}'$. Now truncate both complexes and tensor with M. Maps of complexes induce maps of homology, so we have described maps (a priori depending on $\mathcal{R} \to \mathcal{R}'$)

$$\mathrm{Tor}_i^A(M, N)_{\mathcal{R}} \to \mathrm{Tor}_i^A(M, N')_{\mathcal{R}'}.$$

We say two maps of complexes $f, g : C_* \to C'_*$ are *homotopic* if there is a sequence of maps $w : C_i \to C'_{i+1}$ such that $f - g = dw + wd$. Two homotopic maps give the same map on homology. (Exercise: verify this if you haven't seen this before.)

Crucial Exercise: Show that any two lifts $\mathcal{R} \to \mathcal{R}'$ are homotopic.

We now pull these observations together.

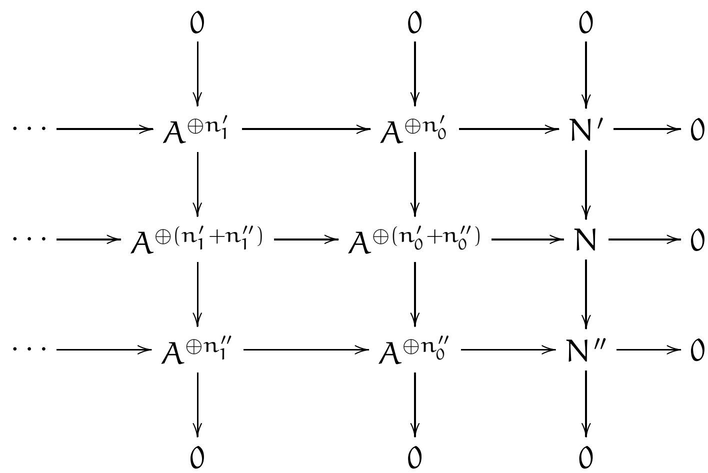
(1) We get a covariant functor from $\mathrm{Tor}_i^A(M, N)_{\mathcal{R}} \to \mathrm{Tor}_i^A(M, N')_{\mathcal{R}'}$ (independent of the lift $\mathcal{R} \to \mathcal{R}'$).

(2) Hence for any two resolutions $\mathcal{R}$ and $\mathcal{R}'$ we get a canonical isomorphism $\mathrm{Tor}_i^A(M,N)_{\mathcal{R}} \cong \mathrm{Tor}_i^1(M,N)_{\mathcal{R}'}$. Here's why. Choose lifts $\mathcal{R} \to \mathcal{R}'$ and $\mathcal{R}' \to \mathcal{R}$. The composition $\mathcal{R} \to \mathcal{R}' \to \mathcal{R}$ is homotopic to the identity (as it is a lift of the identity map $N \to N$). Thus if $f_{\mathcal{R}\to\mathcal{R}'} : \mathrm{Tor}_i^A(M,N)_{\mathcal{R}} \to \mathrm{Tor}_i^1(M,N)_{\mathcal{R}'}$ is the map induced by $\mathcal{R} \to \mathcal{R}'$, and similarly $f_{\mathcal{R}'\to\mathcal{R}}$ is the map induced by $\mathcal{R} \to \mathcal{R}'$, then $f_{\mathcal{R}'\to\mathcal{R}} \circ f_{\mathcal{R}\to\mathcal{R}'}$ is the identity, and similarly $f_{\mathcal{R}\to\mathcal{R}'} \circ f_{\mathcal{R}'\to\mathcal{R}}$ is the identity.

(3) Hence the covariant functor doesn't depend on the resolutions!

Finally:

(4) For any short exact sequence (1) we get a long exact sequence of Tor's (5). Here's why: given a short exact sequence (1), choose resolutions of N' and N''. Then use these to get a resolution for N in the obvious way (see below; the map $A^{\oplus(n_0' \to n_0'')} \to N$ is the composition $A^{\oplus n_0'} \to N' \to N$ along with any lift of $A^{n_0''} \to N''$ to N) so that we have a short exact sequence of resolutions

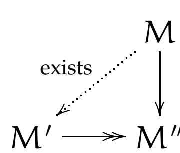


Then truncate (removing the column (1)), tensor with M (obtaining a short exact sequence of complexes) and take cohomology, yielding a long exact sequence.

We have thus established the foundations of Tor!

Note that if N is a free module, then $\mathrm{Tor}_i^A(M,N) = 0$ for all M and all $i > 0$, as N has itself as a resolution.

3.1. *Remark: Projective resolutions.* We used very little about free modules in the above construction; in fact we used only that free modules are *projective*, i.e. those modules M such that for any surjection $M' \to M''$, it is possible to lift any morphism $M \to M''$ to $M \to M'$. This is summarized in the following diagram.



Equivalently, $\mathrm{Hom}(M,\cdot)$ is an *exact functor* ($\mathrm{Hom}(N,\cdot)$ is always left-exact for any N). (More generally, we can define the notion of a *projective object in any abelian category*.)

Hence (i) we can compute$\mathrm{Tor}_i^A(M,N)$ by taking any projective resolution of N, and (ii) $\mathrm{Tor}_i^A(M,N) = 0$ for any projective A-module N.

3.2. *Remark: Generalizing this construction.* The above description was low-tech, but immediately generalizes drastically. All we are using is that $M \otimes_A$ is a right-exact functor. In general, if F is *any* right-exact covariant functor from the category of A-modules to any abelian category, this construction will define a sequence of functors $L_i F$ (called left-derived functors of F) such that $L_0 F = F$ and the L_i's give a long-exact sequence. We can make this more general still. We say that an abelian category *has enough projectives* if for any object N there is a surjection onto it from a projective object. Then if F is any right-exact functor from an abelian category with enough projectives to any abelian category, then F has left-derived functors.

3.3. *Exercise.* The notion of an *injective object* in an abelian category is dual to the notion of a projective object. Define derived functors for (i) covariant left-exact functors (these are called right-derived functors), (ii) contravariant left-exact functors (also right-derived functors), and (iii) contravariant right-exact functors (these are called left-derived functors), making explicit the necessary assumptions of the category having enough injectives or projectives.

Here are two quick practice exercises, giving useful properties of Tor.

Important exercise. If B is A-flat, then we get isomorphism $B \otimes \mathrm{Tor}_i^A(M,N) \cong \mathrm{Tor}_i^B(B \otimes M, B \otimes N)$. (This is tricky rather than hard; it has a clever one-line answer. Here is a fancier fact that experts may want to try: if B is not A-flat, we don't get an isomorphism; instead we get a spectral sequence.)

Exercise- (not too important, but good practice if you haven't played with Tor *before).* If x is not a 0-divisor, show that $\mathrm{Tor}_i^A(A/x, M)$ is 0 for $i > 1$, and for $i = 0$, get M/xM, and for $i = 1$, get $(M : x)$ (those things sent to 0 upon multiplication by x).

3.4. "Symmetry" of Tor. The natural isomorphism $M \otimes N \to N \otimes M$ extends to the following.

3.5. *Theorem. — There is a natural isomorphism* $\mathrm{Tor}_i(M,N) \cong \mathrm{Tor}(N,M)$.

Proof. Take two resolutions of M and N:

$$\cdots \to A^{\oplus m_1} \to A^{\oplus m_0} \to M \to 0$$

and

$$\cdots \to A^{\oplus n_1} \to A^{\oplus n_0} \to N \to 0.$$

Consider the double complex obtained by tensoring their truncations.

$$
\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & A^{\oplus(m_2+n_2)} & \longrightarrow & A^{\oplus(m_1+n_2)} & \longrightarrow & A^{\oplus(m_0+n_2)} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & A^{\oplus(m_2+n_1)} & \longrightarrow & A^{\oplus(m_1+n_1)} & \longrightarrow & A^{\oplus(m_0+n_1)} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & A^{\oplus(m_2+n_0)} & \longrightarrow & A^{\oplus(m_1+n_0)} & \longrightarrow & A^{\oplus(m_0+n_0)} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0. & &
\end{array}
$$

Apply our spectral sequence machinery. We compute the homology of this complex in two ways.

We start by using the vertical arrows. Notice that the ith column is precisely the truncated resolution of N, tensored with $A^{\oplus m_i}$. Thus the homology in the vertical direction in the ith column is 0 except in the bottom element of the column, where it is $N^{\oplus m_i}$. We next take homology in the horizontal direction. In the only non-zero row (the bottom row), we see precisely the complex computing $\operatorname{Tor}_i(N, M)$. After using these second arrows, the spectral sequence has converged. Thus the ith homology of the double complex is (naturally isomorphic to) $\operatorname{Tor}_i(N, M)$.

Similarly, if we began with the arrows in the horizontal direction, we would conclude that the ith homology of the double complex is $\operatorname{Tor}_i(M, N)$. □

This gives us a quick but very useful result. Recall that if $0 \to N' \to N \to N'' \to 0$ is exact, then so is the complex obtained by tensoring with M *if M is flat*. (Indeed that is the definition of flatness!) But in general we have an exact sequence

$$\operatorname{Tor}_1^A(M, N'') \to M \otimes_A N' \to M \otimes_A N \to M \otimes_A N'' \to 0$$

Hence we conclude:

3.6. *Proposition. — If $0 \to N' \to N \to N'' \to 0$ is exact, and N'' is flat, then $0 \to M \otimes_A N' \to M \otimes_A N \to M \otimes_A N'' \to 0$ is exact.*

Note that we would have cared about this result long before learning about Tor. This gives some motivation for learning about Tor. Presumably one can also show this directly by some sort of diagram chase. (Is there an easy proof?)

One important consequence is the following. Suppose we have a short exact sequence of sheaves on Y, and the rightmost element is flat (e.g. locally free). Then if we pull this exact sequence back to X, it remains exact. (I think we may have used this.)

3.7. An ideal-theoretic criterion for flatness. We come now to a useful fact. Observe that $\mathrm{Tor}_1(M, N) = 0$ for all N implies that M is flat; this in turn implies that $\mathrm{Tor}_i(M, N) = 0$ for all $i > 0$.

The following is a very useful variant on this.

3.8. *Key theorem. — M is flat if and only if* $\mathrm{Tor}_1^A(M, A/I) = 0$ *for all ideals* I.

(The interested reader can tweak the proof below a little to show that it suffices to consider *finitely generated ideals* I, but we won't use this fact.)

Proof. [The M's and N's are messed up in this proof.] We have already observed that if N is flat, then $\mathrm{Tor}_1^A(M, R/I) = 0$ for all I. So we assume that $\mathrm{Tor}_1^A(M, A/I) = 0$, and hope to prove that $\mathrm{Tor}_1^A(M, N) = 0$ for all A-modules N, and hence that M is flat.

By induction on the number of generators of N, we can prove that $\mathrm{Tor}_1^A(M, N) = 0$ for all *finitely generated* modules N. (The base case is our assumption, and the inductive step is as follows: if N is generated by $a_1, \ldots, a_n$, then let N' be the submodule generated by $a_1, \ldots, a_{n-1}$, so $0 \to N' \to N \to A/I \to 0$ is exact, where I is some ideal. Then the long exact sequence for Tor gives us $0 = \mathrm{Tor}_1^A(M, N') \to \mathrm{Tor}_1^A(M, N) \to \mathrm{Tor}_1^A(M, A/I) = 0$.)

We conclude by noting that N is the union (i.e. direct limit) of its finitely generated submodules. As $\otimes$ commutes with direct limits, Tor_1 commutes with direct limits as well. (This requires some argument!)

Here is a sketch of an alternate conclusion. We wish to show that for any exact $0 \to N' \to N$, $0 \to M \otimes N' \to M \otimes N$ is also exact. Suppose $\sum m_i \otimes n_i' \mapsto 0$ in $M \otimes N$. Then that equality involves only finitely many elements of N. Work instead in the submodule generated by these elements of N. Within these submodules, we see that $\sum m_i \otimes n_i' = 0$. Thus this equality holds inside $M \otimes N'$ as well.

(I may try to write up a cleaner argument. Joe pointed out that the cleanest thing to do is to show that injectivity commutes with direct limits.) ☐

This has some cheap but important consequences.

Recall (or reprove) that flatness over a domain implies torsion-free.

3.9. *Corollary to Theorem 3.8. — Flatness over principal ideal domain is the* **same** *as torsion-free.*

This follows directly from the proposition.

3.10. *Important Exercise (flatness over the dual numbers).* This fact is important in deformation theory and elsewhere. Show that M is flat over $k[t]/t^2$ if and only if the natural map $M/tM \to tM$ is an isomorphism.

3.11. Flatness in exact sequences.

Suppose $0 \to M' \to M \to M'' \to 0$ is an exact sequence of A-modules.

3.12. *Proposition. — If M and M'' are both flat, then so is M'. If M' and M'' are both flat, then so is M.*

Proof. We use the characterization of flatness that N is flat if and only if $\mathrm{Tor}_i(N, N') = 0$ for all $i > 0$, N'. The result follows immediately from the long exact sequence for Tor. □

3.13. *Unimportant remark.* This begs the question: if M' and M are both flat, is M'' flat? (The argument above breaks down.) The answer is no: over $k[t]$, consider $0 \to tk[t] \to k[t] \to k[t]/t \to 0$ (geometrically: the closed subscheme exact sequence for a point on $\mathbb{A}^1$). The module on the right has torsion, and hence is not flat. The other two modules are free, hence flat.

3.14. *Easy exercise.* (We will use this shortly.) If $0 \to M_0 \to M_1 \to \cdots \to M_n \to 0$ is an exact sequence, and M_i is flat for $i > 0$, show that M_0 is flat too. (Hint: break the exact sequence into short exact sequences.)

We now come to the next result about flatness that will cause us to think hard.

3.15. *Important Theorem (for coherent modules over Noetherian local rings, flat equals free). — Suppose $(A, \mathfrak{m})$ is a local ring, and M is a coherent A-module (e.g. if A is Noetherian, then M is finitely generated). Then M is flat if and only if it is free.*

(It is true more generally, although we won't use those facts: apparently we can replace coherent with finitely presented, which only non-Noetherian people care about; or we can give up coherent completely if A is Artinian, although I haven't defined this notion. Reference: Mumford p. 296. I may try to clean the proof up to work in these cases.)

Proof. Clearly we are going to be using Nakayama's lemma. Now $M/\mathfrak{m}M$ is a finite-dimensional vector space over the field $A/\mathfrak{m}$. Choose a basis, and lift it to elements $m_1, \dots, m_n \in M$. Then consider $A^n \to M$ given by $e_i \mapsto m_i$. We'll show this is an isomorphism. This is surjective by Nakayama's lemma: the image is all of M modulo the maximal ideal, hence is everything. Let K be the kernel, which is finitely generated by coherence:

$$0 \to K \to A^n \to M \to 0.$$

Tensor this with $A/\mathfrak{m}$. As M is flat, the result is still exact (Proposition 3.6):

$$0 \to K/\mathfrak{m}K \to (A/\mathfrak{m})^n \to M/\mathfrak{m}M \to 0.$$

But $(A/\mathfrak{m})^n \to M/\mathfrak{m}M$ is an isomorphism, so $K/\mathfrak{m}K = 0$. As K is finitely generated, $K = 0$. □

Here is an immediate corollary (or really just a geometric interpretation).

3.16. *Corollary. — Suppose $\mathcal{F}$ is coherent over a locally Noetherian scheme* X. *Then $\mathcal{F}$ is flat over* X *if and only if it is locally free.*

(Reason: we have shown that local-freeness can be checked at the stalks.)

This is a useful fact. Here's a consequence that we prove earlier by other means: if $C \to C'$ is a surjective map of nonsingular irreducible projective curves, then $\pi_* \mathcal{O}_C$ is locally free.

In general, this gives us a useful criterion for flatness: Suppose $X \to Y$ finite, and Y integral. Then f is flat if and only if $\dim_{FF(Y)} f_*(\mathcal{O}_X)_y \otimes FF(Y)$ is constant. So the normalization of a node is not flat (I drew a picture here).

3.17. A useful special case: flatness over nonsingular curves. When are morphisms to nonsingular curves flat? Local rings of nonsingular curves are discrete valuation rings, which are principal ideal domains, so for them flat = torsion-free (Prop. 3.9). Thus, any map from a scheme to a nonsingular curve where all associated points go to a generic point is flat. (I drew several pictures of this.)

Here's a version we've seen before: a map from an irreducible curve to a nonsingular curve.

Here is another important consequence, which we can informally state as: we can take flat limits over one-parameter families. More precisely: suppose A is a discrete valuation ring, and let 0 be the closed point of $\operatorname{Spec} A$ and η the generic point. Suppose X is a scheme over A, and Y is a scheme over $X|_\eta$. Let Y' be the scheme-theoretic closure of Y in X. Then Y' is flat over A. Then $Y'|_0$ is often called the *flat limit* of Y.

(Suppose A is a discrete valuation ring, and let η be the generic point of $\operatorname{Spec} A$. Suppose X is proper over A, and Y is a closed subscheme of X_η. *Exercise:* Show that there is only one closed subscheme Y' of X, proper over A, such that $Y'|_\eta = Y$, and Y' is flat over A. Aside for experts: For those of you who know what the Hilbert scheme is, by taking the case of X as projective space, this shows that the Hilbert scheme is proper, using the valuative criterion for properness.)

3.18. *Exercise (an interesting explicit example of a flat limit).* (Here the base is $\mathbb{A}^1$, not a discrete valuation ring. You can either restrict to the discrete valuation ring that is the stalk near 0, or generalize the above discussion appropriately.) Let $X = \mathbb{A}^3 \times \mathbb{A}^1 \to Y = \mathbb{A}^1$ over a field k, where the coordinates on $\mathbb{A}^3$ are x, y, and z, and the coordinates on $\mathbb{A}^1$ are t. Define X away from $t = 0$ as the union of the two lines $y = z = 0$ (the x-axis) and $x = z - t = 0$ (the y-axis translated by t). Find the flat limit at $t = 0$. (Hint: it is *not* the union of the two axes, although it includes it. The flat limit is non-reduced.)

3.19. Stray but important remark: flat morphisms are (usually) open. I'm discussing this here because I have no idea otherwise where to put it.

3.20. *Exercise.* Prove that flat and locally finite type morphisms of locally Noetherian schemes are open. (Hint: reduce to the affine case. Use Chevalley's theorem to show that the image is constructible. Reduce to a target that is the spectrum of a local ring. Show that the generic point is hit.)

3.21. I ended by stating an important consequence of flatness: flat plus projective implies constant Euler characteristic. I'll state this properly in next Tuesday's notes, where I will also give consequences and a proof.

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FOUNDATIONS OF ALGEBRAIC GEOMETRY CLASSES 43 AND 44

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CONTENTS

This week: constancy of Euler characteristic in flat families. The semicontinuity theorem and consequences. Glimpses of the relative Picard scheme.

1. FLAT IMPLIES CONSTANT EULER CHARACTERISTIC

We come to an important consequence of flatness. We'll see that this result implies many answers and examples to questions that we would have asked before we even knew about flatness.

1.1. *Important Theorem. — Suppose* $f : X \to Y$ *is a projective morphism, and* $\mathcal{F}$ *is a coherent sheaf on* X*, flat over* Y*. Suppose* Y *is locally Noetherian. Then* $\sum (-1)^i h^i(X_y \mathcal{F}|_y)$ *is a locally constant function of* $y \in Y$*. In other words, the Euler characteristic of* $\mathcal{F}$ *is constant in the fibers.*

This is first sign that cohomology behaves well in families. (We'll soon see a second: the Semicontinuity Theorem 4.4.) Before getting to the proof, I'll show you some of its many consequences. (A second proof will be given after the semicontinuity discussion.)

The theorem also gives a necessary condition for flatness. It also sufficient if target is integral and locally Noetherian, although we won't use this. (Reference: You can translate Hartshorne Theorem III.9.9 into this.) I seem to recall that both the necessary and sufficient conditions are due to Serre, but I'm not sure of a reference. It is possible that integrality is not necessary, and that reducedness suffices, but I haven't checked.

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1.2. *Corollary. — Assume the same hypotheses and notation as in Theorem 1.1. Then the Hilbert polynomial of $\mathcal{F}$ is locally constant as a function of $y \in Y$.*

Thus for example a flat family of varieties in projective space will all have the same degree and genus (and the same dimension!). Another consequence of the corollary is something remarkably useful.

1.3. *Corollary. — An invertible sheaf on a flat projective family of connected nonsingular curves has locally constant degree on the fibers.*

Proof. An invertible sheaf $\mathcal{L}$ on a flat family of curves is always flat (as locally it is isomorphic to the structure sheaf). Hence $\chi(\mathcal{L}_y)$ is constant. From the Riemann-Roch formula $\chi(\mathcal{L}_y) = \deg(\mathcal{L}_y) - g(X_y) + 1$, using the local constancy of $\chi(\mathcal{L}_y)$, the result follows. □

Riemann-Roch holds in more general circumstances, and hence the corollary does too. Technically, in the example I'm about to give, we need Riemann-Roch for the union of two $\mathbb{P}^1$'s, which I haven't shown. This can be shown in three ways. (i) I'll prove that Riemann-Roch holds for projective generically reduced curves later. (ii) You can prove it by hand, as an exercise. (iii) You can consider this curve C inside $\mathbb{P}^1 \times \mathbb{P}^1$ as the union of a "vertical fiber" and "horizontal fiber". Any invertible sheaf on C is the restriction of some $\mathcal{O}(a,b)$ on $\mathbb{P}^1 \times \mathbb{P}^1$. Use additivity of Euler characteristics on $0 \to \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(a-1, b-1) \to \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(a,b) \to \mathcal{O}_C(a,b) \to 0$, and note that we have earlier computed the $\chi(\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(c,d))$.

This result has a lot of interesting consequences.

1.4. *Example of a proper non-projective surface.* We can use it to show that a certain proper surface is not projective. Here is how.

Fix any field with more than two elements. We begin with a flat projective family of curves whose $X \to \mathbb{P}^1$, such that the fiber X_0 over 0 is isomorphic to $\mathbb{P}^1$, and the fiber X_∞ over ∞ is isomorphic to two $\mathbb{P}^1$'s meeting at a point, $X_\infty = Y_\infty \cup Z_\infty$. For example, consider the family of conics in $\mathbb{P}^2$ (with projective coordinates x, y, z) parameterized by $\mathbb{P}^1$ (with projective coordinates λ and μ given by

$$\lambda xy + \mu z(x + y + z) = 0.$$

This family unfortunately is singular for $[\lambda; \mu] = [0; 1]$ (as well as $[1; 0]$ and one other point), so change coordinates on $\mathbb{P}^1$ so that we obtain a family of the desired form.

We now take a break from this example to discuss an occasionally useful construction.

1.5. *Gluing two schemes together along isomorphic closed subschemes.* Suppose X' and X'' are two schemes, with closed subschemes $W' \hookrightarrow X'$ and $W'' \hookrightarrow X''$, and an isomorphism $W' \to W''$. Then we can glue together X' and X'' along $W' \cong W''$. We define this more formally as the *coproduct*:

$$\begin{array}{ccc} W' \cong W'' & \longrightarrow & X' \\ \downarrow & & \downarrow \\ X'' & \longrightarrow & ?. \end{array}$$

Exercise. Prove that this coproduct exists. Possible hint: work by analogy with our product construction. If the coproduct exists, it is unique up to unique isomorphism. Start with judiciously chosen affine open subsets, and glue.

Warning: You might hope that if you have a scheme X with two disjoint closed subschemes W' and W'', and an isomorphism $W' \to W''$, then you should be able to glue X to itself along $W' \to W''$. This is not always possible! I'll give an example shortly. You can still make sense of the quotient as an *algebraic space*, which I will not define here. If you want to know what it is, ask Jarod, or come to one of the three lectures he'll give later this quarter.

1.6. *Back to the non-projective surface.* Now take two copies of the X we defined above; call them X' and X''. Glue X' to X'' by identifying X'_0 with Y''_∞ (in any way you want) and Y'_∞ with X''_0. (Somewhat more explicitly: we are choosing an isomorphism $X'_0 \cup Y'_\infty$ with $X''_0 \cup Y''_\infty$ that "interchanges the components".) I claim that the resulting surface X is proper and not projective over the base field k. The first is an exercise.

Exercise. Show that X is proper over k. (Hint: show that the union of two proper schemes is also proper.)

Suppose now that X is projective, and is embedded in projective space by an invertible sheaf (line bundle) $\mathcal{L}$. Then the degree of $\mathcal{L}$ on each curve of X is non-negative. For any curve $C \subset X$, let $\deg C$ be the degree of $\mathcal{L}$ on C (or equivalently, the degree of C under this projective embedding). Pull $\mathcal{L}$ back to X'. Then this is a line bundle on a flat projective family, so the degree is constant in fibers. Thus

$$\deg X'_0 = \deg(Y'_\infty \cup Z'_\infty) = \deg Y'_\infty + \deg Z'_\infty > \deg Y'_\infty.$$

(Technically, we have not shown that the middle equality holds, so you should think about why that is clear.) Similarly $\deg X''_0 > \deg Y''_\infty$. But after gluing, $X'_0 = Y''_\infty$ and $X''_0 = Y'_\infty$, so we have a contradiction.

1.7. *Remark.* This is a stripped down version of Hironaka's example in dimension 3. Hironaka's example has the advantage of being nonsingular. I'll present that example (and show how this one comes from Hironaka) when we discuss blow-ups. (I think it is a fact that nonsingular proper surfaces over a field are always projective.)

1.8. *Unimportant remark.* You can do more fun things with this example. For example, we know that projective surfaces can be covered by three affine open sets. This can be used to give an example of (for any N) a proper surface that requires at least N affine open subsets to cover it (see my paper with Mike Roth on my preprints page, Example 4.9).

1.9. *Problematic nature of the notion of "projective morphism".* This example shows that the notion of being projective isn't a great notion. There are four possible definitions that might go with this notion. (1) We are following Grothendieck's definition. This notion is not local on the base. For example, by following the gluing above for the morphisms $X' \to \mathbb{P}^1$ and $X'' \to \mathbb{P}^1$, we obtain a morphism $\pi : X \to \mathbb{P}^1 \cup \mathbb{P}^1$, where the union on the right is obtained by gluing the 0 of the first $\mathbb{P}^1$ to the ∞ of the second, and vice versa. Then away from each node of the target, π is projective. (You could even give some explicit equations if you wanted.) However, we know that π is not projective, as $\rho : \mathbb{P}^1 \cup \mathbb{P}^1 \to \operatorname{Spec} k$ is projective, but we have already shown that $\rho \circ \pi : X \to \operatorname{Spec} k$ is not projective.

(2) Hartshorne's definition is designed for finite type k-schemes, and is definitely the wrong one for schemes in general.

(3) You could make our notion "local on the base" by also requiring more information: e.g. the notion of a projective morphism could be a morphism of schemes $X \to Y$ along with an invertible sheaf $\mathcal{L}$ on X that serves as an $\mathcal{O}(1)$. This is a little unpleasant; when someone says "consider a projective surface", they usually wouldn't want to have any particular projective embedding preferred.

(4) Another possible notion is that of *locally projective*: $\pi : X \to Y$ is locally projective if there is an open cover of Y by U_i such that over each U_i, π is projective (in our original sense (1)). The disadvantage is that this isn't closed under composition, as is shown by our example $X \to \mathbb{P}^1 \cup \mathbb{P}^1 \to \operatorname{Spec} k$.

1.10. *Example: You can't always glue a scheme to itself along isomorphic disjoint subschemes.* In class, we had an impromptu discussion of this, so it is a little rough. I'll use a variation of the above example. We'll see that you can't glue X to itself along an isomorphism $X_0 \cong Y_\infty$. (To make this a precise statement: there is no morphism $\pi : X \to W$ such that there is a curve $C \hookrightarrow W$ such that $\pi^{-1}(W - U) = X - X_0 - Y_\infty$, and π maps both X_0 and Y_∞ isomorphically to W.) A picture here is essential!

If there were such a scheme W, consider the point $\pi(Y_\infty \cap Z_\infty) \in W$. It has an affine neighborhood U; let K be its complement. Consider $\pi^{-1}(K)$. This is a closed subset of X, missing $Y_\infty \cap Z_\infty$. Note that it meets Y_∞ (as the affine open U can contain no $\mathbb{P}^1$'s) and Z_∞. Discard all components of $\pi^{-1}(K)$ that are dimension 0, and that contain components of fibers; call what's left K'. *Caution: I need to make sure that I don't end up discarding the points on Y_∞ and Z_∞. I could show that $\pi^{-1}(K)$ has pure codimension 1, but I'd like to avoid doing that. For now, assume that is the case; I may patch this later.* Then K' is an effective Cartier divisor, inducing an invertible sheaf on the surface X, which in turn is a flat projective family over $\mathbb{P}^1$. Thus the degree of K' is constant on fibers. Then we get the same sort of contradiction:

$$\deg_{K'} Y_\infty = \deg_{K'} X_0 = \deg_{K'} Y_\infty + \deg_{K'} Z_\infty > \deg_{K'} Y_\infty.$$

This led to a more wide-ranging discussion. A surprisingly easy theorem (which you can find in Mumford's Abelian Varieties for example) states that if X is a projective k-scheme with an action by a finite group G, then the quotient X/G exists, and is also a projective scheme. (One first has to define what one means by X/G!) If you are a little careful in choosing the isomorphisms used to build our nonprojective surface (picking

$X_0' \to Y_\infty''$ and $X_0'' \to Y_\infty'$ to be the "same" isomorphisms), then there is a $\mathbb{Z}/2$-action on X ("swapping the $\mathbb{P}^1$'s"), we have shown that the quotient W does *not* exist as a scheme, hence giving another proof (modulo things we haven't shown) that X is not projective.

2. Proof of Important Theorem on constancy of Euler characteristic in flat families

Now you've seen a number of interesting results that seem to have nothing to do with flatness. I find this a good motivation for this motivation: using the concept, we can prove things that were interested in beforehand. It is time to finally prove Theorem 1.1.

Proof. The question is local on the base, so we may reduce to case Y is affine, say $Y = \operatorname{Spec} B$, so $X \hookrightarrow \mathbb{P}^n_B$ for some n. We may reduce to the case $X = \mathbb{P}^n_B$ (as we can consider $\mathcal{F}$ as a sheaf on $\mathbb{P}^n_B$). We may reduce to showing that Hilbert polynomial $\mathcal{F}(m)$ is locally constant for all $m \gg 0$ (as by Serre vanishing for $m \gg 0$, the Hilbert polynomial agrees with the Euler characteristic). Now consider the Cech complex $\mathcal{C}^*$ for $\mathcal{F}$. Note that all the terms in the Cech complex are flat. Twist by $\mathcal{O}(m)$ for $m \gg 0$, so that all the higher push-forwards vanish. Hence $\Gamma(\mathcal{C}^*(m))$ is exact except at the first term, where the cohomology is $\Gamma(\pi_*\mathcal{F}(m))$. We tack on this module to the front of the complex, so it is once again exact. Thus (by an earlier exercise), as we have an exact sequence in which all but the first terms are known to be flat, the first term is flat as well. As it is finitely generated, it is also free by an earlier fact (flat and finitely generated over a Noetherian local ring equals free), and thus has constant rank.

We're interested in the cohomology of the fibers. To obtain that, we tensor the Cech resolution with $k(y)$ (as y runs over Y) and take cohomology. Now the extended Cech resolution (with $\Gamma(\pi_*\mathcal{F}(m))$ tacked on the front) is an exact sequence of flat modules, and hence remains exact upon tensoring with $k(y)$ (or indeed anything else). (Useful translation: cohomology commutes with base change.) Thus $\Gamma(\pi_*\mathcal{F}(m)) \otimes k(y) \cong \Gamma(\pi_*\mathcal{F}(m)|_y)$. Thus the dimension of the Hilbert function is the rank of the locally free sheaf at that point, which is locally constant. □

3. Start of Thursday's class: Review

At this point, you've already seen a large number of facts about flatness. Don't be overwhelmed by them; keep in mind that you care about this concept because we have answered questions we cared about even before knowing about flatness. Here are three examples. (i) If you have a short exact sequence where the last is locally free, then you can tensor with anything and the exact sequence will remain exact. (ii) We described a morphism that is proper but not projective. (iii) We showed that you can't always glue a scheme to itself.

Here is a summary of what we know, highlighting the hard things.

- definition; basic properties (pullback and localization). flat base change commutes with higher pushforwards
- Tor: definition and symmetry. (Hence tensor exact sequences of flats with anything and keep exactness.)
- ideal-theoretic criterion: $\mathrm{Tor}_1(M, A/I) = 0$ for all I. (flatness over PID = torsion-free; over dual numbers) (important special case: DVR)
- for coherent modules over Noetherian local rings, flat=locally free
- flatness is open in good circumstances (flat + lft of lN is open; we should need only weaker hypotheses)
- euler characteristics behave well in projective flat families. In particular, the degree of an invertible sheaf on a flat projective family of curves is locally constant.

4. Cohomology and base change theorems

Here is the type of question we are considering. We'd like to see how higher pushforwards behave with respect to base change. For example, we've seen that higher pushforward commutes with *flat* base change. A special case of base change is the inclusion of a point, so this question specializes to the question: can you tell the cohomology of the fiber from the higher pushforward? The next group of theorems I'll discussed deal with this issue. I'll prove things for projective morphisms. The statements are true for proper morphisms of Noetherian schemes too; the one fact you'll see that I need is the following: that the higher direct image sheaves of coherent sheaves under proper morphisms are also coherent. (I'm largely following Mumford's *Abelian Varieties*. The geometrically interesting theorems all flow from the following neat but unmotivated result.

4.1. *Key theorem. — Suppose $\pi : X \to \operatorname{Spec} B$ is a projective morphism of Noetherian [needed?] schemes, and $\mathcal{F}$ is a coherent sheaf on X, flat over $\operatorname{Spec} B$. Then there is a* finite *complex*

$$0 \to K^0 \to K^1 \to \cdots \to K^n \to 0$$

of finitely generated projective B-modules and an isomorphism of functors

$$H^p(X \times_B A, \mathcal{F} \otimes_B A) \cong H^p(K^* \otimes_B A) \tag{1}$$

for all $p \geq 0$ in the category of B-algebras A.

In fact, K^i will be free for $i > 0$. For $i = 0$, it is projective hence flat hence locally free (by an earlier theorem) on Y.

Translation/idea: Given $\pi : X \to \operatorname{Spec} B$, we will have a complex of vector bundles on the target that computes cohomology (higher-pushforwards), "universally" (even after any base change). The idea is as follows: take the Cech complex, produce a "quasiisomorphic" complex (a complex with the same cohomology) of free modules. For those taking derived category class: we have an isomorphic object in the derived category which is easier to deal with as a complex. We'll first construct the complex so that (1) holds for $B = A$, and then show the result for general A later. Let's put this into practice.

4.2. *Lemma. — Let C^* be a complex of B-modules such that $H^i(C^*)$ are finitely generated B-modules, and that $C^p \neq 0$ only if $0 \leq p \leq n$. Then there exists a complex K^* of finitely generated B-modules such that $K^p \neq 0$ only if $0 \leq p \leq n$ and K^p is free for $p \geq 1$, and a homomorphism of complexes $\phi : K^* \to C^*$ such that ϕ induces isomorphisms $H^i(K^*) \to H^i(C^*)$ for all i.*

Note that K^i is B-flat for $i > 0$. Moreover, if C^p are B-flat, then K^0 is B-flat too.

For all of our purposes except for a side remark, I'd prefer a cleaner statement, where C^* is a complex of B-modules, with $C^p \neq 0$ only if $p \leq n$ (in other words, there could be infinitely many non-zero C^p's). The proof is then about half as long

Proof. Step 1. We'll build this complex inductively, and worry about K^0 when we get there.

$$\begin{array}{ccccccccc} & & K^m & \xrightarrow{\delta^m} & K^{m+1} & \xrightarrow{\delta^{m+1}} & K^{m+2} & \longrightarrow & \cdots \\ & & \downarrow{\scriptstyle \phi_m} & & \downarrow{\scriptstyle \phi_{m+1}} & & \downarrow{\scriptstyle \phi_{m+2}} & & \\ \cdots \longrightarrow & C^{m-1} \longrightarrow & C^m & \xrightarrow[\delta^m]{} & C^{m+1} & \xrightarrow[\delta^{m+1}]{} & C^{m+2} & \longrightarrow & \cdots . \end{array}$$

We assume we've defined (K^p, ϕ_p, δ^p) for $p \geq m+1$ such that these squares commute, and the top row is a complex, and ϕ^p defines an isomorphism of cohomology $H^q(K^*) \to H^q(C^*)$ for $q \geq m+2$ and a surjection $\ker \delta^{m+1} \to H^{m+1}(C^*)$, and the K^p are finitely generated B-modules.

We'll adjust the complex to make ϕ_{m+1} an isomorphism of cohomology, and then again to make ϕ_m a surjection on cohomology. Let $B^{m+1} = \ker(\delta^{m+1} : H^{m+1}(K^*) \to H^{m+1}(C^*))$. Then we choose generators, and make these K_1^m. We have a new complex. We get the 0-maps on cohomology at level m. We then add more in to surject on cohomology on level m.

Now what happens when we get to $m = 0$? We have maps of complexes, where everything in the top row is free, and we have an isomorphism of cohomology everywhere except for K^0, where we have a surjection of cohomology. Replace K^0 by $K^0/\ker \delta^0 \cap \ker \phi_0$. Then this gives an isomorphism of cohomology.

Step 2. We need to check that K^0 is B-flat. Note that everything else in this quasiisomorphism is B-flat. Here is a clever trick: construct the *mapping cylinder* (call it M^*):

$$0 \to K^0 \to C^0 \oplus K^1 \to C^1 \oplus K^2 \to \cdots \to C^{n-1} \oplus K^n \to C^n \to 0.$$

Then we have a short exact sequence of complexes

$$0 \to C^* \to M^* \to K^*[1] \to 0$$

(where $K^*[1]$ is just the same complex as K^*, except slid over by one) yielding isomorphisms of cohomology $H^*(K^*) \to H^*(C^*)$, from which $H^*(M^*) = 0$. (This was an earlier exercise: given a map of complexes induces an isomorphism on cohomology, the mapping cylinder is exact.) Now look back at the mapping cylinder M^*, which we now realize is an exact sequence. All terms in it are flat except possibly K^0. Hence K^0 is flat too (also by an earlier exercise)! □

4.3. *Lemma. — Suppose* $K^* \to C^*$ *is a morphism of finite complexes of* **flat** B*-modules inducing isomorphisms of cohomology (a "quasiisomorphism"). Then for every* B*-algebra* A*, the maps* $H^p(C^* \otimes_B A) \to H^p(K^* \otimes_B A)$ *are isomorphisms.*

Proof. Consider the mapping cylinder M^*, which we know is exact. Then $M^* \otimes_B A$ is still exact! (The reason was our earlier exercise that any exact sequence of flat modules tensored with anything remains flat.) But $M^* \otimes_B A$ is the mapping cylinder of $K^* \otimes_B A \to C^* \otimes_B A$, so this is a quasiisomorphism too. □

Now let's prove the theorem!

Proof of theorem 4.1. Choose a finite covering (e.g. the standard covering). Take the Cech complex C^* for $\mathcal{F}$. Apply the first lemma to get the nicer version K^* of the same complex C^*. Apply the second lemma to see that if you tensor with B and take cohomology, you get the same answer whether you use K^* or C^*. □

We are now ready to put this into use. We will use it to discuss a trio of facts: the Semicontinuity Theorem, Grauert's Theorem, and the Cohomology and Base Change Theorem. (We'll prove the first two.) The theorem of constancy of euler characteristic in flat families also fits in this family.

These theorems involve the following situation. Suppose $\mathcal{F}$ is a coherent sheaf on X, $\pi : X \to Y$ projective, Y (hence X) Noetherian, and $\mathcal{F}$ flat over Y.

Here are two related questions. Is $R^p\pi_*\mathcal{F}$ locally free? Is $\phi^p : R^p\pi_*\mathcal{F}\otimes k(y) \to H^p(X_y, \mathcal{F}_y)$ an isomorphism?

We have shown Key theorem 4.1, that if Y is affine, say $Y = \operatorname{Spec} B$, then we can compute the pushforwards of $\mathcal{F}$ by a complex of locally free modules

$$0 \to M^0 \to M^1 \to \cdots \to M^n \to 0$$

where in fact M^p is free for $p > 1$. Moreover, this computes pushforwards "universally": after a base change, this remains true.

Now the dimension of the left is uppersemicontinuous by uppersemicontinuity of fiber dimension of coherent sheaves. The semicontinuity theorem states that the dimension of the right is also uppersemicontinuous. More formally:

4.4. *Semicontinuity theorem. — Suppose* $X \to Y$ *is a projective morphisms of Noetherian schemes, and* $\mathcal{F}$ *is a coherent sheaf on* X *flat over* Y*. Then for each* $p \geq 0$*, the function* $Y \to \mathbb{Z}$ *given by* $y \mapsto \dim_{k(y)} H^p(X_y, \mathcal{F}_y)$ *is upper semicontinuous on* Y*.*

So "cohomology groups jump in projective flat families". Again, we can replace projective by proper once we've shown finite-dimensionality of higher pushforwards (which we haven't). For pedants: can the Noetherian hypotheses be excised?

Here is an example of jumping in action. Let C be a positive genus nonsingular projective irreducible curve, and consider the projection $\pi : E \times E \to E$. Let $\mathcal{L}$ be the invertible sheaf (line bundle) corresponding to the divisor that is the diagonal, minus the section $p_0 \in E$. then $\mathcal{L}_{p_0}$ is trivial, but $\mathcal{L}_p$ is non-trivial for any $p \neq p_0$ (as we've shown earlier in the "fun with curves" section). Thus $h^0(E, \mathcal{L}_p)$ is 0 in general, but jumps to 1 for $p = p_0$.

Remark. Deligne showed that in the smooth case, at least over $\mathbb{C}$, there is no jumping of cohomology of the structure sheaf.

Proof. The result is local on Y, so we may assume Y is affine. Let K^* be a complex as in the key theorem 4.1. By localizing further, we can assume K^* is locally free. *So we are computing cohomology on any fiber using a complex of vector bundles.*

Then for $y \in Y$

$$\begin{aligned}\dim_{k(y)} H^p(X_y, \mathcal{F}_y) &= \dim_{k(y)} \ker(d^p \otimes_A k(y)) - \dim_{k(y)} \operatorname{im}(d^{p-1} \otimes_A k(y)) \\ &= \dim_{k(y)}(K^p \otimes k(y)) - \dim_{k(y)} \operatorname{im}(d^p \otimes_A k(y)) - \dim_{k(y)} \operatorname{im}(d^{p-1} \otimes_A k(y))\end{aligned}$$

(Side point: by taking alternating sums of these terms, we get a second proof of Theorem 1.1 that $\chi(X_y, \mathcal{F}_y) = \sum (-1)^i h^i(X_y, \mathcal{F}_y)$ is a constant function of y. I mention this because if extended the fact that higher cohomology of coherents is coherent under proper pushforwards, we'd also have Theorem 1.1 in this case.)

Now $\dim_{k(y)} \operatorname{im}(d^p \otimes_A k(y))$ is a lower semicontinuous function on Y. Reason: the locus where the dimension is less than some number q is obtained by setting all $q \times q$ minors of the matrix $K^p \to K^{p+1}$ to 0. So we're done! □

5. Line bundles are trivial in a Zariski-closed locus, and glimpses of the relative Picard scheme

(This was discussed on Thursday May 4, but fits in well here.)

5.1. *Proposition. — Suppose $\mathcal{L}$ is an invertible sheaf on an integral projective scheme* X *such that both $\mathcal{L}$ and $\mathcal{L}^\vee$ have non-zero sections. Then $\mathcal{L}$ is the trivial sheaf.*

As usual, "projective" may be replaced by "proper". The only fact we need (which we haven't proved) is that the only global functions on proper schemes are constants. (We haven't proved that. It follows easily from the valuative criterion of properness — but we haven't proved that either!)

Proof. Suppose s and t are the non-zero sections of $\mathcal{L}$ and $\mathcal{L}^\vee$. Then they are both non-zero at the generic point (or more precisely, in the stalk at the generic point). (Otherwise, they would be the zero-section — this is where we are using the integrality of X.) Under the map $\mathcal{L} \otimes \mathcal{L}^\vee \to \mathcal{O}$, $s \otimes t$ maps to st, which is also non-zero. But the only global functions (global sections of $\mathcal{O}_X$) are the constants, so st is a non-zero constant. But then s

is nowhere 0 (or else st would be somewhere zero), so $\mathcal{L}$ has a nowhere vanishing section, and hence is trivial (isomorphic to $\mathcal{O}_X$). □

Now suppose $X \to Y$ is a flat projective morphism with integral fibers. (It is a "flat family of geometrically integral schemes".) Suppose that $\mathcal{L}$ is an invertible sheaf. Then the locus of $y \in Y$ where $\mathcal{L}_y$ is trivial on X_y is a closed set. Reason: the locus where $h^0(X_y, \mathcal{L}_y) \geq 1$ is closed by the Semicontinuity Theorem 4.4, and the same holes for the locus where $h^0(X_y, \mathcal{L}_y^\vee) \geq 1$.

(Similarly, if $\mathcal{L}'$ and $\mathcal{L}''$ are two invertible sheaves on the family X, the locus of points y where $\mathcal{L}'_y \cong \mathcal{L}''$ is a closed subset: just apply the previous paragraph to $\mathcal{L} := \mathcal{L}' \otimes (\mathcal{L}'')^\vee$.)

In fact, we can jazz this up: for any $\mathcal{L}$, there is in a natural sense a closed *subscheme* where $\mathcal{L}$ is trivial. More precisely, we have the following theorem.

5.2. *Seesaw Theorem. — Suppose $\pi : X \to Y$ is a projective flat morphism to a Noetherian scheme, all of whose fibers are geometrically integral schemes, and $\mathcal{L}$ is an invertible sheaf on X. Then there is a unique closed subscheme $Y' \hookrightarrow Y$ such that for any fiber diagram*

$$\begin{array}{ccc} X \times_Y Z & \xrightarrow{g} & X \\ \downarrow{\scriptstyle\rho} & & \downarrow{\scriptstyle\pi} \\ Z & \xrightarrow{f} & Y \end{array}$$

such that $g^*\mathcal{L} \cong \rho^*\mathcal{M}$ for some invertible sheaf $\mathcal{M}$ on Z, then f factors (uniquely) through $Y' \to Y$.

I want to make three comments before possibly proving this.

- I have no idea why it is called the seesaw theorem.

- As a special case, there is a "largest closed subscheme" on which the invertible sheaf is the pullback of a trivial invertible sheaf.

- Also, this is precisely the statement that the functor is representable $Y' \to Y$, and that this morphism is a closed immersion.

I'm not going to use this, so I won't prove it. But a slightly stripped down version of this appears in Mumford (p. 89), and you should be able to edit his proof so that it works in this generality.

There is a lesson I want to take away from this: this gives evidence for existence of a very important moduli space: the Picard scheme. The Picard scheme $\operatorname{Pic} X/Y \to Y$ is a scheme over Y which represents the following functor: Given any $T \to Y$, we have the set of invertible sheaves on $X \times_Y T$, modulo those invertible sheaves pulled back from T. In other words, there is a natural bijection between diagrams of the form

$$\begin{array}{ccc} \mathcal{L} & & \\ \downarrow & & \\ X \times_T Y & \longrightarrow & X \\ \downarrow & & \downarrow \\ T & \longrightarrow & Y \end{array}$$

and diagrams of the form

$$\begin{array}{ccc} & & \mathrm{Pic}_{X/Y} \\ & \nearrow & \downarrow \\ T & \longrightarrow & Y. \end{array}$$

It is a hard theorem (due to Grothendieck) that (at least if Y is reasonable, e.g. locally Noetherian — I haven't consulted the appropriate references) $\mathrm{Pic}\, X/Y \to Y$ exists, i.e. that this functor is representable. In fact $\mathrm{Pic}\, X/Y$ is of finite type.

We've seen special cases before when talking about curves: if C is a geometrically integral curve over a field k, of genus g, $\mathrm{Pic}\, C = \mathrm{Pic}\, C/k$ is a dimension g projective nonsingular variety.

Given its existence, it is easy to check that $\mathrm{Pic}_{X/Y}$ is a group scheme over Y, using our functorial definition of group schemes.

5.3. *Exercise.* Do this!

The group scheme has a zero-section $0 : Y \to \mathrm{Pic}_{X/Y}$. This turns out to be a closed immersion. The closed subscheme produced by the Seesaw theorem is precisely the pullback of the 0-section. I suspect that you can use the Seesaw theorem to show that the zero-section *is* a closed immersion.

5.4. *Exercise.* Show that the Picard scheme for $X \to Y$ (with our hypotheses: the morphism is flat and projective, and the fibers are geometrically integral) is separated over Y by showing that it satisfies the valuative criterion of separatedness.

Coming up soon: Grauert's Theorem and Cohomology and base change!

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FOUNDATIONS OF ALGEBRAIC GEOMETRY CLASSES 45 AND 46

RAVI VAKIL

CONTENTS

This week: Grauert's theorem and the Cohomology and base change theorem, and applications. The Rigidity Lemma. Proof of Grauert's theorem. Dimensions behave well for flat morphisms. Associated points go to associated points.

1. COHOMOLOGY AND BASE CHANGE THEOREMS

We're in the midst of discussing a family of theorems involving the following situation. Suppose $\mathcal{F}$ is a coherent sheaf on X, $\pi : X \to Y$ projective, Y (hence X) Noetherian, and $\mathcal{F}$ flat over Y.

Here are two related questions. Is $R^p\pi_*\mathcal{F}$ locally free? Is $\phi^p : R^p\pi_*\mathcal{F} \otimes k(y) \to H^p(X_y, \mathcal{F}_y)$ an isomorphism?

We have shown a key intermediate result, that if Y is affine, say $Y = \operatorname{Spec} B$, then we can compute the pushforwards of $\mathcal{F}$ by a complex of locally free modules

$$0 \to M^0 \to M^1 \to \cdots \to M^n \to 0$$

where in fact M^p is free for $p > 1$. Moreover, this computes pushforwards "universally": after a base change, this remains true.

We have already shown the constancy of Euler characteristic, and the semicontinuity theorem. I'm now going to discuss two big theorems, Grauert's theorem and the Cohomology and base change theorem, that are in some sense the scariest in Hartshorne, coming at the end of Chapter III (along with the semicontinuity theorem). I hope you

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agree that semicontinuity isn't that scary (given the key fact). I'd like to discuss applications of these two theorems to show you why you care; then given time I'll give proofs. I've found the statements worth remembering, even though they are a little confusing.

Note that if $R^p\pi_*\mathcal{F}$ is locally free and ϕ^p is an isomorphism, then the right side is locally constant. The following is a partial converse.

1.1. *Grauert's Theorem. — If Y is reduced, then h^p locally constant implies $R^p\pi_*\mathcal{F}$ is locally free and ϕ^p is an isomorphism.*

1.2. *Cohomology and base change theorem. — Assume ϕ^p is surjective. Then the following hold.*

(a) *ϕ^p is an isomorphism, and the same is true nearby. [Note: The hypothesis is trivially satisfied in the common case $H^p = 0$. If $H^p = 0$ at a point, then it is true nearby by semicontinuity.]*
(b) *ϕ^{p-1} is surjective (=isomorphic) if and only if $R^p\pi_*\mathcal{F}$ is locally free. [This in turn implies that h^p is locally constant.]*

Notice that (a) is about just what happens over the reduced scheme, but (b) has a neat twist: you can check things over the reduced scheme, and it has implications over the scheme as a whole!

Here are a couple of consequences.

1.3. *Exercise.* Suppose $H^p(X_y, \mathcal{F}_y) = 0$ for all $y \in Y$. Show that ϕ^{p-1} is an isomorphism for all $y \in Y$. (Hint: cohomology and base change (b).)

1.4. *Exercise.* Suppose $R^p\pi_*\mathcal{F} = 0$ for $p \geq p_0$. Show that $H^p(X_y, \mathcal{F}_y) = 0$ for all $y \in Y$, $k \geq k_0$. (Same hint. You can also do this directly from the key theorem above.)

2. When the pushforward of the functions on X are the functions on Y

Many fun applications happen when a certain hypothesis holds, which I'll now describe.

We say that π satisfies (*) if it is projective, and the natural morphism $\mathcal{O}_Y \to \pi_*\mathcal{O}_X$ is an isomorphism. Here are two statements that will give you a feel for this notion. First:

2.1. *Important Exercise.* Suppose π is a projective flat family, each of whose fibers are (nonempty) integral schemes, or more generally whose fibers satisfy $h^0(X_y) = 1$. Then (*) holds. (Hint: consider

$$\mathcal{O}_Y \otimes k(y) \longrightarrow (\pi_*\mathcal{O}_X) \otimes k(y) \xrightarrow{\phi^0} H^0(X_y, \mathcal{O}_{X_y}) \cong k(y)\ .$$

The composition is surjective, hence ϕ^0 is surjective, hence it is an isomorphism (by the Cohomology and base change theorem 1.2 (a)). Then thanks to the Cohomology and base change theorem 1.2 (b), $\pi_* \mathcal{O}_X$ is locally free, thus of rank 1. If I have a map of invertible sheaves $\mathcal{O}_Y \to \pi_* \mathcal{O}_X$ that is an isomorphism on closed points, it is an isomorphism (everywhere) by Nakayama.)

Note in the previous exercise: we are obtaining things not just about closed points!

Second: we will later prove a surprisingly hard result, that given any projective (proper) morphism of Noetherian schemes satisfying (*) (without any flatness hypotheses!), the fibers are all connected ("Zariski's connectedness lemma").

2.2. *Exercise (the Hodge bundle; important in Gromov-Witten theory).* Suppose $\pi : X \to Y$ is a projective flat family, all of whose geometric fibers are connected reduced curves of arithmetic genus g. Show that $R^1\pi_* \mathcal{O}_X$ is a locally free sheaf of rank g. This is called the *Hodge bundle*. [Hint: use cohomology and base change (b) twice, once with $p = 2$, and once with $p = 1$.]

Here is the question we'll address in this section. Given an invertible sheaf $\mathcal{L}$ on X, we wonder when it is the pullback of an invertible sheaf $\mathcal{M}$ on Y. Certainly it is necessary for it to be trivial on the fibers. We'll see that (*) holds, then this basically suffices. Here is the idea: given $\mathcal{L}$, how can we recover $\mathcal{M}$? Thanks to the next exercise, it must be $\pi_* \mathcal{L}$.

2.3. *Exercise.* Suppose $\pi : X \to Y$ satisfies (*). Show that if $\mathcal{M}$ is any invertible sheaf on Y, then the natural morphism $\mathcal{M} \to \pi_* \pi^* \mathcal{M}$ is an isomorphism. In particular, we can recover $\mathcal{M}$ from $\pi^* \mathcal{M}$ by pushing forward. (Hint: projection formula.)

2.4. *Proposition. — Suppose $\pi : X \to Y$ is a morphism of locally Noetherian integral schemes with geometrically integral fibers (hence by Exercise 2.1 satisfying (*)). Suppose also that Y is reduced, and $\mathcal{L}$ is an invertible sheaf on X that is trivial on the fibers of π (i.e. $\mathcal{L}_y$ is a trivial invertible sheaf on X_y). Then $\pi_* \mathcal{L}$ is an invertible sheaf on Y (call it $\mathcal{M}$), and $\mathcal{L} = \pi^* \mathcal{M}$.*

Proof. To show that there exists such an invertible sheaf $\mathcal{M}$ on Y with $\pi^* \mathcal{M} \cong \mathcal{L}$, it suffices to show that $\pi_* \mathcal{L}$ is an invertible sheaf (call it $\mathcal{M}$) and the natural homomorphism $\pi^* \mathcal{M} \to \mathcal{L}$ is an isomorphism.

Now by Grauert's theorem 1.1, $\pi_* \mathcal{L}$ is locally free of rank 1 (again, call it $\mathcal{M}$), and $\mathcal{M} \otimes_{\mathcal{O}_Y} k(y) \to H^0(X_y, \mathcal{L}_y)$ is an isomorphism. We have a natural map of invertible sheaves $\pi^* \mathcal{M} = \pi^* \pi_* \mathcal{L} \to \mathcal{L}$. To show that it is an isomorphism, we need only show that it is surjective, i.e. show that it is surjective on the fibers, which is done. □

Here are some consequences.

A first trivial consequence: if you have two invertible sheaves on X that agree on the fibers of π, then they differ by a pullback of an invertible sheaf on Y.

2.5. *Exercise.* Suppose X is an integral Noetherian scheme. Show that $\mathrm{Pic}(X \times \mathbb{P}^1) \cong \mathrm{Pic}\, X \times \mathbb{Z}$. (Side remark: If X is non-reduced, this is still true, see Hartshorne Exercise III.12.6(b). It need only be connected of finite type over k. Presumably locally Noetherian suffices.) Extend this to $X \times \mathbb{P}^n$. Extend this to any $\mathbb{P}^n$-bundle over X.

2.6. *Exercise.* Suppose $X \to Y$ is the projectivization of a vector bundle $\mathcal{F}$ over a reduced locally Noetherian scheme (i.e. $X = \underline{\mathrm{Proj}}\, \mathrm{Sym}^* \mathcal{F}$). Then I think we've already shown in an exercise that it is also the projectivization of $\mathcal{F} \otimes \mathcal{L}$. If Y is reduced and locally Noetherian, show that these are the only ways in which it is the projectivization of a vector bundle. (Hint: note that you can recover $\mathcal{F}$ by pushing forward $\mathcal{O}(1)$.)

2.7. *Exercise.* Suppose $\pi : X \to Y$ is a projective flat morphism over a Noetherian integral scheme, all of whose geometric fibers are isomorphic to $\mathbb{P}^n$ (over the appropriate field). Show that this is a projective bundle if and only if there is an invertible sheaf on X that restricts to $\mathcal{O}(1)$ on all the fibers. (One direction is clear: if it is a projective bundle, then it has a projective $\mathcal{O}(1)$. In the other direction, the candidate vector bundle is $\pi_* \mathcal{O}(1)$. Show that it is indeed a locally free sheaf of the desired rank. Show that its projectivization is indeed $\pi : X \to Y$.)

2.8. *Exercise (An example of a Picard scheme).* Show that the Picard scheme of $\mathbb{P}^1_k$ over k is isomorphic to $\mathbb{Z}$.

2.9. *Harder but worthwhile Exercise (An example of a Picard scheme).* Show that if E is an elliptic curve over k (a geometrically integral and nonsingular genus 1 curve with a marked k-point), then $\mathrm{Pic}\, E$ is isomorphic to $E \times \mathbb{Z}$. Hint: Choose a marked point p. (You'll note that this isn't canonical.) Describe the candidate universal invertible sheaf on $E \times \mathbb{Z}$. Given an invertible sheaf on $E \times X$, where X is an arbitrary Noetherian scheme, describe the morphism $X \to E \times \mathbb{Z}$.

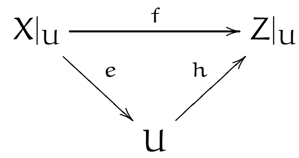
3. The rigidity lemma

The rigidity lemma is another useful fact about morphisms $\pi : X \to Y$ such that $\pi_* \mathcal{O}_X$ (condition (*) of the previous section). It is quite powerful, and quite cheap to prove, so we may as well do it now. (During class, the hypotheses kept on dropping until there was almost nothing left!)

3.1. *Rigidity lemma (first version). — Suppose we have a commutative diagram*

$$\begin{array}{ccc} X & \xrightarrow{\quad f \quad} & Z \\ & {\scriptstyle e \text{ closed},\ e_* \mathcal{O}_X = \mathcal{O}_Y} \searrow \quad \swarrow {\scriptstyle g \text{ quasi-proj.}} & \\ & Y & \end{array}$$

where Y is locally Noetherian, where f takes X_y for some $y \in Y$. Then there is a neighborhood $U \subset Y$ of y on which this is true. Better: over U, f factors through the projection to Y, i.e. the *following diagram commutes for some choice of* h*:*



.

Proof. This proof is very reminiscent of an earlier result, when we showed that a projective morphisms with finite fibers is a finite morphism.

We can take g to be projective. We can take Y to be an affine neighborhood of y. Then $Z \hookrightarrow \mathbb{P}^n_Y$ for some n. Choose a hyperplane of $\mathbb{P}^n_y$ missing $f(X_y)$, and extend it to a hyperplane H of $\mathbb{P}^n_Y$. (If $Y = \operatorname{Spec} B$, and $y = [\mathfrak{n}]$, then we are extending a linear equation with coefficients in $B/\mathfrak{n}$ to an equation with coefficients in B.) Pull back this hyperplane to X; the preimage is a closed subset. The image of this closed subset in Y is also a closed set $K \subset Y$, as e is a closed map. But $y \notin K$, so let $U = Y - K$. Over U, $f(X_y)$ misses our hyperplane H. Thus the map $X_y \to \mathbb{P}^n_U$ factors through $X_y \to \mathbb{A}^n_U$. Thus the map is given by n functions on $X|_U$. But $e_* \mathcal{O}_X \cong \mathcal{O}_Y$, so these are precisely the pullbacks of functions on U, so we are done. □

3.2. *Rigidity lemma (second version). — Same thing, with the condition on* g *changed from "projective" to simply "finite type".*

Proof. Shrink Y so that it is affine. Choose an open affine subset Z' of Z containing the $f(X_y)$. Then the complement the pullback of $K = Z - Z'$ to X is a closed subset of X whose image in Y is thus closed (as again e is a closed map), and misses y. We shrink Y further such that $f(X)$ lies in Z'. But $Z' \to Y$ is quasiprojective, so we can apply the previous version. □

Here is another mild strengthening.

3.3. *Rigidity lemma (third version). — If* X *is reduced and* g *is separated, and* Y *is connected, and there is a section* $Y \to X$*, then we can take* $U = Y$*.*

Proof. We have two morphisms $X \to Z$: f and $f \circ s \circ e$ which agree on the open set U. But we've shown earlier that any two morphisms from a reduced scheme to a separated scheme agreeing on a dense open set are the same. □

Here are some nifty consequences.

3.4. *Corollary (abelian varieties are abelian). — Suppose* A *is a projective integral group variety (an abelian variety) over a field* k*. Then the multiplication map* $m : A \times A \to A$ *is commutative.*

Proof. Consider the commutator map $c : A \times A \to A$ that corresponds to $(x, y) \mapsto xyx^{-1}y^{-1}$. We wish to show that this map sends $A \times A$ to the identity in A. Consider $A \times A$ as a family over the first factor. Then over $x = e$, c maps the fiber to e. Thus by the rigidity lemma (third version), the map c is a function only of the first factor. But then $c(x, y) = c(x, e) = e$. □

3.5. *Exercise.* By a similar argument show that any map $f : A \to A'$ from one abelian variety to another is a group homomorphism followed by a translation. (Hint: reduce quickly to the case where f sends the identity to the identity. Then show that "$f(x+y) - f(x) - f(y) = e$".)

4. Proof of Grauert's theorem

I'll prove Grauert, but not Cohomology and Base Change. It would be wonderful if Cohomology and Base Change followed by just mucking around with maps of free modules over a ring.

4.1. *Exercise++.* Find such an argument.

We'll need a preliminary result.

4.2. *Lemma. — Suppose $Y = \operatorname{Spec} B$ is a reduced Noetherian scheme, and $f : M \to N$ is a homomorphism of coherent free (hence projective, flat) B-modules. If $\dim_{k(y)} \operatorname{im}(f \otimes k(y))$ is locally constant, then there are splittings $M = M_1 \oplus M_2$ and $N = N_1 \oplus N_2$ with f killing M_1, and sending M_2 isomorphically to N_1.*

Proof. Note that $f(M) \otimes k \cong f(M \otimes k)$ from that surjection. From $0 \to f(M) \to N \to N/f(M) \to 0$ we have

$$\begin{array}{ccccccc} f(M)\otimes k & \longrightarrow & N\otimes k & \longrightarrow & N/f(M)\otimes k & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ f(M\otimes k) & \longrightarrow & N\otimes k & \longrightarrow & N\otimes k/f(M\otimes k) & \longrightarrow & 0 \end{array}$$

from which $(N/f(M)) \otimes k \cong (N \otimes k)/f(M \otimes k)$. Now the one on the right has locally constant rank, so the one on the left does too, hence is locally free, and flat, and projective. Hence $0 \to f(M) \to N \to N/f(M) \to 0$ splits, so let $N_2 = N/f(M)$, $N_1 = f(M)$. Also, N and $N/f(M)$ are flat and coherent, hence so is $f(M)$.

We now play the same game with

$$0 \to \ker f \to M \to f(M) \to 0.$$

$f(M)$ is projective, hence this splits. Let $\ker f = M_1$. □

Now let's prove Grauert's theorem 1.1. We can use this lemma to rewrite

$$M^{p-1} \xrightarrow{d^{p-1}} M^p \xrightarrow{d^p} M^{p+1}$$

as $Z^{p-1} \oplus K^{p-1} \longrightarrow B^p \oplus H^p \oplus K^p \longrightarrow B^{p+1} \oplus K^{p+1}$ where d^{p-1} sends K^{p-1} isomorphically onto B^p (and is otherwise 0), and d^p sends K^p isomorphically onto B_{p+1}. Here H^p is a projective module, so we have local freeness. Thus when we tensor with some other ring, this structure is preserved as well; hence we have isomorphism. $\square$

5. Dimensions behave well for flat morphisms

There are a few easier statements about flatness that I could have said much earlier.

Here's a basic statement about how dimensions behave in flat families.

5.1. *Proposition. — Suppose $f : X \to Y$ is a flat morphism of schemes all of whose stalks are localizations of finite type k-algebras, with $f(x) = y$. (For example, X and Y could be finite type k-schemes.) Then the dimension of X_y at x plus the dimension of Y and y is the dimension X at x.*

In other words, there can't be any components contained in a fibers; and you can't have any dimension-jumping.

In class, I first incorrectly stated this with the weaker hypotheses that X and Y are just locally Noetherian. Kirsten pointed out that I used the fact that height = codimension, which is not true for local Noetherian rings in general. However, we have shown it for local rings of finite type k-schemes. Joe suggested that one could work around this problem.

Proof. This is a question about local rings, so we can consider $\operatorname{Spec} \mathcal{O}_{X,x} \to \operatorname{Spec} \mathcal{O}_{Y,y}$. We may assume that Y is reduced. We prove the result by induction on $\dim Y$. If $\dim Y = 0$, the result is immediate, as $X_y = X$ and $\dim_y Y = 0$.

Now for $\dim Y > 0$, I claim there is an element $t \in \mathfrak{m}$ that is not a zero-divisor, i.e. is not contained in any associated prime, i.e. (as Y is reduced) is not contained in any minimal prime. Let $\mathfrak{p}_1, \dots, \mathfrak{p}_n$ be the (finite number of) minimal primes. If $\mathfrak{m} \subset \mathfrak{p}_1 \cup \cdots \cup \mathfrak{p}_n$, then in the first quarter we showed (in an exercise) that $\mathfrak{m} \subset \mathfrak{p}_i$ for some i. But as $\mathfrak{m}$ is maximal, and $\mathfrak{p}_i$ is minimal, we must have $\mathfrak{m} = \mathfrak{p}_i$, and $\dim Y = 0$.

Now by flatness t is not a zero-divisor of $\mathcal{O}_{X,x}$. (Recall that non-zero-divisors pull back to non-zero-divisors.) $\dim \mathcal{O}_{Y,y}/t = \dim \mathcal{O}_{Y,y} - 1$ by Krull's principal ideal theorem (here we use the fact that codimension = height), and $\dim \mathcal{O}_{X,x}/t = \dim \mathcal{O}_{X,x} - 1$ similarly. $\square$.

5.2. *Corollary. — Suppose $f : X \to Y$ is a flat finite-type morphism of locally Noetherian schemes, and Y is irreducible. Then the following are equivalent.*

- *Every irreducible component of* X *has dimension* $\dim Y + n$.
- *For any point* $y \in Y$ *(not necessarily closed!), every irreducible component of the fiber* X_y *has dimension* n.

5.3. *Exercise.* Prove this.

Important definition: If these conditions hold, we say that π is *flat of relative dimension* n. This definition will come up when we define *smooth of relative dimension* n.

5.4. *Exercise.*
(a) Suppose $\pi : X \to Y$ is a finite-type morphism of locally Noetherian schemes, and Y is irreducible. Show that the locus where π is flat of relative dimension n is an open condition.
(b) Suppose $\pi : X \to Y$ is a *flat* finite-type morphism of locally Noetherian schemes, and Y is irreducible. Show that X can be written as the disjoint union of schemes $X_0 \cup X_1 \cup \cdots$ where $\pi|_{X_n} : X_n \to Y$ is flat of relative dimension n.

5.5. *Important Exercise.* Use a variant of the proof of Proposition 5.1 to show that if $f : X \to Y$ is a flat morphism of finite type k-schemes (or localizations thereof), then any associated point of X must map to an associated point of Y. (I find this an important point when visualizing flatness!)

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FOUNDATIONS OF ALGEBRAIC GEOMETRY CLASSES 47 AND 48

RAVI VAKIL

CONTENTS

This week: Local criteria for flatness (statement), (relatively) base-point-free, (relatively) ample, very ample, every ample on a proper has a tensor power that is very ample, Serre's criterion for ampleness, Riemann-Roch for generically reduced curves.

1. THE LOCAL CRITERION FOR FLATNESS

I'll end our discussion of flatness with the statement of two results which can be quite useful. (Translation: I've seen them used.) They are both called the local criterion for flatness.

In both situations, assume that $(B, \mathfrak{n}) \to (A, \mathfrak{m})$ is a local morphism of local Noetherian rings (i.e. a ring homomorphism with $\mathfrak{n}A \subset \mathfrak{m}$), and that M is a finitely generated A-module. Of course we picture this in terms of geometry:

$$\begin{array}{c} \tilde{M} \\ | \\ \operatorname{Spec}(A, \mathfrak{m}) \\ \downarrow \\ \operatorname{Spec}(B, \mathfrak{n}). \end{array}$$

The local criteria for flatness are criteria for when M is flat over A. In practice, these are used in two circumstances: to check when a morphism to a locally Noetherian scheme is flat, or when a coherent sheaf on a locally Noetherian scheme is flat.

We've shown that to check if M is flat, we need check if $\operatorname{Tor}_1^B(B/I, M) = 0$ for all ideals I. The (first) local criterion says we need only deal with the maximal ideal.

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1.1. *Theorem (local criterion for flatness). —* M *is* B*-flat if and only if* $\mathrm{Tor}_1^B(B/\mathfrak{n}, M) = 0$.

(You can see a proof in Eisenbud, p. 168.)

An even more useful variant is the following. Suppose t is a non-zero-divisor of B in $\mathfrak{m}$ (geometrically: a Cartier divisor on the target passing through the generic point). If M is flat over B, then t is not a zero-divisor of M (we've checked this before: tensor $0 \longrightarrow B \xrightarrow{\times t} B \longrightarrow B/(t) \to 0$ with M). Also, M/tM is a flat B/tB-module (flatness commutes with base change). The next result says that this is a characterization.

1.2. *Theorem (local slicing criterion for flatness). —* M *is* B*-flat if and only if* M/tM *is flat over* $B/(t)$.

This is also sometimes called the local criterion for flatness. The proof is short (given the first local criterion). You can read it in Eisenbud (p. 169).

1.3. *Exercise (for those who know what a Cohen-Macaulay scheme is).* Suppose $\pi : X \to Y$ is a map of locally Noetherian schemes, where both X and Y are equidimensional, and Y is nonsingular. Show that if any two of the following hold, then the third does as well:

- π is flat.
- X is Cohen-Macaulay.
- Every fiber X_y is Cohen-Macaulay of the expected dimension.

I concluded the section on flatness by reviewing everything we have learned about flatness, in a good order.

2. Base-point-free, ample, very ample

My goal is to discuss properties of invertible sheaves on schemes (an "absolute" notion), and properties of invertible sheaves on a scheme with a morphism to another scheme (a "relative" notion, meaning that it makes sense in families). The notions fit into this table:

absolute	relative
base-point-free	relatively base-point-free
ample	relatively ample
very ample over a ring	very ample

This is admittedly horrible terminology. Warning: my definitions may have some additional hypotheses not used in EGA. The additional hypotheses exclude some nasty behavior which tends not to come up in nature; indeed, I have only seen these notions used in the circumstances in which I will describe them. There are very few facts to know, and there is fairly little to prove.

2.1. *Definition of base-point-free and relative base-point-free (review from class 22 and class 24, respectively).* Recall that if $\mathcal{F}$ is a quasicoherent sheaf on a scheme X, then $\mathcal{F}$ is generated by global sections if for any $x \in X$, the global sections generate the stalk $\mathcal{F}_x$. Equivalently: $\mathcal{F}$ is the quotient of a free sheaf. If $\mathcal{F}$ is a finite type quasicoherent sheaf, then we just need to check that for any x, the global sections generate the fiber of $\mathcal{F}$, by Nakayama's lemma. If furthermore $\mathcal{F}$ is invertible, we need only check that for any x there is a global section not vanishing there. In the case where $\mathcal{F}$ is invertible, we give "generated by global sections" a special name: *base-point-free*.

2.2. *Exercise (generated $\otimes$ generated = generated for finite type sheaves).* Suppose $\mathcal{F}$ and $\mathcal{G}$ are finite type sheaves on a scheme X that are generated by global sections. Show that $\mathcal{F} \otimes \mathcal{G}$ is also generated by global sections. In particular, if $\mathcal{L}$ and $\mathcal{M}$ are invertible sheaves on a scheme X, and both $\mathcal{L}$ and $\mathcal{M}$ are base-point-free, then so is $\mathcal{L} \otimes \mathcal{M}$. (This is often summarized as "base-point-free + base-point-free = base-point-free". The symbols $+$ is used rather than $\otimes$, because Pic is an abelian group.)

If $\pi : X \to Y$ is a morphism of schemes *that is quasicompact and quasiseparated* (so push-forwards of quasicoherent sheaves are quasicoherent sheaves), and $\mathcal{F}$ is a quasicoherent sheaf on X, we say that $\mathcal{F}$ is *relatively generated by global sections* (or *relatively generated* for short) if $\pi^*\pi_*\mathcal{F} \to \mathcal{F}$ is a surjection of sheaves (class 24). As this is a morphism of quasicoherent sheaves, this can be checked over any affine open subset of the target, and corresponds to "generated by global sections" above each affine. In particular, this notion is affine-local on the target. If $\mathcal{F}$ is locally free, this notion is called *relatively base-point-free*.

2.3. *Definition of very ample.* Suppose $X \to Y$ is a projective morphism. Then $X = \underline{\operatorname{Proj}}\, \mathcal{S}_*$ for some graded algebra, locally generated in degree 1; given this description, X comes with $\mathcal{O}(1)$. Then any invertible sheaf on X of this sort is said to be *very ample* (for the morphism π). The notion of very ample is local on the base. (This is "better" than the notion of projective, which isn't local on the base, as we've seen in class 43/44 p. 4. Recall why: a morphism is projective if there *exists* an $\mathcal{O}(1)$. Thus a morphism $X \to Y \cup Y'$ could be projective over Y and over Y', but not projective over $Y \cup Y'$, as the "$\mathcal{O}(1)$" above Y need not be the same as the "$\mathcal{O}(1)$" above Y'. On the other hand, the notion is "very ample" is precisely the data of "an $\mathcal{O}(1)$".) You'll recall that given such an invertible sheaf, then $X = \underline{\operatorname{Proj}}\, \pi_* \mathcal{L}^{\otimes n}$, where the algebra on the right has the desired form. (It isn't necessarily the same graded algebra as you originally used to construct X.)

Notational remark: If Y is implicit, it is often omitted from the terminology. For example, if X is a complex projective scheme, the phrase "$\mathcal{L}$ is very ample on X" often means that "$\mathcal{L}$ is very ample for the structure morphism $X \to \operatorname{Spec} \mathbb{C}$".

2.4. *Exercise (very ample + very ample = very ample).* If $\mathcal{L}$ and $\mathcal{M}$ are invertible sheaves on a scheme X, and both $\mathcal{L}$ and $\mathcal{M}$ are base-point-free, then so is $\mathcal{L} \otimes \mathcal{M}$. Hint: Segre. In particular, tensor powers of a very ample invertible sheaf are very ample.

2.5. *Tricky exercise+ (very ample + relatively generated = very ample).* Suppose $\mathcal{L}$ is very ample, and $\mathcal{M}$ is relatively generated, both on $X \to Y$. Show that $\mathcal{L} \otimes \mathcal{M}$ is very ample.

(Hint: Reduce to the case where the target is affine. $\mathcal{L}$ induces a map to $\mathbb{P}^n_A$, and this corresponds to $n+1$ sections $s_0, \ldots, s_n$ of $\mathcal{L}$. We also have a finite number m of sections $t_1, \ldots, t_m$ of $\mathcal{M}$ which generate the stalks. Consider the $(n+1)m$ sections of $\mathcal{L} \otimes \mathcal{M}$ given by $s_i t_j$. Show that these sections are base-point-free, and hence induce a morphism to $\mathbb{P}^{(n+1)m-1}$. Show that it is a closed immersion.)

2.6. *Definition of ample and relatively ample.* Suppose X is a quasicompact scheme. We say an invertible sheaf $\mathcal{L}$ on X is *ample* if for all finite type sheaves $\mathcal{F}$, $\mathcal{F} \otimes \mathcal{L}^n$ is generated by global sections for $n \gg 0$. ("After finite twist, it is generated by global sections.") This is an *absolute* notion, not depending on a morphism.

2.7. *Example.* (a) If X is an affine scheme, and $\mathcal{L}$ is any invertible sheaf on X, then $\mathcal{L}$ is ample.
(b) If $X \to \operatorname{Spec} B$ is a projective morphism and $\mathcal{L}$ is a very ample invertible sheaf on X, then $\mathcal{L}$ is ample (by Serre vanishing, Theorem 4.2(ii), class 29, p. 5). (We may need B Noetherian here.)

We now give the relative version of this notion. Suppose $\pi : X \to Y$ is a morphism, and $\mathcal{L}$ is an invertible sheaf on X. Suppose that for every affine open subset $\operatorname{Spec} B$ of Y there is an n_0 such that $\mathcal{F} \otimes \mathcal{L}^{\otimes n}$ restricted to the preimage of $\operatorname{Spec} B$ is relatively generated by global sections for $n \geq n_0$. (In particular, π is quasicompact and quasiseparated — that was a hypothesis for relatively generated.) Then we say that $\mathcal{L}$ is *relatively ample* (with respect to π; although the reference to the morphism is often suppressed when it is clear from the context). It is also sometimes called π-ample. Warning: the n_0 depends on the affine open; we may not be able to take a single n_0 for all affine opens. We can, however, if Y is quasicompact, and hence we'll see this quasicompactness hypothesis on Y often.

Example. The examples of 2.7 naturally generalize.
(a) If $X \to Y$ is an affine morphism, and $\mathcal{L}$ is any invertible sheaf on X, then $\mathcal{L}$ is relatively ample.
(b) If $X \to Y$ is a projective morphism and $\mathcal{L}$ is a very ample invertible sheaf on X, then $\mathcal{L}$ is relatively ample. (We may need Y locally Noetherian here.)

2.8. *Easy Lemma. — Fix a positive integer* n.

(a) *If $\mathcal{L}$ is an invertible sheaf on a scheme X, then $\mathcal{L}$ is ample if and only if $\mathcal{L}^{\otimes n}$ is ample.*
(b) *If $\pi : X \to Y$ is a morphism, and $\mathcal{L}$ is an invertible sheaf on X, then $\mathcal{L}$ is relatively ample if and only if $\mathcal{L}^{\otimes n}$ is relatively ample.*

In general, statements about ample sheaves (such as (a) above) will have immediate analogues for statements about relatively ample sheaves where the target is quasicompact (such as (b) above), and I won't spell them out in the future. [I'm not sure what I meant by this comment about (b); I'll think about it.]

Proof. We prove (a); (b) is then immediate.

Suppose $\mathcal{L}$ is ample. Then for any finite type sheaf $\mathcal{F}$ on X, there is some m_0 such that for $m \geq m_0$, $\mathcal{F} \otimes \mathcal{L}^{\otimes m_0}$ is generated by global sections. Thus for $m' \geq m_0/n$, $\mathcal{F} \otimes (\mathcal{L}^{\otimes n})^{m'}$ is generated by global sections, so $\mathcal{L}^{\otimes n}$ is ample.

Suppose next that $\mathcal{L}^{\otimes n}$ is ample, and let $\mathcal{F}$ by any finite type sheaf. Then there is some m_0 such that $(\mathcal{F}) \otimes (\mathcal{L}^n)^m$, $(\mathcal{F} \otimes \mathcal{L}) \otimes (\mathcal{L}^n)^m$, $(\mathcal{F} \otimes \mathcal{L}^{\otimes 2}) \otimes (\mathcal{L}^n)^m$, $\dots (\mathcal{F} \otimes \mathcal{L}^{\otimes (m-1)}) \otimes (\mathcal{L}^n)^m$, are all generated by global sections for $m \geq m_0$. In other words, for $m' \geq nm_0$, $\mathcal{F} \otimes \mathcal{L}^{\otimes m'}$ is generated by global sections. Hence $\mathcal{L}$ is ample. $\square$

Example: any positive degree invertible sheaf on a curve is ample. Reason: a high tensor power (such that the degree is at least $2g + 1$) is very ample.

2.9. *Proposition. — In each of the following, X is a scheme, $\mathcal{L}$ is an ample invertible sheaf (hence X is quasicompact), and $\mathcal{M}$ is an invertible sheaf.*

(a) *(ample + generated = ample) If $\mathcal{M}$ is generated by global sections, then $\mathcal{L} \otimes \mathcal{M}$ is ample.*
(b) *(ample + ample = ample) If $\mathcal{M}$ is ample, then $\mathcal{L} \otimes \mathcal{M}$ is ample.*

Similar statements hold for quasicompact and quasiseparated morphisms and relatively ample and relatively generated.

Proof. (a) Suppose $\mathcal{F}$ is any finite type sheaf. Then by ampleness of $\mathcal{L}$, there is an n_0 such that for $n \geq n_0$, $\mathcal{F} \otimes \mathcal{L}^{\otimes n}$ is generated by global sections. Hence $\mathcal{F} \otimes \mathcal{L}^{\otimes n} \otimes \mathcal{M}^{\otimes n}$ is generated by global sections. Thus there is an n_0 such that for $n \geq n_0$, $\mathcal{F} \otimes (\mathcal{L} \otimes \mathcal{M})^{\otimes n}$ is generated by global sections. Hence $\mathcal{L} \otimes \mathcal{M}$ is ample.

(b) As $\mathcal{M}$ is ample, $\mathcal{M}^{\otimes n}$ is base-point-free for some $n > 0$. But $\mathcal{L}^{\otimes n}$ is ample, so by (a) $(\mathcal{L} \otimes \mathcal{M})^{\otimes n}$ is ample, so by Lemma 2.8, $\mathcal{L} \otimes \mathcal{M}$ is ample. $\square$

3. Every ample on a proper has a tensor power that is very ample

We'll spend the rest of our discussion of ampleness considering consequences of the following very useful result.

3.1. *Theorem. — Suppose $\pi : X \to Y$ is proper and $Y = \operatorname{Spec} B$ is affine. If $\mathcal{L}$ is ample, then some tensor power of $\mathcal{L}$ is very ample.*

The converse follows from our earlier discussion, that very ample implies ample, Example 2.7(b).

Proof. I hope to type in a short proof at some point. For now, I'll content myself with referring to Hartshorne Theorem II.7.6. (He has more hypotheses, but his argument essentially applies in this more general situation.)

3.2. *Exercise.* Suppose $\pi : X \to Y$ is proper and Y is quasicompact. Show that if $\mathcal{L}$ is relatively ample on X, then some tensor power of $\mathcal{L}$ is very ample.

Serre vanishing holds for any relatively ample invertible sheaf for a proper morphism to a Noetherian base. More precisely:

3.3. *Corollary (Serre vanishing, take two). — Suppose $\pi : X \to Y$ is a proper morphism, Y is quasicompact, and $\mathcal{L}$ is a π-ample invertible sheaf on X. Then for any coherent sheaf $\mathcal{F}$ on X, for $m \gg 0$, $R^i\pi_*\mathcal{F} \otimes \mathcal{L}^{\otimes m} = 0$ for all $i > 0$.*

Proof. By Theorem 3.1, $\mathcal{L}^{\otimes n}$ very ample for some n, so π is projective. Apply Serre vanishing to $\mathcal{F} \otimes \mathcal{L}^{\otimes i}$ for $0 \leq i < n$. □

The converse holds, i.e. this in fact characterizes ampleness. For convenience, we state it for the case of an affine target.

3.4. *Theorem (Serre's criterion for ampleness). — Suppose that $\pi : X \to Y = \operatorname{Spec} B$ is a proper morphism, and $\mathcal{L}$ is an invertible sheaf on X such that for any finite type sheaf $\mathcal{F}$ on X, $\mathcal{F} \otimes \mathcal{L}^{\otimes n}$ is generated by global sections for $n \gg 0$. Then $\mathcal{L}$ is ample.*

Essentially the same statement holds for relatively ample and quasicompact target. *Exercise.* Give and prove the statement. **Whoops! Ziyu and Rob point out that I used Serre's criterion as the *definition* of ampleness (and similarly, relative ampleness). Thus this exercise is nonsense.**

3.5. *Proof of Serre's criterion.* I hope to type in a better proof before long, but for now I'll content myself with referring to Hartshorne, Proposition III.5.3.

3.6. *Exercise.* Use Serre's criterion for ampleness to prove that the pullback of ample sheaf on a projective scheme by a finite morphism is ample. Hence if a base-point-free invertible sheaf on a proper scheme induces a morphism to projective space that is finite onto its image, then it is is ample.

3.7. *Key Corollary. — Suppose $\pi : X \to \operatorname{Spec} B$ is proper, and $\mathcal{L}$ and $\mathcal{M}$ are invertible sheaves on X with $\mathcal{L}$ ample. Then $\mathcal{L}^{\otimes n} \otimes \mathcal{M}$ is very ample for $n \gg 0$.*

3.8. *Exercise.* Give and prove the corresponding statement for a relatively ample invertible sheaf over a quasicompact base.

Proof. The theorem says that $\mathcal{L}^{\otimes n}$ is very ample for $n \gg 0$. By the definition of ampleness, $\mathcal{L}^{\otimes n} \otimes \mathcal{M}$ is generated for $n \gg 0$. Tensor these together, using the above. □

A key implication of the key corollary is:

3.9. *Corollary.* — *Any invertible sheaf on a projective* $X \to \operatorname{Spec} B$ *is a difference of two very ample invertible sheaves.*

Proof. If $\mathcal{M}$ is any invertible sheaf, choose $\mathcal{L}$ very ample. Corollary 3.7 states that $\mathcal{M} \otimes \mathcal{L}^{\otimes n}$ is very ample. As $\mathcal{L}^{\otimes n}$ is very ample (Exercise 2.4), we can write $\mathcal{M}$ as the difference of two very ample sheaves: $\mathcal{M} \cong (\mathcal{M} \otimes \mathcal{L}^{\otimes n}) \otimes (\mathcal{L}^{\otimes n})^*$.

As always, we get a similar statement for relatively ample sheaves over a quasicompact base.

Here are two interesting consequences of Corollary 3.9.

3.10. *Exercise.* Suppose X a projective k-scheme. Show that every invertible sheaf is the difference of two *effective* Cartier divisors. Thus the groupification of the semigroup of effective Cartier divisors is the Picard group. Hence if you want to prove something about Cartier divisors on such a thing, you can study effective Cartier divisors.

(This is false if projective is replaced by proper — ask Sam Payne for an example.)

3.11. *Important exercise.* Suppose C is a generically reduced projective k-curve. Then we can define degree of an invertible sheaf $\mathcal{M}$ as follows. Show that $\mathcal{M}$ has a meromorphic section that is regular at every singular point of C. Thus our old definition (number of zeros minus number of poles, using facts about discrete valuation rings) applies. Prove the Riemann-Roch theorem for generically reduced projective curves. (Hint: our original proof essentially will carry through without change.)

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FOUNDATIONS OF ALGEBRAIC GEOMETRY CLASSES 49 AND 50

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CONTENTS

At the start of class 49, I gave an informal discussion on other criteria for ampleness, and other adjectives for divisors. We discussed the following notions: Kleiman's criterion for ampleness, numerical equivalence, Neron-Severi group, Picard number, nef, the nef cone and the ample cone, Nakai's criterion, the Nakai-Moishezon criterion, big, $\mathbb{Q}$-Cartier, Snapper's theorem.)

1. BLOWING UP A SCHEME ALONG A CLOSED SUBSCHEME

We'll next discuss an important construction in algebraic geometry (and especially the geometric side of the subject), the blow-up of a scheme along a closed subscheme (cut out by a finite type ideal sheaf). We'll start with a motivational example that will give you a picture of the construction in a particularly important case (and the historically earliest case), in Section 2. I'll then give a formal definition, in terms of universal property, Section 3. This definition won't immediately have a clear connection to the motivational example! We'll deduce some consequences of the definition (assuming that the blow-up actually exists). We'll prove that the blow-up always exists, by describing it quite explicitly, in Section 4. As a consequence, the blow-up morphism is projective, and we'll deduce more consequences from this. In Section 5, we'll do a number of explicit computations, and see that in practice, it is possible to compute many things by hand. I'll then mention a couple of useful facts: (i) the blow-up a nonsingular variety in a nonsingular variety is still nonsingular, something we'll have observed in our explicit examples, and (ii) Castelnuovo's criterion, that on a smooth surface, "(-1)-curves" ($\mathbb{P}^1$'s with normal bundle $\mathcal{O}(-1)$) can be "blown down".

Date: Tuesday, May 16 and Tuesday, May 23, 2006.

We're going to generalize the following notion, which will correspond to "blowing up" the origin of $\mathbb{A}^2_k$ (over an algebraically closed field k). Because this is just motivation, I'll be informal. Consider the subset of $\mathbb{A}^2 \times \mathbb{P}^1$ corresponding to the following. We interpret $\mathbb{P}^1$ as the lines through the origin. Consider the subset $\{(p \in \mathbb{A}^2, [\ell] \in \mathbb{P}^1) : p \in \ell)\}$. (I showed you a model in class, admittedly over the non-algebraically-closed field $k = \mathbb{R}$.)

I'll now try to convince you that this is nonsingular (informally). Now $\mathbb{P}^1$ is smooth, and for each point $[\ell]$ in $\mathbb{P}^1$, we have a smooth choice of points on the line ℓ. Thus we are verifying smoothness by way of the fibration over $\mathbb{P}^1$.

Let's make this more algebraic. Let x and y be coordinates on $\mathbb{A}^2$, and X and Y be projective coordinates on $\mathbb{P}^1$ ("corresponding" to x and y); we'll consider the subset $\mathrm{Bl}_{(0,0)} \mathbb{A}^2$ of $\mathbb{A}^2 \times \mathbb{P}^1$ corresponding to $xY - yX = 0$. We could then verify that this is nonsingular (by looking at two covering patches).

Notice that the preimage of $(0, 0)$ is a curve and hence a divisor (an effective Cartier divisor, as the blown-up surface is nonsingular). Also, note that if we have some curve singular at the origin, this could be partially desingularized. (A *desingularization* or a *resolution of singularities* of a variety X is a proper birational morphism $\tilde{X} \to X$ from a nonsingular scheme. We are interested in desingularizations for many reasons. For example, we understand nonsingular curves quite well. and we could hope to understand other curves through their desingularizations. This philosophy holds true in higher dimension as well.) For example, the curve $y^2 = x^3 + x^2$, which is nonsingular except for a node at the origin, then we can take the preimage of the curve minus the origin, and take the closure of this locus in the blow-up, and we'll obtain a nonsingular curve; the two branches of the node downstairs are separated upstairs. (This will later be an exercise, once we've defined things properly. The result will be called the *proper transform* of the curve.)

Let's generalize this. First, we can blow up $\mathbb{A}^n$ at the origin (or more informally, "blow up the origin"), getting a subvariety of $\mathbb{A}^n \times \mathbb{P}^{n-1}$. More algebraically, If $x_1, \ldots, x_n$ are coordinates on $\mathbb{A}^n$, and $X_1, \ldots, X_n$ are projective coordinates on $\mathbb{P}^{n-1}$, then the blow-up $\mathrm{Bl}_{\vec{0}} \mathbb{A}^n$ is given by the equations $x_i X_j - x_j X_i = 0$. Once again, this is smooth: $\mathbb{P}^{n-1}$ is smooth, and for each point $[\ell] \in \mathbb{P}^{n-1}$, we have a smooth choice of $p \in \ell$.

We can extend this further, by blowing up $\mathbb{A}^{n+m}$ along a coordinate m-plane $\mathbb{A}^n$ by adding m more variables $x_{n+1}, \ldots, x_{n+m}$ to the previous example; we get a subset of $\mathbb{A}^{n+m} \times \mathbb{P}^{n-1}$.

Then intuitively, we could extend this to blowing up a nonsingular subvariety of a nonsingular variety. We'll make this more precise. In the course of doing so, we will accidentally generalize this notion greatly, defining the blow-up of any finite type sheaf of ideals in a scheme. In general, blowing up may not have such an intuitive description as in the case of blowing up something nonsingular inside something nonsingular — it does great violence to the scheme — but even then, it is very useful (for example, in developing intersection theory). The result will be very powerful, and will touch on many other useful notions in algebra (such as the Rees algebra) that we won't discuss here.

Our description will depend only the closed subscheme being blown up, and not on coordinates. That remedies a defect was already present in the first baby example, blowing up the plane at the origin. It is not obvious that if we picked different coordinates for the plane (preserving the origin as a closed subscheme) that we wouldn't have two different resulting blow-ups.

As is often the case, there are two ways of understanding this notion, and each is useful in different circumstances. The first is by universal property, which lets you show some things without any work. The second is an explicit construction, which lets you get your hands dirty and compute things (and implies for example that the blow-up morphism is projective).

3. Blowing up, by universal property

I'll start by defining the blow-up using the universal property. The disadvantage of starting here is that this definition won't obviously be the same as the examples I just gave. It won't even look related!

Suppose $X \hookrightarrow Y$ is a closed subscheme corresponding to a finite type sheaf of ideals. (If Y is locally Noetherian, the "finite type" hypothesis is automatic, so Noetherian readers can ignore it.)

The blow-up $X \hookrightarrow Y$ is a *fiber diagram*

$$\begin{array}{ccc} E_X Y & \hookrightarrow & \mathrm{Bl}_X Y \\ \downarrow & & \downarrow \beta \\ X & \hookrightarrow & Y \end{array}$$

such that $E_X Y$ is an *effective Cartier divisor* on $\mathrm{Bl}_X Y$ (and is the scheme-theoretical pullback of X on Y), such any other such fiber diagram

(1)
$$\begin{array}{ccc} D & \hookrightarrow & W \\ \downarrow & & \downarrow \\ X & \hookrightarrow & Y, \end{array}$$

where D is an effective Cartier divisor on W, factors uniquely through it:

$$\begin{array}{ccc} D & \hookrightarrow & W \\ \downarrow & & \downarrow \\ E_X Y & \hookrightarrow & \mathrm{Bl}_X Y \\ \downarrow & & \downarrow \\ X & \hookrightarrow & Y. \end{array}$$

(Recall that an effective Cartier divisor is locally cut out by one equation that is not a zero-divisor; equivalently, it is locally cut out by one equation, and contains no associated points. This latter description will prove crucial.) $\mathrm{Bl}_X Y$ is called *the blow-up* (of Y along X, or of Y with center X). $E_X Y$ is called the *exceptional divisor*. (Bl and β stand for "blow-up", and E stands for "exceptional".)

By a universal property argument, if the blow-up exists, it is unique up to unique isomorphism. (We can even recast this more explicitly in the language of Yoneda's lemma: consider the category of diagrams of the form (1), where morphisms are of the form

$$\begin{array}{ccc} D & \hookrightarrow & W \\ \downarrow & & \downarrow \\ D' & \hookrightarrow & W' \\ \downarrow & & \downarrow \\ X & \hookrightarrow & Y. \end{array}$$

Then the blow-up is a final object in this category, if one exists.)

If $Z \hookrightarrow Y$ is any closed subscheme of Y, then the (scheme-theoretic) pullback $\beta^{-1}Z$ is called the *total transform* of Z. We will soon see that β is an isomorphism away from X (Observation 3.4). $\overline{\beta^{-1}(Z-X)}$ is called the *proper transform* or *strict transform* of Z. (We will use the first terminology. We will also define it in a more general situation.) We'll soon see that the proper transform is naturally isomorphic to $\mathrm{Bl}_{Z\cap X} Z$, where by $Z \cap X$ we mean the scheme-theoretic intersection (the blow-up closure lemma 3.7).

We will soon show that the blow-up always exists, and describe it explicitly. But first, we make a series of observations, assuming that the blow up exists.

3.1. *Observation.* If X is the empty set, then $\mathrm{Bl}_X Y = Y$. More generally, if X is a Cartier divisor, then the blow-up is an isomorphism. (Reason: $\mathrm{id}_Y : Y \to Y$ satisfies the universal property.)

3.2. *Exercise.* If U is an open subset of Y, then $\mathrm{Bl}_{U\cap X} U \cong \beta^{-1}(U)$, where $\beta : \mathrm{Bl}_X Y \to Y$ is the blow-up. (Hint: show $\beta^{-1}(U)$ satisfies the universal property!)

Thus "we can compute the blow-up locally."

3.3. *Exercise.* Show that if Y_α is an open cover of Y (as α runs over some index set), and the blow-up of Y_α along $X \cap Y_\alpha$ exists, then the blow-up of Y along X exists.

3.4. *Observation.* Combining Observation 3.1 and Exercise 3.2, we see that the blow-up is an isomorphism away from the locus you are blowing up:

$$\beta|_{\mathrm{Bl}_X Y - E_X Y} : \mathrm{Bl}_X Y - E_X Y \to Y - X$$

is an isomorphism.

3.5. *Observation.* If $X = Y$, then the blow-up is the empty set: the only map $W \to Y$ such that the pullback of X is a Cartier divisor is $\emptyset \hookrightarrow Y$. In this case we have "blown Y out of existence"!

3.6. *Exercise (blow-up preserves irreducibility and reducedness).* Show that if Y is irreducible, and X doesn't contain the generic point of Y, then $\mathrm{Bl}_X Y$ is irreducible. Show that if Y is reduced, then $\mathrm{Bl}_X Y$ is reduced.

The following blow-up closure lemma is useful in several ways. At first, it is confusing to look at, but once you look closely you'll realize that it is not so unreasonable.

Suppose we have a fibered diagram

$$\begin{array}{ccc} W & \xhookrightarrow{\text{cl. imm.}} & Z \\ \downarrow & & \downarrow \\ X & \xhookrightarrow{\text{cl. imm.}} & Y \end{array}$$

where the bottom closed immersion corresponds to a finite type ideal sheaf (and hence the upper closed immersion does too). The first time you read this, it may be helpful to consider the special case where $Z \to Y$ is a closed immersion.

Then take the fiber product of this square by the blow-up $\beta : \mathrm{Bl}_X Y \to Y$, to obtain

$$\begin{array}{ccc} Z \times_Y E_X Y & \hookrightarrow & Z \times_Y \mathrm{Bl}_X Y \\ \downarrow & & \downarrow \\ E_X Y & \xhookrightarrow{\text{Cartier}} & \mathrm{Bl}_X Y. \end{array}$$

The bottom closed immersion is locally cut out by one equation, and thus the same is true of the top closed immersion as well. However, it need not be a non-zero-divisor, and thus the top closed immersion is not necessarily an effective Cartier divisor.

Let $\overline{Z}$ be the scheme-theoretic closure of $Z \times_Y \mathrm{Bl}_X Y - W \times_Y \mathrm{Bl}_X Y$ in $Z \times_Y \mathrm{Bl}_X Y$. Note that in the special case where $Z \to Y$ is a closed immersion, $\overline{Z}$ is the proper transform, as defined in §3. For this reason, it is reasonable to call $\overline{Z}$ the proper transform of Z even if Z *isn't* a closed immersion. Similarly, it is reasonable to call $Z \times_Z \mathrm{Bl}_X Y$ the total transform even if Z isn't a closed immersion.

Define $E_{\overline{Z}} \hookrightarrow \overline{Z}$ as the pullback of $E_X Y$ to $\overline{Z}$, i.e. by the fibered diagram

$$\begin{array}{ccccc} E_{\overline{Z}} & \hookrightarrow & \overline{Z} & & \text{proper transform} \\ \downarrow \text{cl. imm.} & & \downarrow \text{cl. imm.} & & \\ Z \times_Y E_X Y & \hookrightarrow & Z \times_Y \mathrm{Bl}_X Y & & \text{total transform} \\ \downarrow & & \downarrow & & \\ E_X Y & \xhookrightarrow{\text{Cartier}} & \mathrm{Bl}_X Y. & & \end{array}$$

Note that $E_{\overline{Z}}$ is Cartier on $\overline{Z}$ (as it is locally the zero-scheme of a single function that does not vanish on any associated points of $\overline{Z}$).

3.7. *Blow-up closure lemma. —* $(\mathrm{Bl}_Z W, E_Z W)$ *is canonically isomorphic to* $(\overline{Z}, E_{\overline{Z}})$.

This is very handy.

The first three comments apply to the special case where $Z \to W$ is a closed immersion, and the fourth basically tells us we shouldn't have concentrated on this special case.

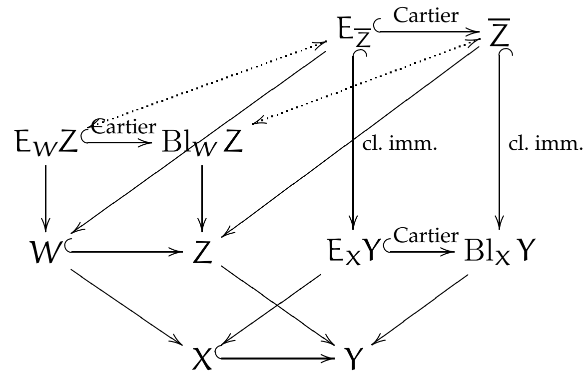
(1) First, note that if $Z \to Y$ is a closed immersion, then this states that the proper transform (as defined in §3) is the blow-up of Z along the scheme-theoretic intersection $W = X \cap Z$.

(2) In particular, it lets you actually compute blow-ups, and we'll do lots of examples soon. For example, suppose C is a plane curve, singular at a point p, and we want to blow up C at p. Then we could instead blow up the plane at p (which we have already described how to do, even if we haven't yet proved that it satisfies the universal property of blowing up), and then take the scheme-theoretic closure of $C - p$ in the blow-up.

(3) More generally, if W is some nasty subscheme of Z that we wanted to blow-up, and Z were a finite type k-scheme, then the same trick would work. We could work locally (Exercise 3.2), so we may assume that Z is affine. If W is cut out by r equations $f_1, \ldots, f_r \in \Gamma(\mathcal{O}_Z)$, then complete the f's to a generating set $f_1, \ldots, f_n$ of $\Gamma(\mathcal{O}_Z)$. This gives a closed immersion $Y \hookrightarrow \mathbb{A}^n$ such that W is the scheme-theoretic intersection of Y with a coordinate linear space $\mathbb{A}^r$.

3.8. (4) Most generally still, this reduces the existence of the blow-up to a specific special case. (If you prefer to work over a fixed field k, feel free to replace $\mathbb{Z}$ by k in this discussion.) Suppose that for each n, $\mathrm{Bl}_{(x_1,\ldots,x_n)} \operatorname{Spec} \mathbb{Z}[x_1, \ldots, x_n]$ exists. Then I claim that the blow-up always exists. Here's why. We may assume that Y is affine, say $\operatorname{Spec} B$, and $X = \operatorname{Spec} B/(f_1, \ldots, f_n)$. Then we have a morphism $Y \to \mathbb{A}^n_{\mathbb{Z}}$ given by $x_i \mapsto f_i$, such that X is the scheme-theoretic pullback of the origin. Hence by the blow-up closure lemma, $\mathrm{Bl}_X Y$ exists.

3.9. *Tricky Exercise+.* Prove the blow-up closure lemma. Hint: obviously, construct maps in both directions, using the universal property. The following diagram may or may not help.



3.10. *Exercise.* If Y and Z are closed subschemes of a given scheme X, show that $\mathrm{Bl}_Y Y \cup Z \cong \mathrm{Bl}_{Y \cap Z} Z$. (In particular, if you blow up a scheme along an irreducible component, the irreducible component is blown out of existence.)

4. The blow-up exists, and is projective

It is now time to show that the blow up always exists. I'll give two arguments, because I find them enlightening in two different ways. Both will imply that the blow-up morphism is projective. Hence the blow-up morphism is projective, hence quasicompact, proper, finite type, separated. In particular, if $Y \to Z$ is projective (resp. quasiprojective, quasicompact, proper, finite type, separated), so is $\mathrm{Bl}_X Y \to Z$. The blow-up of a k-variety is a k-variety (using the fact that irreducibility, reducedness are preserved, Exercise 3.6).

Approach 1. As explained above (§3.8), it suffices to show that $\mathrm{Bl}_{(x_1,\ldots,x_n)} \operatorname{Spec} \mathbb{Z}[x_1, \ldots, x_n]$ exists. But we know what it is supposed to be: the locus in

$$\operatorname{Spec} \mathbb{Z}[x_1, \ldots, x_n] \times \operatorname{Proj} \mathbb{Z}[X_1, \ldots, X_n]$$

such that $x_i X_j - x_j X_i = 0$. We'll show this soon.

Approach 2. We can describe the blow-up all at once as a $\underline{\operatorname{Proj}}$.

4.1. *Theorem ($\underline{\operatorname{Proj}}$ description of the blow-up). — Suppose $X \hookrightarrow Y$ is a closed subscheme cut out by a finite type sheaf of ideals $\mathcal{I} \hookrightarrow \mathcal{O}_Y$. Then*

$$\underline{\operatorname{Proj}}\left(\mathcal{O}_Y \oplus \mathcal{I} \oplus \mathcal{I}^2 \oplus \mathcal{I}^3 \oplus \cdots\right) \to Y$$

satisfies the universal property of blowing up.

We'll prove this soon (Section 4.2), after seeing what this gives us. (The reason we had a finite type requirement is that I wanted this $\underline{\operatorname{Proj}}$ to exist; we needed the sheaf of algebras to satisfy the conditions stated earlier.)

But first, we should make sure that the preimage of X is indeed an effective Cartier divisor. We can work affine-locally (Exercise 3.2), so I'll assume that $Y = \operatorname{Spec} B$, and X is cut out by the finitely generated ideal I. Then

$$\mathrm{Bl}_X Y = \mathrm{Proj}\left(B \oplus I \oplus I^2 \oplus \cdots\right).$$

(We are slightly abusing notation by using the notation $\mathrm{Bl}_X Y$, as we haven't yet shown that this satisfies the universal property. But I hope that by now you trust me.)

The preimage of X isn't just any effective Cartier divisor; it corresponds to the invertible sheaf $\mathcal{O}(1)$ on this $\underline{\mathrm{Proj}}$. Indeed, $\mathcal{O}(1)$ corresponds to taking our graded ring, chopping off the bottom piece, and sliding all the graded pieces to the left by 1; it is the invertible sheaf corresponding to the graded module

$$I \oplus I^2 \oplus I^3 \oplus \cdots$$

(where that first summand I has grading 0). But this can be interpreted as the scheme-theoretic pullback of X, which corresponds to the ideal I of B:

$$I\left(B \oplus I \oplus I^2 \oplus \cdots\right) \hookrightarrow B \oplus I \oplus I^2 \oplus \cdots.$$

Thus the scheme-theoretic pullback of $X \hookrightarrow Y$ to $\underline{\mathrm{Proj}}\, \mathcal{O}_Y \oplus \mathcal{I} \oplus \mathcal{I}^2 \oplus \cdots$, the invertible sheaf corresponding to $\mathcal{I} \oplus \mathcal{I}^2 \oplus \mathcal{I}^3 \oplus \cdots$, is an effective Cartier divisor in class $\mathcal{O}(1)$. Once we have verified that this construction is indeed the blow-up, this divisor will be our exceptional divisor $E_X Y$.

Moreover, we see that the exceptional divisor can be described beautifully as a $\underline{\mathrm{Proj}}$ over X:

$$E_X Y = \underline{\mathrm{Proj}}_X\, B/I \oplus I/I^2 \oplus I^2/I^3 \oplus \cdots. \tag{2}$$

We'll later see that in good circumstances (if X is a local complete intersection in something nonsingular, or more generally a local complete intersection in a Cohen-Macaulay scheme) this is a projective bundle (the "projectivized normal bundle").

4.2. Proof of the universal property, Theorem 4.1. Let's prove that this $\underline{\mathrm{Proj}}$ construction satisfies the universal property. Then approach 1 will also follow, as a special case of approach 2. You may ask why I bothered with approach 1. I have two reasons: one is that you may find it more comfortable to work with this one nice ring, and the picture may be geometrically clearer to you (in the same way that thinking about the blow-up closure lemma in the case where $Z \to Y$ is a closed immersion is more intuitive). The second reason is that, as you'll find in the exercises, you'll see some facts more easily in this explicit example, and you can then pull them back to more general examples.

Proof. Reduce to the case of affine target R with ideal I. Reduce to the case of affine source, with principal effective Cartier divisor t. (A principal effective Cartier divisor is cut out by a single non-zero-divisor. Recall that an effective Cartier divisor is cut out only *locally* by a single non-zero divisor.) Thus we have reduced to the case $\mathrm{Spec}\, S \to \mathrm{Spec}\, R$, corresponding to $f : R \to S$. Say $(x_1, \ldots, x_n) = I$, with $(f(x_1), \ldots, f(x_n)) = (t)$. We'll describe *one* map $\mathrm{Spec}\, S \to \mathrm{Proj}\, R[I]$ that will extend the map on the open set $\mathrm{Spec}\, S_t \to \mathrm{Spec}\, R$. It is then unique: a map to a separated R-scheme is determined by its behavior away from the associated points (proved earlier). We map R[I] to S as follows: the degree one part is $f : R \to S$, and $f(X_i)$ (where X_i corresponds to x_i, except it is in degree 1) goes to $f(x_i)/t$. Hence an element X of degree d goes to $X/(t^d)$. On the open set $D_+(X_1)$, we get the map $R[X_2/X_1, \dots, X_n/X_1]/(x_2 - X_2/X_1 x_1, \dots, x_i X_j - x_j X_i, \dots) \to S$ (where there may be many relations) which agrees with f away from $D(t)$. Thus this map does extend away from $V(I)$. $\square$

Here are some applications and observations arising from this construction of the blow-up.

4.3. *Observation.* We can verify that our initial motivational examples are indeed blow-ups. For example, blowing up $\mathbb{A}^2$ (with co-ordinates x and y) at the origin yields: $B = k[x, y]$, $I = (x, y)$, and $\operatorname{Proj} B \oplus I \oplus I^2 = \operatorname{Proj} B[X, Y]$ where the elements of B have degree 0, and X and Y are degree 1 and correspond to x and y.

4.4. *Observation.* Note that the normal bundle to a Cartier divisor D is the invertible sheaf $\mathcal{O}(D)|_D$, the invertible sheaf corresponding to the D on the total space, then restricted to D. (This was discussed earlier in the section on differentials.) (Reason: if D corresponds to the ideal sheaf $\mathcal{I}$, then recall that $\mathcal{I} = \mathcal{O}(D)^\vee$, and that the conormal sheaf was $\mathcal{I}/\mathcal{I}^2 = \mathcal{I}|_D$.) The ideal sheaf corresponding to the exceptional divisor is $\mathcal{O}(1)$, so the invertible sheaf corresponding to the exceptional divisor is $\mathcal{O}(-1)$. (I prefer to think of this in light of approach 1, but there is no real difference.) Thus for example in the case of the blow-up of a point in the plane, the exceptional divisor has normal bundle $\mathcal{O}(-1)$. In the case of the blow-up of a nonsingular subvariety of a nonsingular variety, the blow up turns out to be nonsingular (a fact discussed soon in §6.1), and the exceptional divisor is a projective bundle over X, and the normal bundle to the exceptional divisor restricts to $\mathcal{O}(-1)$.

4.5. More serious application: dimensional vanishing for quasicoherent sheaves on quasiprojective schemes. Here is something promised long ago. I want to point out something interesting here: in proof I give below, we will need to potentially blow up arbitrary closed schemes. We won't need to understand precisely what happens when we do so; all we need is the fact that the exceptional divisor is indeed a (Cartier) divisor.

5. Explicit computations

In this section you will do a number of explicit of examples, to get a sense of how blow-ups behave, how they are useful, and how one can work with them explicitly. For convenience, all of the following are over an algebraically closed field k of characteristic 0.

5.1. *Example: Blowing up the plane along the origin.* Let's first blow up the plane $\mathbb{A}^2_k$ along the origin, and see that the result agrees with our discussion in §2. Let x and y be the coordinates on $\mathbb{A}^2_k$. The the blow-up is $\operatorname{Proj} k[x, y, X, Y]$ where $xY - yX = 0$. This is naturally a closed subscheme of $\mathbb{A}^2_k \times \mathbb{P}^1_k$, cut out (in terms of the projective coordinates X and Y on $\mathbb{P}^1_k$) by $xY - yX = 0$. We consider the two usual patches on $\mathbb{P}^1_k$: $[X; Y] = [s; 1]$ and $[1; t]$. The first patch yields $\operatorname{Spec} k[x, y, s]/(sy - x)$, and the second gives $\operatorname{Spec} k[x, y, t]/(y -$

xt). Notice that both are nonsingular: the first is naturally $\operatorname{Spec} k[y, s] \cong \mathbb{A}_k^2$, the second is $\operatorname{Spec} k[x, t] \cong \mathbb{A}_k^2$.

Let's describe the exceptional divisor. We first consider the first (s) patch. The ideal is generated by (x, y), which in our ys-coordinates is $(ys, y) = (y)$, which is indeed principal. Thus on this patch the exceptional divisor is generated by y. Similarly, in the second patch, the exceptional divisor is cut out by x. (This can be a little confusing, but there is no contradiction!)

5.2. *The proper transform of a nodal curve.* Consider next the curve $y^2 = x^3 + x^2$ inside the plane $\mathbb{A}_k^2$. Let's blow up the origin, and compute the total and proper transform of the curve. (By the blow-up closure lemma, the latter is the blow-up of the nodal curve at the origin.) In the first patch, we get $y^2 - s^2y^2 - s^3y^3 = 0$. This factors: we get the exceptional divisor y with multiplicity two, and the curve $1 - s^2 - y^3 = 0$. Easy exercise: check that the proper transform is nonsingular. Also, notice where the proper transform meets the exceptional divisor: at two points, $s = \pm 1$. This corresponds to the two tangent directions at the origin. (Notice that $s = y/x$.)

5.3. *Exercise.* Describe both the total and proper transform of the curve C given by $y = x^2 - x$ in $\operatorname{Bl}_{(0,0)} \mathbb{A}^2$. Verify that the proper transform of C is isomorphic to C. Interpret the intersection of the proper transform of C with the exceptional divisor E as the slope of C at the origin.

5.4. *Exercise: blowing up a cuspidal plane curve.* Describe the proper transform of the cuspidal curve C' given by $y^2 = x^3$ in the plane $\mathbb{A}_k^2$. Show that it is nonsingular. Show that the proper transform of C meets the exceptional divisor E at one point, and is tangent to E there.

5.5. *Exercise.* (a) Desingularize the tacnode $y^2 = x^4$ by blowing up the plane at the origin (and taking the proper transform), and then blowing up the resulting surface once more.
(b) Desingularize $y^8 - x^5 = 0$ in the same way. How many blow-ups do you need?
(c) Do (a) instead in one step by blowing up (y, x^2).

5.6. *Exercise.* Blowing up a nonreduced subscheme of a nonsingular scheme can give you something singular, as shown in this example. Describe the blow up of the ideal (x, y^2) in $\mathbb{A}_k^2$. What singularity do you get? (Hint: it appears in a nearby exercise.)

5.7. *Exercise.* Blow up the cone point $z^2 = x^2 + y^2$ at the origin. Show that the resulting surface is nonsingular. Show that the exceptional divisor is isomorphic to $\mathbb{P}^1$.

5.8. *Harder but enlightening exercise.* If $X \hookrightarrow \mathbb{P}^n$ is a projective scheme, show that the exceptional divisor of the blow up the affine cone over X at the origin is isomorphic to X, and that its normal bundle is $\mathcal{O}_X(-1)$. (I prefer approach 1 here, but both work.)

In the case $X = \mathbb{P}^1$, we recover the blow-up of the plane at a point. In particular, we again recover the important fact that the normal bundle to the exceptional divisor is $\mathcal{O}(-1)$.

5.9. *Exercise.* Show that the multiplicity of the exceptional divisor in the total transform of a subscheme of $\mathbb{A}^n$ when you blow up the origin is the lowest degree that appears in a defining equation of the subscheme. (For example, in the case of the nodal and cuspidal curves above, Example 5.2 and Exercise 5.4 respectively, the exceptional divisor appears with multiplicity 2.) This is called the *multiplicity* of the singularity.

5.10. *Exercise.* Suppose Y is the cone $x^2 + y^2 = z^2$, and X is the ruling of the cone $x = 0$, $y = z$. Show that $\mathrm{Bl}_X Y$ is nonsingular. (In this case we are blowing up a codimension 1 locus that is not a Cartier divisor. Note that it *is* Cartier away from the cone point, so you should expect your answer to be an isomorphism away from the cone point.)

5.11. *Harder but useful exercise (blow-ups resolve base loci of rational maps to projective space).* (I find this easier via method 1.) Suppose we have a scheme Y, an invertible sheaf $\mathcal{L}$, and a number of sections $s_0, \dots, s_n$ of $\mathcal{L}$. Then away from the closed subscheme X cut out by $s_0 = \cdots = s_n = 0$, these sections give a morphism to $\mathbb{P}^n$. Show that this morphism extends to a morphism $\mathrm{Bl}_X Y \to \mathbb{P}^n$, where this morphism corresponds to the invertible sheaf $(\pi^*\mathcal{L})(-E_X Y)$, where $\pi : \mathrm{Bl}_X Y \to Y$ is the blow-up morphism. In other words, "blowing up the base scheme resolves this rational map". (Hint: it suffices to consider an affine open subset of Y where $\mathcal{L}$ is trivial.)

6. Two stray facts

There are two stray facts I want to mention.

6.1. Blowing up a nonsingular in a nonsingular. The first is that if you blow up a nonsingular subscheme of a nonsingular locally Noetherian scheme, the result is nonsingular. I didn't have the time to prove this, but I discussed some of the mathematics behind it. (This is harder than our previous discussion. Also, it uses a flavor of argument that in general I haven't gotten to, about local complete intersections and Cohen-Macaulayness.) Moreover, for a local complete intersection $X \hookrightarrow Y$ cut out by ideal sheaf $\mathcal{I}$, I/I^2 is locally free (class 39/40, Theorem 2.20, p. 10). Then it is a fact (unproved here) that for a local complete intersection, the natural map $\mathrm{Sym}^n \mathcal{I}/\mathcal{I}^2 \to \mathcal{I}^n/\mathcal{I}^{n+1}$ is an isomorphism. Of course it suffices to prove this for affine open sets. More generally, if A is Cohen-Macaulay (recall that I've stated that nonsingular schemes are Cohen-Macaulay), and $x_1, \dots, x_r \in \mathfrak{m}$ is a regular sequence, with $I = (x_1, \dots, x_r)$, then the natural map is an isomorphism. You can read about this at p. 110 of Matsumura's Commutative Algebra.

Assuming this fact, we conclude that if $X \hookrightarrow Y$ is a complete intersection in a nonsingular scheme (or more generally cut out by a regular sequence in a Cohen-Macaulay scheme), the exceptional divisor is the projectivized normal bundle (by (2)). (Exercise:

Blow up (xy, z) in $\mathbb{A}^3$, and verify that the exceptional divisor is indeed the projectivized normal bundle.)

In particular, in the case where we blow up a nonsingular subvariety in a nonsingular variety, the exceptional divisor is nonsingular. We can then show that the blow-up is nonsingular as follows. The blow-up $\mathrm{Bl}_X Y$ remains nonsingular away from $E_X Y$, as it is here isomorphic to the nonsingular space $Y - X$. Thus we need check only the exceptional divisor. Fix any point of the exceptional divisor p. Then the dimension of $E_X Y$ at p is precisely the dimension of the Zariski tangent space (by nonsingularity). Moreover, the dimension of $\mathrm{Bl}_X Y$ at p is one more than that of $E_X Y$ (by Krull's Principal Ideal Theorem), as the latter is an effective Cartier divisor), and the dimension of the Zariski tangent space of $\mathrm{Bl}_X Y$ at p is at most one more than that of $E_X Y$. But the first of these is at most as big as the second, so we must have equality, which means that $\mathrm{Bl}_X Y$ is nonsingular at p.

6.2. *Exercise.* Suppose X is an irreducible nonsingular subvariety of a nonsingular variety Y, of codimension at least 2. Describe a natural isomorphism $\mathrm{Pic}\,\mathrm{Bl}_X Y \cong \mathrm{Pic}\, Y \oplus \mathbb{Z}$. (Hint: compare divisors on $\mathrm{Bl}_X Y$ and Y. Show that the exceptional divisor $E_X Y$ gives a non-torsion element of $\mathrm{Pic}(\mathrm{Bl}_X Y)$ by describing a $\mathbb{P}^1$ on $\mathrm{Bl}_X Y$ which has intersection number -1 with $E_X Y$.)

(If I had more time, I would have used this to give Hironaka's example of a nonprojective proper nonsingular threefold. If you are curious and have ten minutes, please ask me! It includes our nonprojective proper surface as a closed subscheme, and indeed that is how we can show nonprojectivity.)

6.3. Castelnuovo's criterion.

A curve in a nonsingular surface that is isomorphic to $\mathbb{P}^1$ with normal bundle $\mathcal{O}(-1)$ is called a (-1)-curve. We've shown that if we blow up a nonsingular point of a surface at a (reduced) point, the exceptional divisor is a (-1)-curve. Castelnuovo's criterion is the converse: if we have a quasiprojective surface containing a (-1)-curve, that surface is obtained by blowing up another surface at a reduced nonsingular point. (We say that we can "blow down" the (-1)-curve.)

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FOUNDATIONS OF ALGEBRAIC GEOMETRY CLASSES 51 AND 52

RAVI VAKIL

CONTENTS

1. SMOOTH, ÉTALE, UNRAMIFIED

We will next describe analogues of some important notions in differential geometry — the following particular types of maps of manifolds. They naturally form a family of three.

- *Submersions* are maps that induce surjections of tangent spaces everywhere. They are useful in the notion of a fibration.
- *Covering spaces* are maps that induce isomorphisms of tangent spaces, or equivalently, are local isomorphisms.
- *Immersions* are maps that induce injections of tangent spaces.

Warning repeated from earlier: "immersion" is often used in algebraic geometry with a different meaning. We won't use this word in an algebro-geometric context (without an adjective such as "open" or "closed") in order to avoid confusion. I drew pictures of the three. (A fourth notion is related to these three: a map of manifolds is an *embedding* if it is an immersion that is an inclusion of sets, where the source has the subspace topology. This is analogous to *locally closed immersion* in algebraic geometry.)

We will define algebraic analogues of these three notions: smooth, étale, and unramified. In the case of nonsingular varieties over an algebraically closed field, we could take the differential geometric definition. We would like to define these notions more generally. Indeed, one of the points of algebraic geometry is to generalize "smooth" notions to singular situations. Also, we'll want to make arguments by "working over" the generic point, and also over nonreduced subschemes. We may even want to do things over non-algebraically closed fields, or over the integers.

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Our definitions will be combinations of notions we've already seen, and thus we'll see that they have many good properties. We'll see (§2.1) that in the category of nonsingular varieties over algebraically closed fields, we recover the differential geometric definition. Our three definitions won't be so obviously a natural triplet, but I'll mention the definition given in EGA (§4.1), and in this context once again the definitions are very similar.

Let's first consider some examples of things we want to be analogues of "covering space" and "submersion", and see if they help us make good definitions.

We'll start with something we would want to be a covering space. Consider the parabola $x = y^2$ projecting to the x-axis, over the complex numbers. (This example has come up again and again!) We might reasonably want this to be a covering space away from the origin. We might also want the notion of covering space to be an open condition: the locus where a morphism is a covering space should be open on the source. This is true for the differential geometric definition. (More generally, we might want this notion to be preserved by base change.) But then this should be a "covering space" over the generic point, and here we get a non-trivial residue field extension ($\mathbb{C}(y)/\mathbb{C}(y^2)$), not an isomorphism. Thus we are forced to consider (the Spec's of) certain finite extensions of fields to be covering spaces. (We'll see soon that we just want separable extensions.)

Note also in this example there are no (non-empty) Zariski-open subsets $U \subset X$ and $V \subset V$ where the map sends U into V isomorphically. This will later lead to the notion of the étale topology, which is a bizarre sort of topology (not even a topology in the usual sense, but a "Grothendieck topology").

1.1. Here is an issue with smoothness: we would certainly want the fibers to be smooth, so reasonably we would want the fibers to be nonsingular. But we know that nonsingularity over a field does not behave well over a base change (consider $\operatorname{Spec} k(t)[u]/(u^p - t) \to \operatorname{Spec} k(t)$ and base change by $\operatorname{Spec} k(t)[v]/(v^p - t) \to \operatorname{Spec} k(t)$, where $\operatorname{char} k = p$). We can patch that by noting that nonsingularity behaves well over algebraically closed fields, and hence we could require that all the geometric fibers are nonsingular. But that isn't quite enough. For example, a horrible map from a scheme X to a curve Y that maps a different nonsingular variety to a each point Y (X is an infinite disjoint union of these) should not be considered a submersion in any reasonable sense. Also, we might reasonably not want to consider $\operatorname{Spec} k \to \operatorname{Spec} k[\epsilon]/\epsilon^2$ to be a submersion (for example, this isn't surjective on tangent spaces, and more generally the picture "doesn't look like a fibration"). (I drew pictures of these two pathologies.) Both problems are failures of $\pi : X \to Y$ to be a nice, "continuous" family. Whenever we are looking for some vague notion of "niceness" we know that "flatness" will be in the definition. (This is the reason we waited so long before introducing the notion of smoothness — we needed to develop flatness first!)

One last issue: we will require the geometric fibers to be varieties, so we can think of them as "smooth" in the old-fashioned intuitive sense. We could impose this by requiring our morphisms to be locally of finite type, or (a stronger condition) locally of finite presentation.

I should have defined "locally of finite presentation" back when we defined "locally of finite type" and the many other notions satisfying the affine covering lemma. It isn't any harder. A morphism of affine schemes $\operatorname{Spec} A \to \operatorname{Spec} B$ is *locally of finite presentation* if it corresponds to $B \to B[x_1, \dots, x_n]/(f_1, \dots, f_r)$ — A should be finitely generated over B, and also have a finite number of relations. This notion satisfies the hypotheses of the affine covering lemma. A morphism of schemes $\pi : X \to Y$ is *locally of finite presentation* if every map of affine open sets $\operatorname{Spec} A \to \operatorname{Spec} B$ induced by π is locally of finite presentation. If you work only with locally Noetherian schemes, then these two notions are the same.

I haven't thought through why Grothendieck went with the stricter condition of "locally of finite presentation" in his definition of smooth etc., rather than "locally of finite type".

Finally, we define our three notions!

1.2. Definition. A morphism $\pi : X \to Y$ is *smooth of relative dimension* n provided that it is locally of finite presentation and flat of relative dimension n, and $\Omega_{X/Y}$ is locally free of rank n.

A morphism $\pi : X \to Y$ is *étale* provided that it is locally of finite presentation and flat, and $\Omega_{X/Y} = 0$.

A morphism $\pi : X \to Y$ is *unramified* provided that it is locally of finite presentation, and $\Omega_{X/Y} = 0$.

1.3. Examples.

- $\mathbb{A}^n_Y \to Y$, $\mathbb{P}^n_Y \to Y$ are smooth morphisms of relative dimension n.
- Locally finitely presented open immersions are étale.
- *Unramified.* Locally finitely presented locally closed immersions are unramified.

1.4. Quick observations and comments.

1.5. All three notions are local on the target, and local on the source, and are preserved by base change. That's because all of the terms arising in the definition have these properties. *Exercise.* Show that all three notions are open conditions. State this rigorously and prove it. (Hint: Given $\pi : X \to Y$, then there is a largest open subset of X where π is smooth of relative dimension n, etc.)

1.6. Note that π is étale if and only if π is smooth and unramified, if and only if π is flat and unramified.

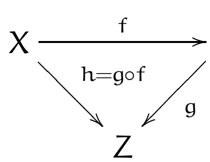
1.7. *Jacobian criterion.* The smooth and étale definitions are perfectly set up to use a Jacobian criterion. *Exercise.* Show that $\operatorname{Spec} B[x_1, \dots, x_n]/(f_1, \dots, f_r) \to \operatorname{Spec} B$ is smooth of relative dimension n (resp. étale) if it is flat of relative dimension n (resp. flat) and the corank of Jacobian matrix is n (resp. the Jacobian matrix is full rank).

1.8. *Exercise: smoothness etc. over an algebraically closed field.* Show that if k is an algebraically closed field, $X \to \operatorname{Spec} k$ is smooth of relative dimension n if and only if X is a disjoint union of nonsingular k-varieties of dimension n. (Hint: use the Jacobian criterion.) Show that $X \to \operatorname{Spec} k$ is étale if and only if it is unramified if and only if X is a union of points isomorphic to $\operatorname{Spec} k$. More generally, if k is a field (not necessarily algebraically closed), show that $X \to \operatorname{Spec} k$ is étale if and only if it is unramified if and only if X is the disjoint union of Spec's of finite separable extensions of k.

1.9. A morphism $\pi : X \to Y$ is *smooth* if it is locally of finite presentation and flat, and in an open neighborhood of every point $x \in X$ in which π is of constant relative dimension, $\Omega_{X/Y}$ is locally free of that relative dimension. (I should have shown earlier that the locus where a locally of finite presentation morphism is flat of a given relative dimension is open, but I may not have. We indeed showed the fact without the "relative dimension" statement, and the argument is essentially the same with this condition added.) (*Exercise.* Show that π is smooth if X can be written as a disjoint union $X = \coprod_{n \geq 0} X_n$ where $\pi|_{X_n}$ is smooth of relative dimension n.) This notion isn't really as "clean" as "smooth of relative dimension n", but people often use the naked adjective "smooth" for simplicity.

Exercise. Show that étale is the same as smooth of relative dimension 0. In other words, show that étale implies relative dimension 0. (Hint: if there is a point $x \in X$ where π has positive relative dimension, show that $\Omega_{X/Y}$ is not 0 at x. You may want to base change, to consider just the fiber above $\pi(x)$.)

1.10. Note that unramified doesn't have a flatness hypothesis, and indeed we didn't expect it, as we would want the inclusion of the origin into $\mathbb{A}^1$ to be unramified. Thus seemingly pathological things of the sort we excluded from the notion of "smooth" and "unramified" morphisms are unramified. For example, if $X = \coprod_{z \in \mathbb{C}} \operatorname{Spec} \mathbb{C}$, then the morphism $X \to \mathbb{A}^1_{\mathbb{C}}$ sending the point corresponding to z to the point $z \in \mathbb{A}^1_{\mathbb{C}}$ is unramified. Such is life.

Exercise. Suppose 

are locally finitely presented morphisms.

(a) Show that if h is unramified, then so is f. (Hint: property P exercise.)
(b) Suppose g is étale. Show that f is smooth (resp. étale, unramified) if and only if h is. (Hint: Observe that $\Omega_{X/Y} \to \Omega_{X/Y}$ is an isomorphism from the relative cotangent sequence, see 2.3 for a reminder.)

Regularity vs. smoothness. Suppose $\operatorname{char} k = p$, and consider the morphism $\operatorname{Spec} k(u) \to \operatorname{Spec} k(u^p)$. Then the source is nonsingular, but the morphism is not étale (or smooth, or unramified).

In fact, if k is not algebraically closed, "nonsingular" isn't a great notion, as we saw in the fall when we had to work hard to develop the theory of nonsingularity. Instead, "smooth (of some dimension)" over a field is much better. You should almost go back in your notes and throw out our discussion of nonsingularity. But don't — there were a couple of key concepts that have been useful: discrete valuation rings (nonsingularity in codimension 1) and nonsingularity at closed points of a variety (nonsingularity in top codimension).

2. Harder facts

I want to segregate three facts which require more effort, to emphasize that the earlier facts are automatic given what we know.

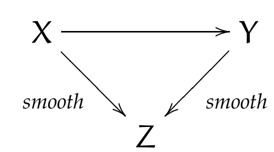
2.1. Connection to differential-geometric notion of smoothness.

The following exercise makes the connection to the differential-geometric notion of smoothness. Unfortunately, we will need this fact in the next section on generic smoothness.

2.2. *Trickier Exercise.* Suppose $\pi : X \to Y$ is a morphism of smooth (pure-dimensional) varieties over a field k. Let $n = \dim X - \dim Y$. Suppose that for each closed point $x \in X$, the induced map on the Zariski tangent space $T_f : T_x \to T_y$ is surjective. Show that f is smooth of relative dimension n. (Hint: The trickiest thing is to show flatness. Use the (second) local criterion for flatness.)

I think this is the easiest of the three "harder" facts, and it isn't so bad.

For pedants: I think the same argument works over a more arbitrary base. In other words, suppose in the following diagram of pure-dimensional Noetherian schemes, Y is reduced.



Let $n = \dim X - \dim Y$. Suppose that for each closed point $x \in X$, the induced map on the Zariski tangent space $T_f : T_x \to T_y$ is surjective. Show that f is smooth of relative dimension n. I think the same argument works, with a twist at the end using Exercise 1.10(b). Please correct me if I'm wrong!

2.3. The relative cotangent sequence is left-exact in good circumstances.

Recall the relative cotangent sequence. Suppose $X \xrightarrow{f} Y \xrightarrow{g} Z$ be morphisms of schemes. Then there is an exact sequence of quasicoherent sheaves on X

$$f^*\Omega_{Y/Z} \to \Omega_{X/Z} \to \Omega_{X/Y} \to 0.$$

We have been always keeping in mind that if you see a right-exact sequence, you should expect that this is the tail end of a long exact sequence. In this case, you should expect that the next term to the left (the "H_1 term") should depend just on X/Y, and not on Z, because the last term on the right does. Indeed this is the case: these "homology" groups are called André-Quillen homology groups. You might also hope then that in some mysteriously "good" circumstances, this first "H_1" on the left should vanish, and hence the relative cotangent sequence should be exact on the left. Indeed that is the case, as is hinted by the following exercise.

2.4. *Exercise on differentials.* If $X \to Y$ is a smooth morphism, show that the relative cotangent sequence is exact on the left as well.

This exercise is the reason this discussion is in the "harder" section — the rest is easy. Can someone provide a clean proof of this fact?

2.5. *Unimportant exercise.* Predict a circumstance in which the relative conormal sequence is left-exact.

2.6. *Corollary.* Suppose f is étale. Then the pullback of differentials $f^*\Omega_{Y/Z} \to \Omega_{X/Z}$ is an isomorphism. (This should be very believable to you from the picture you should have in your head!)

2.7. *Exercise.* Show that all three notions are preserved by composition. (More precisely, in the smooth case, smooth of relative dimension m composed with smooth of relative dimension n is smooth of relative dimension $n + m$.) You'll need Exercise 2.4 in the smooth case.

2.8. *Easy exercise.* Show that all three notions are closed under products. (More precisely, in the case of smoothness: If $X, Y \to Z$ are smooth of relative dimension m and n respectively, then $X \times_Z Y \to Z$ is smooth of relative dimension $m + n$.) (Hint: This is a consequence of base change and composition, as we have discussed earlier. Consider $X \times_Z Y \to Y \to Z$.)

2.9. *Exercise: smoothness implies surjection of tangent sheaves.* Continuing the terminology of the above, Suppose $X \to Y$ is a smooth morphism of Z-schemes. Show that $0 \to T_{X/Y} \to T_{X/Z} \to f^*T_{Y/Z} \to 0$ is an exact sequence of sheaves, and in particular, $T_{X/Z} \to f^*T_{Y/Z}$ is surjective, paralleling the notion of submersion in differential geometry. (Recall $T_{X/Y} = \underline{\mathrm{Hom}}(\Omega_{X/Y}, \mathcal{O}_X)$ and similarly for $T_{X/Z}$, $T_{Y/Z}$.)

2.10. Characterization of smooth and etale in terms of fibers.

By Exercise 1.8, we know what the fibers look like for étale and unramified morphisms; and what the geometric fibers look like for smooth morphisms. There is a good characterization of these notions in terms of the geometric fibers, and this is a convenient way of thinking about the three definitions.

2.11. *Exercise: characterization of etale and unramified morphisms in terms of fibers.* Suppose $\pi : X \to Y$ is a morphism locally of finite presentation. Prove that π is étale if and only if it is flat, and the geometric fibers (above $\operatorname{Spec} \overline{k} \to Y$, say) are unions of Spec's of fields (with discrete topology), each a finite separable extension of the field $\overline{k}$. Prove that π is unramified if and only if the geometric fibers (above $\operatorname{Spec} \overline{k} \to Y$, say) are unions of Spec's of fields (with discrete topology), each a finite separable extension of the field $\overline{k}$. (Hint: a finite type sheaf that is 0 at all points must be the 0-sheaf.)

There is an analogous statement for smooth morphisms, that is harder. (That's why this discussion is in the "harder" section.)

2.12. *Harder exercise.* Suppose $\pi : X \to Y$ is locally of finite presentation. Show that π is smooth of relative dimension n if and only if π is flat, and the geometric fibers are disjoint unions of n-dimensional nonsingular varieties (over the appropriate field).

3. Generic smoothness in characteristic 0

We will next see a number of important results that fall under the rubric of "generic smoothness". All will require working over a field of characteristic 0 in an essential way. So far in this course, we have had to add a few caveats here and there for people encountering positive characteristic. This is probably the first case where positive characteristic people should just skip this section.

Our first result is an algebraic analog of Sard's theorem.

3.1. *Proposition (generic smoothness in the source). — Let k be a field of characteristic 0, and let $\pi : X \to Y$ be a dominant morphism of integral finite-type k-schemes. Then there is a non-empty (=dense) open set $U \subset X$ such that $\pi|_U$ is smooth.*

We've basically seen this argument before, when we showed that a variety has an open subset that is nonsingular.

Proof. Define $n = \dim X - \dim Y$ (the "relative dimension"). Now $FF(X)/FF(Y)$ is a finitely generated field extension of transcendence degree n. It is separably generated by n elements (as we are in characteristic 0). Thus Ω has rank n at the generic point. Its rank is at least n everywhere. By uppersemicontinuity of fiber rank of a coherent sheaf, it is rank n for every point in a dense open set. Recall that on a reduced scheme, constant rank implies locally free of that rank (Class 15, Exercise 5.2); hence Ω is locally free of rank n on that set. Also, by openness of flatness, it is flat on a dense open set. Let U be the intersection of these two open sets. $\square$

For pedants: In class, I retreated to this statement above. However, I think the following holds. Suppose $\pi : X \to Y$ is a dominant finite type morphism of integral schemes, where $\operatorname{char} FF(Y) = 0$ (and hence $\operatorname{char} FF(X) = 0$ from $FF(Y) \hookrightarrow FF(X)$). Then there is a non-empty open set $U \subset X$ such that $\pi|_U$ is smooth.

The proof above needs the following tweak. Define $n = \dim X - \dim Y$. Let η be the generic point of Y, and let X_η be fiber of π above η; it is non-empty by the dominant hypothesis. Then X_η is a finite type scheme over $FF(Y)$. I claim $\dim X_\eta = n$. Indeed, π is flat near X_η (everything is flat over a field, and flatness is an open condition), and we've shown for a flat morphism the dimension of the fiber is the dimension of the source minus the dimension of the target. Then proceed as above.

Please let me know if I've made a mistake!

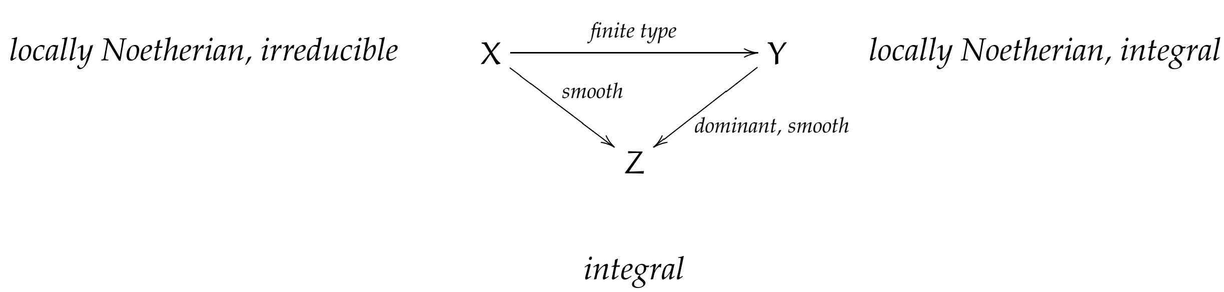
3.2. In §1.1, we saw an example where this result fails in positive characteristic, involving an inseparable extension of fields. Here is another example, over an *algebraically closed* field of characteristic p: $\mathbb{A}^1_k = \operatorname{Spec} k[t] \to \operatorname{Spec} k[u] = \mathbb{A}^1_k$, given by $u \mapsto t^p$. The earlier example (§1.1) is what is going on at the generic point.

If the source of π is smooth over a field, the situation is even nicer.

3.3. *Theorem (generic smoothness in the target).* — *Suppose $f : X \to Y$ is a morphism of k-varieties, where $\operatorname{char} k = 0$, and X is smooth over k. Then there is a dense open subset of Y such that $f|_{f^{-1}(U)}$ is a smooth morphism.*

(Note: $f^{-1}(U)$ may be empty! Indeed, if f is not dominant, we will have to take such a U.)

For pedants: I think the following generalization holds, assuming that my earlier notes to pedants aren't bogus. Generalize the above hypotheses to the following morphisms of $\mathbb{Q}$-schemes. (Requiring a scheme to be defined over $\mathbb{Q}$ is precisely the same as requiring it to "live in characteristic 0", i.e. the morphism to $\operatorname{Spec} \mathbb{Z}$ has image precisely $[(0)]$.)



To prove this, we'll use a neat trick.

3.4. *Lemma.* — *Suppose $\pi : X \to Y$ is a morphism of schemes that are finite type over k, where $\operatorname{char} k = 0$. Define*

$$X_r = \{\textit{closed points } x \in X : \textit{rank } T_{\pi,x} \le r\}.$$

Then $\dim f(X_r) \leq r$. *(Note that* X_r *is a closed subset; it is cut out by determinantal equations. Hence by Chevalley's theorem, its image is constructible, and we can take its dimension.)*

For pedants: I think the only hypotheses we need are that π *is a finite type morphism of locally Noetherian schemes over* $\mathbb{Q}$*. The proof seems to work as is, after an initial reduction to verifying it on an arbitrary affine open subset of* Y*.*

Here is an example of the lemma, to help you find it believable. Suppose X is a nonsingular surface, and Y is a nonsingular curve. Then for each $x \in x$, the tangent map $T_{\pi,x} : T_x \to T_{\pi(x)}$ is a map from a two-dimensional vector space to a one-dimensional vector space, and thus has rank 1 or 0. I then drew some pictures. If π is dominant, then we have a picture like this [omitted]. The tangent map has rank 0 at this one point. The image is indeed rank 0. The tangent map has rank at most 1 everywhere. The image indeed has rank 1.

Now imagine that π contracted X to a point. Then the tangent map has rank 0 everywhere, and indeed the image has dimension 0.

Proof of lemma. We can replace by X by an irreducible component of X_r, and Y by the closure of that component's image of X in Y. (The resulting map will have all of X contained in X_r. This boils down to the following linear algebra observation: if a linear map $\rho : V_1 \to V_2$ has rank at most r, and V_i' is a subspace of V_i, with ρ sending V_1' to V_2', then the restriction of ρ to V_1' has rank at most that of ρ itself.) Thus we have a dominant morphism $f : X \to Y$, and we wish to show that $\dim Y \leq r$. By generic smoothness on the source (Proposition 3.1), there is a nonempty open subset $U \subset X$ such that $f : U \to Y$ is smooth. But then for any $x \in X$, the tangent map $T_{x,X} \to T_{\pi(x),Y}$ is surjective (by smoothness), and has rank at most r, so $\dim Y = \dim_{\pi(x)} Y \leq \dim T_{\pi(x),Y} \leq r$. □

There's not much left to prove the theorem.

Proof of Theorem 3.3. Reduce to the case Y smooth over k (by restricting to a smaller open set, using generic smoothness of Y, Proposition 3.1). Say $n = \dim Y$. $\dim \overline{f(X_{n-1})} \leq n-1$ by the lemma, so remove this as well. Then the rank of T_f is at least r for each closed point of X. But as Y is nonsingular of dimension r, we have that T_f is surjective for every closed point of X, hence surjective. Thus f is smooth by Hard Exercise 2.2. □

3.5. The Kleiman-Bertini theorem. The Kleiman-Bertini theorem is elementary to prove, and extremely useful, for example in enumerative geometry.

Throughout this discussion, we'll work in the category of k-varieties, where k is an algebraically closed field of characteristic 0. The definitions and results generalize easily to the non-algebraically closed case, and I'll discuss this parenthetically.

3.6. Suppose G is a group variety. Then I claim that G is smooth over k. Reason: It is generically smooth (so it has a dense open set U that is smooth), and G acts transitively on itself (so we can cover G with translates of U).

We can generalize this. We say that a G-action $a : G \times X \to X$ on a variety X is *transitive* if it is transitive on closed points. (If k is not algebraically closed, we replace this by saying that it is transitive on $\overline{k}$-valued points. In other words, we base change to the algebraic closure, and ask if the resulting action is transitive. Note that in characteristic 0, reduced = geometrically reduced, so G and X both remain reduced upon base change to $\overline{k}$.)

In other words, if U is a non-empty open subset of X, then we can cover X with translates of X. (Translation: $G \times U \to X$ is surjective.) Such an X (with a transitive G-action) is called a *homogeneous space* for G.

3.7. *Exercise.* Paralleling §3.6, show that a homogeneous space X is smooth over k.

3.8. *The Kleiman-Bertini theorem. — Suppose* X *is homogeneous space for group variety* G *(over an algebraically closed field* k *of characteristic* 0*). Suppose* $f : Y \to X$ *and* $g : Z \to X$ *be morphisms from smooth* k*-varieties* Y, Z*. Then there is a nonempty open subset* $V \subset G$ *such that for every* $\sigma \in V(k)$, $Y \times_X Z$ *defined by*

$$\begin{array}{ccc} Y \times_X Z & \longrightarrow & Z \\ \downarrow & & \downarrow g \\ Y & \xrightarrow{\sigma \circ f} & X \end{array}$$

(i.e. Y *is "translated by* σ*") is smooth over* k *of dimension exactly* $\dim Y + \dim Z - \dim X$*. Better: there is an open subset of* $V \subset G$ *such that*

(1) $$(G \times_k Y) \times_X Z \to G$$

is a smooth morphism of relative dimension $\dim Y + \dim Z - \dim X$.

(The statement and proof will carry through even if k is not algebraically closed.)

The first time you hear this, you should think of the special case where $Y \to X$ and $Z \to X$ are closed immersions (hence "smooth subvarieties"). In this case, the Kleiman-Bertini theorem says that the second subvariety will meet a "general translate" of the first transversely.

Proof. It is more pleasant to describe this proof "backwards", by considering how we would prove it ourselves. We will end up using generic smoothness twice, as well as many facts we now know and love.

In order to show that the morphism (1) is generically smooth on the target, it would suffice to apply Theorem 3.3), so we wish to show that $(G \times_k Y) \times_X Z$ is a smooth k-variety. Now Z is smooth over k, so it suffices to show that $(G \times_k Y) \times_X Z \to Z$ is a smooth morphism (as the composition of two smooth morphisms is smooth). But this is obtained by base changed from $G \times_k Y \to X$, so it suffices to show that this latter morphism is smooth (as smoothness is preserved by base change).

This is a G-equivariant morphism $G \times_k Y \xrightarrow{a \circ f} X$. (By "G-equivariant", we mean that G action on both sides respects the morphism.) By generic smoothness of the target (Theorem 3.3), this is smooth over a dense open subset X. But then by transitivity of the G action, this morphism is smooth (everywhere). (*Exercise:* verify the relative dimension statement.) □

3.9. *Corollary (Bertini's theorem, improved version). Suppose X is a smooth k-variety, where k is algebraically closed of characteristic 0. Let δ be a finite-dimensional base-point-free linear system, i.e. a finite vector space of sections of some invertible sheaf $\mathcal{L}$. Then almost every element of δ, considered as a closed subscheme of X, is nonsingular. (More explicitly: each element $s \in H^0(X, \mathcal{L})$ gives a closed subscheme of X. For a general s, considered as a point of $\mathbb{P}H^0(X, \mathcal{L})$, the closed subscheme is smooth over k.)*

(Again, the statement and proof will carry through even if k is not algebraically closed.)

This is a good improvement on Bertini's theorem. For example, we don't actually need $\mathcal{L}$ to be very ample, or X to be projective.

3.10. *Exercise.* Prove this!

3.11. *Easy Exercise.* Interpret the old version of Bertini's theorem (over a characteristic 0 field) as a corollary of this statement.

Note that this fails in positive characteristic, as shown by the one-dimensional linear system $\{pP : P \in \mathbb{P}^1\}$. This is essentially Example 3.2.

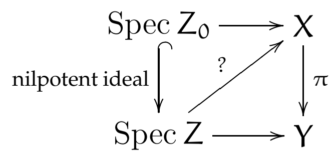
4. Formal interpretations

For those of you who like complete local rings, or who want to make the connection to complex analytic geometry, here are some useful reformulations, which I won't prove.

Suppose $(B, \mathfrak{n}) \to (A, \mathfrak{m})$ is a map of Noetherian local rings, inducing an isomorphism of residue fields, and a morphism of completions at the maximal ideals $\hat{B} \to \hat{A}$ (the "hat" terminology arose first in class 13, immediately after the statement of Theorem 2.2). Then the induced map of schemes $\operatorname{Spec} A \to \operatorname{Spec} B$ is:

- *étale* if $\hat{B} \to \hat{A}$ is a bijection.
- *smooth* if $\hat{B} \to \hat{A}$ is isomorphic to $\hat{B} \to \hat{B}[[x_1, \dots, x_n]]$. In other words, formally, smoothness involves adding some free variables. (In case I've forgotten to say this before: "Formally" means "in the completion".)
- *unramified* if $\hat{B} \to \hat{A}$ is surjective.

4.1. Formally unramified, smooth, and étale. EGA has defines these three notions differently. The definitions there make clear that these three definitions form a family, in a way that is quite similar to the differential-geometric definition. (You should largely ignore what follows, unless you find later in life that you really care. I won't prove anything.) We say that $\pi : X \to Y$ is *formally smooth* (resp. *formally unramified*, *formally étale*) if for all affine schemes Z, and every closed subscheme Z_0 defined by a nilpotent ideal, and every morphism $Z \to Y$, the canonical map $\mathrm{Hom}_Y(Z, X) \to \mathrm{Hom}_Y(Z_0, X)$ is surjective (resp. injective, bijective). This is summarized in the following diagram, which is reminiscent of the valuative criteria for separatedness and properness.



(Exercise: show that this is the same as the definition we would get by replacing "nilpotent" by "square-zero". This is sometimes an easier formulation to work with.)

EGA defines smooth as morphisms that are formally smooth and locally of finite presentation (and similarly for the unramified and étale).

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FOUNDATIONS OF ALGEBRAIC GEOMETRY CLASSES 53 AND 54

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CONTENTS

1. SERRE DUALITY

Our last topic is Serre duality. Recall that Serre duality arose in our section on "fun with curves" (classes 33–36). We'll prove the statement used there, and generalize it greatly.

Our goal is to rigorously prove everything we needed for curves, and to generalize the statement significantly. Serre duality can be generalized beyond belief, and we'll content ourselves with the version that is most useful. For the generalization, we will need a few facts that we haven't proved, but that we came close to proving.

(i) The existence (and behavior) of the cup product in (Cech) cohomology. For any quasicoherent sheaves $\mathcal{F}$ and $\mathcal{G}$, there is a natural map $H^i(X, \mathcal{F}) \times H^j(X, \mathcal{G}) \to H^{i+j}(X, \mathcal{F} \otimes \mathcal{G})$ satisfying all the properties you might hope. From the Cech cohomology point of view this isn't hard. For those of you who prefer derived functors, I haven't thought through why it is true. For $i = 0$ or $j = 0$, the meaning of the cup product is easy. (For example, if $i = 0$, the map involves the following. The j-cocycle of $\mathcal{G}$ is the data of sections of $\mathcal{G}$ of $(j+1)$-fold intersections of affine open sets. The cup product corresponds to "multiplying

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each of these by the (restriction of the) global section of $\mathcal{F}''$.) This version is all we'll need for nonsingular projective curves (as if $i, j > 0$, $i + j > 1$).

(ii) The Cohen-Macaulay/flatness theorem. I never properly defined Cohen-Macaulay, so I didn't have a chance to prove that nonsingular schemes are Cohen-Macaulay, and if $\pi : X \to Y$ is a morphism from a pure-dimensional Cohen-Macaulay scheme to a pure-dimensional nonsingular scheme, then π is flat if all the fibers are of the expected dimension. (I stated this, however.)

We'll take these two facts for granted.

Here now is the statement of Serre duality.

Suppose X is a Cohen-Macaulay projective k-scheme of pure dimension n. A *dualizing sheaf* for X over k is a coherent sheaf ω_X (or $\omega_{X/k}$) on X along with a *trace map* $H^n(X, \omega_W) \to k$, such that for all finite rank locally free sheaves $\mathcal{F}$ on X,

$$H^i(X, \mathcal{F}) \times H^{n-i}(X, \mathcal{F}^\vee \otimes \omega_X) \longrightarrow H^n(X, \omega_X) \xrightarrow{t} k \tag{1}$$

is a perfect pairing. In terms of the cup product, the first map in (1) is the composition

$$H^i(X, \mathcal{F}) \times H^{n-i}(X, \mathcal{F}^\vee \otimes \omega_X) \to H^n(X, (\mathcal{F} \otimes \mathcal{F}^\vee) \otimes \omega_X) \to H^n(X, \omega_X).$$

1.1. *Theorem (Serre duality). — A dualizing sheaf always exists.*

We will proceed as follows.

- We'll partially extend this to coherent sheaves in general: $\operatorname{Hom}(\mathcal{F}, \omega_X) \to H^n(\mathcal{F})^\vee$ is an isomorphism for all $\mathcal{F}$.
- Using this, we'll show by a Yoneda argument that (ω_X, t) is unique up to unique isomorphism.
- We will then prove the Serre duality theorem 1.1. This will take us some time. We'll first prove that the dualizing sheaf exists for projective space. We'll then prove it for anything admitting a finite flat morphism to projective space. Finally we'll show that every projective Cohen-Macaulay k-scheme admits a finite flat morphism to projective space.
- We'll prove the result in families (i.e. we'll define a "relative dualizing sheaf" in good circumstances). This is useful in the theory of moduli of curves, and Gromov-Witten theory.
- The existence of a dualizing sheaf will be straightforward to show — surprisingly so, at least to me. However, it is also surprisingly slippery — getting a hold of it in concrete circumstances is quite difficult. For example, on the open subset where X is smooth, ω_X is an invertible sheaf. We'll show this. Furthermore, on this locus, $\omega_X = \det \Omega_X$. (Thus in the case of curves, $\omega_X = \Omega_X$. In the "fun with curves" section, we needed the fact that Ω_X is dualizing because we wanted to prove the Riemann-Hurwitz formula.)

1.2. *Warm-up trivial exercise.* Show that if $h^0(X, \mathcal{O}_X) = 1$ (e.g. if X is geometrically integral), then the trace map is an isomorphism, and conversely.

2. EXTENSION TO COHERENT SHEAVES; UNIQUENESS OF THE DUALIZING SHEAF

2.1. *Proposition. — If (ω_X, t) exists, then for any coherent sheaf $\mathcal{F}$ on X, the natural map $\operatorname{Hom}(\mathcal{F}, \omega_X) \times H^n(X, \mathcal{F}) \to H^n(X, \omega_X) \to k$ is a perfect pairing.*

In other words, (1) holds for $i = n$ and any coherent sheaf (not just locally free coherent sheaves). You might reasonably ask if it holds for general i, and it is true that these other cases are very useful, although not as useful as the case we're proving here. In fact the naive generalization does not hold. The correct generalization involves Ext groups, which we have not defined. The precise statement is the following. For any quasicoherent sheaves $\mathcal{F}$ and $\mathcal{G}$, there is a natural map $\operatorname{Ext}^i(\mathcal{F}, \mathcal{G}) \times H^j(X, \mathcal{F}) \to H^{i+j}(\mathcal{G})$. Via this morphism,

$$\operatorname{Ext}^i(\mathcal{F}, \omega_X) \times H^{n-i}(X, \mathcal{F}) \longrightarrow H^n(X, \omega_X) \xrightarrow{t} k$$

is a perfect pairing.

Proof of Proposition 2.1. Given any coherent $\mathcal{F}$, take a partial locally free resolution

$$\mathcal{E}^1 \to \mathcal{E}^0 \to \mathcal{F} \to 0.$$

(Recall that we find a locally free resolution as follows. $\mathcal{E}^0$ is a direct sum of line bundles. We then find $\mathcal{E}^1$ that is also a direct sum of line bundles that surjects onto the kernel of $\mathcal{E}^0 \to \mathcal{F}$.)

Then applying the left-exact functor $\operatorname{Hom}(\cdot, \omega_X)$, we get

$$0 \to \operatorname{Hom}(\mathcal{F}, \omega_X) \to \operatorname{Hom}(\mathcal{E}^0, \omega_X) \to \operatorname{Hom}(\mathcal{E}^1, \omega_X)$$

$$\text{i.e.} \quad 0 \to \operatorname{Hom}(\mathcal{F}, \omega_X) \to (\mathcal{E}^0)^\vee \otimes \omega_X \to (\mathcal{E}^1)^\vee \otimes \omega_X$$

Also

$$H^n(\mathcal{E}^1) \to H^n(\mathcal{E}^0) \to H^n(\mathcal{F}) \to 0$$

from which

$$0 \to H^n(\mathcal{F})^\vee \to H^n(\mathcal{E}^0)^\vee \to H^n(\mathcal{E}^1)^\vee$$

There is a natural map $\operatorname{Hom}(\mathcal{H}, \omega_X) \times H^n(\mathcal{H}) \to H^n(\omega_X) \to k$ for all coherent sheaves, which by assumption (that ω_X is dualizing) is an isomorphism when $\mathcal{H}$ is locally free. Thus we have morphisms (where all squares are commuting)

$$\begin{array}{ccccccc} 0 & \longrightarrow & \operatorname{Hom}(\mathcal{F}, \omega) & \longrightarrow & (\mathcal{E}^0)^\vee(\omega) & \longrightarrow & (\mathcal{E}^1)^\vee(\omega) \\ \downarrow \sim & & \downarrow & & \downarrow \sim & & \downarrow \sim \\ 0 & \longrightarrow & H^n(\mathcal{F})^\vee & \longrightarrow & H^n(\mathcal{E}^0)^\vee & \longrightarrow & H^n(\mathcal{E}^1)^\vee \end{array}$$

where all vertical maps but one are known to be isomorphisms. Hence by the Five Lemma, the remaining map is also an isomorphism. □

We can now use Yoneda's lemma to prove:

2.2. *Proposition.* — If a dualizing sheaf (ω_X, t) exists, it is unique up to unique isomorphism.

Proof. Suppose we have two dualizing sheaves, (ω_X, t) and (ω'_X, t'). From the two morphisms

$$\mathrm{Hom}(\mathcal{F}, \omega_X) \times H^n(X, \mathcal{F}) \longrightarrow H^n(X, \omega_X) \xrightarrow{t} k \tag{2}$$

$$\mathrm{Hom}(\mathcal{F}, \omega'_X) \times H^n(X, \mathcal{F}) \longrightarrow H^n(X, \omega'_X) \xrightarrow{t'} k,$$

we get a natural bijection $\mathrm{Hom}(\mathcal{F}, \omega_X) \cong \mathrm{Hom}(\mathcal{F}, \omega'_X)$, which is functorial in $\mathcal{F}$. By Yoneda's lemma, this induces a (unique) isomorphism $\omega_X \cong \omega'_X$. From (2), under this isomorphism, the two trace maps must be the same too. $\square$

3. Proving Serre duality for projective space over a field

3.1. *Exercise.* Prove (1) for $\mathbb{P}^n$, and $\mathcal{F} = \mathcal{O}(m)$, where $\omega_{\mathbb{P}^n} = \mathcal{O}(-n-1)$. (Hint: do this by hand!) Hence (1) holds for direct sums of $\mathcal{O}(m)$'s.

3.2. *Proposition.* — *Serre duality (Theorem 1.1) holds for projective space.*

Proof. We now prove (1) for any locally free $\mathcal{F}$ on $\mathbb{P}^n$. As usual, take

$$0 \to \mathcal{K} \to \oplus\mathcal{O}(m) \to \mathcal{F} \to 0. \tag{3}$$

Note that $\mathcal{K}$ is flat (as $\mathcal{O}(m)$ and $\mathcal{F}$ are flat and coherent), and hence $\mathcal{K}$ is also locally free of finite rank (flat coherent sheaves on locally Noetherian schemes are locally free — this was one of the important facts about flatness). For convenience, set $\mathcal{G} = \oplus\mathcal{O}(m)$.

Take the long exact sequence in cohomology, and dualize, to obtain

$$0 \to H^n(\mathbb{P}^n, \mathcal{F})^\vee \to H^n(\mathbb{P}^n, \mathcal{G})^\vee \to \cdots \to H^0(\mathbb{P}^n, \mathcal{H})^\vee \to 0. \tag{4}$$

Now instead take (3), tensor with $\omega_{\mathbb{P}^n} \cong \mathcal{O}_{\mathbb{P}^n}(-n-1)$ (which preserves exactness, as $\mathcal{O}_{\mathbb{P}^n}(-n-1)$ is locally free), and take the corresponding long exact sequence

$$0 \longrightarrow H^0(\mathbb{P}^n, \mathcal{F}^\vee \otimes \omega_{\mathbb{P}^n}) \longrightarrow H^0(\mathbb{P}^n, \mathcal{G}^\vee \otimes \omega_{\mathbb{P}^n}) \longrightarrow H^0(\mathbb{P}^n, \mathcal{H}^\vee \otimes \omega_{\mathbb{P}^n})$$

$$\longrightarrow H^1(\mathbb{P}^n, \mathcal{F}^\vee \otimes \omega_{\mathbb{P}^n}) \longrightarrow \cdots$$

Using the trace morphism, this exact sequence maps to the earlier one (4):

$$\begin{array}{ccccccc}
H^i(\mathbb{P}^n, \mathcal{F}^\vee \otimes \omega_{\mathbb{P}^n}) & \longrightarrow & H^i(\mathbb{P}^n, \mathcal{G}^\vee \otimes \omega_{\mathbb{P}^n}) & \longrightarrow & H^i(\mathbb{P}^n, \mathcal{H}^\vee \otimes \omega_{\mathbb{P}^n}) & \longrightarrow & H^{i+1}(\mathbb{P}^n, \mathcal{F}^\vee \otimes \omega_{\mathbb{P}^n}) \\
\downarrow {\scriptstyle \alpha^i_{\mathcal{F}}} & & \downarrow {\scriptstyle \alpha^i_{\mathcal{G}}} & & \downarrow {\scriptstyle \alpha^i_{\mathcal{H}}} & & \downarrow {\scriptstyle \alpha^{i+1}_{\mathcal{F}}} \\
H^{n-i}(\mathbb{P}^n, \mathcal{F})^\vee & \longrightarrow & H^i(\mathbb{P}^n, \mathcal{G})^\vee & \longrightarrow & H^i(\mathbb{P}^n, \mathcal{H})^\vee & \longrightarrow & H^{i+1}(\mathbb{P}^n, \mathcal{F})^\vee
\end{array}$$

(At some point around here, I could simplify matters by pointing out that $H^i(\mathcal{G}) = 0$ for all $i \neq 0, n$, as $\mathcal{G}$ is the direct sum of line bundles, but then I'd still need to deal with the ends, so I'll prefer not to.) All squares here commute. This is fairly straightforward check for those not involving the connecting homomorphism. (*Exercise.* Check this.) It is longer and more tedious (but equally straightforward) to check that

$$\begin{array}{ccc}
H^i(\mathbb{P}^n, \mathcal{H}^\vee \otimes \omega_{\mathbb{P}^n}) & \longrightarrow & H^{i+1}(\mathbb{P}^n, \mathcal{F}^\vee \otimes \omega_{\mathbb{P}^n}) \\
\downarrow {\scriptstyle \alpha^i_{\mathcal{H}}} & & \downarrow {\scriptstyle \alpha^{i+1}_{\mathcal{F}}} \\
H^i(\mathbb{P}^n, \mathcal{H})^\vee & \longrightarrow & H^{i+1}(\mathbb{P}^n, \mathcal{F})^\vee
\end{array}$$

commutes. This requires the definition of the cup product, which we haven't done, so this is one of the arguments I promised to omit.

We then induct our way through the sequence as usual: $\alpha^{-1}_{\mathcal{G}}$ is surjective (vacuously), and $\alpha^{-1}_{\mathcal{H}}$ and $\alpha^0_{\mathcal{G}}$ are injective, hence by the "subtle" Five Lemma (class 32, page 10), $\alpha^0_{\mathcal{F}}$ is injective for all locally free $\mathcal{F}$. In particular, $\alpha^0_{\mathcal{H}}$ is injective (as $\mathcal{H}$ is locally free). But then $\alpha^0_{\mathcal{H}}$ is injective, and $\alpha^{-1}_{\mathcal{H}}$ and $\alpha^0_{\mathcal{G}}$ are surjective, hence $\alpha^0_{\mathcal{F}}$ is surjective, and thus an isomorphism for all locally free $\mathcal{F}$. Thus $\alpha^0_{\mathcal{H}}$ is also an isomorphism, and we continue inductively to show that $\alpha^i_{\mathcal{F}}$ is an isomorphism for all i. □

4. Proving Serre duality for finite flat covers of other spaces for which duality holds

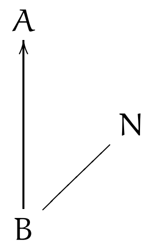
We're now going to make a new construction. It will be relatively elementary to describe, but the intuition is very deep. (Caution: here "cover" doesn't mean covering space as in differential geometry; it just means "surjective map". The word "cover" is often used in this imprecise way in algebraic geometry.)

Suppose $\pi : X \to Y$ is an *affine* morphism, and $\mathcal{G}$ is a quasicoherent sheaf on Y:

$$\begin{array}{ccc}
X & & \\
\downarrow {\scriptstyle \pi} & & \mathcal{G} \\
 & \diagup & \\
Y. & &
\end{array}$$

Observe that $\underline{\mathrm{Hom}}_Y(\pi_* \mathcal{O}_X, \mathcal{G})$ is a sheaf of $\pi_* \mathcal{O}_X$-modules. (The subscript Y is included to remind us where the sheaf lives.) The reason is that affine-locally on Y, over an affine open set $\operatorname{Spec} B$ (on which $\mathcal{G}$ corresponds to B-module N, and with preimage $\operatorname{Spec} A \subset X$)

(5)



this is the statement that $\operatorname{Hom}_B(A, N)$ is naturally an A-module (i.e. the A-module structure behaves well with respect to localization by $b \in B$, and hence these modules glue together to form a quasicoherent sheaf).

In our earlier discussion of affine morphisms, we saw that quasicoherent $\pi_*\mathcal{O}_X$-modules correspond to quasicoherent sheaves on X. Hence $\underline{\operatorname{Hom}}_Y(\pi_*\mathcal{O}_X, \mathcal{G})$ corresponds to some quasicoherent sheaf $\pi'\mathcal{G}$ on X.

Notational warning. This notation π' is my own, and solely for the purposes of this section. If π is finite, then this construction is called $\pi^!$ (pronounced "upper shriek"). You may ask why I'm introducing this extra notation "upper shriek". That's because this notation is standard, while my π' notation is just made up. $\pi^!$ is one of the "six operations" on sheaves defined Grothendieck. It is the most complicated one, and is complicated to define for general π. Those of you attending Young-Hoon Kiem's lectures on the derived category may be a little perplexed, as there he defined $\pi^!$ for elements of the derived category of sheaves, not for sheaves themselves. In the finite case, we can define this notion at the level of sheaves, but we can't in general.

Here are some important observations about this notion.

4.1. By construction, we have an isomorphism of quasicoherent sheaves on Y

$$\pi_*\pi'\mathcal{G} \cong \underline{\operatorname{Hom}}_Y(\pi_*\mathcal{O}_X, \mathcal{G}).$$

4.2. π' is a covariant functor from the category of quasicoherent sheaves on Y to quasicoherent sheaves on X.

4.3. If π is a finite morphism, and Y (and hence X) is locally Noetherian, then π' is a covariant functor from the category of *coherent* sheaves on Y to *coherent* sheaves on X. We show this affine locally, see (5). As A and N are both coherent B-modules, $\operatorname{Hom}_B(A, N)$ is a coherent B-module, hence a finitely generated B-module, and hence a finitely generated A-module, hence a coherent A-module.

4.4. If $\mathcal{F}$ is a quasicoherent sheaf on X, then there is a natural map

$$\pi_* \underline{\operatorname{Hom}}_X(\mathcal{F}, \pi'\mathcal{G}) \to \underline{\operatorname{Hom}}_Y(\pi_*\mathcal{F}, \mathcal{G}). \tag{6}$$

Reason: if M is an A-module, we have a natural map

$$\operatorname{Hom}_A(M, \operatorname{Hom}_B(A, N)) \to \operatorname{Hom}_B(M, N) \tag{7}$$

defined as follows. Given $m \in M$, and an element of $\mathrm{Hom}_A(M, \mathrm{Hom}_B(A, N))$, send m to $\phi_m(1)$. This is clearly a homomorphism of B-modules. Moreover, this morphism behaves well with respect to localization of B with respect to an element of B, and hence this description yields a morphism of quasicoherent sheaves.

4.5. *Lemma. The morphism* (6) *is an isomorphism.*

Is there an obvious reason why the map is an isomorphism? There should be...

Proof. We will show that the natural map (7) is an isomorphism. Fix a presentation of M:

$$A^{\oplus m} \to A^{\oplus n} \to M \to 0$$

(where the direct sums needn't be finite). Applying $\mathrm{Hom}_A(\cdot, \mathrm{Hom}_B(A, N))$ to this sequence yields the top row of the following diagram, and applying $\mathrm{Hom}_B(\cdot, N)$ yields the bottom row, and the vertical morphisms arise from the morphism (7).

$$\begin{array}{ccccccc}
0 & \longrightarrow & \mathrm{Hom}_A(M, \mathrm{Hom}_B(A, N)) & \longrightarrow & \mathrm{Hom}_A(A, \mathrm{Hom}_B(A, N))^{\oplus n} & \longrightarrow & \mathrm{Hom}_A(A, \mathrm{Hom}_B(A, N))^{\oplus m} \\
\downarrow\wr & & \downarrow & & \downarrow\wr & & \downarrow\wr \\
0 & \longrightarrow & \mathrm{Hom}_B(M, N) & \longrightarrow & \mathrm{Hom}_B(A, N)^{\oplus n} & \longrightarrow & \mathrm{Hom}_B(A, N)^{\oplus m}
\end{array}$$

(The squares clearly commute.) Be sure to convince yourself that

$$\mathrm{Hom}_B(A, N)^{\oplus n} \cong \mathrm{Hom}_B(A^{\oplus n}, N)$$

even when n isn't finite (and ditto for the three similar terms)! Then all but one of the vertical homomorphisms are isomorphisms, and hence by the Five Lemma the remaining morphism is an isomorphism. ☐

Hence $\pi^!$ is right-adjoint to π_* for affine morphisms and quasicoherent sheaves. (Also, by Observation 4.3, it is right-adjoint for finite morphisms and coherent sheaves on locally Noetherian schemes.) In particular, there is a natural morphism $\pi_*\pi^!\mathcal{G} \to \mathcal{G}$.

4.6. *Proposition. — Suppose $X \to Y$ is a finite flat morphism of projective k-schemes of pure dimension n, and (ω_Y, t_Y) is a dualizing sheaf for Y. Then $\pi^!\omega_Y$ along with the trace morphism*

$$t_X : H^n(X, \pi^!\omega_Y) \xrightarrow{\sim} H^n(Y, \pi_*\pi^!\omega_Y) \longrightarrow H^n(Y, \omega_Y) \xrightarrow{t_Y} k$$

is a dualizing sheaf for X.

(That first isomorphism arises because $X \to Y$ is affine.)

Proof.

$$\begin{aligned} H^{n-i}(X, \mathcal{F}^\vee(\pi^! \omega_Y)) &\cong H^{n-i}(Y, \pi_*(\mathcal{F}^\vee \otimes \pi^! \omega_Y)) \quad \text{as } \pi \text{ is affine} \\ &\cong H^{n-i}(Y, \pi_*(\underline{\mathrm{Hom}}(\mathcal{F}, \pi^! \omega_Y))) \\ &\cong H^{n-i}(Y, \underline{\mathrm{Hom}}(\pi_* \mathcal{F}, \omega_Y)) \quad \text{by 4.5} \\ &\cong H^{n-i}(Y, (\pi_* \mathcal{F})^\vee(\omega_Y)) \\ &\cong H^i(Y, \pi_* \mathcal{F})^\vee \quad \text{by Serre duality for } Y \\ &\cong H^i(X, \mathcal{F})^\vee \quad \text{as } \pi \text{ is affine} \end{aligned}$$

At the third-last and second-last steps, we are using the fact that $\pi_*\mathcal{F}$ is locally free, and it is here that we are using flatness! ☐

5. All projective Cohen-Macaulay k-schemes of pure dimension n are finite flat covers of $\mathbb{P}^n$

We conclude the proof of the Serre duality theorem 1.1 by establishing the result in the title of this section.

Assume $X \hookrightarrow \mathbb{P}^N$ is projective Cohen-Macaulay of pure dimension n (e.g. smooth).

First assume that k is an infinite field. Then long ago in an exercise that I promised would be important (and has repeatedly been so), we showed that there is a linear space of dimension $N - n - 1$ (one less than complementary dimension) missing X. Project from that linear space, to obtain $\pi : X \to \mathbb{P}^n$. Note that the fibers are finite (the fibers are all closed subschemes of affine space), and hence π is a finite morphism. I've stated the "Cohen-Macaulay/flatness theorem" that a morphism from a equidimensional Cohen-Macaulay scheme to a smooth k-scheme is flat if and only if the fibers are of the expected dimension. Hence π is flat.

5.1. *Exercise.* Prove the result in general, if k is not necessarily infinite. Hint: show that there is some d such that there is an intersection of $N - n - 1$ degree d hypersurfaces missing X. Then try the above argument with the dth Veronese of $\mathbb{P}^N$.

6. Serre duality in families

6.1. *Exercise: Serre duality in families.* Suppose $\pi : X \to Y$ is a flat projective morphism of locally Noetherian schemes, of relative dimension n. Assume all of the geometric fibers are Cohen-Macaulay. Then there exists a coherent sheaf $\omega_{X/Y}$ on X, along with a trace map $R^n\pi_*\omega_{X/Y} \to \mathcal{O}_Y$ such that, for every finite rank locally free sheaves $\mathcal{F}$ on X, each of whose higher pushforwards are locally free on Y,

$$R^i\pi_*\mathcal{F} \times R^{n-i}\pi_*(\mathcal{F}^\vee \otimes \omega_X) \longrightarrow R^n\pi_*\omega_X \xrightarrow{t} \mathcal{O}_Y \tag{8}$$

is a perfect pairing. (Hint: follow through the same argument!)

Note that the hypothesis, that all higher pushforwards are locally free on Y, is the sort of thing provided by the cohomology and base change theorem. (In the solution to Exercise 6.1, you will likely show that $R^{n-i}\pi_*(\mathcal{F}^\vee \otimes \omega_X)$ is a locally free sheaf for all $\mathcal{F}$ such that $R^i\pi_*\mathcal{F}$ is a locally free sheaf.)

You will need the *fibral flatness theorem* (EGA IV(3).11.3.10–11), which you should feel free to use: if $g : X \to S$, $h : Y \to S$ are locally of finite presentation, and $f : X \to Y$ is an S-morphism, then the following are equivalent:

(a) g is flat and $f_s : X_s \to Y_s$ is flat for all $s \in s$,
(b) h is flat at all points of $f(X)$ and f is flat.

7. WHAT WE STILL WANT

There are three or four more facts I want you to know.

- On the locus of X where k is smooth, there is an isomorphism $\omega_{X/k} \cong \det \Omega_{X/k}$. (Note for experts: it isn't canonical!) We define $\det \Omega_{X/k}$ to be $\mathcal{K}_X$. We used this in the case of smooth curves over k (proper, geometrically integral). This is surprisingly hard, certainly harder than the mere existence of the canonical sheaf!
- *The adjunction formula.* If D is a Cartier divisor on X (so D is also Cohen-Macaulay, by one of the facts about Cohen-Macaulayness I've mentioned), then $\omega_{D/k} = (\omega_{X/k} \otimes \mathcal{O}_X(D))\,|_D$.

On can show this using Ext groups, but I haven't established their existence or properties. So instead, I'm going to go as far as I can without using them, and then I'll tell you a little about them.

But first, here are some exercises *assuming* that ω is isomorphic to $\det \Omega$ on the smooth locus.

7.1. *Exercise (Serre duality gives a symmetry of the Hodge diamond).* Suppose X is a smooth projective k-variety of dimension n. Define $\Omega^p_X = \wedge^p \Omega X$. Show that we have a natural isomorphism $H^q(X, \Omega^p) \cong H^{n-q}(X, \Omega^{n-p})^\vee$.

7.2. *Exercise (adjunction for smooth subvarieties of smooth varieties).* Suppose X is a smooth projective k-scheme, and and D is a smooth effective Cartier divisor. Show that $\mathcal{K}_D \cong \mathcal{K}_X(D)|_D$. Hence if we knew that $\mathcal{K}_X \cong \omega_X$ and $\mathcal{K}_D \cong \omega_D$, this would let us compute ω_D in terms of ω_X. We will use this shortly.

7.3. *Exercise.* Compute $\mathcal{K}$ for a smooth complete intersection in $\mathbb{P}^N$ of hypersurfaces of degree $d_1, \dots, d_n$. Compute ω for a complete intersection in $\mathbb{P}^N$ of hypersurfaces of degree $d_1, \dots, d_n$. (This will be the same calculation!) Find all possible cases where $\mathcal{K} \cong \mathcal{O}$. These are examples of *Calabi-Yau varieties* (or *Calabi-Yau manifolds* if $k = \mathbb{C}$), at least when they have dimension at least 2. If they have dimension precisely 2, they are called K3 surfaces.

8. The dualizing sheaf is an invertible sheaf on the smooth locus

(I didn't do this in class, but promised it in the notes. A simpler proof in the case where X is a curve is given in §9.)

We begin with some preliminaries.

(0) If $f : U \to U$ is the identity, and $\mathcal{F}$ is a quasicoherent sheaf on U, then $f'\mathcal{F} \cong \mathcal{F}$.

(i) The $'$ construction behaves well with respect to flat base change, as the pushforward does. In other words, if

$$\begin{array}{ccc} X' & \xrightarrow{h} & X \\ \downarrow g & & \downarrow e \\ Y' & \xrightarrow{f} & Y \end{array}$$

is a fiber diagram, where f (and hence h) is flat, and $\mathcal{F}$ is any quasicoherent sheaf on Y, then there is a canonical isomorphism $h^* e' \mathcal{F} \cong g' f^* \mathcal{F}$.

(ii) The $'$ construction behaves well with respect to disjoint unions of the source. In other words, if $f_i : X_i \to Y$ $(i = 1, 2)$ are two morphisms, $f : X_1 \cup X_2 \to Y$ is the induced morphism from the disjoint union, and $\mathcal{F}$ is a quasicoherent sheaf on Y, then $f'\mathcal{F}$ is $f_1'\mathcal{F}$ on X_1 and $f_2'\mathcal{F}$ on X_2. The reason again is that pushforward behaves well with respect to disjoint union.

Exercise. Prove both these facts, using abstract nonsense.

Given a smooth point $x \in X$, we can choose our projection so that $\pi : X \to \mathbb{P}^n$ is etale at that point. *Exercise.* Prove this. (Hint: We need only check isomorphisms of tangent spaces.)

So hence we need only check our desired result on the etale locus U for $X \to \mathbb{P}^n$. (This is an open set, as etaleness is an open condition.) Consider the base change.

$$\begin{array}{ccc} X \times_{\mathbb{P}^n_k} U & \xrightarrow{h} & X \\ \downarrow g & & \downarrow e \\ U & \xrightarrow{f} & \mathbb{P}^n_k. \end{array}$$

There is a section $U \to X \times_{\mathbb{P}^n} U$ of the vertical morphism on the left. *Exercise.* Show that it expresses U as a connected component of $X \times_{\mathbb{P}^n} U$. (Hint: Show that a section of an etale morphism always expresses the target as a component of the source as follows. Check that s is a homeomorphism onto its image. Use Nakayama's lemma.) The dualizing sheaf $\omega_{\mathbb{P}^n_k}$ is invertible, and hence $f^*\omega_{\mathbb{P}^n_k}$ is invertible on U. Hence $g^! f^* \omega_{\mathbb{P}^n_k}$ is invertible on $s(U)$ (by observation (0)). By observation (i) then, $h^* g^* \omega_{\mathbb{P}^n_k} \cong h^* \omega_X$ is an invertible sheaf.

We are now reduced to showing the following. Suppose $h : U \to X$ is an etale morphism. (In the etale topology, this is called an "etale open set", even though it isn't an open set in any reasonable sense.) Its image is an open subset of X (as etale morphisms are open maps). Suppose $\mathcal{F}$ is a coherent sheaf on X such that $h^*\mathcal{F}$ is an invertible sheaf on U. Then $\mathcal{F}$ is an invertible sheaf on the image of U.

(Experts will notice that this is a special case of *faithfully flat descent*.)

Exercise. Prove this. Hint: it suffices to check that the stalks of $\mathcal{F}$ are isomorphic to the stalks of the structure sheaf. Hence reduce the question to a map of local rings: suppose $(B, \mathfrak{n}) \to (A, \mathfrak{m})$ is etale, and N is a coherent B-module such that $M := N \otimes_B A$ is isomorphic to A. We wish to show that N is isomorphic to B. Use Nakayama's lemma to show that N has the same minimal number of generators (over B) as M (over A), by showing that $\dim_{B/\mathfrak{n}} N - \dim_{A/\mathfrak{m}} M$. Hence this number is 1, so $N \cong B/I$ for some ideal I. Then show that $I = 0$ — you'll use flatness here.

9. An easier proof that the dualizing sheaf of a smooth curve is invertible

Here is another proof that for curves, the dualizing sheaf is invertible. We'll show that it is torsion-free, and rank 1.

First, here is why it is rank 1 at the generic point. We have observed that $f^!$ behaves well with respect to flat base change. Suppose L/K is a finite extension of degree n. Then $\mathrm{Hom}_K(L, K)$ is an L-module. What is its rank? As a K-module, it has rank n. Hence as an L-module it has rank 1. Applying this to $C \to \mathbb{P}^1$ at the generic point ($L = FF(C)$, $K = FF(\mathbb{P}^1)$) gives us the desired result. (Side remark: its structure as an L-module is a little mysterious. You can see that some sort of duality is relevant here. Illuminating this module's structure involves the norm map.)

Conclusion: the dualizing sheaf is rank 1 at the generic point.

Here is why it is torsion free. Let ω_t be the torsion part of ω, and ω_{nt} be the torsion-free part, so we have an exact sequence

$$0 \to \omega_t \to \omega \to \omega_{nt} \to 0.$$

9.1. *Exercise.* Show that this splits: $\omega = \omega_t \oplus \omega_{nt}$. (Hint: It suffices to find a splitting map $\omega \to \omega_t$. As ω_t is supported at a finite set of points, it suffices to find this map in a neighborhood of one of the points in the support. Restrict to a small enough affine open set where ω_{nt} is free. Then on this there is a splitting $\omega_{nt} \to \omega$, from which on that open set we have a splitting $\omega \to \omega_t$.)

Notice that ω_{nt} is rank 1 and torsion-free, hence an invertible sheaf. By Serre duality, for any invertible sheaf $\mathcal{L}$, $h^0(\mathcal{L}) = h^1(\omega_{nt} \otimes \mathcal{L}^\vee)$ and $h^1(\mathcal{L}) = h^0(\omega_{nt} \otimes \mathcal{L}^\vee) + h^0(\omega_t \otimes \mathcal{L})$. Substitute $\mathcal{L} = \mathcal{O}_X$ in the first of these equations and $\mathcal{L} = \omega_X$ in the second, to obtain that $h^0(X, \omega_t) = 0$. But the only skyscraper sheaf with no sections is the 0 sheaf, hence $\omega_t = 0$.

10. The sheaf of differentials is dualizing for a smooth projective curve

One can show that the determinant of the sheaf of differentials is the dualizing sheaf using Ext groups, but this involves developing some more machinery, without proof. Instead, I'd like to prove it directly for curves, using what we already have proved. (Note again that our proof of Serre duality for curves was rigorous — the cup product was already well-defined for dimension 1 schemes.)

I'll do this in a sequence of exercises.

Suppose C is an geometrically irreducible, smooth projective k-curve.

We wish to show that $\Omega_C \cong \omega_C$. Both are invertible sheaves. (Proofs that ω_C is invertible were given in §8 and §9.)

Define the genus of a curve as $g = h^1(C, \mathcal{O}_C)$. By Serre duality, this is $h^0(C, \omega_C)$. Also, $h^0(C, \mathcal{O}_C) = h^1(C, \omega_C) = 1$.

Suppose we knew that $h^0(C, \Omega_C) = h^0(C, \omega_C)$, and $h^1(C, \Omega_C) = h^1(\omega_C)$ $(= 1)$. Then $\deg \Omega_C = \deg \omega_C$. Also, by Serre duality $h^0(C, \Omega_C^\vee \otimes \omega_C) = h^1(\Omega_C) = 1$. Thus $\Omega_C^\vee \otimes \omega_C$ is a degree 0 invertible sheaf with a nonzero section. We have seen that this implies that the sheaf is trivial, so $\Omega_C \cong \omega_C$.

Thus it suffices to prove that $h^1(C, \Omega_C) = 1$, and $h^0(C, \Omega_C) = h^0(C, \omega_C)$. By Serre duality, we can restate the latter equality without reference to ω: $h^0(C, \Omega) = h^1(C, \mathcal{O}_C)$. Note that we can assume $k = \overline{k}$: all three cohomology group dimensions $h^i(C, \Omega_C)$, $h^0(C, \mathcal{O}_C)$ are preserved by field extension (shown earlier).

Until this point, the argument is slick and direct. What remains is reasonably pleasant, but circuitous. Can you think of a faster way to proceed, for example using branched covers of $\mathbb{P}^1$?

10.1. *Exercise.* Show that C can be expressed as a plane curve with only nodes as singularities. (Hint: embed C in a large projective space, and take a general projection. The Kleiman-Bertini theorem, or at least its method of proof, will be handy.)

Let the degree of this plane curve be d, and the number of nodes be δ. We then blow up $\mathbb{P}^2$ at the nodes (let $S = \mathrm{Bl}\,\mathbb{P}^2$), obtaining a closed immersion $C \hookrightarrow S$. Let H be the divisor class that is the pullback of the line ($\mathcal{O}(1)$) on $\mathbb{P}^2$. Let $E_1, \ldots, E_\delta$ be the classes of the exceptional divisors.

10.2. *Exercise.* Show that the class of C on $\mathbb{P}^2$ is $dH - 2\sum E_i$. (Reason: the total transform has class dH. Each exceptional divisor appears in the total transform with multiplicity two.)

10.3. *Exercise.* Use long exact sequences to show that $h^1(C, \mathcal{O}_C) = \binom{d-1}{2} - \delta$. (Hint: Compute $\chi(C, \mathcal{O}_C)$ instead. One possibility is to compute $\chi(C', \mathcal{O}_{C'})$ where C' is the image of C in $\mathbb{P}^2$, and use the Leray spectral sequence for $C \to C'$. Another possibility is to work on S directly.)

10.4. *Exercise.* Show that $\Omega_C = \mathcal{K}_S(C)|_C$. Show that this is

$$(-3H + \sum E_i) + (dH - \sum 2E_i).$$

Show that this has degree $2g - 2$ where $g = h^1(\mathcal{O}_C)$. (Possible hint: use long exact sequences.)

10.5. *Exercise.* Show that $h^0(\Omega_C) > 2g - 2 - g + 1 = g - 1$ from

$$0 \to H^0(S, \mathcal{K}_S) \to H^0(S, \mathcal{K}_S(C)) \to H^0(C, \Omega_C).$$

10.6. *Exercise.* Show that $\Omega_C \cong \omega_C$.

11. Ext groups, and adjunction

Let me now introduce Ext groups and their properties, without proof. Suppose i is a non-negative integer. Given two quasicoherent sheaves, $\operatorname{Ext}^i(\mathcal{F}, \mathcal{G})$ is a quasicoherent sheaf. $\operatorname{Ext}^0(\mathcal{F}, \mathcal{G}) = \operatorname{Hom}(\mathcal{F}, \mathcal{G})$. Then there are long exact sequences in both arguments. In other words, if

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$$

is a short exact sequence, then there is a long exact sequence starting

$$0 \to \operatorname{Ext}^0(\mathcal{F}'', \mathcal{G}) \to \operatorname{Ext}^0(\mathcal{F}, \mathcal{G}) \to \operatorname{Ext}^0(\mathcal{F}', \mathcal{G}) \to \operatorname{Ext}^1(\mathcal{F}'', \mathcal{G}) \to \cdots,$$

and if

$$0 \to \mathcal{G}' \to \mathcal{G} \to \mathcal{G}'' \to 0$$

is a short exact sequence, then there is a long exact sequence starting

$$0 \to \operatorname{Ext}^0(\mathcal{F}, \mathcal{G}') \to \operatorname{Ext}^0(\mathcal{F}, \mathcal{G}) \to \operatorname{Ext}^0(\mathcal{F}, \mathcal{G}'') \to \operatorname{Ext}^0(\mathcal{F}, \mathcal{G}') \to \cdots.$$

Also, if $\mathcal{F}$ is locally free, there is a canonical isomorphism $\operatorname{Ext}^i(\mathcal{F}, \mathcal{G}) \cong H^i(X, \mathcal{G} \otimes \mathcal{F}^\vee)$.

For any quasicoherent sheaves $\mathcal{F}$ and $\mathcal{G}$, there is a natural map $\operatorname{Ext}^i(\mathcal{F}, \mathcal{G}) \times H^j(X, \mathcal{F}) \to H^{i+j}(\mathcal{G})$.

For any coherent sheaf on X, there is a natural morphism ("cup product") $\operatorname{Ext}^i(\mathcal{F}, \mathcal{G}) \times H^j(X, \mathcal{F}) \to H^{i+j}(X, \mathcal{G})$.

11.1. *Exercise.* Suppose X is Cohen-Macaulay, and finite type and projective over k (so Serre duality holds). Via this morphism, show that

$$\operatorname{Ext}^i(\mathcal{F}, \omega_X) \times H^{n-i}(X, \mathcal{F}) \longrightarrow H^n(X, \omega_X) \xrightarrow{t} k$$

is a perfect pairing. Feel free to assume whatever nice properties of Ext-groups you need (as we haven't proven any of them anyway).

Hence Serre duality yields a natural extension to coherent sheaves. This is sometimes called Serre duality as well. This more general statement is handy to prove the adjunction formula.

11.2. *Adjunction formula.* — If X is a Serre duality space (i.e. a space where Serre duality holds), and D is an effective Cartier divisor, then $\omega_D = (\omega_X \otimes \mathcal{O}(D))|_D$.

We've seen that if X and D were smooth, and we knew that $\omega_X \cong \det \Omega_X$ and $\omega_D \cong \det \Omega_D$, we would be able to prove this easily (Exercise 7.2).

But we get more. For example, complete intersections in projective space have invertible dualizing sheaves, no matter how singular or how nonreduced. Indeed, complete intersections in *any* smooth projective k-scheme have invertible dualizing sheaves.

A projective k-schemes with invertible dualizing sheaf is so nice that it has a name: it is said to be *Gorenstein*. (Gorenstein has a more general definition, that also involves a dualizing sheaf. It is a local definition, like nonsingularity and Cohen-Macaulayness.)

11.3. *Exercise.* Prove the adjunction formula. (Hint: Consider $0 \to \omega_X \to \omega_X(D) \to \omega_X(D)|_D \to 0$. Apply $\operatorname{Hom}_X(\mathcal{F}, \cdot)$ to this, and take the long exact sequence in Ext-groups.) As before, feel free to assume whatever facts about Ext groups you need.

The following exercise is a bit distasteful, but potentially handy. Most likely you should skip it, and just show that $\omega_X \cong \det \Omega_X$ using the theory of Ext groups.

11.4. *Exercise.* We make a (temporary) definition inductively by definition. A k-variety is "nice" if it is smooth, and (i) it has dimension 0 or 1, or (ii) for any nontrivial invertible sheaf $\mathcal{L}$ on X, there is a nice divisor D such that $\mathcal{L}|_D \neq 0$. Show that for any nice k-variety, $\omega_X \cong \det \Omega_X$. (Hint: use the adjunction formula, and the fact that we know the result for curves.)

11.5. *Remark.* You may wonder if ω_X is always an invertible sheaf. In fact it isn't, for example if $X = \operatorname{Spec} k[x,y]/(x,y)^2$. I think I can give you a neat and short explanation of this fact. If you are curious, just ask.

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FOUNDATIONS OF ALGEBRAIC GEOMETRY PROBLEM SET 1

RAVI VAKIL

This set is due Monday, October 17. It covers classes 1 and 2. Hand in five of these problems. If you are pressed for time, try more A problems. If you are ambitious, try more B problems.

I intend there to be weekly problem sets, to be given out each Monday and handed in the following Monday (although this set is an exception). If you are taking this course for a grade, you'll have to hand in all but one of the sets. These problems are not intended to be (too) onerous, but they are intended to help you get practical experience with ideas that may be new to you. Even if you are not taking the course for a grade, I strongly encourage you to try these problems, and if you are handing in problems, I encourage you to try more than the minimum number. Choose problems that stretch your knowledge, and not problems that you already know how to do. Feedback on the problems would be appreciated.

You are encouraged to talk to each other about the problems. (Write up your solutions individually.) You are also encouraged to talk to me about them. Ideally, you should find out who did problems that you didn't do.

I will be away Wednesday, October 5 until Thursday, October 13. The next class after Monday, October 3 will be Friday, October 14*. The week after we will meet Monday, Wednesday, and Friday (Oct. 17, 19, 21). Then we will be only one class behind.*

A1. A category in which each morphism is an isomorphism is called a *groupoid*. A perverse definition of a group is: a groupoid with one element. Make sense of this. (The notion of "groupoid" isn't important for this course. The point of this exercise is to give you some practice with categories, by relating them to an object you know well.)

A2 (if you haven't seen tensor products before). Calculate $\mathbb{Z}/10 \otimes_{\mathbb{Z}} \mathbb{Z}/12$. (The point of this exercise is to give you a very little hands-on practice with tensor products.)

A3. Interpret fibered product in the category of sets: If we are given maps from sets X and Y to the set Z, interpret $X \times_Z Y$. (This will help you build intuition about this concept.)

A4. A morphism $f : X \to Y$ is said to be a a **monomorphism** if any two morphisms $g_1, g_2 : Z \to X$ such that $f \circ g_1 = f \circ g_2$ must satisfy $g_1 = g_2$. This is the generalization of an injection of sets. Suppose $X \to Y$ is a monomorphism, and $W, Z \to X$ are two morphisms. Show that $W \times_X Z$ and $W \times_Y Z$ are canonically isomorphic. (We will use this later when talking about fibered products.)

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A5. Given $X \to Y \to Z$, show that there is a natural morphism $X \times_Y X \to X \times_Z X$, assuming these fibered products exist. (This is trivial once you figure out what it is saying. The point of this exercise is to see why it is trivial.)

A6. Define coproduct in a category by reversing all the arrows in the definition of product. Show that coproduct for sets is disjoint union.

A7. If Z is the final object in a category $\mathcal{C}$, and $X, Y \in \mathcal{C}$, then "$X \times_Z Y = X \times Y$" ("the" fibered product over Z is canonically isomorphic to "the" product). (This is an exercise about unwinding the definition.)

A8 ("A presheaf is the same as a contravariant functor"). Given any topological space X, we can get a category, which I will call the "category of open sets". The objects are the open sets. The morphisms are the inclusions $U \hookrightarrow V$. (What is the initial object? What is the final object?) Verify that the data of a presheaf is precisely the data of a contravariant functor from the category of open sets of X to the category of sets, plus the final object axiom, that there is one section over $\emptyset$. (This exercise is intended for people wanting practice with categories.)

A9. (a) Let X be a topological space, and S a set with more than one element, and define $\mathcal{F}(U) = S$ for all open sets U. Show that this forms a presheaf (with the obvious restriction maps), and even satisfies the identity axiom. Show that this needn't form a sheaf. (Here we need the axiom that $\mathcal{F}(\emptyset)$ must be the final object, not S. Without this patch, the constant presheaf *is* a sheaf.) This is called the *constant presheaf with values in* S. We will denote this presheaf $\underline{S}^{\text{pre}}$.
(b) Now let $\mathcal{F}(U)$ be the maps to S that are *locally constant*, i.e. for any point x in U, there is a neighborhood of x where the function is constant. (Better description is this: endow S with the discrete topology, and let $\mathcal{F}(U)$ be the continuous maps $U \to S$.) Show that this is a *sheaf*. (Here we need $\mathcal{F}(\emptyset)$ to be the final object again.) We will try to call this the *locally constant sheaf*. (In the real world, this is called the *constant sheaf*. I don't understand why.) We will denote this sheaf $\underline{S}$.

B1 (Yoneda's lemma). Pick an object in your favorite category $A \in \mathcal{C}$. For any object $C \in \mathcal{C}$, we have a set of morphisms $\mathrm{Mor}(C, A)$. If we have a morphism $f : B \to C$, we get a map of sets

$$\mathrm{Mor}(C, A) \to \mathrm{Mor}(B, A), \tag{1}$$

just by composition: given a map from C to A, we immediately get a map from B to A by precomposing with f. Yoneda's lemma, or at least part of it, says that this functor determines A up to unique isomorphism. Translation: If we have two objects A and A', and isomorphisms

$$i_C : \mathrm{Mor}(C, A) \to \mathrm{Mor}(C, A') \tag{2}$$

that commute with the maps (1), then the i_C must be induced from a unique isomorphism $A \to A'$. Prove this.

B2. Prove that a morphism is a monomorphism if and only if the natural morphism $X \to X \times_Y X$ is an isomorphism. (We may then take this as the definition of monomorphism.)

(Monomorphisms aren't very central to future discussions, although they will come up again. This exercise is just good practice.)

B3 (tensor product). (This will be important later!) Suppose $T \to R, S$ are two ring morphisms. Let I be an ideal of R. Let I^e be the extension of I to $R \otimes_T S$. These are the elements $\sum_j i_j \otimes s_j$ where $i_j \in I$, $s_j \in S$. Show that there is a natural isomorphism

$$R/I \otimes_T S \cong (R \otimes_T S)/I^e.$$

Hence the natural morphism $R \otimes_T S \to R/I \otimes_T S$ is a surjection. As an application, we can compute tensor products of finitely generated k algebras over k. For example,

$$k[x_1, x_2]/(x_1^2 - x_2) \otimes_k k[y_1, y_2]/(y_1^3 + y_2^3) \cong k[x_1, x_2, y_1, y_2]/(x_1^2 - x_2, y_1^3 + y_2^3).$$

B4 (direct limits). We say a partially ordered set I is a *directed set* if for $i, j \in I$, there is some $k \in I$ with $i, j \leq k$. In this exercise, you will show that the direct limit of any system of A-modules indexed by I exists, by constructing it. Say the system is given by M_i ($i \in I$), and $f_{ij} : M_i \to M_j$ ($i \leq j$ in I). Let $M = \oplus_i M_i$, where each M_i is identified with its image in M, and let R be the submodule generated by all elements of the form $m_i - f_{ij}(m_i)$ where $m_i \in M_i$ and $i \leq j$. Show that M/R (with the projection maps from the M_i) is $\lim_{\to} M_i$. You will notice that the same argument works in other interesting categories, such as: sets; groups; and abelian groups. (This example came up in interpreting/defining stalks as direct limits.)

B5 (practice with universal properties). The purpose of this exercise is to give you some practice with "adjoints of forgetful functors", the means by which we get groups from semigroups, and sheaves from presheaves. Suppose R is a ring, and S is a multiplicative subset. Then $S^{-1}R$-modules are a fully faithful subcategory of the category of R-modules (meaning: the objects of the first category are a subset of the objects of the second; and the morphisms between any two objects of the second that are secretly objects of the first are just the morphisms from the first). Then $M \to S^{-1}M$ satisfies a universal property. Figure out what the universal property is, and check that it holds. In other words, describe the universal property enjoyed by $M \to S^{-1}M$, and prove that it holds.

(Here is the larger story. Let $S^{-1}R$-Mod be the category of $S^{-1}R$-modules, and R-Mod be the category of R-modules. Every $S^{-1}R$-module is an R-module, so we have a (covariant) forgetful functor $F : S^{-1}R\text{-Mod} \to R\text{-Mod}$. In fact this is a fully faithful functor: it is injective on objects, and the morphisms between any two $S^{-1}R$-modules *as R-modules* are just the same when they are considered as $S^{-1}R$-modules. Then there is a functor $G : R\text{-Mod} \to S^{-1}R\text{-Mod}$, which might reasonably be called "localization with respect to S", which is left-adjoint to the forgetful functor. Translation: If A is an R-module, and B is an $S^{-1}R$-module, then $\mathrm{Hom}(GA, B)$ (morphisms as $S^{-1}R$-modules, which is incidentally the same as morphisms as R-modules) are in natural bijection with $\mathrm{Hom}(A, FB)$ (morphisms as R-modules).)

B6 (good examples of sheaves). Suppose Y is a topological space. Show that "continuous maps to Y" form a sheaf of sets on X. More precisely, to each open set U of X, we associate the set of continuous maps to Y. Show that this forms a sheaf.

(b) Suppose we are given a continuous map $f : Y \to X$. Show that "sections of f" form a sheaf. More precisely, to each open set U of X, associate the set of continuous maps s to Y such that $f \circ s = id|_U$. Show that this forms a sheaf. (A classical construction of sheaves in general is to interpret them in precisely this way. See Serre's revolutionary article *Faisceaux Algébriques Cohérents*.)

B7 (an important construction, the pushforward sheaf). (a) Suppose $f : X \to Y$ is a continuous map, and $\mathcal{F}$ is a sheaf on X. Then define $f_*\mathcal{F}$ by $f_*\mathcal{F}(V) = \mathcal{F}(f^{-1}(V))$, where V is an open subset of Y. Show that $f_*\mathcal{F}$ is a sheaf. This is called a *pushforward sheaf*. More precisely, $f_*\mathcal{F}$ is called the *pushforward of $\mathcal{F}$ by f*.
(b) Assume $\mathcal{F}$ is a sheaf of sets (or rings or R-modules), so stalks exist. If $f(x) = y$, describe the natural morphism of stalks $(f_*\mathcal{F})_y \to \mathcal{F}_x$.

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FOUNDATIONS OF ALGEBRAIC GEOMETRY PROBLEM SET 2

This set is due Monday, October 24. It covers classes 3 and 4. Read all of these problems, and hand in six solutions. The problems are arranged roughly in "chronological order", not by difficulty. Try to solve problems on a range of topics. If you are pressed for time, try more straightforward problems. If you are ambitious, push the envelope a bit. You are encouraged to talk to each other about the problems. (Write up your solutions individually.) You are also encouraged to talk to me about them. Ideally, you should find out who did problems that you didn't do.

1. Suppose

$$0 \xrightarrow{d^0} A^1 \xrightarrow{d^1} \cdots \xrightarrow{d^{n-1}} A^n \xrightarrow{d^n} 0$$

is a complex of vector spaces (often called $A^\bullet$ for short), i.e. $d^i \circ d^{i-1} = 0$. Show that $\sum (-1)^i \dim A^i = \sum (-1)^i h^i(A^\bullet)$. (Recall that $h^i(A^\bullet) = \dim \ker(d^i)/\operatorname{im}(d^{i-1})$.) In particular, if $A^\bullet$ is exact, then $\sum(-1)^i \dim A^i = 0$. (If you haven't dealt much with cohomology, this will give you some practice. If you have, you shouldn't do this problem.)

Problems on presheaves and sheaves

2. Suppose $\phi : \mathcal{F} \to \mathcal{G}$ is a morphism of presheaves of abelian groups or $\mathcal{O}_X$-modules. If $\mathcal{H}$ is defined by the collection of data $\mathcal{H}(U) = \mathcal{G}(U)/\phi(\mathcal{F}(U))$ for all open U, show that $\mathcal{H}$ is a presheaf, and show that it is a cokernel in the category of presheaves. (I stated this as a fact in class, but you aren't allowed to appeal to authority.)

3. Suppose that $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \cdots \to \mathcal{F}_n \to 0$ is an *exact sequence of presheaves* of groups or $\mathcal{O}_X$-modules. Show that $0 \to \mathcal{F}_1(U) \to \mathcal{F}_2(U) \to \cdots \to \mathcal{F}_n(U) \to 0$ is also an exact sequence for all U.

4. (This problem sounds more confusing than it is!) Show that the presheaf kernel of a morphism of sheaves (of abelian groups, or $\mathcal{O}_X$-modules) is also sheaf. Show that it is the sheaf kernel (a kernel in the category of sheaves) as well. (This is one reason that kernels are easier than cokernels.)

5. The presheaf cokernel was defined in problem 2. Show that the sheafification of the presheaf cokernel is in fact the sheaf cokernel, by verifying that it satisfies the universal property.

6. Suppose $f : \mathcal{F} \to \mathcal{G}$ is a morphism of sheaves of abelian groups or $\mathcal{O}_X$-modules. Let $\operatorname{im} f$ be the sheafification of the "presheaf image". Show that there are natural isomorphisms $\operatorname{im} f \cong \mathcal{F}/\ker f$ and $\operatorname{coker} f \cong \mathcal{G}/\operatorname{im} f$. (This problem shows that this construction deserves to be called the "image".)

Date: Monday, October 17, 2005. (Corrected version October 20, 2005.)

7. Suppose $\mathcal{O}_X$ is a sheaf of rings on X. Define (categorically) what we should mean by tensor product of two presheaves or sheaves of $\mathcal{O}_X$-modules. Give an explicit construction, and show that it satisfies your categorical definition. *Hint:* take the "presheaf tensor product" — which needs to be defined — and sheafify. (This is admittedly a vague problem. If it is confusing, just ask. But it is good practice to turn your rough intuition into precise statements.)

8. Suppose $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H}$ is an exact sequence of sheaves (of abelian groups) on X. If $f : X \to Y$ is a continuous map, show that

$$0 \to f_*\mathcal{F} \to f_*\mathcal{G} \to f_*\mathcal{H}$$

is exact. Translation: pushforward is a left-exact functor. (The case of left-exactness of the global section functor can be interpreted as a special case of this, in the case where Y is a point.) Show that it needn't be exact on the right, i.e. that $f_*\mathcal{G} \to f_*\mathcal{H}$ needn't be surjective (= an epimorphism). (Hint: see the previous parenthetical comment, and think of your favorite short exact sequence of sheaves.)

The next three problems present some new concepts: gluing sheaves, sheaf homomorphisms, and flasque sheaves. I will feel comfortable using these concepts in class.

9. Suppose $X = \cup U_i$ is an open cover of X, and we have sheaves $\mathcal{F}_i$ on U_i along with isomorphisms $\phi_{ij} : \mathcal{F}_i|_{U_i \cap U_j} \to \mathcal{F}_j|_{U_i \cap U_j}$ that agree on triple overlaps (i.e. $\phi_{ij} \circ \phi_{jk} = \phi_{ij}$ on $U_i \cap U_j \cap U_k$). Show that these sheaves can be glued together into a unique sheaf $\mathcal{F}$ on X, such that $\mathcal{F}_i = \mathcal{F}|_{U_i}$, and the isomorphisms over $U_i \cap U_j$ are the obvious ones. (Thus we can "glue sheaves together", using limited patching information.)

10. Suppose $\mathcal{F}$ and $\mathcal{G}$ are two sheaves on X. Let $\underline{\mathrm{Hom}}(\mathcal{F}, \mathcal{G})$ be the collection of data

$$\underline{\mathrm{Hom}}(\mathcal{F}, \mathcal{G})(U) := \mathrm{Hom}(\mathcal{F}|_U, \mathcal{G}|_U).$$

Show that this is a sheaf. (This is called the "sheaf $\underline{\mathrm{Hom}}$". If $\mathcal{F}$ and $\mathcal{G}$ are sheaves of sets, $\underline{\mathrm{Hom}}(\mathcal{F}, \mathcal{G})$ is a sheaf of sets. If $\mathcal{G}$ is a sheaf of abelian groups, then $\underline{\mathrm{Hom}}(\mathcal{F}, \mathcal{G})$ is a sheaf of abelian groups.) I've decided to call this $\underline{\mathrm{Hom}}$ rather than $\mathcal{H}\mathrm{om}$ because of the convention that "underlining often denotes sheaf". (Of course, the calligraphic font also often denotes sheaf.)

11. A sheaf $\mathcal{F}$ is said to be *flasque* if for every $U \subset V$, the restriction map $\mathrm{res}_{V,U} : \mathcal{F}(V) \to \mathcal{F}(U)$ is surjective. In other words, every section over U extends to a section over V. This is a very strong condition, but it comes up surprisingly often.
(a) Show that $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ is exact, and $\mathcal{F}'$ and $\mathcal{F}''$ are flasque, then so is $\mathcal{F}$.
(b) Suppose $f : X \to Y$ is a continuous map, and $\mathcal{F}$ is a flasque sheaf on X. Show that $f_*\mathcal{F}$ is a flasque sheaf on Y.
(If $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ is exact, and $\mathcal{F}'$ is flasque, then $0 \to \mathcal{F}'(U) \to \mathcal{F}(U) \to \mathcal{F}''(U) \to 0$ is exact, i.e. the global section functor is exact here, even on the right. Similarly, for any continuous map $f : X \to Y$, $0 \to f_*\mathcal{F}' \to f_*\mathcal{F} \to f_*\mathcal{F}'' \to 0$ is exact. I haven't thought about how hard this is yet, so I haven't made this part of the exercise. But it is good to know, and gives a reason to like flasque sheaves.)

Understanding sheaves via stalks

12. Prove that a section of a sheaf is determined by its germs, i.e.

$$\mathcal{F}(U) \to \coprod_{x \in U} \mathcal{F}_x$$

is injective. Hint: you won't use the gluability axiom. So this is true of morphisms of "separated presheaves". (This exercise is important, as you've seen!) Corollary: If a sheaf has all stalks 0, then it is the 0-sheaf.

13. Show that a morphism of sheaves on a topological space X induces a morphism of stalks. More precisely, if $\phi : \mathcal{F} \to \mathcal{G}$ is a morphism of sheaves on X, describe a natural map $\phi_x : \mathcal{F}_x \to \mathcal{G}_x$.

14. Show that morphisms of sheaves are determined by morphisms of stalks. Hint # 1: you won't use the gluability axiom. Hint # 2: study the following diagram.

(1)
$$\begin{array}{ccc} \mathcal{F}(U) & \longrightarrow & \mathcal{G}(U) \\ \big\downarrow & & \big\downarrow \\ \prod_{x \in U} \mathcal{F}_x & \longrightarrow & \prod_{x \in U} \mathcal{G}_x \end{array}$$

15. Show that a morphism of sheaves is an isomorphism if and only if it induces an isomorphism of all stalks. Hint: Use (1). Injectivity of $\mathcal{F}(U) \to \mathcal{G}(U)$ uses the previous exercise. Surjectivity requires gluability. (I largely did this in class, so you should try this mainly if you want to make sure you are clear on the concept.)

16. Show that problems 12, 14, and 15 are false for presheaves in general. (Hint: take a 2-point space with the discrete topology, i.e. every subset is open.)

17. Show that for any morphism of presheaves $\phi : \mathcal{F} \to \mathcal{G}$, we get a natural induced morphism of sheaves $\phi^{sh} : \mathcal{F}^{sh} \to \mathcal{G}^{sh}$.

18. Show that the stalks of $\mathcal{F}^{sh}$ are the same as ("are naturally isomorphic to") the stalks of $\mathcal{F}$. Hint: Use the concrete description of the stalks.

Sheaves on a nice base

19. Suppose $\{B_i\}$ is a "nice base" for the topology of X.
(a) Verify that a morphism of sheaves is determined by the induced morphism of sheaves on the base.
(b) Show that a morphism of sheaves on the base (i.e. such that the diagram

$$\begin{array}{ccc} \Gamma(B_i, \mathcal{F}) & \longrightarrow & \Gamma(B_i, \mathcal{G}) \\ \big\downarrow & & \big\downarrow \\ \Gamma(B_j, \mathcal{F}) & \longrightarrow & \Gamma(B_j, \mathcal{G}) \end{array}$$

commutes for all $B_j \hookrightarrow B_i$) gives a morphism of the induced sheaves.

The inverse image sheaf

Suppose we have a continuous map $f : X \to Y$. If $\mathcal{F}$ is a sheaf on X, we have defined the pushforward $f_*\mathcal{F}$, which is a sheaf on Y. There is also a notion of inverse image. If $\mathcal{G}$ is a sheaf on Y, then there is a sheaf on X, denoted $f^{-1}\mathcal{G}$. This gives a covariant functor from sheaves on Y to sheaves on X. For example, if we have a morphism of sheaves on Y, we'll get an induced morphism of their inverse image sheaves on X.

Here is a concrete but unmotivated (and frankly unpleasant) definition: temporarily define $f^{-1}\mathcal{G}^{\mathrm{pre}}(U) = \lim_{\to V \supset f(U)} \mathcal{G}(V)$. (Recall explicit description of direct limit: sections are sections on open sets containing $f(U)$, with an equivalence relation.)

20. Show that this defines a presheaf on X.

Now define the *inverse image sheaf* $f^{-1}\mathcal{G} := (f^{-1}\mathcal{G}^{\mathrm{pre}})^{\mathrm{sh}}$.

21. Show that the stalks of $f^{-1}\mathcal{G}$ are the same as the stalks of $\mathcal{G}$. More precisely, if $f(x) = y$, describe a natural isomorphism $\mathcal{G}_y \cong (f^{-1}\mathcal{G})_x$. (Hint: use the concrete description of the stalk, as a direct limit.)

22. Show that f^{-1} is an exact functor from sheaves of abelian groups on Y to sheaves of abelian groups on X. (Hint: exactness can be checked on stalks, and by the previous exercise, stalks are the same.) The identical argument will show that f^{-1} is an exact functor from sheaves of $\mathcal{O}_Y$-modules on Y to sheaves of $f^{-1}\mathcal{O}_Y$-modules on X, but don't bother writing that down.

Here is a categorical definition of inverse image: it is left-adjoint to f_*. More precisely, suppose $f : X \to Y$ is a continuous map (= morphism) of topological spaces, and $\mathcal{F}$ is a sheaf of sets on X, and $\mathcal{G}$ is a sheaf of sets on Y. There is a natural bijection between $\mathrm{Hom}(f^{-1}(\mathcal{G}), \mathcal{F})$ and $\mathrm{Hom}(\mathcal{G}, f_*\mathcal{F})$. (The same argument will apply for sheaves of abelian groups etc.)

23. Show that the explicit definition of inverse image satisfies this universal property. (Just describe the bijection. One should also check that this bijection is natural, i.e. that for any $\mathcal{F}_1 \to \mathcal{F}_2$, the diagram

$$\begin{array}{ccc} \mathrm{Hom}(f^{-1}(\mathcal{G}), \mathcal{F}_2) & \longrightarrow & \mathrm{Hom}(\mathcal{G}, f_*\mathcal{F}_2) \\ \downarrow & & \downarrow \\ \mathrm{Hom}(f^{-1}(\mathcal{G}), \mathcal{F}_1) & \longrightarrow & \mathrm{Hom}(\mathcal{G}, f_*\mathcal{F}_1) \end{array}$$

commutes, and something similar for the "left argument", but don't worry about that.) This problem requires some elbow grease.

A small exercise on a small affine scheme

24. Describe the set $\mathrm{Spec}\, k[x]/x^2$. This seems like a very boring example, but it will grow up to be very important indeed! (This is not hard.)

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FOUNDATIONS OF ALGEBRAIC GEOMETRY PROBLEM SET 3

RAVI VAKIL

This set is due Monday, October 31. It covers classes 5, 6, and 7. Read all of these problems, and hand in six solutions. Try to solve problems on a range of topics. If you are pressed for time, try more straightforward problems. If you are ambitious, push the envelope a bit. You are encouraged to talk to each other about the problems. (Write up your solutions individually.) You are also encouraged to talk to me about them. Ideally, you should find out who did problems that you didn't do. Make sure you read all the problems, because we will be be making use of many of these results.

Facts we'll use (short proofs)

Three of these count for one problem.

A1. Show that if (S) is the ideal generated by S, then $V(S) = V((S))$. Thus when looking at vanishing sets, it suffices to consider vanishing sets of ideals.

A2. (a) Show that $\emptyset$ and $\operatorname{Spec} R$ are both open.
(b) (The intersection of two open sets is open.) Check that $V(I_1 I_2) = V(I_1) \cup V(I_2)$.
(c) (The union of any collection of open sets is open.) If I_i is a collection of ideals (as i runs over some index set), check that $V(\sum_i I_i) = \cap_i V(I_i)$.

A3. If $I \subset R$ is an ideal, show that $V(\sqrt{I}) = V(I)$.

A4. Show that if R is an integral domain, then $\operatorname{Spec} R$ is an irreducible topological space. (Hint: look at the point $[(0)]$.)

A5. Show that the closed points of $\operatorname{Spec} R$ correspond to the maximal ideals.

A6. If $X = \operatorname{Spec} R$, show that $[\mathfrak{p}]$ is a specialization of $[\mathfrak{q}]$ if and only if $\mathfrak{q} \subset \mathfrak{p}$.

A7. If X is a finite union of quasicompact spaces, show that X is quasicompact.

A8. Suppose $f_i \in R$ for $i \in I$. Show that $\cup_{i \in I} D(f_i) = \operatorname{Spec} R$ if and only if $(f_i) = R$.

A9. Show that $D(f) \cap D(g) = D(fg)$. Hence the distinguished base is a *nice* base.

A10. Show that if $D(f) \subset D(g)$, then $f^n \in (g)$ for some n.

A11. Show that $f \in \mathfrak{N}$ if and only if $D(f) = \emptyset$.

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A12. Suppose $f \in R$. Show that under the identification of $D(f)$ in $\operatorname{Spec} R$ with $\operatorname{Spec} R_f$, there is a natural isomorphism of sheaves $(D(f), \mathcal{O}_{\operatorname{Spec} R}|_{D(f)}) \cong (\operatorname{Spec} R_f, \mathcal{O}_{\operatorname{Spec} R_f})$.

A13. Show that the disjoint union of a *finite* number of affine schemes is also an affine scheme. (Hint: say what the ring is.)

A14. An infinite disjoint union of (non-empty) affine schemes is not an affine scheme. (One-word hint: quasicompactness.)

A15. If X is a scheme, and U is any open subset, then prove that $(U, \mathcal{O}_X|_U)$ is also a scheme.

A16. Show that if X is a scheme, then the affine open sets form a base for the Zariski topology. (Warning: they don't form a nice base, as we'll see in a different exercise on this problem set.) However, in "most nice situations" this will be true, as we will later see, when we define the analogue of "Hausdorffness", called separatedness.)

Facts we'll use

B1. Show that $\operatorname{Spec} R$ is quasicompact.

B2. Suppose that $I, S \subset R$ are an ideal and multiplicative subset respectively. Show that the Zariski topology on $\operatorname{Spec} R/I$ (resp. $\operatorname{Spec} S^{-1}R$) is the subspace topology induced by inclusion in $\operatorname{Spec} R$. (Hint: compare closed subsets.)

B3. (a) Show that $V(I(S)) = \overline{S}$. Hence $V(I(S)) = S$ for a closed set S. (b) Show that if $I \subset R$ is an ideal, then $I(V(I)) = \sqrt{I}$.

B4. (Important!) Show that V and I give a bijection between *irreducible closed subsets* of $\operatorname{Spec} R$ and *prime* ideals of R. From this conclude that in $\operatorname{Spec} R$ there is a bijection between points of $\operatorname{Spec} R$ and irreducible closed subsets of $\operatorname{Spec} R$ (where a point determines an irreducible closed subset by taking the closure). Hence each irreducible closed subset has precisely one generic point.

B5. (Important!) Show that the distinguished opens form a base for the Zariski topology.

B6. (a) Recall that sections of the structure sheaf on the base were defined by $\mathcal{O}_{\operatorname{Spec} R}(D(f)) = R_f$. Verify that this is well-defined, i.e. if $D(f) = D(f')$ then $R_f \cong R_{f'}$.
(b) Recall that restriction maps on the base were defined as follows. If $D(f) \subset D(g)$, then we have shown that $f^n \in (g)$, i.e. we can write $f^n = ag$, so there is a natural map $R_g \to R_f$ given by $r/g^m \mapsto (ra^m)/(f^{mn})$, and we define

$$\operatorname{res}_{D(g),D(f)} : \mathcal{O}_{\operatorname{Spec} R(D(g))} \to \mathcal{O}_{\operatorname{Spec} R(D(f))}$$

to be this map. Show that $\operatorname{res}_{D(g),D(f)}$ is well-defined, i.e. that it is independent of the choice of a and n, and if $D(f) = D(f')$ and $D(g) = D(g')$, then

$$\begin{array}{ccc} R_g & \xrightarrow{\operatorname{res}_{D(g),D(f)}} & R_f \\ \downarrow{\scriptstyle\sim} & & \downarrow{\scriptstyle\sim} \\ R_{g'} & \xrightarrow{\operatorname{res}_{D(g),D(f)}} & R_{f'} \end{array}$$

commutes.

B7. Show that the structure sheaf satisfies "identity on the distinguished base". Show that it satisfies "gluability on the distinguished base". (We used this to show that the structure sheaf is actually a sheaf.)

B8. Suppose M is an R-module. Show that the following construction describes a sheaf $\tilde{M}$ on the distinguished base. To $D(f)$ we associate $M_f = M \otimes_R R_f$; the restriction map is the "obvious" one.

B9. Show that the stalk of $\mathcal{O}_{\operatorname{Spec} R}$ at the point $[\mathfrak{p}]$ is the ring $R_\mathfrak{p}$. (Hint: use distinguished open sets in the direct limit you use to define the stalk. In the course of doing this, you'll discover a useful principle. In the concrete definition of stalk, the elements were sections of the sheaf over *some* open set containing our point, and two sections over different open sets were considered the same if they agreed on some smaller open set. In fact, you can just consider elements of your base when doing this. I think this is called a cofinal system in the directed set, but I might be mistaken.) This is yet another reason to like the notion of a sheaf on a base.

B10. (Important!) Figure out how to define projective n-space $\mathbb{P}^n_k$. Glue together $n+1$ opens each isomorphic to $\mathbb{A}^n_k$. Show that the only global sections of the structure sheaf are the constants, and hence that $\mathbb{P}^n_k$ is not affine if $n > 0$. (Hint: you might fear that you will need some delicate interplay among all of your affine opens, but you will only need two of your opens to see this. There is even some geometric intuition behind this: the complement of the union of two opens has codimension 2. But "Hartogs' Theorem" says that any function defined on this union extends to be a function on all of projective space. Because we're expecting to see only constants as functions on all of projective space, we should already see this for this union of our two affine open sets.)

Practice with the concepts

C1. Verify that $[(y - x^2)] \in \mathbb{A}^2_k$ is a generic point for $V(y - x^2)$.

C2. Suppose $X \subset \mathbb{A}^3_k$ is the union of the three axes. Give generators for the ideal $I(X)$.

C3. Describe a natural isomorphism $(k[x,y]/(xy))_x \cong k[x]_x$.

C4. Suppose we have a polynomial $f(x) \in k[x]$. Instead, we work in $k[x,\epsilon]/\epsilon^2$. What then is $f(x+\epsilon)$? (Do a couple of examples, and you will see the pattern. For example, if $f(x) = 3x^3 + 2x$, we get $f(x+\epsilon) = (3x^3 + 2x) + \epsilon(9x^2 + 2)$. Prove the pattern!) Useful tip: the dual numbers are a good source of (counter)examples, being the "smallest ring with nilpotents". They will also end up being important in defining differential information.

C5. Show that the affine base of the Zariski topology isn't necessarily a nice base. (Hint: look at the affine plane with the doubled origin.)

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FOUNDATIONS OF ALGEBRAIC GEOMETRY PROBLEM SET 4

RAVI VAKIL

This set is due Monday, November 7. It covers (roughly) classes 8 and 9. Read all of these problems, and hand in six solutions. Two A problems count for one solution. One B problem counts for one solution. Try to solve problems on a range of topics. If you are pressed for time, try more straightforward problems. If you are ambitious, push the envelope a bit. You are encouraged to talk to each other about the problems. (Write up your solutions individually.) You are also encouraged to talk to me about them. Ideally, you should find out who did problems that you didn't do. Make sure you read all the problems, because we will be be making use of many of these results.

A1. Show that $\mathbb{P}^n_k$ is irreducible.

A2. You showed earlier that for affine schemes, there is a bijection between irreducible closed subsets and points. Show that this is true of schemes as well.

A3. Prove the following. If R is Noetherian, then $\operatorname{Spec} R$ is a Noetherian topological space. If X is a scheme that has a finite cover $X = \cup_{i=1}^n \operatorname{Spec} R_i$ where R_i is Noetherian, then X is a Noetherian topological space. Thus $\mathbb{P}^n_k$ and $\mathbb{P}^n_{\mathbb{Z}}$ are Noetherian topological spaces: we built them by gluing together a finite number of Spec's of Noetherian rings.

A4. If R is any ring, show that the irreducible components of $\operatorname{Spec} R$ are in bijection with the minimal primes of R. (Here minimality is with respect to inclusion.)

A5. Show that an irreducible topological space is connected.

A6. Show that a finite union of affine schemes is quasicompact. (Hence $\mathbb{P}^n_k$ is quasicompact.) Show that every closed subset of an affine scheme is quasicompact. Show that every closed subset of a quasicompact scheme is quasicompact.

A7. Show that a scheme is reduced if and only if none of the stalks have nilpotents. Hence show that if f and g are two functions on a reduced scheme that agree at all points, then $f = g$.

A8. Show that an affine scheme $\operatorname{Spec} R$ is integral if and only if R is an integral domain.

A9. Show that a scheme X is integral if and only if it is irreducible and reduced.

A10. Suppose X is an integral scheme. Then X (being irreducible) has a generic point η. Suppose $\operatorname{Spec} R$ is any non-empty affine open subset of X. Show that the stalk at η,

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$\mathcal{O}_{X,\eta}$, is naturally $\operatorname{Frac} R$. This is called the *function field* of X. It can be computed on any non-empty open set of X (as any such open set contains the generic point).

A11. Suppose X is an integral scheme. Show that the restriction maps $\operatorname{res}_{U,V} : \mathcal{O}_X(U) \to \mathcal{O}_X(V)$ are inclusions so long as $V \neq \emptyset$. Suppose $\operatorname{Spec} R$ is any non-empty affine open subset of X (so R is an integral domain). Show that the natural map $\mathcal{O}_X(U) \to \mathcal{O}_{X,\eta} = \operatorname{Frac} R$ (where U is any non-empty open set) is an inclusion.

A12. Suppose $f(x, y)$ and $g(x, y)$ are two complex polynomials ($f, g \in \mathbb{C}[x, y]$). Suppose f and g have no common factors. Show that the system of equations $f(x, y) = g(x, y) = 0$ has a finite number of solutions.

A13. If R is a finitely generated domain over k, show that $\dim R[x] = \dim R + 1$. (In fact this is true if R is Noetherian. You're welcome to try to prove that. We'll prove it later in the class, and you may use this fact in later problem sets.)

A14. Show that the underlying topological space of a Noetherian scheme is Noetherian. Show that a Noetherian scheme has a finite number of irreducible components.

A15. Suppose X is an integral scheme, that can be covered by open subsets of the form $\operatorname{Spec} R$ where R is a finitely generated domain over k. Then $\dim X$ is the transcendence degree of the function field (the stalk at the generic point) $\mathcal{O}_{X,\eta}$ over k. Thus (as the generic point lies in all non-empty open sets) the dimension can be computed in any open set of X.

A16. What is the dimension of $\operatorname{Spec} k[w, x, y, z]/(wx - yz, x^{17} + y^{17})$? (Be careful to check the hypotheses before invoking Krull!)

A17. Suppose that R is a finitely generated domain over k, and $\mathfrak{p}$ is a prime ideal. Show that $\dim R_{\mathfrak{p}} = \dim R - \dim R/\mathfrak{p}$.

A18. Show that all open subsets of a Noetherian topological space (hence of a Noetherian scheme) are quasicompact.

A19. Check that our new definition of reduced (in terms of affine covers) agrees with our earlier definition. This definition is advantageous: our earlier definition required us to check that the ring of functions over *any* open set is nilpotent free. This lets us check in an affine cover. Hence for example $\mathbb{A}^n_k$ and $\mathbb{P}^n_k$ are reduced.

A20. If R is a unique factorization domain, show that R is integrally closed (in its fraction field $\operatorname{Frac}(R)$). Hence $\mathbb{A}^n_k$ and $\mathbb{P}^n_k$ are both normal.

A21. Suppose R is a ring, and $(f_1, \ldots, f_n) = R$. Show that if R has no nonzero nilpotents (i.e. 0 is a radical ideal), then R_{f_i} also has no nonzero nilpotents. Show that if no R_{f_i} has a nonzero nilpotent, then neither does R.

A22. Suppose R is an integral domain. Show that if R is integrally closed, then so is R_f.

A23. Suppose X is a quasicompact scheme, and f is a function vanishing on all the points of X. Show that $f^n = 0$ for some n. Show that this can be false without the quasicompact hypothesis.

B1. Show that $\left(k[x,y]/(xy,x^2)\right)_y$ has no nilpotents. (Hint: show that it is isomorphic to another ring, by considering the geometric picture.)

B2. Give (with proof!) an example of a scheme that is connected but reducible.

B3. Show that $\dim \mathbb{A}^1_{\mathbb{Z}} = 2$.

B4. Suppose that R is a Unique Factorization Domain containing $1/2$, $f \in R$ has no repeated prime factors, and $z^2 - f$ is irreducible in $R[z]$. Show that $\operatorname{Spec} R[z]/(z^2 - f)$ is normal. (Hint: one of Gauss' Lemmas.) Show that the following schemes are normal: $\operatorname{Spec} \mathbb{Z}[x]/(x^2 - n)$ where n is a square-free integer congruent to $3 \pmod 4$; $\operatorname{Spec} k[x_1, \dots, x_n]/x_1^2 + x_2^2 + \cdots + x_m^2$ where $\operatorname{char} k \neq 2$, $m \geq 3$; $\operatorname{Spec} k[w,x,y,z]/(wx - yz)$ where $\operatorname{char} k \neq 2$ and k is algebraically closed. Show that if f has repeated prime factors, then $\operatorname{Spec} R[z]/(z^2 - f)$ is *not* normal.

B5. Show that $\operatorname{Spec} k[w,x,y,z]/(wz - xy, wy - x^2, xz - y^2)$ is an irreducible surface. (It is no harder to show that it is an integral surface.) We will see next week that this is the affine cone over the twisted cubic.

B6. Suppose $X = \operatorname{Spec} R$ where R is a Noetherian domain, and Z is an irreducible component of $V(r_1, \dots, r_n)$, where $r_1, \dots, r_n \in R$. Show that the height of (the prime associated to) Z is at most n. Conversely, suppose $X = \operatorname{Spec} R$ where R is a Noetherian domain, and Z is an irreducible subset of height n. Show that there are $f_1, \dots, f_n \in R$ such that Z is an irreducible component of $V(f_1, \dots, f_n)$.

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FOUNDATIONS OF ALGEBRAIC GEOMETRY PROBLEM SET 5

RAVI VAKIL

This set is due Monday, November 14. It covers (roughly) classes 10, 11, and 12.

As you might have noticed, last week there were a **lot** of interesting problems worth trying — too many to do! (This is just because we've gone far enough that we can really explore interesting questions.) So please *read all of the problems*, and ask me about any statements that you are unsure of, even of the many problems you won't try. Hand in six solutions. If you are ambitious (and have the time), go for more. **Problems marked with "-" count for half a solution.** Problems marked with "+" may be harder or more fundamental, but still count for one solution. Try to solve problems on a range of topics. You are encouraged to talk to each other, and to me, about the problems.

Class 8:

1. (a) Use dimension theory to prove a microscopically stronger version of the weak Nullstellensatz: Suppose $R = k[x_1, \dots, x_n]/I$, where k is an algebraically closed field and I is some ideal. Then the maximal ideals are precisely those of the form $(x_1 - a_1, \dots, x_n - a_n)$, where $a_i \in k$.
(b) Suppose $R = k[x_1, \dots, x_n]/I$ where k is not necessarily algebraically closed. Show that every maximal ideal of R has a residue field that is a finite extension of k. [Hint for both: the maximal ideals correspond to dimension 0 points, which correspond to transcendence degree 0 extensions of k, i.e. finite extensions of k. If k is algebraically closed, the maximal ideals correspond to surjections $f : k[x_1, \dots, x_n] \to k$. Fix one such surjection. Let $a_i = f(x_i)$, and show that the corresponding maximal ideal is $(x_1 - a_1, \dots, x_n - a_n)$.]

Class 10:

2+. Suppose R is a ring, and $(f_1, \dots, f_n) = R$. Suppose A is a ring, and R is an A-algebra. Show that if each R_{f_i} is a finitely-generated A-algebra, then so is R.

3. Show that an irreducible homogeneous polynomial in $n + 1$ variables (over a field k) describes an integral scheme of dimension $n - 1$. We think of this as a "hypersurface in $\mathbb{P}^n_k$".

4. Show that $wx = yz, x^2 = wy, y^2 = xz$ describes an irreducible curve in $\mathbb{P}^3_k$ (the twisted cubic!).

5. Suppose S_* is a graded ring (with grading $\mathbb{Z}^{\geq 0}$). It is automatically a module over S_0. Now $S_+ := \oplus_{i>0} S_i$ is an ideal, which we will call the *irrelevant ideal*; suppose that it is a finitely generated ideal. Show that S_* is a finitely-generated S_0-algebra.

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6+. Recall the definition of the distinguished open subset $D(f)$ on $\operatorname{Proj} S_*$, where f is homogeneous of positive degree. Show that

$$(D(f), \mathcal{O}_{\operatorname{Proj} S_*}) \cong \operatorname{Spec}(S_f)_0$$

defines a sheaf on $\operatorname{Proj} S_*$. (We used this to define the structure sheaf $\mathcal{O}_{\operatorname{Proj} S_*}$ on $\operatorname{Proj} S_*$.)

7-. Show that $\operatorname{Proj} k[x_0, \ldots, x_n]$ is isomorphic to our earlier definition of $\mathbb{P}^n$.

8-. Show that $Y = \mathbb{P}^2 - (x^2 + y^2 + z^2 = 0)$ is affine, and find its corresponding ring (= find its ring of global sections).

Class 11:

9-. Show that $\mathbb{P}^0_A = \operatorname{Proj} A[T] \cong A$. Thus "$\operatorname{Spec} A$ is a projective A-scheme".

10. Show that all projective A-schemes are quasicompact. (Translation: show that any projective A-scheme is covered by a finite number of affine open sets.) Show that $\operatorname{Proj} S_*$ is finite type over $A = S_0$. If S_0 is a Noetherian ring, show that $\operatorname{Proj} S_*$ is a Noetherian scheme, and hence that $\operatorname{Proj} S_*$ has a finite number of irreducible components. Show that any quasiprojective scheme is locally of finite type over A. If A is Noetherian, show that any quasiprojective A-scheme is quasicompact, and hence of finite type over A.

11. Give an example of a quasiprojective A-scheme that is not quasicompact (necessarily for some non-Noetherian A).

12-. Show that $\mathbb{P}^n_k$ is normal. More generally, show that $\mathbb{P}^n_R$ is normal if R is a Unique Factorization Domain.

13+. Show that the projective cone of $\operatorname{Proj} S_*$ has an open subscheme $D(T)$ that is the affine cone, and that its complement $V(T)$ can be identified with $\operatorname{Proj} S_*$ (as a topological space). (More precisely, setting $T = 0$ "cuts out" a scheme isomorphic to $\operatorname{Proj} S_*$ — see if you can make that precise.)

14. If S_* is a finitely generated domain over k, and $\operatorname{Proj} S_*$ is non-empty show that $\dim \operatorname{Spec} S_* = \dim \operatorname{Proj} S_* + 1$.

15. Show that the irreducible subsets of dimension $n-1$ of $\mathbb{P}^n_k$ correspond to homogeneous irreducible polynomials modulo multiplication by non-zero scalars.

16+.

(a) Suppose I is any homogeneous ideal, and f is a homogeneous element. Suppose f vanishes on $V(I)$. Show that $f^n \in I$ for some n. (Hint: Mimic the proof in the affine case.)
(b) If $Z \subset \operatorname{Proj} S_*$, define $I(\cdot)$. Show that it is a homogeneous ideal. For any two subsets, show that $I(Z_1 \cup Z_2) = I(Z_1) \cap I(Z_2)$.
(c) For any homogeneous ideal I with $V(I) \neq \emptyset$, show that $I(V(I)) = \sqrt{I}$.
(d) For any subset $Z \subset \operatorname{Proj} S_*$, show that $V(I(Z)) = \overline{Z}$.

17. Show that the following are equivalent. (a) $V(I) = \emptyset$ (b) for any f_i (i in some index set) generating I, $\cup D(f_i) = \operatorname{Proj} S_*$ (c) $\sqrt{I} \supset S_+$.

18+. Show that $\operatorname{Proj} S_{n\cdot}$ is isomorphic to $\operatorname{Proj} S_*$.

For problems 19-21, suppose $S_* = k[x,y]$ (with the usual grading).

19. Show that $S_{2\cdot} \cong k[u,v,w]/(uw - v^2)$. (Thus the 2-uple Veronese embedding of $\mathbb{P}^1$ is as a conic in $\mathbb{P}^2$.)

20. Show that $\operatorname{Proj} S_{3\cdot}$ is the *twisted cubic* "in" $\mathbb{P}^3$. (The equations of the twisted cubic turn up in problems 4 and 39.)

21+. Show that $\operatorname{Proj} S_{d\cdot}$ is given by the equations that

$$\begin{pmatrix} y_0 & y_1 & \cdots & y_{d-1} \\ y_1 & y_2 & \cdots & y_d \end{pmatrix}$$

is rank 1 (i.e. that all the 2×2 minors vanish). This is called the *degree* d *rational normal curve* "in" $\mathbb{P}^d$.

22. Find the equations cutting out the *Veronese surface* $\operatorname{Proj} S_{2\cdot}$ where $S_* = k[x_0, x_1, x_2]$, which sits naturally in $\mathbb{P}^5$.

23. Show that $\mathbb{P}(m,n)$ is isomorphic to $\mathbb{P}^1$. Show that $\mathbb{P}(1,1,2) \cong \operatorname{Proj} k[u,v,w,z]/(uw - v^2)$. Hint: do this by looking at the even-graded parts of $k[x_0, x_1, x_2]$. (Picture: this is a projective cone over a conic curve.)

24+. (This is a handy exercise for later.) (a) *(Hypersurfaces meet everything of dimension at least* 1 *in projective space — unlike in affine space.)* Suppose X is a closed subset of $\mathbb{P}^n_k$ of dimension at least 1, and H a nonempty hypersurface in $\mathbb{P}^n_k$. Show that H meets X. (Hint: consider the affine cone, and note that the cone over H contains the origin. Use Krull's Principal Ideal Theorem.)
(b) (Definition: Subsets in $\mathbb{P}^n$ cut out by linear equations are called *linear subspaces*. Dimension 1, 2 linear subspaces are called *lines* and *planes* respectively.) Suppose $X \hookrightarrow \mathbb{P}^n_k$ is a closed subset of dimension r. Show that any codimension r linear space meets X. (Hint: Refine your argument in (a).)
(c) Show that there is a codimension $r+1$ complete intersection (codimension $r+1$ set that is the intersection of $r+1$ hypersurfaces) missing X. (The key step: show that there is a hypersurface that doesn't contain every generic point of X.) If k is infinite, show that there is a codimension $r+1$ linear subspace missing X. (The key step: show that there is a hyperplane not containing any generic point of a component of X.)

25. Describe all the lines on the quadric surface $wx - yz = 0$ in $\mathbb{P}^3_k$. (Hint: they come in two "families", called the *rulings* of the quadric surface.)

26. (This is intended for people who already know what derivations are.) In differential geometry, the tangent space at a point is sometimes defined as the vector space of derivations at that point. A derivation is a function that takes in functions near the point that vanish at the point, and gives elements of the field k, and satisfies the Leibniz rule

$(fg)' = f'g + g'f$. Show that this agrees with our definition of tangent space. (One direction was shown in class 11.)

27+. *(Nakayama's lemma version 3)* Suppose R is a ring, and I is an ideal of R contained in all maximal ideals. Suppose M is a *finitely generated* R-module, and $N \subset M$ is a submodule. If $N/IN \hookrightarrow M/IM$ an isomorphism, then $M = N$.

28+. *(Nakayama's lemma version 4)* Suppose $(R, \mathfrak{m})$ is a local ring. Suppose M is a finitely-generated R-module, and $f_1, \dots, f_n \in M$, with (the images of) $f_1, \dots, f_n$ generating $M/\mathfrak{m}M$. Then $f_1, \dots, f_n$ generate M. (In particular, taking $M = \mathfrak{m}$, if we have generators of $\mathfrak{m}/\mathfrak{m}^2$, they also generate $\mathfrak{m}$.)

Class 12:

29-. Show that if A is a Noetherian local ring, then A has finite dimension. (*Warning:* Noetherian rings in general could have infinite dimension.)

30+. *(the Jacobian criterion for checking nonsingularity)* Suppose k is an algebraically closed field, and X is a finite type k-scheme. Then locally it is of the form $\operatorname{Spec} k[x_1, \dots, x_n]/(f_1, \dots, f_r)$. Show that the Zariski tangent space at the closed point p (with residue field k, by the Nullstellensatz) is given by the cokernel of the Jacobian map $k^r \to k^n$ given by the Jacobian matrix

$$J = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(p) & \cdots & \frac{\partial f_r}{\partial x_1}(p) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_1}{\partial x_n}(p) & \cdots & \frac{\partial f_r}{\partial x_n}(p) \end{pmatrix}. \tag{1}$$

(This is just making precise our example of a curve in $\mathbb{A}^3$ cut out by a couple of equations, where we picked off the linear terms .) Possible hint: "translate p to the origin," and consider linear terms.

31. Show that the singular *closed* points of the hypersurface $f(x_1, \dots, x_n) = 0$ in $\mathbb{A}^n_k$ are given by the equations $f = \frac{\partial f}{\partial x_1} = \cdots = \frac{\partial f}{\partial x_n} = 0$.

32. Show that $\mathbb{A}^1$ and $\mathbb{A}^2$ are nonsingular. (Make sure to check nonsingularity at the non-closed points! Fortunately you know what all the points of $\mathbb{A}^2$ are; this is trickier for $\mathbb{A}^3$.) You are not allowed to use the fact that regular local rings remain regular under localization.

33. Show that $\operatorname{Spec} \mathbb{Z}$ is a nonsingular curve.

34. Note that $\mathbb{Z}[i]$ is dimension 1, as $\mathbb{Z}[x]$ has dimension 2 (problem set exercise), and is a domain, and $x^2 + 1$ is not 0, so $\mathbb{Z}[x]/(x^2 + 1)$ has dimension 1 by Krull. Show that $\operatorname{Spec} \mathbb{Z}[i]$ is a nonsingular curve. (This exercise is intended for people who know about the primes in the Gaussian integers $\mathbb{Z}[i]$.)

35. Show that there is one singular point of $\operatorname{Spec} \mathbb{Z}[2i]$, and describe it.

36. *(the Euler test for projective hypersurfaces)* There is an analogous Jacobian criterion for hypersurfaces $f = 0$ in $\mathbb{P}^n_k$. Show that the singular *closed* points correspond to the locus $f = \frac{\partial f}{\partial x_1} = \cdots = \frac{\partial f}{\partial x_n} = 0$. If the degree of the hypersurface is not divisible by the characteristic of any of the residue fields (e.g. if we are working over a field of characteristic 0), show that it suffices to check $\frac{\partial f}{\partial x_1} = \cdots = \frac{\partial f}{\partial x_n} = 0$. (Hint: show that f lies in the ideal $(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n})$). (Fact: this will give the singular points in general. I don't want to prove this, and I won't use it.)

37-. Suppose k is algebraically closed. Show that $y^2 z = x^3 - xz^2$ in $\mathbb{P}^2_k$ is an irreducible nonsingular curve. (This is for practice.) Warning: I didn't say $\operatorname{char} k = 0$.

38-. Find all the singular closed points of the following plane curves. Here we work over a field of characteristic 0 for convenience.

(a) $y^2 = x^2 + x^3$. This is called a *node*.
(b) $y^2 = x^3$. This is called a *cusp*.
(c) $y^2 = x^4$. This is called a *tacnode*.

39. Show that the twisted cubic $\operatorname{Proj} k[w, x, y, z]/(wz - xy, wy - x^2, xz - y^2)$ is nonsingular. (You can do this by using the fact that it is isomorphic to $\mathbb{P}^1$. I'd prefer you to do this with the explicit equations, for the sake of practice.)

40-. Show that the only dimension 0 Noetherian regular local rings are fields. (Hint: Nakayama.)

41-. Consider the following two examples:
(i) *(the 5-adic valuation)* $K = \mathbb{Q}$, $\nu(r)$ is the "power of 5 appearing in r", e.g. $\nu(35/2) = 1$, $\nu(27/125) = -3$.
(ii) $K = k(x)$, $\nu(f)$ is the "power of x appearing in f.
Describe the valuation rings in those two examples.

42. Show that $0 \cup \{x \in K^* : \nu(x) \geq 1\}$ is the unique maximal ideal of the valuation ring. (Hint: show that everything in the complement is invertible.) Thus the valuation ring is a local ring.

43+. Show that every discrete valuation ring is a Noetherian regular local ring of dimension 1. (This was part of our long theorem showing that many things were equivalent.)

44-. Suppose R is a Noetherian local domain of dimension 1. Show that R is a principal ideal domain if and only if it is a discrete valuation ring.

45-. Show that there is only one discrete valuation on a discrete valuation ring.

46. Suppose X is a regular integral Noetherian scheme, and $f \in \operatorname{Frac}(\Gamma(X, \mathcal{O}_X))^*$ is a nonzero element of its function field. Show that f has a finite number of zeros and poles.

47+. Suppose A is a subring of a ring B, and $x \in B$. Suppose there is a faithful $A[x]$-module M that is finitely generated as an A-module. Show that x is integral over A. (Hint: look carefully at the proof of Nakayama's Lemma version 1 in the Class 11 notes, and change a few words.)

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FOUNDATIONS OF ALGEBRAIC GEOMETRY PROBLEM SET 6

RAVI VAKIL

This set is due Wednesday, November 30. It covers (roughly) classes 13 and 14.

Please *read all of the problems*, and ask me about any statements that you are unsure of, even of the many problems you won't try. Hand in six solutions. If you are ambitious (and have the time), go for more. Problems marked with "-" count for half a solution. Problems marked with "+" may be harder or more fundamental, but still count for one solution. Try to solve problems on a range of topics. You are encouraged to talk to each other, and to me, about the problems. I'm happy to give hints, and some of these problems require hints!

Class 13:

1. Show that $(x, z) \subset k[w, x, y, z]/(wz - xy)$ is a height 1 ideal that is not principal. (Make sure you have a picture of this in your head!)

2. Suppose X is an integral Noetherian scheme, and $f \in \operatorname{Frac}(\Gamma(X, \mathcal{O}_X))^*$ is a non-zero element of its function field. Show that f has a finite number of zeros and poles. (Hint: reduce to $X = \operatorname{Spec} R$. If $f = f_1/f_2$, where $f_i \in R$, prove the result for f_i.)

3. Let R be the subring $k[x^3, x^2, xy, y] \subset k[x, y]$. (The idea behind this example: I'm allowing all monomials in $k[x, y]$ except for x.) Show that it is not integrally closed (easy — consider the "missing x"). Show that it is regular in codimension 1 (hint: show it is dimension 2, and when you throw out the origin you get something nonsingular, by inverting x^2 and y respectively, and considering R_{x^2} and R_y).

4. You have checked that if $k = \mathbb{C}$, then $k[w, x, y, z]/(wx - yz)$ is integrally closed (PS4, problem B5). Show that it is not a unique factorization domain. (The most obvious possibility is to do this "directly", but this might be hard. Another possibility, faster but less intuitive, is to prove the intermediate result that *in a unique factorization domain, any height* 1 *prime is principal*, and considering Exercise 1.)

5. Show that on a Noetherian scheme, you can check nonsingularity by checking at closed points. (Caution: a scheme in general needn't have any closed points!) You are allowed to use the unproved fact from the notes, that any localization of a regular local ring is regular.

6. Show that a nonsingular locally Noetherian scheme is irreducible if and only if it is connected. (I'm not sure if this fact requires Noetherianness.)

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7-. Show that there is a nonsingular hypersurface of degree d. Show that there is a Zariski-open subset of the space of hypersurfaces of degree d. The two previous sentences combine to show that the nonsingular hypersurfaces form a Zariski-open set. Translation: almost all hypersurfaces are smooth.

8-. Suppose $(R, \mathfrak{m}, k)$ is a regular Noetherian local ring of dimension n. Show that $\dim_k(\mathfrak{m}^i/\mathfrak{m}^{i+1}) = \binom{n+i-1}{i}$.

9. Show that fact 2 in the "good facts to know about regular local rings" implies that $(R, \mathfrak{m})$ is a domain. (Hint: show that if $f, g \neq 0$, then $fg \neq 0$, by considering the leading terms.)

Note that we have proved this fact (referred to in the previous problem) if $(R, \mathfrak{m})$ is a Noetherian local ring containing its residue field k. The next three exercises fill out the proof in the notes. Do them only if you are fairly happy with other things.

10. If S is a Noetherian ring, show that $S[[t]]$ is Noetherian. (Hint: Suppose $I \subset S[[t]]$ is an ideal. Let $I_n \subset S$ be the coefficients of t^n that appear in the elements of I form an ideal. Show that $I_n \subset I_{n+1}$, and that I is determined by $(I_0, I_1, I_2, \dots)$.)

11. Show that $\dim k[[t_1, \dots, t_n]]$ is dimension n. (Hint: find a chain of $n+1$ prime ideals to show that the dimension is at least n. For the other inequality, use Krull.)

12. If R is a Noetherian local ring, show that $\hat{R} := \lim_{\leftarrow} R/\mathfrak{m}^n$ is a Noetherian local ring. (Hint: Suppose $\mathfrak{m}/\mathfrak{m}^2$ is finite-dimensional over k, say generated by $x_1, \dots, x_n$. Describe a surjective map $k[[t_1, \dots, t_n]] \to \hat{R}$.)

13. Show that a section of a sheaf on the distinguished affine base is determined by the section's germs.

14+. Recall Theorem 4.2(a) in the class 13 notes, which states that a sheaf on the distinguished affine base $\mathcal{F}^b$ determines a unique sheaf $\mathcal{F}$, which when restricted to the affine base is $\mathcal{F}^b$. We defined

$$\mathcal{F}(U) := \{(f_x \in \mathcal{F}^b_x)_{x \in U} : \forall x \in U, \exists U_X \text{ with } x \subset U_x \subset U, F^x \in \mathcal{F}^b(U_x) : F^x_y = f_y \forall y \in U_x\}$$

where each U_x is in our base. In class I claimed that if U is in our base, that $\mathcal{F}(U) = \mathcal{F}^b(U)$. We clearly have a map $\mathcal{F}^b(U) \to \mathcal{F}(U)$. Prove that it is an isomorphism.

15+. Show that a sheaf of $\mathcal{O}_X$-modules on "the distinguished affine base" yields an $\mathcal{O}_X$-module.

Class 14:

16+. (a first example of the total complex of a double complex) Suppose $0 \to A \to B \to C$ is exact. Define the total complex

$$\begin{array}{ccccccc} 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C \\ & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle -\mathrm{id}} & & \\ 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C \end{array}$$

as

$$0 \to A \to A \oplus B \to B \oplus C$$

in the "obvious" way. Show that the total complex is also exact.

17. (a) Suppose $X = \operatorname{Spec} k[t]$. Let $\mathcal{F}$ be the skyscraper sheaf supported at the origin $[(t)]$, with group $k(t)$. Give this the structure of an $\mathcal{O}_X$-module. Show that this is not a quasicoherent sheaf. (More generally, if X is an integral scheme, and $p \in X$ that is not the generic point, we could take the skyscraper sheaf at p with group the function field of X. Except in a silly circumstances, this sheaf won't be quasicoherent.)
(b) Suppose $X = \operatorname{Spec} k[t]$. Let $\mathcal{F}$ be the skyscraper sheaf supported at the generic point $[(0)]$, with group $k(t)$. Give this the structure of an $\mathcal{O}_X$-module. Show that this *is* a quasicoherent sheaf. Describe the restriction maps in the distinguished topology of X.

18+. (Important Exercise for later) Suppose X is a Noetherian scheme. Suppose $\mathcal{F}$ is a quasicoherent sheaf on X, and let $f \in \Gamma(X, \mathcal{O}_X)$ be a function on X. Let $R = \Gamma(X, \mathcal{O}_X)$ for convenience. Show that the restriction map $\operatorname{res}_{X_f \subset X} : \Gamma(X, \mathcal{F}_X) \to \Gamma(X_f, \mathcal{F}_X)$ (here X_f is the open subset of X where f doesn't vanish) is precisely localization. In other words show that there is an isomorphism $\Gamma(X, \mathcal{F})_f \to \Gamma(X_f, \mathcal{F})$ making the following diagram commute.

$$\begin{array}{ccc} \Gamma(X, \mathcal{F}) & \xrightarrow{\operatorname{res}_{X_f \subset X}} & \Gamma(X_f, \mathcal{F}) \\ & {\scriptstyle \otimes_R R_f} \searrow \quad \nearrow {\scriptstyle \sim} & \\ & \Gamma(X, \mathcal{F})_f & \end{array}$$

All that you should need in your argument is that X admits a cover by a finite number of open sets, and that their pairwise intersections are each quasicompact. We will later rephrase this as saying that X is quasicompact and quasiseparated. (Hint: cover by affine open sets. Use the sheaf property. A nice way to formalize this is the following. Apply the exact functor $\otimes_R R_f$ to the exact sequence

$$0 \to \Gamma(X, \mathcal{F}) \to \oplus_i \Gamma(U_i, \mathcal{F}) \to \oplus \Gamma(U_{ijk}, \mathcal{F})$$

where the U_i form a finite cover of X and U_{ijk} form an affine cover of $U_i \cap U_j$.)

19-. Give a counterexample to show that the above statement need not hold if X is not quasicompact. (Possible hint: take an infinite disjoint union of affine schemes.)

20. (This is for arithmetically-minded people only — I won't define my terms.) Prove that a fractional ideal on a ring of integers in a number field yields an invertible sheaf. Show that any two that differ by a principal ideal yield the same invertible sheaf. (Thus we have described a map from the class group of the number field to the Picard group of its ring of integers. We will later see that this is an isomorphism.)

21+. Show that you can check exactness of a sequence of quasicoherent sheaves on an affine cover. (In particular, taking sections over an affine open $\operatorname{Spec} R$ is an exact functor from the category of quasicoherent sheaves on X to the category of R-modules. Recall that taking sections is only left-exact in general. Similarly, you can check surjectivity on an affine cover unlike sheaves in general.)

22+. If $\mathcal{F}$ and $\mathcal{G}$ are quasicoherent sheaves, show that $\mathcal{F} \otimes \mathcal{G}$ is given by the following information: If $\operatorname{Spec} R$ is an affine open, and $\Gamma(\operatorname{Spec} R, \mathcal{F}) = M$ and $\Gamma(\operatorname{Spec} R, \mathcal{G}) = N$, then $\Gamma(\operatorname{Spec} R, \mathcal{F} \otimes \mathcal{G}) = M \otimes N$, and the restriction map $\Gamma(\operatorname{Spec} R, \mathcal{F} \otimes \mathcal{G}) \to \Gamma(\operatorname{Spec} R_f, \mathcal{F} \otimes \mathcal{G})$ is precisely the localization map $M \otimes_R N \to (M \otimes_R N)_f \cong M_f \otimes_{R_f} N_f$. (We are using the algebraic fact that that $(M \otimes_R N)_f \cong M_f \otimes_{R_f} N_f$. You can prove this by universal property if you want, or by using the explicit construction.)

23. If $\mathcal{F}$ and $\mathcal{G}$ are locally free sheaves, show that $\mathcal{F} \otimes \mathcal{G}$ is locally free. (Possible hint for this, and later exercises: check on sufficiently small affine open sets.)

24. Prove the following.
(a) Tensoring by a quasicoherent sheaf is right-exact. More precisely, if $\mathcal{F}$ is a quasicoherent sheaf, and $\mathcal{G}' \to \mathcal{G} \to \mathcal{G}'' \to 0$ is an exact sequence of quasicoherent sheaves, then so is $\mathcal{G}' \otimes \mathcal{F} \to \mathcal{G} \otimes \mathcal{F} \to \mathcal{G}'' \otimes \mathcal{F} \to 0$ is exact.
(b) Tensoring by a locally free sheaf is exact. More precisely, if $\mathcal{F}$ is a quasicoherent sheaf, and $\mathcal{G}' \to \mathcal{G} \to \mathcal{G}''$ is an exact sequence of quasicoherent sheaves, then then so is $\mathcal{G}' \otimes \mathcal{F} \to \mathcal{G} \otimes \mathcal{F} \to \mathcal{G}'' \otimes \mathcal{F}$.
(c) The stalk of the tensor product of quasicoherent sheaves at a point is the tensor product of the stalks.
(d) Invertible sheaves on a scheme X (up to isomorphism) form a group. This is called the Picard group of X, and is denoted $\operatorname{Pic} X$. For arithmetic people: this group, for the Spec of the ring of integers R in a number field, is the class group of R.

25. Show that sheaf Hom, $\underline{\mathrm{Hom}}$, is quasicoherent, and is what you think it might be. (Describe it on affine opens, and show that it behaves well with respect to localization with respect to f. To show that $\operatorname{Hom}_A(M, N)_f \cong \operatorname{Hom}_{A_f}(M_f, N_f)$, take a "partial resolution" $A^q \to A^p \to M \to 0$, and apply $\operatorname{Hom}(\cdot, N)$ and localize.) ($\underline{\mathrm{Hom}}$ was defined earlier, and was the subject of a homework problem.) Show that $\underline{\mathrm{Hom}}$ is a left-exact functor in both variables.

26+. Show that if $\mathcal{F}$ is locally free then $\mathcal{F}^\vee$ is locally free, and that there is a canonical isomorphism $(\mathcal{F}^\vee)^\vee \cong \mathcal{F}$. (Caution: your argument showing that if there is a canonical isomorphism $(\mathcal{F}^\vee)^\vee \cong \mathcal{F}$ better not also show that there is a canonical isomorphism $\mathcal{F}^\vee \cong \mathcal{F}$! We'll see an example soon of a locally free $\mathcal{F}$ that is not isomorphic to its dual. The example will be the line bundle $\mathcal{O}(1)$ on $\mathbb{P}^1$.)

27. The direct sum of quasicoherent sheaves is what you think it is.

For the next exercises, recall the following. If M is an A-module, then the *tensor algebra* $T^*(M)$ is a non-commutative algebra, graded by $\mathbb{Z}^{\geq 0}$, defined as follows. $T^0(M) = A$, $T^n(M) = M \otimes_A \cdots \otimes_A M$ (where n terms appear in the product), and multiplication is what you expect. The *symmetric algebra* $\operatorname{Sym}^* M$ is a symmetric algebra, graded by $\mathbb{Z}^{\geq 0}$, defined as the quotient of $T^*(M)$ by the (two-sided) ideal generated by all elements of the form $x \otimes y - y \otimes x$ for all $x, y \in M$. Thus $\operatorname{Sym}^n M$ is the quotient of $M \otimes \cdots \otimes M$ by the relations of the form $m_1 \otimes \cdots \otimes m_n - m_1' \otimes \cdots \otimes m_n'$ where $(m_1', \dots, m_n')$ is a rearrangement of $(m_1, \dots, m_n)$. The *exterior algebra* $\wedge^* M$ is defined to be the quotient of $T^* M$ by the (two-sided) ideal generated by all elements of the form $x \otimes y + y \otimes x$ for all $x, y \in M$. Thus $\wedge^n M$ is the quotient of $M \otimes \cdots \otimes M$ by the relations of the form $m_1 \otimes \cdots \otimes m_n - (-1)^{\operatorname{sgn}} m_1' \otimes \cdots \otimes m_n'$ where $(m_1', \dots, m_n')$ is a rearrangement of $(m_1, \dots, m_n)$, and the sgn is even if the rearrangement is an even permutation, and odd if the rearrangement is an odd permutation. (It is a "skew-commutative" A-algebra.) It is most correct to write $T_A^*(M)$, $\operatorname{Sym}_A^*(M)$, and $\wedge_A^*(M)$, but the "base ring" is usually omitted for convenience.

28. If $\mathcal{F}$ is a quasicoherent sheaf, then define the quasicoherent sheaves $T^n \mathcal{F}$, $\operatorname{Sym}^n \mathcal{F}$, and $\wedge^n \mathcal{F}$. If $\mathcal{F}$ is locally free of rank m, show that $T^n \mathcal{F}$, $\operatorname{Sym}^n \mathcal{F}$, and $\wedge^n \mathcal{F}$ are locally free, and find their ranks.

29+. If $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ is an exact sequence of locally free sheaves, then for any r, there is a filtration of $\operatorname{Sym}^r \mathcal{F}$:

$$\operatorname{Sym}^r \mathcal{F} = F^0 \supseteq F^1 \supseteq \cdots \supseteq F^r \supset F^{r+1} = 0$$

with quotients

$$F^p / F^{p+1} \cong (\operatorname{Sym}^p \mathcal{F}') \otimes (\operatorname{Sym}^{r-p} \mathcal{F}'')$$

for each p.

30. Suppose $\mathcal{F}$ is locally free of rank n. Then $\wedge^n \mathcal{F}$ is called the *determinant (line) bundle*. Show that $\wedge^r \mathcal{F} \times \wedge^{n-r} \mathcal{F} \to \wedge^n \mathcal{F}$ is a perfect pairing for all r.

31+. If $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ is an exact sequence of locally free sheaves, then for any r, there is a filtration of $\wedge^r \mathcal{F}$:

$$\wedge^r \mathcal{F} = F^0 \supseteq F^1 \supseteq \cdots \supseteq F^r \supset F^{r+1} = 0$$

with quotients

$$F^p / F^{p+1} \cong (\wedge^p \mathcal{F}') \otimes (\wedge^{r-p} \mathcal{F}'')$$

for each p. In particular, $\det \mathcal{F} = (\det \mathcal{F}') \otimes (\det \mathcal{F}'')$.

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FOUNDATIONS OF ALGEBRAIC GEOMETRY PROBLEM SET 7

RAVI VAKIL

This set is due Wednesday, December 7. It covers (roughly) classes 15 and 16.

Please *read all of the problems*, and ask me about any statements that you are unsure of, even of the many problems you won't try. Hand in six solutions, *including # 23*. If you are ambitious (and have the time), go for more. Problems marked with "-" count for half a solution. Problems marked with "+" may be harder or more fundamental, but still count for one solution. Try to solve problems on a range of topics. You are encouraged to talk to each other, and to me, about the problems. I'm happy to give hints, and some of these problems require hints!

Class 15:

You are not allowed to try the next four problems if you already know how to do them!

1. M Noetherian implies that any submodule of M is a finitely generated R-module. Hence for example if R is a Noetherian ring then finitely generated = Noetherian.

2. If $0 \to M' \to M \to M'' \to 0$ is exact, then M' and M'' are Noetherian if and only if M is Noetherian. (Hint: Given an ascending chain in M, we get two simultaneous ascending chains in M' and M''.)

3. A Noetherian as an A-module implies A^n is a Noetherian A-module.

4. If A is a Noetherian ring and M is a finitely generated A-module, then any submodule of M is finitely generated. (Hint: suppose $M' \hookrightarrow M$ and $A^n \twoheadrightarrow M$. Construct N with

$$\begin{array}{ccc} N & \hookrightarrow & A^n \\ \downarrow & & \downarrow \\ M' & \hookrightarrow & M \end{array}$$

.)

5-. Show A is coherent (as an A-module) if and only if the notion of finitely presented agrees with the notion of coherent.

6. If $f \in A$, show that if M is a finitely generated (resp. finitely presented, coherent) A-module, then M_f is a finitely generated (resp. finitely presented, coherent) A_f-module. (Hint: localization is exact.)

7. If $(f_1, \ldots, f_n) = A$, and M_{f_i} is finitely generated (resp. coherent) A_{f_i}-module for all i, then M is a finitely generated (resp. coherent) A-module.

Date: Monday, November 28, 2005. Minor update January 23, 2006.

8. *(Exercise on support of a sheaf)* Show that the support of a finite type quasicoherent sheaf on a scheme is a closed subset. (Hint: Reduce to an affine open set. Choose a finite set of generators of the corresponding module.) Show that the support of a quasicoherent sheaf need not be closed. (Hint: If $A = \mathbb{C}[t]$, then $\mathbb{C}[t]/(t-a)$ is an A-module supported at a. Consider $\oplus_{a \in \mathbb{C}} \mathbb{C}[t]/(t-a)$.)

9. *(Exercise on rank)*

(a) If $m_1, \dots, m_n$ are generators at P, they are generators in an open neighborhood of P. (Hint: Consider $\operatorname{coker} A^n \xrightarrow{(f_1,\dots,f_n)} M$ and Exercise 8.)

(b) Show that at any point, $\operatorname{rank}(\mathcal{F} \oplus \mathcal{G}) = \operatorname{rank}(\mathcal{F}) + \operatorname{rank}(\mathcal{G})$ and $\operatorname{rank}(\mathcal{F} \otimes \mathcal{G}) = \operatorname{rank} \mathcal{F} \operatorname{rank} \mathcal{G}$ at any point. (Hint: Show that direct sums and fibered products commute with ring quotients and localizations, i.e. $(M \oplus N) \otimes_R (R/I) \cong M/IM \oplus N/IN$, $(M \otimes_R N) \otimes_R (R/I) \cong (M \otimes_R R/I) \otimes_{R/I} (N \otimes_R R/I) \cong M/IM \otimes_{R/I} N/IM$, etc.) Thanks to Jack Hall for improving this problem.

(c) Show that rank is an upper semicontinuous function on X. (Hint: Generators at P are generators nearby.)

10. If X is reduced, $\mathcal{F}$ is coherent, and the rank is constant, show that $\mathcal{F}$ is locally free. (Hint: choose a point $p \in X$, and choose generators of the stalk $\mathcal{F}_p$. Let U be an open set where the generators are sections, so we have a map $\phi : \mathcal{O}_U^{\oplus n} \to \mathcal{F}|_U$. The cokernel and kernel of ϕ are supported on closed sets by Exercise 8. Show that these closed subsets don't include p. Make sure you use the reduced hypothesis!) Thus coherent sheaves are locally free on a dense open set. Show that this can be false if X is not reduced. (Hint: $\operatorname{Spec} k[x]/x^2$, $M = k$.)

11. *(Geometric Nakayama)* Suppose X is a scheme, and $\mathcal{F}$ is a finite type quasicoherent sheaf. Show that if $\mathcal{F}_x \otimes k(x) = 0$, then there exists V such that $\mathcal{F}|_V = 0$. Better: if I have a set that generates the fiber, it defines the stalk.

12. *(Reason for the name "invertible" sheaf)* Suppose $\mathcal{F}$ and $\mathcal{G}$ are finite type sheaves such that $\mathcal{F} \otimes \mathcal{G} \cong \mathcal{O}_X$. Then $\mathcal{F}$ and $\mathcal{G}$ are both invertible (Hint: Nakayama.) This is the reason for the adjective "invertible" these sheaves are the invertible elements of the monoid of finite type sheaves. This exercise is a little less important.

13. *(A non-quasicoherent sheaf of ideals)* Let $X = \operatorname{Spec} k[x]_{(x)}$, the germ of the affine line at the origin, which has two points, the closed point and the generic point η. Define $\mathcal{I}(X) = \{0\} \subset \mathcal{O}_X(X) = k[x]_{(x)}$, and $\mathcal{I}(\eta) = k(x) = \mathcal{O}_X(\eta)$. Show that $\mathcal{I}$ is not a quasicoherent sheaf of ideals.

14. *(Sections of locally free sheaves cut out closed subschemes)* Suppose $\mathcal{F}$ is a locally free sheaf on a scheme X, and s is a section of $\mathcal{F}$. Describe how $s = 0$ "cuts out" a closed subscheme.

15. *(Reduction of a scheme)*

(a) X^{red} has the same underlying topological space as X: there is a natural homeomorphism of the underlying topological spaces $X^{\mathrm{red}} \cong X$. Picture: taking the reduction may be interpreted as shearing off the fuzz on the space.

(b) Give an example to show that it is *not* true that $\Gamma(X^{\mathrm{red}}, \mathcal{O}_{X^{\mathrm{red}}}) = \Gamma(X, \mathcal{O}_X)/\sqrt{\Gamma(X, \mathcal{O}_X)}$. (Hint: $\coprod_{n>0} \operatorname{Spec} k[t]/(t^n)$ with global section $(t, t, t, \dots)$.) Motivation for this exercise: this *is* true on each affine open set.

Class 16:

16. Describe the scheme-theoretic intersection of $(y - x^2)$ and y in $\mathbb{A}^2$. Draw a picture.

17. Suppose we have an effective Cartier divisor, a closed subscheme locally cut out by a single equation. As described in class, this gives an invertible sheaf with a canonical section. Show that this section vanishes along our actual effective Cartier divisor.

18. Describe the invertible sheaf corresponding to an effective Cartier divisor in terms of transition functions. More precisely, on any affine open set where the effective Cartier divisor is cut out by a single equation, the invertible sheaf is trivial. Determine the transition functions between two such overlapping affine open sets. Verify that there is indeed a canonical section of this invertible sheaf, by describing it.

19. Show that $\widetilde{M_*} \otimes \widetilde{N_*} \cong \widetilde{M_* \otimes_{S_*} N_*}$. (Hint: describe the isomorphism of sections over each $D(f)$, and show that this isomorphism behaves well with respect to smaller distinguished opens.)

20. *(Closed immersions in projective S_0-schemes)* Show that if I_* is a graded ideal of S_*, show that we get a closed immersion $\operatorname{Proj} S_*/I_* \hookrightarrow \operatorname{Proj} S_*$.

21. Suppose S_* is generated over S_0 by $f_1, \dots, f_n$. Suppose $d = \operatorname{lcm}(\deg f_1, \dots, \deg f_n)$. Show that S_{d*} is generated in "new" degree 1 (= "old" degree d). (Hint: I like to show this by induction on the size of the set $\{\deg f_1, \dots, \deg f_n\}$.) This is handy, because we can stick every Proj in some projective space via the construction of previous exercise.

22. If S_* is generated in degree 1, show that $\mathcal{O}_{\operatorname{Proj} S_*}(n)$ is an invertible sheaf.

23. (Mandatory exercise — I am happy to walk you through it, see the notes.) Calculate $\dim_k \Gamma(\mathbb{P}^m_k, \mathcal{O}_{\mathbb{P}^m_k}(n))$.

24. Show that $\mathcal{F}(n) \cong \mathcal{F} \otimes \mathcal{O}(n)$.

25. Show that $\mathcal{O}(m+n) \cong \mathcal{O}(m) \otimes \mathcal{O}(n)$.

26. Show that if $m \neq n$, then $\mathcal{O}_{\mathbb{P}^l_k}(m)$ is not isomorphic to $\mathcal{O}_{\mathbb{P}^l_k}(n)$ if $l > 0$. (Hence we have described a countable number of invertible sheaves (line bundles) that are non-isomorphic. We will see later that these are *all* the line bundles on projective space $\mathbb{P}^n_k$.)

27. If quasicoherent sheaves $\mathcal{L}$ and $\mathcal{M}$ are generated by global sections at a point p, then so is $\mathcal{L} \otimes \mathcal{M}$. (This exercise is less important, but is good practice for the concept.)

28. An invertible sheaf $\mathcal{L}$ on X is generated by global sections if and only if for any point $x \in X$, there is a section of $\mathcal{L}$ not vanishing at x. (Hint: Nakayama.)

29+. (Important! A theorem of Serre. See the notes for extensive hints.) Suppose S_0 is a Noetherian ring, and S_* is generated in degree 1. Let $\mathcal{F}$ be any finite type sheaf on $\operatorname{Proj} S_*$. Then for some integer n_0, for all $n \geq n_0$, $\mathcal{F}(n)$ can be generated by a finite number of global sections.

30. Show that Γ_* gives a functor from the category of quasicoherent sheaves on $\operatorname{Proj} S_*$ to the category of graded S_*-modules. (In other words, show that if $\mathcal{F} \to \mathcal{G}$ is a morphism of quasicoherent sheaves on $\operatorname{Proj} S_*$, describe the natural map $\Gamma_*(\mathcal{F}) \to \Gamma_*(\mathcal{G})$, and show that such natural maps respect the identity and composition.)

31. Show that the canonical map $M_* \to \Gamma_* \widetilde{M_*}$ need not be injective, nor need it be surjective. (Hint: $S_* = k[x]$, $M_* = k[x]/x^2$ or $M_* = \{$ polynomials with no constant terms $\}$.)

32. Describe the natural map $\widetilde{\Gamma_* \mathcal{F}} \to \mathcal{F}$ as follows. First describe it over $D(f)$. Note that sections of the left side are of the form m/f^n where $m \in \Gamma_{n \deg f} \mathcal{F}$, and $m/f^n = m'/f^{n'}$ if there is some N with $f^N(f^{n'} m - f^n m') = 0$. Show that your map behaves well on overlaps $D(f) \cap D(g) = D(fg)$.

33+. Show that the natural map $\widetilde{\Gamma_* \mathcal{F}} \to \mathcal{F}$ is an isomorphism, by showing that it is an isomorphism over $D(f)$ for any f. Do this by first showing that it is surjective. This will require following some of the steps of the proof of Serre's theorem (a previous exercise on this set). Then show that it is injective. (This is longer, but worth it.)

34. *(Γ_* and $\sim$ are adjoint functors)* Prove part of the statement that Γ_* and $\sim$ are adjoint functors, by describing a natural bijection $\operatorname{Hom}(M_*, \Gamma_*(\mathcal{F})) \cong \operatorname{Hom}(\widetilde{M_*}, \mathcal{F})$. For the map from left to right, start with a morphism $M_* \to \Gamma_*(\mathcal{F})$. Apply $\sim$, and postcompose with the isomorphism $\widetilde{\Gamma_* \mathcal{F}} \to \mathcal{F}$, to obtain

$$\widetilde{M_*} \to \widetilde{\Gamma_* \mathcal{F}} \to \mathcal{F}.$$

Do something similar to get from right to left. Show that "both compositions are the identity in the appropriate category". (Is there a clever way to do that?)

Coherence: These twenty problems are only for people who are curious about the notion of coherence for general rings. Others should just skip these. (This is the one exception of my injunction to read all problems.)

A. Show that coherent implies finitely presented implies finitely generated.

B. Show that 0 is coherent.

Suppose for problems C–I that

$$0 \to M \to N \to P \to 0 \tag{1}$$

is an exact sequence of A-modules.

Hint ⋆. Here is a *hint* which applies to several of the problems: try to write

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A^p & \longrightarrow & A^{p+q} & \longrightarrow & A^q & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & M & \longrightarrow & N & \longrightarrow & P & \longrightarrow & 0 \end{array}$$

and possibly use the snake lemma.

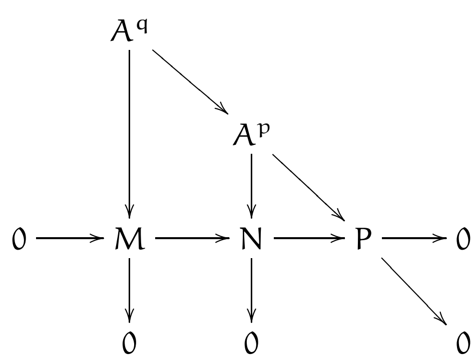
C. Show that N finitely generated implies P finitely generated. (You will only need right-exactness of (1).)

D. Show that M, P finitely generated implies N finitely generated. (Possible hint: ⋆.) (You will only need right-exactness of (1).)

E. Show that N, P finitely generated need not imply M finitely generated. (Hint: if I is an ideal, we have $0 \to I \to A \to A/I \to 0$.)

F. Show that N coherent, M finitely generated implies M coherent. (You will only need left-exactness of (1).)

G. Show that N, P coherent implies M coherent. Hint for (i) in the definition of coherence:



(You will only need left-exactness of (1).)

H. Show that M finitely generated and N coherent implies P coherent. (Hint for (ii) in the definition of coherence: ⋆. You will only right-exactness of (1).)

I. Show that M, P coherent implies N coherent. (Hint: ⋆.)

At this point, we have shown that if two of (1) are coherent, the third is as well.

J. Show that a finite direct sum of coherent modules is coherent.

K. Suppose M is finitely generated, N coherent. Then if $\phi : M \to N$ is any map, then show that $\operatorname{Im} \phi$ is coherent.

L. Show that the kernel and cokernel of maps of coherent modules are coherent.

At this point, we have verified that coherent A-modules form an abelian subcategory of the category of A-modules. (Things you have to check: 0 should be in this set; it should be closed under finite sums; and it should be closed under taking kernels and cokernels.)

M. Suppose M and N are coherent submodules of the coherent module P. Show that $M + N$ and $M \cap N$ are coherent. (Hint: consider the right map $M \oplus N \to P$.)

N. Show that if A is coherent (as an A-module) then finitely presented modules are coherent. (Of course, if finitely presented modules are coherent, then A is coherent, as A is finitely presented!) (This is also # 5.)

O. If M is finitely presented and N is coherent, show that $\mathrm{Hom}(M, N)$ is coherent. (Hint: Hom is left-exact in its first entry.)

P. If M is finitely presented, and N is coherent, show that $M \otimes N$ is coherent.

Q. If $f \in A$, show that if M is a finitely generated (resp. finitely presented, coherent) A-module, then M_f is a finitely generated (resp. finitely presented, coherent) A_f-module. Hint: localization is exact. (This is also # 6.)

R. Suppose $(f_1, \dots, f_n) = A$. Show that if M_{f_i} is finitely generated for all i, then M is too. (Hint: Say M_{f_i} is generated by $m_{ij} \in M$ as an A_{f_i}-module. Show that the m_{ij} generate M. To check surjectivity $\oplus_{i,j} A \to M$, it suffices to check "on $D(f_i)$" for all i.) (This is half of # 7.)

S. Suppose $(f_1, \dots, f_n) = A$. Show that if M_{f_i} is coherent for all i, then M is too. (Hint from Rob Easton: if $\phi : A^2 \to M$, then $(\ker \phi)_{f_i} = \ker(\phi_{f_i})$, which is finitely generated for all i. Then apply the previous exercise.) (This is the other half of # 7.)

T. Show that the ring $A := k[x_1, x_2, \dots]$ is not coherent over itself. (Hint: consider $A \to A$ with $x_1, x_2, \dots \mapsto 0$.) Thus we have an example of a finitely presented module that is not coherent; a surjection of finitely presented modules whose kernel is not even finitely generated; hence an example showing that finitely presented modules don't form an abelian category.

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FOUNDATIONS OF ALGEBRAIC GEOMETRY PROBLEM SET 8

RAVI VAKIL

This set is due Wednesday, December 14, in my mailbox. (I will accept it, and other older sets, until Friday, December 16. That will likely be a hard deadline, because I may not be around to pick up any later sets.) It covers (roughly) classes 17 and 18.

Please *read all of the problems*, and ask me about any statements that you are unsure of, even of the many problems you won't try. Hand in four solutions. If you are ambitious (and have the time), go for more. Problems marked with "-" count for half a solution. Problems marked with "+" may be harder or more fundamental, but still count for one solution. Try to solve problems on a range of topics. You are encouraged to talk to each other, and to me, about the problems. I'm happy to give hints, and some of these problems require hints!

Class 17:

1. Show that if $\mathfrak{q}$ is primary, then $\sqrt{\mathfrak{q}}$ is prime.

2-. Show that if $\mathfrak{q}$ and $\mathfrak{q}'$ are $\mathfrak{p}$-primary, then so is $\mathfrak{q} \cap \mathfrak{q}'$.

3-. (reality check) Find all the primary ideals in $\mathbb{Z}$.

4+. Suppose A is a Noetherian ring. Show that every proper ideal $I \neq A$ has a primary decomposition. (Hint: Noetherian induction.)

5. Find a minimal primary decomposition of (x^2, xy).

6+. (a) If $\mathfrak{p}$, $\mathfrak{p}_1$, ..., $\mathfrak{p}_n$ are prime ideals, and $\mathfrak{p} = \cap \mathfrak{p}_i$, show that $\mathfrak{p} = \mathfrak{p}_i$ for some i. (Hint: assume otherwise, choose $f_i \in \mathfrak{p}_i - \mathfrak{p}$, and consider $\prod f_i$.)
(b) If $\mathfrak{p} \supset \cap \mathfrak{p}_i$, then $\mathfrak{p} \supset \mathfrak{p}_i$ for some i.
(c) Suppose $I \subseteq \cup^n \mathfrak{p}_i$. Show that $I \subset \mathfrak{p}_i$ for some i. (Hint: by induction on n.)

7. Show that these associated primes behave well with respect to localization. In other words if A is a Noetherian ring, and S is a multiplicative subset (so, as we've seen, there is an inclusion-preserving correspondence between the primes of $S^{-1}A$ and those primes of A not meeting S), then the associated primes of $S^{-1}A$ are just the associated primes of A not meeting S.

8. Show that the minimal primes of 0 are associated primes. (We have now proved important fact **(1)**.) (Hint: suppose $\mathfrak{p} \supset \cap_{i=1}^n \mathfrak{q}_i$. Then $\mathfrak{p} = \sqrt{\mathfrak{p}} \supset \sqrt{\cap_{i=1}^n \mathfrak{q}_i} = \cap_{i=1}^n \sqrt{\mathfrak{q}_i} = \cap_{i=1}^n \mathfrak{p}_i$, so by Exercise 6(b), $\mathfrak{p} \supset \mathfrak{p}_i$ for some i. If $\mathfrak{p}$ is minimal, then as $\mathfrak{p} \supset \mathfrak{p}_i \supset (0)$, we must have

Date: Monday, December 9, 2005. One-character update December 19.

$\mathfrak{p} = \mathfrak{p}_i$.) Show that there can be other associated primes that are not minimal. (Hint: see an earlier exercise.)

9. Show that if A is reduced, then the only associated primes are the minimal primes.

10. Verify the inclusions and equalities

$$D = \cup_{x \neq 0}(0 : x) \subseteq \cup_{x \neq 0}\sqrt{(0 : x)} \subseteq D.$$

11. Suppose f and g are two global sections of a Noetherian normal scheme with the same poles and zeros. Show that each is a unit times the other.

Class 18:

12. If $W \subset X$ and $Y \subset Z$ are both open immersions of ringed spaces, show that any morphism of ringed spaces $X \to Y$ induces a morphism of ringed spaces $W \to Z$.

13. Show that morphisms of ringed spaces glue. In other words, suppose X and Y are ringed spaces, $X = \cup_i U_i$ is an open cover of X, and we have morphisms of ringed spaces $f_i : U_i \to Y$ that "agree on the overlaps", i.e. $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$. Show that there is a unique morphism of ringed spaces $f : X \to Y$ such that $f|_{U_i} = f_i$. (Long ago we had an exercise proving this for topological spaces.)

14. (Easy but important) Given a morphism of ringed spaces $f : X \to Y$ with $f(p) = q$, show that there is a map of stalks $(\mathcal{O}_Y)_q \to (\mathcal{O}_X)_p$.

15. If $f^\# : S \to R$ is a morphism of rings, verify that $R_{f^\# s} \cong R \otimes_S S_s$.

16. Show that morphisms of locally ringed spaces glue (Hint: Basically, the proof of the corresponding exercise for ringed spaces works.)

17+ (easy but important) (a) Show that $\operatorname{Spec} R$ is a locally ringed space. (b) The morphism of ringed spaces $f : \operatorname{Spec} R \to \operatorname{Spec} S$ defined by a ring morphism $f^\# S \to R$ is a morphism of locally ringed spaces.

18++ (Important practice!) Make sense of the following sentence: "$\mathbb{A}^{n+1} - \vec{0} \to \mathbb{P}^n$ given by $(x_0, x_1, \dots, x_{n+1}) \mapsto [x_0; x_1; \dots; x_n]$ is a morphism of schemes." Caution: you can't just say where points go; you have to say where functions go. So you'll have to divide these up into affines, and describe the maps, and check that they glue.

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FOUNDATIONS OF ALGEBRAIC GEOMETRY PROBLEM SET 9

RAVI VAKIL

This set is due Tuesday, January 24, in Jarod Alper's mailbox. It covers (roughly) classes 19 through 22. (This is a long one, because I'm giving you the option of doing some problems from the end of last quarter.)

Please *read all of the problems*, and ask me about any statements that you are unsure of, even of the many problems you won't try. Hand in four solutions. If you are ambitious (and have the time), go for more. Problems marked with "-" count for half a solution. Problems marked with "+" may be harder or more fundamental, but still count for one solution. Try to solve problems on a range of topics. You are encouraged to talk to each other, and to me, about the problems. I'm happy to give hints, and some of these problems require hints!

Class 19:

1+. Show that morphisms $X \to \operatorname{Spec} A$ are in natural bijection with ring morphisms $A \to \Gamma(X, \mathcal{O}_X)$. (Hint: Show that this is true when X is affine. Use the fact that morphisms glue.)

2. Show that $\operatorname{Spec} \mathbb{Z}$ is the final object in the category of schemes. In other words, if X is any scheme, there exists a unique morphism to $\operatorname{Spec} \mathbb{Z}$. (Hence the category of schemes is isomorphic to the category of $\mathbb{Z}$-schemes.)

3. Show that morphisms $X \to \operatorname{Spec} \mathbb{Z}[t]$ correspond to global sections of the structure sheaf.

4. Show that global sections of $\mathcal{O}_X^*$ correspond naturally to maps to $\operatorname{Spec} \mathbb{Z}[t, t^{-1}]$. ($\operatorname{Spec} \mathbb{Z}[t, t^{-1}]$ is a *group scheme*.)

5+. Suppose X is a finite type k-scheme. Describe a natural bijection $\operatorname{Hom}(\operatorname{Spec} k[\epsilon]/\epsilon^2, X)$ to the data of a k-valued point (a point whose residue field is k, necessarily closed) and a tangent vector at that point.

6. Suppose $i : U \to Z$ is an open immersion, and $f : Y \to Z$ is any morphism. Show that $U \times_Z Y$ exists. (Hint: I'll even tell you what it is: $(f^{-1}(U), \mathcal{O}_Y|_{f^{-1}(U)})$.)

7-. Show that open immersions are monomorphisms.

8+. Suppose $Y \to Z$ is a closed immersion, and $X \to Z$ is any morphism. Show that the fibered product $X \times_Y Z$ exists, by explicitly describing it. Show that the projection $X \times_Y Z \to Y$ is a closed immersion. We say that "closed immersions are preserved by

Date: Monday, January 16, 2005. Updated January 23. Minor update January 25.

base change" or "closed immersions are preserved by fibered product". (Base change is another word for fibered products.)

9. Show that closed immersions are monomorphisms.

10. *(quasicompactness is affine-local on the target)* Show that a morphism $f : X \to Y$ is quasicompact if there is cover of Y by open affine sets U_i such that $f^{-1}(U_i)$ is quasicompact. (Hint: affine communication lemma!)

11. Show that the composition of two quasicompact morphisms is quasicompact.

12. *(the notions "locally of finite type" and "finite type" is affine-local on the target)* Show that a morphism $f : X \to Y$ is locally of finite type if there is a cover of Y by open affine sets $\operatorname{Spec} R_i$ such that $f^{-1}(\operatorname{Spec} R_i)$ is locally of finite type over R_i.

13-. Show that a closed immersion is a morphism of finite type.

14-. Show that an open immersion is locally of finite type. Show that an open immersion into a Noetherian scheme is of finite type. More generally, show that a quasicompact open immersion is of finite type.

15-. Show that a composition of two morphisms of finite type is of finite type.

16. Suppose we have a composition of morphisms $X \xrightarrow{f} Y \xrightarrow{g} Z$, where f is quasicompact, and $g \circ f$ is finite type. Show that f is finite type.

17-. Suppose $f : X \to Y$ is finite type, and Y is Noetherian. Show that X is also Noetherian.

18. Suppose X is an affine scheme, and Y is a closed subscheme locally cut out by one equation (e.g. if X is an effective Cartier divisor). Show that $X - Y$ is affine. (This is clear if Y is globally cut out by one equation f; then if $X = \operatorname{Spec} R$ then $Y = \operatorname{Spec} R_f$. However, this is not always true.) Hint: affine locality of the notion of "affine morphism".

19. Here is an explicit consequence of the previous exercise. We showed earlier that on the cone over the smooth quadric surface $\operatorname{Spec} k[w, x, y, z]/(wz - xy)$, the cone over a ruling $w = x = 0$ is not cut out scheme-theoretically by a single equation, by considering Zariski-tangent spaces. We now show that it isn't even cut out set-theoretically by a single equation. For if it were, its complement would be affine. But then the closed subscheme of the complement cut out by $y = z = 0$ would be affine. But this is the scheme $y = z = 0$ (also known as the wx-plane) minus the point $w = x = 0$, which we've seen is non-affine. (For comparison, on the cone $\operatorname{Spec} k[x, y, z]/(xy - z^2)$, the ruling $x = z = 0$ is not cut out scheme-theoretically by a single equation, but it *is* cut out set-theoretically by $x = 0$.) Verify all of this!

20. *(the property of finiteness is affine-local on the target)* Show that a morphism $f : X \to Y$ is finite if there is a cover of Y by open affine sets $\operatorname{Spec} R$ such that $f^{-1}(\operatorname{Spec} R)$ is the spectrum of a finite R-algebra. (Hint: Use that $f_*\mathcal{O}_X$ is finite type.)

21-. Show that closed immersions are finite morphisms.

22. (a) Show that if a morphism is finite then it is quasifinite. (b) Show that the converse is not true. (Hint: $\mathbb{A}^1 - \{0\} \to \mathbb{A}^1$.)

23. Suppose X is a Noetherian scheme. Show that a subset of X is constructable if and only if it is the finite disjoint union of locally closed subsets.

24-. Show that the image of an irreducible scheme is irreducible.

Class 20:

25. Let $f : \operatorname{Spec} A \to \operatorname{Spec} B$ be a morphism of affine schemes, and suppose M is an A-module, so $\tilde{M}$ is a quasicoherent sheaf on $\operatorname{Spec} A$. Show that $f_*\tilde{M} \cong \widetilde{M_B}$. (Hint: There is only one reasonable way to proceed: look at distinguished opens!)

26. Give an example of a morphism of schemes $\pi : X \to Y$ and a quasicoherent sheaf $\mathcal{F}$ on X such that $\pi_*\mathcal{F}$ is not quasicoherent. (Answer: $Y = \mathbb{A}^1$, $X =$ countably many copies of $\mathbb{A}^1$. Then let $f = t$. X_t has a global section $(1/t, 1/t^2, 1/t^3, \dots)$. The key point here is that infinite direct sums do not commute with localization.)

27. Suppose $f : X \to Y$ is a finite morphism of Noetherian schemes. If $\mathcal{F}$ is a coherent sheaf on X, show that $f_*\mathcal{F}$ is a coherent sheaf. (Hint: Show first that $f_*\mathcal{O}_X$ is finite type = locally finitely generated.)

28. Verify that the following is an example showing that pullback is not left-exact: consider the exact sequence of sheaves on $\mathbb{A}^1$, where p is the origin:

$$0 \to \mathcal{O}_{\mathbb{A}^1}(-p) \to \mathcal{O}_{\mathbb{A}^1} \to \mathcal{O}_p \to 0.$$

(This is a closed subscheme exact sequence; also an effective Cartier exact sequence. Algebraically, we have $k[t]$-modules $0 \to tk[t] \to k[t] \to k \to 0$.) Restrict to p.

Class 21:

29. The notion of integral morphism is well behaved with localization and quotient of B, and quotient of A (but not localization of A, witness $k[t] \to k[t]$, but $k[t] \to k[t]_{(t)}$). The notion of integral extension is well behaved with respect to localization and quotient of B, but not quotient of A (same example, $k[t] \to k[t]/(t)$).

30+. (a) Show that if B is an integral extension of A, and C is an integral extension of B, then C is an integral extension of A.
(b) Show that if B is a finite extension of A, and C is a finite extension of B, then C is an finite extension of A.

31-. Show that the special case of the going-up theorem where A is a field translates to: if $B \subset A$ is a subring with A integral over B, then B is a field. Prove this. (Hint: all you need to do is show that all nonzero elements in B have inverses in B. Here is the start: If $b \in B$, then $1/b \in A$, and this satisfies some integral equation over B.)

32+. *(sometimes also called the going-up theorem)* Show that if $\mathfrak{q}_1 \subset \mathfrak{q}_2 \subset \cdots \subset \mathfrak{q}_n$ is a chain of prime ideals of B, and $\mathfrak{p}_1 \subset \cdots \subset \mathfrak{p}_m$ is a chain of prime ideals of A such that $\mathfrak{p}_i$ "lies over" $\mathfrak{q}_i$ (and $m < n$), then the second chain can be extended to $\mathfrak{p}_1 \subset \cdots \subset \mathfrak{p}_n$ so that this remains true.

33+. Show that if $f : \operatorname{Spec} A \to \operatorname{Spec} B$ corresponds to an integral *extension of rings*, then $\dim \operatorname{Spec} A = \dim \operatorname{Spec} B$.

34. Show that finite morphisms are *closed*, i.e. the image of any closed subset is closed.

35. Show that integral ring extensions induce a surjective map of spectra.

36. Suppose X is a Noetherian scheme. Show that a subset of X is constructable if and only if it is the finite disjoint union of locally closed subsets. (This is admittedly the same as 23.)

37. Show that a dominant morphism of integral schemes $X \to Y$ induces an inclusion of function fields in the other direction.

38. If $\phi : A \to B$ is a ring morphism, show that the corresponding morphism of affine schemes $\operatorname{Spec} B \to \operatorname{Spec} A$ is dominant iff ϕ has nilpotent kernel.

39+. Reduce the proof of Chevalley's theorem to the following case: suppose $f : X = \operatorname{Spec} A \to Y = \operatorname{Spec} B$ is a dominant morphism, where A and B are domains, and f corresponds to $\phi : B \to B[x_1, \dots, x_n]/I \cong A$. Show that the image of f contains a dense open subset of $\operatorname{Spec} B$. (See the class notes.)

Class 22:

40. Let $\phi : X \to \mathbb{P}^n_A$ be a morphism of A-schemes, corresponding to an invertible sheaf $\mathcal{L}$ on X and sections $s_0, \dots, s_n \in \Gamma(X, \mathcal{L})$ as above. Then ϕ is a closed immersion iff (1) each open set $X_i = X_{s_i}$ is affine, and (2) for each i, the map of rings $A[y_0, \dots, y_n] \to \Gamma(X_i, \mathcal{O}_{X_i})$ given by $y_j \mapsto s_j/s_i$ is surjective.

41. *(Automorphisms of projective space)* Show that all the automorphisms of projective space $\mathbb{P}^n_k$ correspond to $(n+1)\times(n+1)$ invertible matrices over k, modulo scalars (also known as $PGL_{n+1}(k)$). (Hint: Suppose $f : \mathbb{P}^n_k \to \mathbb{P}^n_k$ is an automorphism. Show that $f^*\mathcal{O}(1) \cong \mathcal{O}(1)$. Show that $f^* : \Gamma(\mathbb{P}^n, \mathcal{O}(1)) \to \Gamma(\mathbb{P}^n, \mathcal{O}(1))$ is an isomorphism.)

42. Show that any map from projective space to a smaller projective space is constant. (Fun!)

43. Prove that $\mathbb{A}^n_R \cong \mathbb{A}^n_{\mathbb{Z}} \times_{\operatorname{Spec} \mathbb{Z}} \operatorname{Spec} R$. Prove that $\mathbb{P}^n_R \cong \mathbb{P}^n_{\mathbb{Z}} \times_{\operatorname{Spec} \mathbb{Z}} \operatorname{Spec} R$.

44. Show that for finite-type schemes over $\mathbb{C}$, the complex-valued points of the fibered product correspond to the fibered product of the complex-valued points. (You will just use the fact that $\mathbb{C}$ is algebraically closed.)

45-. Describe $\operatorname{Spec} \mathbb{C} \times_{\operatorname{Spec} \mathbb{R}} \operatorname{Spec} \mathbb{C}$.

46. Consider the morphism of schemes $X = \operatorname{Spec} k[t] \to Y = \operatorname{Spec} k[u]$ corresponding to $k[u] \to k[t]$, $u = t^2$ (where the characteristic of k is not 2). Show that $X \times_Y X$ has 2 irreducible components. Compare what is happening above the generic point of Y to the previous exercise.

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FOUNDATIONS OF ALGEBRAIC GEOMETRY PROBLEM SET 10

RAVI VAKIL

This set is due Thursday, February 2, in Jarod Alper's mailbox. It covers (roughly) classes 23 and 24.

Please *read all of the problems*, and ask me about any statements that you are unsure of, even of the many problems you won't try. Hand in six solutions. If you are ambitious (and have the time), go for more. Problems marked with "-" count for half a solution. Problems marked with "+" may be harder or more fundamental, but still count for one solution. Try to solve problems on a range of topics. You are encouraged to talk to each other, and to me, about the problems. I'm happy to give hints, and some of these problems require hints!

0. Here is something I would like to see worked out. Show that the points of $\operatorname{Spec} \overline{\mathbb{Q}} \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}$ are in natural bijection with $\operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, and the Zariski topology on the former agrees with the profinite topology on the latter.

Class 23:

1-. Show that for the morphism $\operatorname{Spec} \mathbb{C} \to \operatorname{Spec} \mathbb{R}$, all geometric fibers consist of two reduced points.

2+. Show that the notion of "morphism locally of finite type" is preserved by base change. Show that the notion of "affine morphism" is preserved by base change. Show that the notion of "finite morphism" is preserved by base change.

3+. Show that the notion of "morphism of finite type" is preserved by base change.

4. Show that the notion of "quasicompact morphism" is preserved by base change.

5. Show that the notion of "quasifinite morphism" (= finite type + finite fibers) is preserved by base change. (Note: the notion of "finite fibers" is not preserved by base change. $\operatorname{Spec} \overline{\mathbb{Q}} \to \operatorname{Spec} \mathbb{Q}$ has finite fibers, but $\operatorname{Spec} \overline{\mathbb{Q}} \otimes_{\mathbb{Q}} \overline{\mathbb{Q}} \to \operatorname{Spec} \overline{\mathbb{Q}}$ has one point for each element of $\operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$.)

6. Show that surjectivity is preserved by base change (or fibered product). In other words, if $X \to Y$ is a surjective morphism, then for any $Z \to Y$, $X \times_Y Z \to Z$ is surjective. (You may end up using the fact that for any fields k_1 and k_2 containing k_3, $k_1 \otimes_{k_3} k_2$ is non-zero, and also the axiom of choice.)

Date: Tuesday, January 24, 2006. Minor update October 26, 2006.

7-. Show that the notion of "irreducible" is not necessarily preserved by base change. Show that the notion of "connected" is not necessarily preserved by base change. (Hint: $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$, $\mathbb{Q}[i] \otimes_{\mathbb{Q}} \mathbb{Q}[i]$.)

8. Show that $\operatorname{Spec} \mathbb{C}$ is not a geometrically irreducible $\mathbb{R}$-scheme. If $\operatorname{char} k = p$, show that $\operatorname{Spec} k(u)$ is not a geometrically reduced $\operatorname{Spec} k(u^p)$-scheme.

9. Show that the notion of geometrically irreducible (resp. connected, reduced, integral) fibers behaves well with respect to base change.

10. Suppose that l/k is a finite field extension. Show that a k-scheme X is normal if and only if $X \times_{\operatorname{Spec} k} \operatorname{Spec} l$ is normal. Hence deduce that if k is any field, then $\operatorname{Spec} k[w,x,y,z]/(wz-xy)$ is normal. Hint: we showed earlier (Problem B4 on set 4) that $\operatorname{Spec} k[a,b,c,d]/(a^2+b^2+c^2+d^2)$ is normal. (This is less important, but helps us understand this example.)

11. Show that the Segre scheme (the image of the Segre morphism) is cut out by the equations corresponding to

$$\operatorname{rank} \begin{pmatrix} a_{00} & \cdots & a_{0n} \\ \vdots & \ddots & \vdots \\ a_{m0} & \cdots & a_{mn} \end{pmatrix} = 1,$$

i.e. that all 2×2 minors vanish. (Hint: suppose you have a polynomial in the a_{ij} that becomes zero upon the substitution $a_{ij} = x_i y_j$. Give a recipe for subtracting polynomials of the form monomial times 2×2 minor so that the end result is 0.)

12. Show that $X^{red} \to X$ satisfies the following universal property: any morphism from a reduced scheme Y to X factors uniquely through X^{red}.

$$\begin{array}{ccc} Y & \overset{\exists!}{\dashrightarrow} & X^{red} \\ & \searrow \quad \swarrow & \\ & X & \end{array}.$$

(Do this exercise if you want to see how this sort of argument works in general.)

13. Show that $\nu : \operatorname{Spec} \tilde{R} \to \operatorname{Spec} R$ satisfies the universal property of normalization. We used this to show that normalization exists.

14. Show that normalizations exist for any quasiaffine X (i.e. any X that can be expressed as an open subset of an affine scheme). Show that normalizations exist in general.

Class 24:

15. Show that the normalization morphism is surjective. (Hint: Going-up!)

16. Show that $\dim \tilde{X} = \dim X$ (hint: see our going-up discussion).

17. Show that if X is an integral finite-type k-scheme, then its normalization $\nu : \tilde{X} \to X$ is a finite morphism.

18. Explain how to generalize the notion of normalization to the case where X is a reduced Noetherian scheme (with possibly more than one component). This basically requires defining a universal property. I'm not sure what the "perfect" definition, but all reasonable universal properties should lead to the same space.

19. Show that if X is an integral finite type k-scheme, then its non-normal points form a closed subset. (This is a bit trickier. Hint: consider where $\nu_* \mathcal{O}_{\tilde{X}}$ has rank 1.) I haven't thought through all the details recently, so I hope I've stated this correctly.

20. (Good practice with the concept.) Suppose $X = \operatorname{Spec} \mathbb{Z}[15i]$. Describe the normalization $\tilde{X} \to X$. (Hint: it isn't hard to find an integral extension of $\mathbb{Z}[15i]$ that is integrally closed. By the above discussion, you've then found the normalization!) Over what points of X is the normalization not an isomorphism?

21. (This is an important generalization!) Suppose X is an integral scheme. Define the *normalization of* X, $\nu : \tilde{X} \to X$, *in a given finite field extension of the function field of* X. Show that $\tilde{X}$ is normal. (This will be hard-wired into your definition.) Show that if either X is itself normal, or X is finite type over a field k, then the normalization in a finite field extension is a finite morphism.

22. Suppose $X = \operatorname{Spec} \mathbb{Z}$ (with function field $\mathbb{Q}$). Find its integral closure in the field extension $\mathbb{Q}(i)$.

23. (a) Suppose $X = \operatorname{Spec} k[x]$ (with function field $k(x)$). Find its integral closure in the field extension $k(y)$, where $y^2 = x^2 + x$. (We get a Dedekind domain.)
(b) Suppose $X = \mathbb{P}^1$, with distinguished open $\operatorname{Spec} k[x]$. Find its integral closure in the field extension $k(y)$, where $y^2 = x^2 + x$. (Part (a) involves computing the normalization over one affine open set; now figure out what happens over the "other".)

24. Show that if $f : Z \to X$ is an affine morphism, then we have a natural isomorphism $Z \cong \underline{\operatorname{Spec}}\, f_* \mathcal{O}_Z$ of X-schemes.

25. ($\underline{\operatorname{Spec}}$ behaves well with respect to base change) Suppose $f : Z \to X$ is any morphism, and $\mathcal{A}$ is a quasicoherent sheaf of algebras on X. Show that there is a natural isomorphism $Z \times_X \underline{\operatorname{Spec}}\, \mathcal{A} \cong \underline{\operatorname{Spec}}\, f^* \mathcal{A}$.

26. If $\mathcal{F}$ is a locally free sheaf, show that $\underline{\operatorname{Spec}} \operatorname{Sym} \mathcal{F}^*$ is a vector bundle, i.e. that given any point $p \in X$, there is a neighborhood $p \in U \subset X$ such that $\underline{\operatorname{Spec}} \operatorname{Sym} \mathcal{F}^*|_U \cong \mathbb{A}^1_U$. Show that $\mathcal{F}$ is a sheaf of sections of it.

27. Suppose $f : \underline{\operatorname{Spec}}\, \mathcal{A} \to X$ is a morphism. Show that the category of quasicoherent sheaves on $\underline{\operatorname{Spec}\, A}$ is "essentially the same" (=equivalent) as the category of quasicoherent sheaves on $\overline{X}$ with the structure of $\mathcal{A}$-modules (quasicoherent $\mathcal{A}$-modules on X).

28. Complete this argument that if $X = \operatorname{Spec} A$, then $(\operatorname{Proj} \mathcal{S}_*, \mathcal{O}(1))$ satisfies the universal property.

29. Show that $(\mathrm{Proj}\,\mathcal{S}_*, \mathcal{O}(1))$ exists in general, by following the analogous universal property argument: show that it exists for X quasiaffine, then in general.

30. ($\underline{\mathrm{Proj}}$ behaves well with respect to base change) Suppose $\mathcal{S}_*$ is a quasicoherent sheaf of graded algebras on X satisfying the required hypotheses above for $\underline{\mathrm{Proj}}\mathcal{S}_*$ to exist. Let $f : Y \to X$ be any morphism. Give a natural isomorphism

$$(\underline{\mathrm{Proj}} f^*\mathcal{S}_*, \mathcal{O}_{\underline{\mathrm{Proj}} f^*\mathcal{S}_*}(1)) \cong (Y \times_X \underline{\mathrm{Proj}}\mathcal{S}_*, g^*\mathcal{O}_{\underline{\mathrm{Proj}}\mathcal{S}_*}(1)) \cong$$

where g is the natural morphism in the base change diagram

$$\begin{array}{ccc} Y \times_X \underline{\mathrm{Proj}}\mathcal{S}_* & \xrightarrow{g} & \underline{\mathrm{Proj}}\mathcal{S}_* \\ \downarrow & & \downarrow \\ Y & \longrightarrow & X. \end{array}$$

31. $\underline{\mathrm{Proj}}(\mathcal{S}_*[t]) \cong \underline{\mathrm{Spec}}\,\mathcal{S}_* \coprod \underline{\mathrm{Proj}}\mathcal{S}_*$, where $\underline{\mathrm{Spec}}\,\mathcal{S}_*$ is an open subscheme, and $\underline{\mathrm{Proj}}\mathcal{S}_*$ is a closed subscheme. Show that $\underline{\mathrm{Proj}}\mathcal{S}*$ is an effective Cartier divisor, corresponding to the invertible sheaf $\mathcal{O}_{\underline{\mathrm{Proj}} N}(1)$. (This is the generalization of the projective and affine cone. At some point I should give an explicit reference to our earlier exercise on this.)

32. Suppose $\mathcal{L}$ is an invertible sheaf on X, and $\mathcal{S}_*$ is a quasicoherent sheaf of graded algebras on X satisfying the required hypotheses above for $\underline{\mathrm{Proj}}\mathcal{S}_*$ to exist. Define $\mathcal{S}'_* = \oplus_{n=0}\mathcal{S}_n \otimes \mathcal{L}_n$. Give a natural isomorphism of X-schemes

$$(\underline{\mathrm{Proj}}\mathcal{S}'_*, \mathcal{O}_{\underline{\mathrm{Proj}}\mathcal{S}'_*}(1)) \cong (\underline{\mathrm{Proj}}\mathcal{S}_*, \mathcal{O}_{\underline{\mathrm{Proj}}\mathcal{S}_*}(1) \otimes \pi^*\mathcal{L}),$$

where $\pi : \underline{\mathrm{Proj}}\mathcal{S}_* \to X$ is the structure morphism. In other words, informally speaking, the $\underline{\mathrm{Proj}}$ is the same, but the $\mathcal{O}(1)$ is twisted by $\mathcal{L}$.

33. Show that closed immersions are projective morphisms. (Hint: Suppose the closed immersion $X \to Y$ corresponds to $\mathcal{O}_Y \to \mathcal{O}_X$. Consider $\mathcal{S}_0 = \mathcal{O}_X$, $\mathcal{S}_i = \mathcal{O}_Y$ for $i > 1$.)

34. (suggested by Kirsten) Suppose $f : X \hookrightarrow \mathbb{P}^n_S$ where S is some scheme. Show that the structure morphism $\pi : X \to S$ is a projective morphism as follows: let $\mathcal{L} = f^*\mathcal{O}_{\mathbb{P}^n_S}(1)$, and show that $X = \underline{\mathrm{Proj}}\pi_*\mathcal{L}^{\otimes n}$.

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FOUNDATIONS OF ALGEBRAIC GEOMETRY PROBLEM SET 11

RAVI VAKIL

This set is due Thursday, February 9, in Jarod Alper's mailbox. It covers (roughly) classes 25 and 26.

Please *read all of the problems*, and ask me about any statements that you are unsure of, even of the many problems you won't try. Hand in six solutions. If you are ambitious (and have the time), go for more. Problems marked with "-" count for half a solution. Problems marked with "+" may be harder or more fundamental, but still count for one solution. Try to solve problems on a range of topics. You are encouraged to talk to each other, and to me, about the problems. I'm happy to give hints, and some of these problems require hints!

Class 25:

1. Verify that the following definition of "induced reduced subscheme structure" is well-defined. Suppose X is a scheme, and S is a *closed subset* of X. Then there is a unique reduced closed subscheme Z of X "supported on S". More precisely, it can be defined by the following universal property: for any morphism from a *reduced* scheme Y to X, whose image lies in S (as a set), this morphism factors through Z uniquely. Over an affine $X = \operatorname{Spec} R$, we get $\operatorname{Spec} R/I(S)$. (For example, if S is the entire underlying set of X, we get X^{red}.)

2+. Show that open immersions and closed immersions are separated. (Answer: Show that monomorphisms are separated. Open and closed immersions are monomorphisms, by earlier exercises. Alternatively, show this by hand.)

3+. Show that every morphism of affine schemes is separated. (Hint: this was essentially done in the notes if you know where to look.)

4. Complete the proof that $\mathbb{P}^n_{\mathbb{Z}} \to \operatorname{Spec} \mathbb{Z}$ is separated, by verifying the last sentence in the proof.

5. Show that the line with doubled origin X is not separated, by verifying that the image of the diagonal morphism is not closed. (Another argument is given below, in Exercise 12.)

6. Show that any morphism from a Noetherian scheme is quasicompact. Hence show that any morphism from a Noetherian scheme is quasiseparated.

Date: Tuesday, January 31, 2006. Update February 27 (changes to 18). Update February 12 (change to 15).

7+. Show that $f : X \to Y$ is quasiseparated if and only if for any affine open $\operatorname{Spec} R$ of Y, and two affine open subsets U and V of X mapping to $\operatorname{Spec} R$, $U \cap V$ is a *finite* union of affine open sets.

8. Here is an example of a nonquasiseparated scheme. Let $X = \operatorname{Spec} k[x_1, x_2, \ldots]$, and let U be $X - \mathfrak{m}$ where $\mathfrak{m}$ is the maximal ideal $(x_1, x_2, \ldots)$. Take two copies of X, glued along U. Show that the result is not quasiseparated.

9. Prove that the condition of being quasiseparated is local on the target. (Hint: the condition of being quasicompact is local on the target by an earlier exercise.)

10-. Show that a k-scheme is separated (over k) iff it is separated over $\mathbb{Z}$.

11+ (the locus where two morphisms agree) We can now make sense of the following statement. Suppose

$$f, g : \quad \begin{array}{ccc} X & \longrightarrow & Y \\ & \searrow \quad \swarrow & \\ & Z & \end{array}$$

are two morphisms over Z. Then the locus on X where f and g agree is a locally closed subscheme of X. If $Y \to Z$ is separated, then the locus is a closed subscheme of X. More precisely, define V to be the following fibered product:

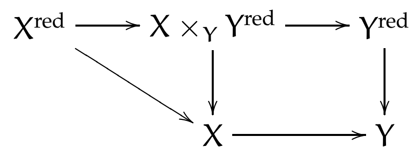
$$\begin{array}{ccc} V & \longrightarrow & Y \\ \downarrow & & \downarrow \delta \\ X & \xrightarrow{(f,g)} & Y \times_Z Y. \end{array}$$

As δ is a locally closed immersion, $V \to X$ is too. Then if $h : W \to X$ is any scheme such that $g \circ h = f \circ h$, then h factors through V. (Put differently: we are describing $V \hookrightarrow X$ by way of a universal property. Taking this as the definition, it is not a priori clear that V is a locally closed subscheme of X, or even that it exists.) Now we come to the exercise: prove this (the sentence before the parentheses). (Hint: we get a map $g \circ h = f \circ h : W \to Y$. Use the definition of fibered product to get $W \to V$.)

12. Show that the line with doubled origin X is not separated, by finding two morphisms $f_1, f_2 : W \to X$ whose domain of agreement is not a closed subscheme. (Another argument was given above, in Exercise 5.)

13. Suppose $\pi : Y \to X$ is a morphism, and $s : X \to Y$ is a *section* of a morphism, i.e. $\pi \circ s$ is the identity on X. Show that s is a locally closed immersion. Show that if π is separated, then s is a closed immersion. (This generalizes Proposition 1.12 in the Class 25 notes.)

14-. Suppose P is a class of morphisms such that closed immersions are in P, and P is closed under fibered product and composition. Show that if $X \to Y$ is in P then $X^{red} \to Y^{red}$ is in P. (Two examples are the classes of separated morphisms and quasiseparated morphisms.) (Hint:



)

15. Suppose $\pi : X \to Y$ is a morphism or a ring R, Y is a separated R-scheme, U is an affine open subset of X, and V is an affine open subset of Y. Show that $U \cap \pi^{-1}V$ is an affine open subset of X. (Hint: this generalizes Proposition 1.9 of the Class 25 notes. Use Proposition 1.12 or 1.13.) This will be used in the proof of the Leray spectral sequence.

16. Make this precise: show that the line with the doubled origin fails the valuative criterion for separatedness.

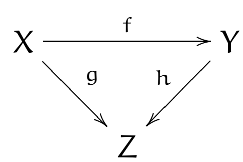
Class 26:

17-. Show that $\mathbb{A}^1_{\mathbb{C}} \to \mathbb{C}$ is not proper.

18. Show that finite morphisms are projective. (There was something that I didn't check in the notes.) More explicitly, if $X \to Y$ is finite, then I described a sheaf of graded algebras $\mathcal{S}_*$ on Y, and claimed that $X = \underline{\text{Proj}}\mathcal{S}_*$. Verify that this is indeed the case. What is $\mathcal{O}_{\underline{\text{Proj}}\mathcal{S}_*}(1)$?

19-. Suppose (1) is a commutative diagram, and f is surjective, g is proper, and h is separated and finite type. Show that h is proper.

(1)



(I'm not sure that this is useful, but I know that if I forget to mention it, it will come back to haunt me!)

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FOUNDATIONS OF ALGEBRAIC GEOMETRY PROBLEM SET 12

RAVI VAKIL

This set is due Thursday, February 16, in Jarod Alper's mailbox. It covers (roughly) classes 27 and 28.

Please *read all of the problems*, and ask me about any statements that you are unsure of, even of the many problems you won't try. Hand in five solutions. If you are ambitious (and have the time), go for more. Problems marked with "-" count for half a solution. Problems marked with "+" may be harder or more fundamental, but still count for one solution. Try to solve problems on a range of topics. You are encouraged to talk to each other, and to me, about the problems. I'm happy to give hints, and some of these problems require hints!

Class 27:

1+. (Scheme-theoretic closure and scheme-theoretic image) If $f : W \to Y$ is any morphism, define the scheme-theoretic image as the smallest closed subscheme $Z \to Y$ so that f factors through $Z \hookrightarrow Y$. Show that this is well-defined. (One possible hint: use a universal property argument.) If Y is affine, the ideal sheaf corresponds to the functions on Y that are zero when pulled back to Z. Show that the closed set underlying the image subscheme may be strictly larger than the closure of the set-theoretic image: consider $\operatorname{Spec} k(t) \to \operatorname{Spec} k[t]$. (We define the scheme-theoretic closure of a locally closed subscheme $W \hookrightarrow Y$ as the scheme-theoretic image of the morphism.)

2-. Show that rational functions on an integral scheme correspond to rational maps to $\mathbb{A}^1_{\mathbb{Z}}$.

3-. Show that you can compose two rational maps $f : X \dashrightarrow Y$, $g : Y \dashrightarrow Z$ if f is dominant.

4. We define the *graph* of a rational map $f : X \dashrightarrow Y$ as follows: let (U, f') be any representative of this rational map (so $f' : U \to Y$ is a morphism). Let Γ_f be the scheme-theoretic closure of $\Gamma_{f'} \hookrightarrow U \times Y \hookrightarrow X \times Y$, where the first map is a closed immersion, and the second is an open immersion. Show that this is independent of the choice of U.

5. Let K be a finitely generated field extension of transcendence degree m over k. Show there exists an irreducible k-variety W with function field K. (Hint: let $x_1, \ldots, x_n$ be generators for K over k. Consider the map $\operatorname{Spec} K \to \operatorname{Spec} k[t_1, \ldots, t_n]$ given by the ring map $t_i \mapsto x_i$. Take the scheme-theoretic closure of the image.)

Date: Tuesday, February 7, 2006. Updated March 8.

6+. Prove the following. Suppose X and Y are integral and separated (our standard hypotheses from last day). Then X and Y are birational if and only if there is a dense=non-empty open subscheme U of X and a dense=non-empty open subscheme V of Y such that $U \cong Y$. (Feel free to consult Iitaka, or Hartshorne Chapter I Corollary 4.5.)

7. Use the class discussion to find a "formula" for all Pythagorean triples.

8. Show that the conic $x^2+y^2 = z^2$ in $\mathbb{P}^2_k$ is isomorphic to $\mathbb{P}^1_k$ for any field k of characteristic not 2. (Presumably this is true for any ring in which 2 is invertible too...)

9. Find all rational solutions to the $y^2 = x^3 + x^2$, by finding a birational map to $\mathbb{A}^1$, mimicking what worked with the conic.

10. Find a birational map from the quadric $Q = \{x^2 + y^2 = w^2 + z^2\}$ to $\mathbb{P}^2$. Use this to find all rational points on Q. (This illustrates a good way of solving Diophantine equations. You will find a dense open subset of Q that is isomorphic to a dense open subset of $\mathbb{P}^2$, where you can easily find all the rational points. There will be a closed subset of Q where the rational map is not defined, or not an isomorphism, but you can deal with this subset in an ad hoc fashion.)

11. *(a first view of a blow-up)* Let k be an algebraically closed field. (We make this hypothesis in order to not need any fancy facts on nonsingularity.) Consider the rational map $\mathbb{A}^2_k \dashrightarrow \mathbb{P}^1_k$ given by $(x, y) \mapsto [x; y]$. I think you have shown earlier that this rational map cannot be extended over the origin. Consider the graph of the birational map, which we denote $\mathrm{Bl}_{(0,0)} \mathbb{A}^2_k$. It is a subscheme of $\mathbb{A}^2_k \times \mathbb{P}^1_k$. Show that if the coordinates on $\mathbb{A}^2$ are x, y, and the coordinates on $\mathbb{P}^1$ are u, v, this subscheme is cut out in $\mathbb{A}^2 \times \mathbb{P}^1$ by the single equation $xv = yu$. Show that $\mathrm{Bl}_{(0,0)} \mathbb{A}^2_k$ is nonsingular. Describe the fiber of the morphism $\mathrm{Bl}_{(0,0)} \mathbb{A}^2_k \to \mathbb{P}^1_k$ over each closed point of $\mathbb{P}^1_k$. Describe the fiber of the morphism $\mathrm{Bl}_{(0,0)} \mathbb{A}^2_k \to \mathbb{A}^2_k$. Show that the fiber over $(0, 0)$ is an effective Cartier divisor. It is called the *exceptional divisor*.

12. *(the Cremona transformation, a useful classical construction)* Consider the rational map $\mathbb{P}^2 \dashrightarrow \mathbb{P}^2$, given by $[x; y; z] \to [1/x; 1/y; 1/z]$. What is the the domain of definition? (It is bigger than the locus where $xyz \neq 0$!) You will observe that you can extend it over codimension 1 sets. This will again foreshadow a result we will soon prove.

Class 28:

13. *(Useful practice!)* Suppose X is a Noetherian k-scheme, and Z is an irreducible codimension 1 subvariety whose generic point is a nonsingular point of X (so the local ring $\mathcal{O}_{X,Z}$ is a discrete valuation ring). Suppose $X \dashrightarrow Y$ is a rational map to a projective k-scheme. Show that the domain of definition of the rational map includes a dense open subset of Z. In other words, rational maps from Noetherian k-schemes to projective k-schemes can be extended over nonsingular codimension 1 sets. See problem 12 to see this principle in action. (By the easy direction of the valuative criterion of separatedness, or the theorem of uniqueness of extensions of maps from reduced schemes to separated schemes — Theorem 3.3 of Class 27 — this map is unique.)

14. Show that all nonsingular proper curves are projective. (We may eventually see that all reduced proper curves over k are projective, but I'm not sure; this will use the Riemann-Roch theorem, and I may just prove it for projective curves.)

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FOUNDATIONS OF ALGEBRAIC GEOMETRY PROBLEM SET 13

RAVI VAKIL

This set is due Thursday, February 23, in Jarod Alper's mailbox. It covers (roughly) classes 29 and 30.

Please *read all of the problems*, and ask me about any statements that you are unsure of, even of the many problems you won't try. Hand in five solutions. If you are ambitious (and have the time), go for more. Problems marked with "-" count for half a solution. Problems marked with "+" may be harder or more fundamental, but still count for one solution. Try to solve problems on a range of topics. You are encouraged to talk to each other, and to me, about the problems. Some of these problems require hints, and I'm happy to give them!

Class 29:

1+. (This was discussed in class 29, but I've put it in the class 27 notes, because it belongs more naturally there.) Suppose $W \hookrightarrow Y$ is a locally closed immersion. The scheme-theoretic closure is the smallest closed subscheme of Y containing W. Show that this notion is well-defined. More generally, if $f : W \to Y$ is any morphism, define the scheme-theoretic image as the smallest closed subscheme $Z \to Y$ so that f factors through $Z \hookrightarrow Y$. Show that this is well-defined. (One possible hint: use a universal property argument.) If Y is affine, the ideal sheaf corresponds to the functions on Y that are zero when pulled back to Z. Show that the closed set underlying the image subscheme may be strictly larger than the closure of the set-theoretic image: consider $\coprod_{n \geq 0} \operatorname{Spec} k[t]/t^n \to \operatorname{Spec} k[t]$. (I suspect that such a pathology cannot occur for finite type morphisms of Noetherian schemes, but I haven't investigated.)

2. Suppose $f : C \to C'$ is a degree d morphism of integral projective nonsingular curves, and $\mathcal{L}$ is an invertible sheaf on C'. Show that $\deg_C f^*\mathcal{L} = d \deg_{C'} \mathcal{L}$.

3. (for those who like working with non-Noetherian schemes) Suppose R is a ring that is coherent over itself. (In other words, R is a coherent R-module.) Show that for any coherent sheaf $\mathcal{F}$ on a projective R-scheme where R is Noetherian, $h^i(X, \mathcal{F})$ is a finitely generated R-module. (Hint: induct downwards as before. The order is as follows: $H^n(\mathbb{P}^n_R, \mathcal{F})$ finitely generated, $H^n(\mathbb{P}^n_R, \mathcal{G})$ finitely generated, $H^n(\mathbb{P}^n_R, \mathcal{F})$ coherent, $H^n(\mathbb{P}^n_R, \mathcal{G})$ coherent, $H^{n-1}(\mathbb{P}^n_R, \mathcal{F})$ finitely generated, $H^{n-1}(\mathbb{P}^n_R, \mathcal{G})$ finitely generated, etc.)

4+ (This is important!) Suppose $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ is a short exact sequence of sheaves on a topological space, and $\mathcal{U}$ is an open cover such that on any intersection the sections of $\mathcal{F}_2$ surject onto $\mathcal{F}_3$. Show that we get a long exact sequence of cohomology. (Note that this applies in our case!)

Date: Tuesday, February 14, 2006. Updated March 8, 2006.

5. If D is an effective Cartier divisor on a projective nonsingular curve, say $D = \sum n_i p_i$, prove that $\deg D = \sum n_i \deg p_i$, where $\deg p_i$ is the degree of the field extension of the residue field at p_i over k.

Class 30:

6. Suppose $V \subset U$ are open subsets of X. Show that we have restriction morphisms $H^i(U, \mathcal{F}) \to H^i(V, \mathcal{F})$ (if U and V are quasicompact, and U hence V is separated). Show that restrictions commute. Hence if X is a Noetherian space, $H^i(\dot{,} \mathcal{F})$ this is a contravariant functor from the category $\text{Top}(X)$ to abelian groups. (The same argument will show more generally that for any map $f : X \to Y$, there exist natural maps $H^i(X, \mathcal{F}) \to H^i(X, f^*\mathcal{F})$; I should have asked this instead.)

7. Show that if $\mathcal{F} \to \mathcal{G}$ is a morphism of quasicoherent sheaves on separated and quasicompact X then we have natural maps $H^i(X, \mathcal{F}) \to H^i(X, \mathcal{G})$. Hence $H^i(X, \cdot)$ is a covariant functor from quasicoherent sheaves on X to abelian groups (or even R-modules).

8. Verify that $H^{n-1}(\mathbb{P}^{n-1}_R, \mathcal{F}') \to H^n(\mathbb{P}^n_R, \mathcal{F})$ is injective. (Hint: one possibility is by verifying that it is the map on Laurent monomials we claimed when proving that cohomology of $\mathcal{O}(m)$ is what we wanted it to be. In particular, this fact was used in that proof, so you can't use that theorem!)

9. Suppose X is a projective k-scheme. Show that Euler characteristic is additive in exact sequences. In other words, if $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$ is an exact sequence of coherent sheaves on X, then $\chi(X, \mathcal{G}) = \chi(X, \mathcal{F}) + \chi(X, \mathcal{H})$. (Hint: consider the long exact sequence in cohomology.) More generally, if

$$0 \to \mathcal{F}_1 \to \cdots \to \mathcal{F}_n \to 0$$

is an exact sequence of sheaves, show that

$$\sum_{i=1}^{n} (-1)^i \chi(X, \mathcal{F}_i) = 0.$$

10. *The Riemann-Roch theorem for line bundles on nonsingular projective curves over* k. Suppose $\mathcal{F}$ is an invertible sheaf on C. Show that $\chi(\mathcal{L}) = \deg \mathcal{L} + \chi(C, \mathcal{O}_C)$. (Possible hint: Write $\mathcal{L}$ as the difference of two effective Cartier divisors, $\mathcal{L} \cong \mathcal{O}(Z - P)$. Describe two exact sequences $0 \to \mathcal{L}(-Z) \to \mathcal{L} \to \mathcal{O}_Z \otimes \mathcal{L} \to 0$ and $0 \to \mathcal{O}_C(-P) \to \mathcal{O}_C \to \mathcal{O}_P \to 0$, where $\mathcal{L}(-Z) \cong \mathcal{O}_C(P)$.)

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FOUNDATIONS OF ALGEBRAIC GEOMETRY PROBLEM SET 14

RAVI VAKIL

This set is due Thursday, March 2, in Jarod Alper's mailbox. It covers (roughly) classes 31 and 32.

Please *read all of the problems*, and ask me about any statements that you are unsure of, even of the many problems you won't try. Hand in six solutions. If you are ambitious (and have the time), go for more. Problems marked with "-" count for half a solution. Problems marked with "+" may be harder or more fundamental, but still count for one solution. Try to solve problems on a range of topics. You are encouraged to talk to each other, and to me, about the problems. Some of these problems require hints, and I'm happy to give them!

Class 31:

1-. Prove the base case of Theorem 1.1 of Class 31. If you choose to do the case $k = -1$, explain precisely why what you are proving is the base case!

2-. Consider the short exact sequence of A-modules $0 \longrightarrow M \xrightarrow{\times f} M \longrightarrow K' \longrightarrow 0$. Show that $\operatorname{Supp} K' = \operatorname{Supp}(M) \cap \operatorname{Supp}(f)$.

3-. Show that the twisted cubic (in $\mathbb{P}^3$) has Hilbert polynomial $3m + 1$.

4. (a) Find the Hilbert polynomial for the dth Veronese embedding of $\mathbb{P}^n$ (i.e. the closed immersion of $\mathbb{P}^n$ in a bigger projective space by way of the line bundle $\mathcal{O}(d)$)
(b) Find the degree of the dth Veronese embedding of $\mathbb{P}^n$.

5-. Show that the degree of a degree d hypersurface is d (preventing a notational crisis).

6. Suppose a curve C is embedded in projective space via an invertible sheaf of degree d. (In other words, this line bundle determines a closed immersion.) Show that the degree of C under this embedding is d (preventing another notational crisis). (Hint: Riemann-Roch.)

7+. *(Bezout's theorem)* Suppose X is a projective scheme of dimension at least 1, and H is a degree d hypersurface not containing any associated points of X. (For example, if X is a projective variety, then we are just requiring H not to contain any irreducible components of X.) Show that $\deg H \cap X = d \deg X$.

8-. Determine the degree of the d-fold Veronese embedding of $\mathbb{P}^n$ in a different way as follows. Let $v_d : \mathbb{P}^n \to \mathbb{P}^N$ be the Veronese embedding. To find the degree of the image,

Date: Tuesday, February 21, 2006. Minor update April 9.

we intersect it with n hyperplanes in $\mathbb{P}^N$ (scheme-theoretically), and find the number of intersection points (counted with multiplicity). But the pullback of a hyperplane in $\mathbb{P}^N$ to $\mathbb{P}^n$ is a degree d hypersurface. Perform this intersection in $\mathbb{P}^n$, and use Bezout's theorem. (If already you know the answer by the earlier exercise on the degree of the Veronese embedding, this will be easier.)

9+. Show that if X is a complete intersection of dimension r in $\mathbb{P}^n$, then $H^i(X, \mathcal{O}_X(m)) = 0$ for all $0 < i < r$ and all m. Show that if $r > 0$, then $H^0(\mathbb{P}^n, \mathcal{O}(m)) \to H^0(X, \mathcal{O}(m))$ is surjective.

10-. Show that complete intersections of positive dimension are connected. (Hint: show $h^0(X, \mathcal{O}_X) = 1$.)

11-. Find the genus of the intersection of 2 quadrics in $\mathbb{P}^3$. (We get curves of more genera by generalizing this!)

12-. Show that the rational normal curve of degree d in $\mathbb{P}^d$ is *not* a complete intersection if $d > 2$.

13-. Show that the union of 2 distinct planes in $\mathbb{P}^4$ is not a complete intersection. (This is the first appearance of another universal counterexample!) Hint: it is connected, but you can slice with another plane and get something not connected.

14. Show that if π is affine, then for $i > 0$, $R^i\pi_*\mathcal{F} = 0$. Moreover, if Y is quasicompact and separated, show that the natural morphism $H^i(X, \mathcal{F}) \to H^i(Y, f_*\mathcal{F})$ is an isomorphism. (A special case of the first sentence is a special case we showed earlier, when π is a closed immersion. Hint: use any affine cover on Y, which will induce an affine cover of X.)

Class 32:

15+. *(Important algebra exercise)* Suppose $M_1 \xrightarrow{\alpha} M_2 \xrightarrow{\beta} M_3$ is a complex of A-modules (i.e. $\beta \circ \alpha = 0$), and N is an A-module. (a) Describe a natural homomorphism of the cohomology of the complex, tensored with N, with the cohomology of the complex you get when you tensor with N $H(M_*) \otimes_A B \to H(M_* \otimes_A N)$, i.e.

$$\left(\frac{\ker \beta}{\operatorname{im} \alpha}\right) \otimes_A N \to \frac{\ker(\beta \otimes N)}{\operatorname{im}(\alpha \otimes N)}.$$

I always forget which way this map is supposed to go.
(b) If N is *flat*, i.e. $\otimes N$ is an exact functor, show that the morphism defined above is an isomorphism. (Hint: This is actually a categorical question: if M_* is an exact sequence in an abelian category, and F is a right-exact functor, then (a) there is a natural morphism $FH(M_*) \to H(FM_*)$, and (b) if F is an exact functor, this morphism is an isomorphism.)

16+. *(Higher pushforwards and base change)* (a) Suppose $f : Z \to Y$ is any morphism, and $\pi : X \to Y$ as usual is quasicompact and separated. Suppose $\mathcal{F}$ is a quasicoherent sheaf on X. Let

$$\begin{array}{ccc} W & \xrightarrow{f'} & X \\ \downarrow{\scriptstyle \pi'} & & \downarrow{\scriptstyle \pi} \\ Z & \xrightarrow{f} & Y \end{array}$$

is a fiber diagram. Describe a natural morphism $f^*(R^i\pi_*\mathcal{F}) \to R^i\pi'_*(f')^*\mathcal{F}$.
(b) If $f : Z \to Y$ is an affine morphism, and for a cover $\operatorname{Spec} A_i$ of Y, where $f^{-1}(\operatorname{Spec} A_i) = \operatorname{Spec} B_i$, B_i is a *flat* A-algebra, show that the natural morphism of (a) is an isomorphism. (You can likely generalize this immediately, but this will lead us into the concept of flat morphisms, and we'll hold off discussing this notion for a while.)

17+. *(The projection formula)* Suppose $\pi : X \to Y$ is quasicompact and separated, and $\mathcal{E}$, $\mathcal{F}$ are quasicoherent sheaves on X and Y respectively. (a) Describe a natural morphism

$$(R^i\pi_*\mathcal{E}) \otimes \mathcal{F} \to R^i\pi_*(\mathcal{E} \otimes \pi^*\mathcal{F}).$$

(b) If $\mathcal{F}$ is locally free, show that this natural morphism is an isomorphism.

18. Consider the open immersion $\pi : \mathbb{A}^n - 0 \to \mathbb{A}^n$. By direct calculation, show that $R^{n-1}f_*\mathcal{O}_{\mathbb{A}^n - 0} \neq 0$.

19+. *(Semicontinuity of fiber dimension of projective morphisms)* Suppose $\pi : X \to Y$ is a projective morphism where $\mathcal{O}_Y$ is coherent. Show that $\{y \in Y : \dim f^{-1}(y) > k\}$ is a Zariski-closed subset. In other words, the dimension of the fiber "jumps over Zariski-closed subsets". (You can interpret the case $k = -1$ as the fact that projective morphisms are closed.) This exercise is rather important for having a sense of how projective morphisms behave! (Hint: see the notes.)

20. Suppose $f : X \to Y$ is a projective morphism, with $\mathcal{O}(1)$ on X. Suppose Y is quasicompact and $\mathcal{O}_Y$ is coherent. Let $\mathcal{F}$ be coherent on X. Show that

(a) $f^*f_*\mathcal{F}(n) \to \mathcal{F}(n)$ is surjective for $n \gg 0$. (First show that there is a natural map for any n! Hint: by adjointness of f_* with f_*.) Translation: for $n \gg 0$, $\mathcal{F}(n)$ is relatively generated by global sections.
(b) For $i > 0$ and $n \gg 0$, $R^if_*\mathcal{F}(n) = 0$.

21-. Show that $H^0(A^*) = E_\infty^{0,0} = E_2^{0,0}$ and

$$0 \to E_2^{1,0} \to H^1(A^*) \to E_2^{0,1} \to E_2^{2,0} \to H^2(A^*).$$

(Here take the spectral sequence starting with the vertical arrows.)

22. Suppose we are working in the category of vector spaces over a field k, and $\oplus_{p,q}E_2^{p,q}$ is a finite-dimensional vector space. Show that $\chi(H^*(A^*))$ is well-defined, and equals $\sum_{p,q}(-1)^{p+q}E_2^{p,q}$. (It will sometimes happen that $\oplus E_0^{p,q}$ will be an infinite-dimensional vector space, but that $E_2^{p,q}$ will be finite-dimensional!)

23. By looking at our spectral sequence proof of the five lemma, prove a subtler version of the five lemma, where one of the isomorphisms can instead just be required to be an injection, and another can instead just be required to be a surjection. (I'm deliberately not telling you which ones, so you can see how the spectral sequence is telling you how to improve the result.) I've heard this called the "subtle five lemma", but I like calling it the $4\frac{1}{2}$-lemma.

24. If β and δ (in (1)) are injective, and α is surjective, show that γ is injective. State the dual statement. (The proof of the dual statement will be essentially the same.)

(1)
$$\begin{array}{ccccccccc} F & \longrightarrow & G & \longrightarrow & H & \longrightarrow & I & \longrightarrow & J \\ \alpha\uparrow & & \beta\uparrow & & \gamma\uparrow & & \delta\uparrow & & \epsilon\uparrow \\ A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & D & \longrightarrow & E \end{array}$$

25. Use spectral sequences to show that a short exact sequence of complexes gives a long exact sequence in cohomology.

26. Suppose $\mu : A^* \to B^*$ is a morphism of complexes. Suppose C^* is the single complex associated to the double complex $A^* \to B^*$. (C^* is called the *mapping cone* of μ.) Show that there is a long exact sequence of complexes:

$$\cdots \to H^{i-1}(C^*) \to H^i(A^*) \to H^i(B^*) \to H^i(C^*) \to H^{i+1}(A^*) \to \cdots .$$

(There is a slight notational ambiguity here; depending on how you index your double complex, your long exact sequence might look slightly different.) In particular, people often use the fact μ induces an isomorphism on cohomology if and only if the mapping cone is exact.

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FOUNDATIONS OF ALGEBRAIC GEOMETRY PROBLEM SET 15

RAVI VAKIL

This set is due Thursday, March 9, in Jarod Alper's mailbox. It covers (roughly) classes 33 and 34.

Please *read all of the problems*, and ask me about any statements that you are unsure of, even of the many problems you won't try. Hand in five solutions. If you are ambitious (and have the time), go for more. Problems marked with "-" count for half a solution. Problems marked with "+" may be harder or more fundamental, but still count for one solution. Try to solve problems on a range of topics. You are encouraged to talk to each other, and to me, about the problems. Some of these problems require hints, and I'm happy to give them!

Class 33:

1. *(for people who like non-algebraically closed fields)* Suppose that X is a quasicompact separated k-scheme, where k is a field. Suppose $\mathcal{F}$ is a quasicoherent sheaf on X. Let $X_{\overline{k}} = X \times_{\operatorname{Spec} k} \operatorname{Spec} \overline{k}$, and $f : X_{\overline{k}} \to X$ the projection. Describe a natural isomorphism $H^i(X, \mathcal{F}) \otimes_k \overline{k} \to H^i(X_{\overline{k}}, f^*\mathcal{F})$. Recall that a k-scheme X is *geometrically integral* if $X_{\overline{k}}$ is integral. Show that if X is geometrically integral, then $H^0(X, \mathcal{O}_X) \cong k$. (This is a clue that $\mathbb{P}^1_{\mathbb{C}}$ is not a geometrically integral $\mathbb{R}$-scheme.)

2. Suppose Y is any scheme, and $\pi : \mathbb{P}^n_Y \to Y$ is the trivial projective bundle over Y. Show that $\pi_* \mathcal{O}_{\mathbb{P}^n_Y} \cong \mathcal{O}_Y$. More generally, show that $R^j \pi_* \mathcal{O}(m)$ is a finite rank free sheaf on Y, and is 0 if $j \neq 0, n$. Find the rank otherwise.

3. Let A be any ring. Suppose a is a negative integer and b is a positive integer. Show that $H^i(\mathbb{P}^m_A \times_A \mathbb{P}^n_A, \mathcal{O}(a, b))$ is 0 unless $i = m$, in which case it is a free A-module. Find the rank of this free A-module. (Hint: Use the previous exercise, and the projection formula, which was Exercise 1.3 of class 32, and exercise 17 of problem set 14.)

4. (a) Find the genus of a curve in class $(2, n)$ on $\mathbb{P}^1_k \times_k \mathbb{P}^1_k$. (A curve in class $(2, n)$ is any effective Cartier divisor corresponding to invertible sheaf $\mathcal{O}(2, n)$. Equivalently, it is a curve whose ideal sheaf is isomorphic to $\mathcal{O}(-2, -n)$. Equivalently, it is a curve cut out by a non-zero form of bidegree $(2, n)$.)
(b) Suppose for convenience that k is algebraically closed of characteristic not 2. Show that there exists an integral nonsingular curve in class $(2, n)$ on $\mathbb{P}^1_k \times \mathbb{P}^1_k$ for each $n > 0$.

5. Suppose X and Y are projective k-schemes, and $\mathcal{F}$ and $\mathcal{G}$ are coherent sheaves on X and Y respectively. Recall that if $\pi_1 : X \times Y \to X$ and $\pi_2 : X \times Y \to Y$ are the two projections, then $\mathcal{F} \boxtimes \mathcal{G} := \pi_1^* \mathcal{F} \otimes \pi_2^* \mathcal{G}$. Prove the following, adding additional hypotheses if you find

Date: Tuesday, February 28, 2006. Updated March 6.

them necessary.
(a) Show that $H^0(X \times Y, \mathcal{F} \boxtimes \mathcal{G}) = H^0(X, \mathcal{F}) \otimes H^0(Y, \mathcal{G})$.
(b) Show that $H^{\dim X + \dim Y}(X \times Y, \mathcal{F} \boxtimes \mathcal{G}) = H^{\dim X}(X, \mathcal{F}) \otimes_k H^{\dim Y}(Y, \mathcal{G})$.
(c) Show that $\chi(X \times Y, \mathcal{F} \boxtimes \mathcal{G}) = \chi(X, \mathcal{F})\chi(Y, \mathcal{G})$.

Class 34:

6-. Show that the following two morphisms are projective morphisms that are injective on points, but that are not injective on tangent vectors.
(a) the normalization of the cusp $y^2 = x^3$ in the plane
(b) the Frobenius morphism from $\mathbb{A}^1$ to $\mathbb{A}^1$, given by $k[t] \to k[u]$, $u \to t^p$, where k has characteristic p.

7. Suppose $\mathcal{L}$ is a degree $2g-2$ invertible sheaf. Show that it has $g-1$ or g sections, and it has g sections if and only if $\mathcal{L} \cong \mathcal{K}$.

8. Suppose C is a genus 0 curve (projective, geometrically integral and nonsingular). Show that C has a point of degree at most 2.

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FOUNDATIONS OF ALGEBRAIC GEOMETRY PROBLEM SET 16

RAVI VAKIL

This set is due Thursday, March 16, in Jarod Alper's mailbox. It covers (roughly) classes 35 and 36.

Please *read all of the problems*, and ask me about any statements that you are unsure of, even of the many problems you won't try. Hand in six solutions. If you are ambitious (and have the time), go for more. Problems marked with "-" count for half a solution. Problems marked with "+" may be harder or more fundamental, but still count for one solution. Try to solve problems on a range of topics. You are encouraged to talk to each other, and to me, about the problems. Some of these problems require hints, and I'm happy to give them!

Class 35:

1-. Show that a curve C of genus at least 1 admits a degree 2 cover of $\mathbb{P}^1$ if and only if it has a degree 2 invertible sheaf with precisely 2 sections.

2. Show that the nonhyperelliptic curves of genus 3 form a family of dimension 6. (Hint: Count the dimension of the family of nonsingular quartics, and quotient by $\operatorname{Aut}\mathbb{P}^2 = PGL(3)$.) This (and all other moduli dimension-counting arguments) should be interpreted as: "make a plausibility argument", as we haven't yet defined these moduli spaces.

3. Suppose C is a genus g curve. Show that if C is not hyperelliptic, then the canonical bundle gives a closed immersion $C \hookrightarrow \mathbb{P}^{g-1}$. (In the hyperelliptic case, we have already seen that the canonical bundle gives us a double cover of a rational normal curve.) Hint: follow the genus 3 case. Such a curve is called a *canonical curve*.

4-. Suppose C is a curve of genus $g > 1$, over a field k that is not algebraically closed. Show that C has a closed point of degree at most $2g-2$ over the base field. (For comparison: if $g = 1$, there is no such bound!)

5. Suppose $X \subset Y \subset \mathbb{P}^n$ are a sequence of closed subschemes, where X and Y have the same Hilbert polynomial. Show that $X = Y$. (Hint: consider the exact sequence

$$0 \to \mathcal{I}_{X/Y} \to \mathcal{O}_Y \to \mathcal{O}_X \to 0.$$

Show that if the Hilbert polynomial of $\mathcal{I}_{X/Y}$ is 0, then $\mathcal{I}_{X/Y}$ must be the 0 sheaf.)

6. Suppose that C is a complete intersection of a quadric surface with a cubic surface. Show that $\mathcal{O}_C(1)$ has 4 sections. (Hint: long exact sequences!)

Date: Tuesday, February 28, 2006. Superficial update June 26.

7. Show that nonhyperelliptic curves of genus 4 "form a family of dimension $9 = 3g - 3$". (Again, this isn't a mathematically well-formed question. So just give a plausibility argument.)

8. Suppose C is a nonhyperelliptic genus 5 curve. The canonical curve is degree 8 in $\mathbb{P}^4$. Show that it lies on a three-dimensional vector space of quadrics (i.e. it lies on 3 independent quadrics). Show that a nonsingular complete intersection of 3 quadrics is a canonical genus 5 curve.

9. Show that the complete intersections of 3 quadrics in $\mathbb{P}^4$ form a family of dimension $12 = 3 \times 5 - 3$.

10-. Show that if $C \subset \mathbb{P}^{g-1}$ is a canonical curve of genus $g \geq 6$, then C is *not* a complete intersection. (Hint: Bezout.)

Class 36:

11. (a) Suppose C is a projective curve. Show that $C - p$ is affine. (Hint: show that $n \gg 0$, $\mathcal{O}(np)$ gives an embedding of C into some projective space $\mathbb{P}^m$, and that there is some hyperplane H meeting C precisely at p. Then $C - p$ is a closed subscheme of $\mathbb{P}^n - H$.)
(b) If C is a geometrically integral nonsingular curve over a field k (i.e. all of our standing assumptions, minus projectivity), show that it is projective or affine.

12. Suppose (E, p) is an elliptic curve. Show that $\mathcal{O}(4p)$ embeds E in $\mathbb{P}^3$ as the complete intersection of two quadrics.

13+. Verify that the axiomatic definition and the functorial definition of a group object in a category are the same.

14+. Suppose (E, p) is an elliptic curve. Show that (E, p) is a group scheme. You may assume that we've defined the multiplication morphism, as sketched in class and in the notes. (Caution! we've stated that only the closed points form a group — the group Pic^0. So there is something to show here. The main idea is that with varieties, lots of things can be checked on closed points. First assume that $k = \overline{k}$, so the closed points are dimension 1 points. Then the associativity diagram is commutative on closed points; argue that it is hence commutative. Ditto for the other categorical requirements. Finally, deal with the case where k is not algebraically closed, by working over the algebraic closure.)

15-. Show that $\mathbb{A}^1_k$ is a group scheme under addition, and $\mathbb{G}_m$ is a group scheme under multiplication. You'll see that the functorial description trumps the axiomatic description here! (Recall that $\mathrm{Hom}(X, \mathbb{A}^1_k)$ is canonically $\Gamma(X, \mathcal{O}_X)$, and $\mathrm{Hom}(X, \mathbb{G}_m)$ is canonically $\Gamma(X, \mathcal{O}_X)^*$.)

16. Define the group scheme $GL(n)$ over the integers.

17-. Define μ_n to be the kernel of the map of group schemes $\mathbb{G}_m \to \mathbb{G}_m$ that is "taking nth powers". In the case where n is a prime p, which is also $\mathrm{char}\, k$, describe μ_p. (I.e. how many points? How "big" = degree over k?)

18-. Define a *ring scheme*. Show that $\mathbb{A}^1_k$ is a ring scheme.

19. Because $\mathbb{A}^1_k$ is a group scheme, $k[t]$ is a Hopf algebra. Describe the comultiplication map $k[t] \to k[t] \otimes_k k[t]$.

20. Suppose X is a scheme, and L is the total space of a line bundle corresponding to invertible sheaf $\mathcal{L}$, so $L = \operatorname{Spec} \oplus_{n \geq 0} (\mathcal{L}^\vee)^{\otimes n}$. Show that $H^0(L, \mathcal{O}_L) = \oplus H^0(X, (\mathcal{L}^\vee)^{\otimes n})$.

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FOUNDATIONS OF ALGEBRAIC GEOMETRY PROBLEM SET 17

RAVI VAKIL

This set is due Thursday, April 20. You can hand it in to Rob Easton, in class or via his mailbox. It covers (roughly) classes 37 and 38.

Please *read all of the problems*, and ask me about any statements that you are unsure of, even of the many problems you won't try. Hand in five solutions. If you are ambitious (and have the time), go for more. Problems marked with "-" count for half a solution. Problems marked with "+" may be harder or more fundamental, but still count for one solution. Try to solve problems on a range of topics. You are encouraged to talk to each other, and to me, about the problems. Some of these problems require hints, and I'm happy to give them!

Class 37:

1+. In class I stated the following. Note that if A is generated over B (as an algebra) by $x_i \in A$ (where i lies in some index set, possibly infinite), subject to some relations r_j (where j lies in some index set, and each is a polynomial in some finite number of the x_i), then the A-module $\Omega_{A/B}$ is generated by the dx_i, subject to the relations (i)—(iii) and $dr_j = 0$. In short, we needn't take every single element of A; we can take a generating set. And we needn't take every single relation among these generating elements; we can take generators of the relations. Verify this.

2. *(localization of differentials)* If S is a multiplicative set of A, show that there is a natural isomorphism $\Omega_{S^{-1}A/B} \cong S^{-1}\Omega_{A/B}$. (Again, this should be believable from the intuitive picture of "vertical cotangent vectors".) If T is a multiplicative set of B, show that there is a natural isomorphism $\Omega_{S^{-1}A/T^{-1}B} \cong S^{-1}\Omega_{A/B}$ where S is the multiplicative set of A that is the image of the multiplicative set $T \subset B$.

3+. (a) *(pullback of differentials)* If

$$\begin{array}{ccc} A' & \longleftarrow & A \\ \uparrow & & \uparrow \\ B' & \longleftarrow & B \end{array}$$

is a commutative diagram, show that there is a natural homomorphism of A'-modules $\Omega_{A/B} \otimes_A A' \to \Omega_{A'/B'}$. An important special case is $B = B'$.
(b) *(differentials behave well with respect to base extension, affine case)* If furthermore the above diagram is a tensor diagram (i.e. $A' \cong B' \otimes_B A$) then show that $\Omega_{A/B} \otimes_A A' \to \Omega_{A'/B'}$ is an isomorphism.

4. Suppose k is a field, and K is a separable algebraic extension of k. Show that $\Omega_{K/k} = 0$.

Date: Tuesday, April 11, 2006. Minor update May 15.

5. *(Jacobian description of $\Omega_{A/B}$)* Suppose $A = B[x_1, \dots, x_n]/(f_1, \dots, f_r)$. Then $\Omega_{A/B} = \{\oplus_i B dx_i\}/\{df_j = 0\}$ maybe interpreted as the cokernel of the Jacobian matrix $J : A^{\oplus r} \to A^{\oplus n}$.

Class 38:

6. *(normal bundles to effective Cartier divisors)* Suppose $D \subset X$ is an effective Cartier divisor. Show that the conormal sheaf $\mathcal{N}^\vee_{D/X}$ is $\mathcal{O}(-D)|_D$ (and in particular is an invertible sheaf), and hence that the normal sheaf is $\mathcal{O}(D)|_D$. It may be surprising that the normal sheaf should be locally free if $X \cong \mathbb{A}^2$ and D is the union of the two axes (and more generally if X is nonsingular but D is singular), because you may be used to thinking that the normal bundle is isomorphic to a "tubular neighborhood".

7-. Suppose $f : X \to Y$ is locally of finite type, and X is locally Noetherian. Show that $\Omega_{X/Y}$ is a coherent sheaf on X.

8+. *(differentials on hyperelliptic curves)* Consider the double cover $f : C \to \mathbb{P}^1_k$ branched over $2g + 2$ distinct points. (We saw earlier that this curve has genus g.) Then $\Omega_{C/k}$ is again an invertible sheaf. What is its degree? (Hint: let x be a coordinate on one of the coordinate patches of $\mathbb{P}^1_k$. Consider $f^* dx$ on C, and count poles and zeros.) In class I gave a sketch showing that you should expect the answer to be $2g - 2$.

9. *(differentials on non-singular plane curves)* Suppose C is a nonsingular plane curve of degree d in $\mathbb{P}^2_k$, where k is algebraically closed. By considering coordinate patches, find the degree of $\Omega_{C/k}$. Make any reasonable simplifying assumption (so that you believe that your result still holds for "most" curves).

10. Suppose that C is a nonsingular projective curve over k such that $\Omega_{C/k}$ is an invertible sheaf. (We'll see that for nonsingular curves, the sheaf of differentials is always locally free. But we don't yet know that.) Let $C_{\overline{k}} = C \times_{\operatorname{Spec} k} \operatorname{Spec} \overline{k}$. Show that $\Omega_{C_{\overline{k}}/\overline{k}}$ is locally free, and that

$$\deg \Omega_{C_{\overline{k}}/\overline{k}} = \deg \Omega_{C/k}.$$

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FOUNDATIONS OF ALGEBRAIC GEOMETRY PROBLEM SET 18

RAVI VAKIL

This set is due Thursday, May 4. You can hand it in to Rob Easton, in class or via his mailbox. It covers (roughly) classes 39, 40, 41, and 42.

Please *read all of the problems*, and ask me about any statements that you are unsure of, even of the many problems you won't try. Hand in seven solutions. If you are ambitious (and have the time), go for more. Problems marked with "-" count for half a solution. Problems marked with "+" may be harder or more fundamental, but still count for one solution. Try to solve problems on a range of topics. You are encouraged to talk to each other, and to me, about the problems. Some of these problems require hints, and I'm happy to give them!

Classes 39–40:

1. Show that $H^1(\mathbb{P}^n_A, T_{\mathbb{P}^n_A}) = 0$. (This later turns out to be an important calculation for the following reason. If X is a nonsingular variety, $H^1(X, T_X)$ parametrizes deformations of the variety. Thus projective space can't deform, and is "rigid".)

2. I discussed the Grassmannian, which "parametrizes" the space of vector spaces of dimension m in an $(n+1)$-dimensional vector space V (over our base field k). The case $m = 1$ is $\mathbb{P}^n$. Over $G(m, n+1)$ we have a short exact sequence of locally free sheaves

$$0 \to S \to V \otimes \mathcal{O}_{G(m,n+1)} \to Q \to 0$$

where $V \otimes \mathcal{O}_{G(m,n+1)}$ is a trivial bundle, and S is the "universal subbundle" (such that over a point $[V' \subset V]$ of the Grassmannian $G(m, n+1)$, $S|_{[V' \subset V]}$ is V). Then

$$\Omega_{G(m,n+1)/k} \cong \underline{\mathrm{Hom}}(Q, S). \tag{1}$$

In the case of projective space, $m = 1$, $S = \mathcal{O}(-1)$. Verify (1) in this case.

3+. Show that if k is separably closed, then $X_{\overline{k}}$ is nonsingular if and only if X is nonsingular.

4-. Show that Bertini's theorem still holds even if the variety X is singular in dimension 0.

5. Suppose $C \subset \mathbb{P}^2$ is a nonsingular conic over a field of characteristic not 2. Show that the dual variety is also a conic. (More precisely, suppose C is cut out by $f(x_0, x_1, x_2) = 0$. Show that $\{(a_0, a_1, a_2) : a_0x_0 + a_1x_1 + a_2x_2 = 0\}$ is cut out by a quadratic equation.) Thus for example, through a general point in the plane, there are two tangents to C. (The points on a line in the dual plane corresponds to those lines through a point of the original plane.)

Date: Tuesday, April 25, 2006. Updated June 26.

6. *(interpreting the ramification divisor in terms of number of preimages)* Suppose all the ramification above $y \in Y$ is tame. Show that the degree of the branch divisor at y is $\deg(f : X \to Y) - \#f^{-1}(y)$. Thus the multiplicity of the branch divisor counts the extent to which the number of preimages is less than the degree.

7. *(degree of dual curves)* Describe the degree of the dual to a nonsingular degree d plane curve C as follows. Pick a general point $p \in \mathbb{P}^2$. Find the number of tangents to C through p, by noting that projection from p gives a degree d map to $\mathbb{P}^1$ (why?) by a curve of known genus (you've calculated this before), and that ramification of this cover of $\mathbb{P}^1$ corresponds to a tangents through p. (Feel free to make assumptions, e.g. that for a general p this branched cover has the simplest possible branching — this should be a back-of-an-envelope calculation.)

8. *(Artin-Schreier covers)* In characteristic 0, the only connected unbranched cover of $\mathbb{A}^1$ is the isomorphism $\mathbb{A}^1 \xrightarrow{\sim} \mathbb{A}^1$; that was an earlier example/exercise, when we discussed Riemann-Hurwitz the first time. In positive characteristic, this needn't be true, because of wild ramification over ∞. Show that the morphism corresponding to $k[x] \to k[x,y]/(y^p - x^p - y)$ is such a map. (Once the theory of the algebraic fundamental group is developed, this translates to: "$\mathbb{A}^1$ is not simply connected in characteristic p.")

Classes 41–42:

9-. If $N' \to N \to N''$ is exact and M is a flat A-module, show that $M \otimes_A N' \to M \otimes_A N \to M \otimes_A N''$ is exact. Hence *any* exact sequence of A-modules remains exact upon tensoring with M. (We've seen things like this before, so this should be fairly straightforward.)

10-. *(localizations are flat)*. Suppose that S is a multiplicative subset of B. Show that $B \to S^{-1}B$ is a flat ring morphism.

11-. Suppose that A is a ring, $\mathfrak{p}$ is a prime ideal, M is an $A_{\mathfrak{p}}$-module, and N is an A-module. Show that $M \otimes_A N$ is canonically isomorphic to $M \otimes_{A_{\mathfrak{p}}} N_{\mathfrak{p}}$.

12. (a) Prove that flatness is preserved by chase of base ring: If M flat A-module, $A \to B$ is a homomorphism, then $M \otimes_A B$ is a flat B-module.

(b) Prove transitivity of flatness: If B is a flat A-algebra, and M is B-flat, then it is also A-flat. (Hint: consider the natural isomorphism $(M \otimes_A B) \otimes_B \cdot \cong M \otimes_B (B \otimes_A \cdot)$.)

13. If X is a scheme, and η is the generic point for an irreducible component, show that the natural morphism $\operatorname{Spec} \mathcal{O}_{X,\eta} \to X$ is flat. (Hint: localization is flat.)

14. Show that $B \to A$ is faithfully flat if and only if $\operatorname{Spec} A \to \operatorname{Spec} B$ is faithfully flat. (Use the definitions in the notes!)

15. Show that two homotopic maps of complexes induce the same map on homology. (Do this only if you haven't seen this before!)

16. Show that any two lifts of resolutions of modules are homotopic (see the notes for a more precise statement).

17. The notion of an *injective object* in an abelian category is dual to the notion of a projective object. Define derived functors for (i) covariant left-exact functors (these are called right-derived functors), (ii) contravariant left-exact functors (also right-derived functors), and (iii) contravariant right-exact functors (these are called left-derived functors), making explicit the necessary assumptions of the category having enough injectives or projectives.

18+. If B is A-flat, then we get isomorphism $B \otimes \mathrm{Tor}_i^A(M, N) \cong \mathrm{Tor}_i^B(B \otimes M, B \otimes N)$. (Here is a fancier fact that experts may want to try: if B is not A-flat, we don't get an isomorphism; instead we get a spectral sequence.)

19. *(not too important, but good practice if you haven't played with* Tor *before)* If x is not a 0-divisor, show that $\mathrm{Tor}_i^A(A/x, M)$ is 0 for $i > 1$, and for $i = 0$, get M/xM, and for $i = 1$, get $(M : x)$ (those things sent to 0 upon multiplication by x).

20+. *(flatness over the dual numbers)* This fact is important in deformation theory and elsewhere. Show that M is flat over $k[t]/t^2$ if and only if the natural map $M/tM \to tM$ is an isomorphism.

21-. If $0 \to M_0 \to M_1 \to \cdots \to M_n \to 0$ is an exact sequence, and M_i is flat for $i > 0$, show that M_0 is flat too. (Hint: as always, break into short exact sequences.)

22+. *(flat limits are unique)* Suppose A is a discrete valuation ring, and let η be the generic point of $\operatorname{Spec} A$. Suppose X is proper over A, and Y is a closed subscheme of X_η. Show that there is only one closed subscheme Y' of X, proper over A, such that $Y'|_\eta = Y$, and Y' is flat over A.

23. *(an interesting explicit example of a flat limit)* Let $X = \mathbb{A}^3 \times \mathbb{A}^1 \to Y = \mathbb{A}^1$ over a field k, where the coordinates on $\mathbb{A}^3$ are x, y, and z, and the coordinates on $\mathbb{A}^1$ are t. Define X away from $t = 0$ as the union of the two lines $y = z = 0$ (the x-axis) and $x = z - t = 0$ (the y-axis translated by t). Find the flat limit at $t = 0$. (Hint: it is *not* the union of the two axes, although it includes it. The flat limit is non-reduced.)

24. Prove that flat and locally finite type morphisms of locally Noetherian schemes are open. (Hint: reduce to the affine case. Use Chevalley's theorem to show that the image is constructible. Reduce to target that is the spectrum of a local ring. Show that the generic point is hit.)

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FOUNDATIONS OF ALGEBRAIC GEOMETRY PROBLEM SET 19

RAVI VAKIL

This set is due Thursday, May 18. You can hand it in to Rob Easton, in class or via his mailbox. It covers (roughly) classes 43, 44, 45, and 46.

Please *read all of the problems*, and ask me about any statements that you are unsure of, even of the many problems you won't try. Hand in eight solutions. If you are ambitious (and have the time), go for more. *Unlike previous sets, problems marked with "+" count for two solutions.* Try to solve problems on a range of topics. You are encouraged to talk to each other, and to me, about the problems. Some of these problems require hints, and I'm happy to give them!

In lieu of completing this problems, you can prove the Cohomology and base change theorem.

Classes 43–44:

1. Prove the Riemann-Roch theorem for two $\mathbb{P}^1$'s glued together at a (reduced) point. (We needed this for our proof that a certain proper surface was nonprojective.)

2. *Gluing two schemes together along isomorphic closed subschemes.* Suppose X' and X'' are two schemes, with closed subschemes $W' \hookrightarrow X'$ and $W'' \hookrightarrow X''$, and an isomorphism $W' \to W''$. Show that we can glue together X' and X'' along $W' \cong W''$. More precisely, show that the following *coproduct* exists:

$$\begin{array}{ccc} W' \cong W'' & \longrightarrow & X' \\ \downarrow & & \downarrow \\ X'' & \longrightarrow & ?. \end{array}$$

Hint: work by analogy with our product construction. If the coproduct exists, it is unique up to unique isomorphism. Start with judiciously chosen affine open subsets, and glue.

3. I alleged that a certain surface is proper over k (see the notes). Prove this. (Possible hint: show that the union of two proper schemes is proper.)

4. The Picard scheme $\operatorname{Pic} X/Y \to Y$ is a scheme over Y which represents the following functor: Given any $T \to Y$, we have the set of invertible sheaves on $X \times_Y T$, modulo those invertible sheaves pulled back from T. In other words, there is a natural bijection between

Date: Tuesday, May 9, 2006.

diagrams of the form

$$\begin{array}{ccc} \mathcal{L} & & \\ \downarrow & & \\ X \times_T Y & \longrightarrow & X \\ \downarrow & & \downarrow \\ T & \longrightarrow & Y \end{array}$$

and diagrams of the form

$$\begin{array}{ccc} & \nearrow & \mathrm{Pic}_{X/Y} \\ & & \downarrow \\ T & \longrightarrow & Y. \end{array}$$

It is a hard theorem (due to Grothendieck) that (at least if Y is reasonable, e.g. locally Noetherian — I haven't consulted the appropriate references) $\mathrm{Pic}\, X/Y \to Y$ exists, i.e. that this functor is representable. In fact $\mathrm{Pic}\, X/Y$ is of finite type. Problem: Given its existence, check that $\mathrm{Pic}_{X/Y}$ is a group scheme over Y, using our functorial definition of group schemes.

5. Show that the Picard scheme for $X \to Y$, where the morphism is flat and projective, and the fibers are geometrically integral, is separated over Y by showing that it satisfies the valuative criterion of separatedness.

Classes 45–46:

6. Suppose $\mathcal{F}$ is a coherent sheaf on X, $\pi : X \to Y$ projective, Y (hence X) Noetherian, and $\mathcal{F}$ flat over Y. Let $\phi^p : R^p\pi_*\mathcal{F} \otimes k(y) \to H^p(X_y, \mathcal{F}_y)$ be the natural morphism. Suppose $H^p(X_y, \mathcal{F}_y) = 0$ for all $y \in Y$. Show that ϕ^{p-1} is an isomorphism for all $y \in Y$. (Hint: cohomology and base change (b).)

7. With the same hypotheses as the previous problem, suppose $R^p\pi_*\mathcal{F} = 0$ for $p \geq p_0$. Show that $H^p(X_y, \mathcal{F}_y) = 0$ for all $y \in Y$, $k \geq k_0$. (Same hint. You can also do this directly from the key theorem presented in class.)

8+. *(Important!)* Suppose π is a projective flat family, each of whose fibers are (nonempty) integral schemes, or more generally whose fibers satisfy $h^0(X_y) = 1$. Then (*) holds. (Hint: consider

$$\mathcal{O}_Y \otimes k(y) \longrightarrow (\pi_*\mathcal{O}_X) \otimes k(y) \xrightarrow{\phi^0} H^0(X_y, \mathcal{O}_{X_y}) \cong k(y) \ .$$

The composition is surjective, hence ϕ^0 is surjective, hence it is an isomorphism (by the Cohomology and base change theorem (a)). Then thanks to the Cohomology and base change theorem (b), $\pi_*\mathcal{O}_X$ is locally free, thus of rank 1. If I have a map of invertible sheaves $\mathcal{O}_Y \to \pi_*\mathcal{O}_X$ that is an isomorphism on closed points, it is an isomorphism (everywhere) by Nakayama.)

9. *(the Hodge bundle; important in Gromov-Witten theory)* Suppose $\pi : X \to Y$ is a projective flat family, all of whose geometric fibers are connected reduced curves of arithmetic genus

g. Show that $R^1\pi_*\mathcal{O}_X$ is a locally free sheaf of rank g. This is called the *Hodge bundle*. [Hint: use cohomology and base change (b) twice, once with $p = 2$, and once with $p = 1$.]

10. Suppose $\pi : X \to Y$ satisfies (*). Show that if $\mathcal{M}$ is any invertible sheaf on Y, then the natural morphism $\mathcal{M} \to \pi_*\pi^*\mathcal{M}$ is an isomorphism. In particular, we can recover $\mathcal{M}$ from $\pi^*\mathcal{M}$ by pushing forward. (Hint: projection formula.)

11. Suppose X is an integral Noetherian scheme. Show that $\operatorname{Pic}(X \times \mathbb{P}^1) \cong \operatorname{Pic} X \times \mathbb{Z}$. (Side remark: If X is non-reduced, this is still true, see Hartshorne Exercise III.12.6(b). It need only be connected of finite type over k. Presumably locally Noetherian suffices.) Extend this to $X \times \mathbb{P}^n$. Extend this to any $\mathbb{P}^n$-bundle over X.

12. Suppose $X \to Y$ is the projectivization of a vector bundle $\mathcal{F}$ over a reduced locally Noetherian scheme (i.e. $X = \underline{\operatorname{Proj}} \operatorname{Sym}^* \mathcal{F}$). Then I think we've already shown in an exercise that it is also the projectivization of $\mathcal{F} \otimes \mathcal{L}$. If Y is reduced and locally Noetherian, show that these are the only ways in which it is the projectivization of a vector bundle. (Hint: note that you can recover $\mathcal{F}$ by pushing forward $\mathcal{O}(1)$.)

13. Suppose $\pi : X \to Y$ is a projective flat morphism over a Noetherian integral scheme, all of whose geometric fibers are isomorphic to $\mathbb{P}^n$ (over the appropriate field). Show that this is a projective bundle if and only if there is an invertible sheaf on X that restricts to $\mathcal{O}(1)$ on all the fibers. (One direction is clear: if it is a projective bundle, then it has a projective $\mathcal{O}(1)$. In the other direction, the candidate vector bundle is $\pi_*\mathcal{O}(1)$. Show that it is indeed a locally free sheaf of the desired rank. Show that its projectivization is indeed $\pi : X \to Y$.)

14. *An example of a Picard scheme* Show that the Picard scheme of $\mathbb{P}^1_k$ over k is isomorphic to $\mathbb{Z}$.

15+. *An example of a Picard scheme* Show that if E is an elliptic curve over k (a geometrically integral and nonsingular genus 1 curve with a marked k-point), then $\operatorname{Pic} E$ is isomorphic to $E \times \mathbb{Z}$. Hint: Choose a marked point p. (You'll note that this isn't canonical.) Describe the candidate universal invertible sheaf on $E \times \mathbb{Z}$. Given an invertible sheaf on $E \times X$, where X is an arbitrary Noetherian scheme, describe the morphism $X \to E \times \mathbb{Z}$.

16. By a similar argument as we showed that abelian varieties are commutative, show that any map $f : A \to A'$ from one abelian variety to another is a group homomorphism followed by a translation. (Hint: reduce quickly to the case where f sends the identity to the identity. Then show that "$f(x + y) - f(x) - f(y) = e$".)

17. Prove the following. Suppose $f : X \to Y$ is a flat finite-type morphism of locally Noetherian schemes, and Y is irreducible. Then the following are equivalent.

- Every irreducible component of X has dimension $\dim Y + n$.
- For any point $y \in Y$ (not necessarily closed!), every irreducible component of the fiber X_y has dimension n.

18+. Show that if $f : X \to Y$ is a flat morphism of finite type k-schemes (or localizations thereof), then any associated point of X must map to an associated point of Y. (I find this an important point when visualizing flatness!) Hint: use a variant of an argument in the notes. (See the statement of this problem in the notes for more details.)

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FOUNDATIONS OF ALGEBRAIC GEOMETRY PROBLEM SET 20

RAVI VAKIL

This set is due Thursday, May 25. You can hand it in to Rob Easton, in class or via his mailbox. It covers (roughly) classes 47 and 48.

Please *read all of the problems*, and ask me about any statements that you are unsure of, even of the many problems you won't try. Hand in five solutions. If you are ambitious (and have the time), go for more. Try to solve problems on a range of topics. You are encouraged to talk to each other, and to me, about the problems. Some of these problems require hints, and I'm happy to give them!

1. *(for those who know what a Cohen-Macaulay scheme is)* Suppose $\pi : X \to Y$ is a map of locally Noetherian schemes, where both X and Y are equidimensional, and Y is nonsingular. Show that if any two of the following hold, then the third does as well:

- π is flat.
- X is Cohen-Macaulay.
- Every fiber X_y is Cohen-Macaulay of the expected dimension.

2. *(generated ⊗ generated = generated for finite type sheaves)* Suppose $\mathcal{F}$ and $\mathcal{G}$ are finite type sheaves on a scheme X that are generated by global sections. Show that $\mathcal{F} \otimes \mathcal{G}$ is also generated by global sections. In particular, if $\mathcal{L}$ and $\mathcal{M}$ are invertible sheaves on a scheme X, and both $\mathcal{L}$ and $\mathcal{M}$ are base-point-free, then so is $\mathcal{L} \otimes \mathcal{M}$. (This is often summarized as "base-point-free + base-point-free = base-point-free". The symbols $+$ is used rather than $\otimes$, because Pic is an abelian group.)

3. *(very ample + very ample = very ample)* If $\mathcal{L}$ and $\mathcal{M}$ are invertible sheaves on a scheme X, and both $\mathcal{L}$ and $\mathcal{M}$ are base-point-free, then so is $\mathcal{L} \otimes \mathcal{M}$. Hint: Segre. In particular, tensor powers of a very ample invertible sheaf are very ample.

4+. *(very ample + relatively generated = very ample)*. Suppose $\mathcal{L}$ is very ample, and $\mathcal{M}$ is relatively generated, both on $X \to Y$. Show that $\mathcal{L} \otimes \mathcal{M}$ is very ample. (Hint: Reduce to the case where the target is affine. $\mathcal{L}$ induces a map to $\mathbb{P}^n_A$, and this corresponds to $n+1$ sections $s_0, \dots, s_n$ of $\mathcal{L}$. We also have a finite number m of sections $t_1, \dots, t_m$ of $\mathcal{M}$ which generate the stalks. Consider the $(n+1)m$ sections of $\mathcal{L} \otimes \mathcal{M}$ given by $s_i t_j$. Show that these sections are base-point-free, and hence induce a morphism to $\mathbb{P}^{(n+1)m-1}$. Show that it is a closed immersion.)

5. Suppose $\pi : X \to Y$ is proper and Y is quasicompact. Show that if $\mathcal{L}$ is relatively ample on X, then some tensor power of $\mathcal{L}$ is very ample.

Date: Tuesday, May 9, 2006. Updated June 19.

6. State and prove Serre's criterion for relative ampleness (where the target is quasicompact) by adapting the statement of Serre's criterion for ampleness. **Whoops! Ziyu and Rob point out that I used Serre's criterion as the *definition* of ampleness (and similarly, relative ampleness). Thus this exercise is nonsense.**

7. Use Serre's criterion for ampleness to prove that the pullback of ample sheaf on a projective scheme by a finite morphism is ample. Hence if a base-point-free invertible sheaf on a proper scheme induces a morphism to projective space that is finite onto its image, then it is is ample.

8. In class, we proved the following: Suppose $\pi : X \to \operatorname{Spec} B$ is proper, $\mathcal{L}$ ample, and $\mathcal{M}$ invertible. Then $\mathcal{L}^{\otimes n} \otimes \mathcal{M}$ is very ample for $n \gg 0$. Give and prove the corresponding statement for a relatively ample invertible sheaf over a quasicompact base.

9. Suppose X a projective k-scheme. Show that every invertible sheaf is the difference of two *effective* Cartier divisors. Thus the groupification of the semigroup of effective Cartier divisors is the Picard group. Hence if you want to prove something about Cartier divisors on such a thing, you can study effective Cartier divisors. (This is false if projective is replaced by proper — ask Sam Payne for an example.)

10. Suppose C is a generically reduced projective k-curve. Then we can define degree of an invertible sheaf $\mathcal{M}$ as follows. Show that $\mathcal{M}$ has a meromorphic section that is regular at every singular point of C. Thus our old definition (number of zeros minus number of poles, using facts about discrete valuation rings) applies. Prove the Riemann-Roch theorem for generically reduced projective curves. (Hint: our original proof essentially will carry through without change.)

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FOUNDATIONS OF ALGEBRAIC GEOMETRY PROBLEM SET 21

RAVI VAKIL

This set is due Thursday June 8. You can hand it in to Rob Easton, for example via his mailbox. It covers (roughly) classes 49 and 50.

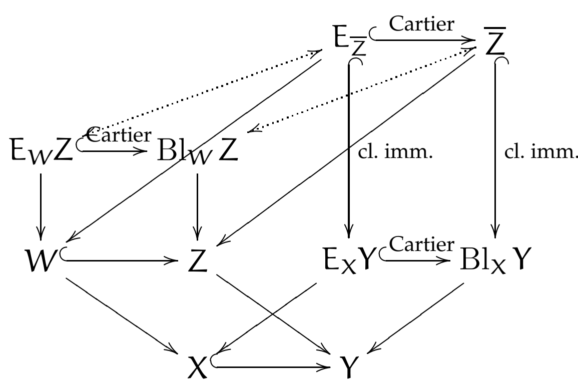
Please *read all of the problems*, and ask me about any statements that you are unsure of, even of the many problems you won't try. Hand in five solutions. If you are ambitious (and have the time), go for more. Try to solve problems on a range of topics. You are encouraged to talk to each other, and to me, about the problems. Some of these problems require hints, and I'm happy to give them!

1. Suppose X is an open subscheme of Y, cut out by a finite type sheaf of ideals. If U is an open subset of Y, show that $\mathrm{Bl}_{U \cap X} U \cong \beta^{-1}(U)$, where $\beta : \mathrm{Bl}_X Y \to Y$ is the blow-up. (Hint: show $\beta^{-1}(U)$ satisfies the universal property!)

2. *(The blow up can be computed locally.)* Show that if Y_α is an open cover of Y (as α runs over some index set), and the blow-up of Y_α along $X \cap Y_\alpha$ exists, then the blow-up of Y along X exists.

3. *(The blow-up preserves irreducibility and reducedness.)* Show that if Y is irreducible, and X doesn't contain the generic point of Y, then $\mathrm{Bl}_X Y$ is irreducible. Show that if Y is reduced, then $\mathrm{Bl}_X Y$ is reduced.

4+. Prove the blow-up closure lemma (see the class notes). Hint: obviously, construct maps in both directions, using the universal property. The following diagram may or may not help.



5. If Y and Z are closed subschemes of a given scheme X, show that $\mathrm{Bl}_Y Y \cup Z \cong \mathrm{Bl}_{Y \cap Z} Z$. (In particular, if you blow up a scheme along an irreducible component, the irreducible component is blown out of existence.)

Date: Tuesday, May 30, 2006.

6. Consider the curve $y^2 = x^3 + x^2$ inside the plane $\mathbb{A}^2_k$. Blow up the origin, and compute the total and proper transform of the curve. (By the blow-up closure lemma, the latter is the blow-up of the nodal curve at the origin.) Check that the proper transform is nonsingular. (All but the last sentence were done in class.)

7. Describe both the total and proper transform of the curve C given by $y = x^2 - x$ in $\mathrm{Bl}_{(0,0)} \mathbb{A}^2$. Verify that the proper transform of C is isomorphic to C. Interpret the intersection of the proper transform of C with the exceptional divisor E as the slope of C at the origin.

8. *(blowing up a cuspidal plane curve)* Describe the proper transform of the cuspidal curve C' given by $y^2 = x^3$ in the plane $\mathbb{A}^2_k$. Show that it is nonsingular. Show that the proper transform of C meets the exceptional divisor E at one point, and is tangent to E there.

9. (a) Desingularize the tacnode $y^2 = x^4$ by blowing up the plane at the origin (and taking the proper transform), and then blowing up the resulting surface once more.
(b) Desingularize $y^8 - x^5 = 0$ in the same way. How many blow-ups do you need?
(c) Do (a) instead in one step by blowing up (y, x^2).

10. Blowing up something nonreduced in nonsingular can give you something singular, as shown in this example. Describe the blow up of the ideal (x, y^2) in $\mathbb{A}^2_k$. What singularity do you get? (Hint: it appears in a nearby exercise.)

11. Blow up the cone point $z^2 = x^2 + y^2$ at the origin. Show that the resulting surface is nonsingular. Show that the exceptional divisor is isomorphic to $\mathbb{P}^1$.

12+. If $X \hookrightarrow \mathbb{P}^n$ is a projective scheme, show that the exceptional divisor of the blow up the affine cone over X at the origin is isomorphic to X, and that its normal bundle is $\mathcal{O}_X(-1)$. (In the case $X = \mathbb{P}^1$, we recover the blow-up of the plane at a point. In particular, we again recover the important fact that the normal bundle to the exceptional divisor is $\mathcal{O}(-1)$.)

13. Show that the multiplicity of the exceptional divisor in the total transform of a subscheme of $\mathbb{A}^n$ when you blow up the origin is the lowest degree that appears in a defining equation of the subscheme. (For example, in the case of the nodal and cuspidal curves above, Example **??** and Exercise respectively, the exceptional divisor appears with multiplicity 2.) This is called the *multiplicity* of the singularity.

14. Suppose Y is the cone $x^2 + y^2 = z^2$, and X is the ruling of the cone $x = 0, y = z$. Show that $\mathrm{Bl}_X Y$ is nonsingular. (In this case we are blowing up a codimension 1 locus that is not a Cartier divisor. Note that it *is* Cartier away from the cone point, so you should expect your answer to be an isomorphism away from the cone point.)

15+. *(blow-ups resolve base loci of rational maps to projective space)* Suppose we have a scheme Y, an invertible sheaf $\mathcal{L}$, and a number of sections $s_0, \ldots, s_n$ of $\mathcal{L}$. Then away from the closed subscheme X cut out by $s_0 = \cdots = s_n = 0$, these sections give a morphism to $\mathbb{P}^n$. Show that this morphism extends to a morphism $\mathrm{Bl}_X Y \to \mathbb{P}^n$, where this morphism corresponds to the invertible sheaf $(\pi^* \mathcal{L})(-E_X Y)$, where $\pi : \mathrm{Bl}_X Y \to Y$ is the blow-up morphism. In other words, "blowing up the base scheme resolves this rational map". (Hint: it suffices to consider an affine open subset of Y where $\mathcal{L}$ is trivial.)

16. Blow up (xy, z) in $\mathbb{A}^3$, and verify that the exceptional divisor is indeed the projectivized normal bundle.

17. Suppose X is an irreducible nonsingular subvariety of a nonsingular variety Y, of codimension at least 2. Describe a natural isomorphism $\operatorname{Pic} \operatorname{Bl}_X Y \cong \operatorname{Pic} Y \oplus \mathbb{Z}$. (Hint: compare divisors on $\operatorname{Bl}_X Y$ and Y. Show that the exceptional divisor $E_X Y$ gives a non-torsion element of $\operatorname{Pic}(\operatorname{Bl}_X Y)$ by describing a $\mathbb{P}^1$ on $\operatorname{Bl}_X Y$ which has intersection number -1 with $E_X Y$.)

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