FOUNDATIONS OF ALGEBRAIC GEOMETRY CLASS 21

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CONTENTS

Today: integral extensions, Going-up theorem, Noether Normalization, proof that transcendence degree = Krull dimension, proof of Chevalley's theorem. Invertible sheaves and morphisms to (quasi)projective schemes

Welcome back everyone! This is the second quarter in a three-quarter experimental sequence on algebraic geometry.

We know what schemes are, their properties, quasicoherent sheaves on them, and morphisms between them. This quarter, we're going to talk about fancier concepts: fibered products; normalization; separatedness and the definition of a variety; rational maps; classification of curves; cohomology; differentials; and Riemann-Roch.

I'd like to start with some notions that I now think I should have done in the middle of last quarter. They are some notions that I think are easier than are usually presented.

1. Integral extensions, the going-up theorem, Noether normalization, and a proof of the big dimension theorem (that transcendence degree = Krull dimension)

Recall the maps of sets corresponding to a map of rings. If we have $\phi : B \to A$, we get a map $\operatorname{Spec} A \to \operatorname{Spec} B$ as sets (and indeed as topological spaces, and schemes), which sends $\mathfrak{p} \subset A$ to $\phi^{-1}\mathfrak{p} \subset B$. The notion behaves well under quotients and localization of both the source and target affine scheme.

A ring homomorphism $\phi : B \to A$ is *integral* if every element of A is integral over $\phi(B)$. (Thanks to Justin for pointing out that this notation is not just my invention — it is in Atiyah-Macdonald, p. 60.) In other words, if a is any element of A, then a satisfies some

Date: Tuesday, January 10, 2006. Updated January 28, 2007.

monic polynomial $a^n + \cdots = 0$ where all the coefficients lie in $\phi(B)$. We call it an *integral extension* if ϕ is an inclusion of rings.

1.1. Exercise. The notion of integral morphism is well behaved with respect to localization and quotient of B, and quotient of A (but not localization of A, witness $k[t] \to k[t]$, but $k[t] \to k[t]_{(t)}$). The notion of integral extension is well behaved with respect to localization and quotient of B, but not quotient of A (same example, $k[t] \to k[t]/(t)$).

1.2. Exercise. Show that if B is an integral extension of A, and C is an integral extension of B, then C is an integral extension of A.

1.3. *Proposition. — If* A *is finitely generated as a* B*-module, then* ϕ *is an integral morphism.*

Proof. (If B is Noetherian, this is easiest: suppose $a \in B$. Then A is a Noetherian B-module, and hence the ascending chain of B-submodules of A $(1) \subset (1, a) \subset (1, a, a^2) \subset (1, a, a^2, a^3) \subset \cdots$ eventually stabilizes, say $(1, a, \ldots, a^{n-1}) = (1, a, \ldots, a^{n-1}, a^n)$. Hence a^n is a B-linear combination of $1, \ldots, a^{n-1}$, i.e. is integral over B. So Noetherian-minded readers can stop reading.) We use a trick we've seen before. Choose a finite generating set $m_1, \ldots, m_n$ of A as a B-module. Then $am_i = \sum a_{ij}m_j$, where $a_{ij} \in B$. Thus

$$(aI_{n\times n} - [a_{ij}]_{ij}) \begin{pmatrix} m_1 \\ \vdots \\ m_n \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Multiplying this equation by the adjoint of the left side, we get

$$\det(aI_{n\times n} - [a_{ij}]_{ij}) \begin{pmatrix} m_1 \\ \vdots \\ m_n \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}.$$

(We saw this trick when discussing Nakayama's lemma.) So $\det(aI - M)$ annihilates A, i.e. $\det(aI - M) = 0$. □

1.4. Exercise (cf. Exercise 1.2). Show that if B is a finite extension of A, and C is a finite extension of B, then C is an finite extension of A. (Recall that if we have a ring homomorphism $A \to B$ such that B is a finitely-generated A-module (*not* necessarily A-algebra) then we say that B is a finite extension of A.)

We now recall the Going-up theorem.

1.5. *Cohen-Seidenberg Going up theorem. — Suppose* $\phi : B \to A$ *is an integral extension. Then for any prime ideal* $\mathfrak{q} \subset B$*, there is a prime ideal* $\mathfrak{p} \subset A$ *such that* $\mathfrak{p} \cap B = \mathfrak{q}$.

Although this is a theorem in algebra, the name reflects its geometric motivation: the theorem asserts that the corresponding morphism of schemes is surjective, and that "above" every prime $\mathfrak{q}$ "downstairs", there is a prime $\mathfrak{q}$ "upstairs". (I drew a picture here.) For this reason, it is often said that $\mathfrak{q}$ is "above" $\mathfrak{p}$ if $\mathfrak{p} \cap B = \mathfrak{q}$. (Joe points out that my speculation on the origin of the name "going up" is wrong.)

As a reality check: note that the morphism $k[t] \to k[t]_{(t)}$ is not integral, so the conclusion of the Going-up theorem 1.5 fails. (I drew a picture again.)

Proof of the Cohen-Seidenberg Going-Up theorem 1.5. This proof is eminently readable, but could be skipped on first reading. We start with an exercise.

1.6. Exercise. Show that the special case where A is a field translates to: if $B \subset A$ is a subring with A integral over B, then B is a field. Prove this. (Hint: all you need to do is show that all nonzero elements in B have inverses in B. Here is the start: If $b \in B$, then $1/b \in A$, and this satisfies some integral equation over B.)

We're ready to prove the Going-Up Theorem 1.5.

We first make a reduction: by localizing at $\mathfrak{q}$, so we can assume that $(B, \mathfrak{q})$ is a local ring.

Then let $\mathfrak{p}$ be any *maximal* ideal of A. We will see that $\mathfrak{p} \cap B = \mathfrak{q}$. Consider the following diagram.

$$\begin{array}{ccc} A & \twoheadrightarrow & A/\mathfrak{p} \\ \uparrow & & \uparrow \\ B & \twoheadrightarrow & B/(B \cap \mathfrak{p}) \end{array} \qquad \text{field}$$

By the Exercise above, the lower right is a field too, so $B \cap \mathfrak{p}$ is a maximal ideal, hence $\mathfrak{q}$. □

1.7. Important but straightforward exercise (sometimes also called the going-up theorem). Show that if $\mathfrak{q}_1 \subset \mathfrak{q}_2 \subset \cdots \subset \mathfrak{q}_n$ is a chain of prime ideals of B, and $\mathfrak{p}_1 \subset \cdots \subset \mathfrak{p}_m$ is a chain of prime ideals of A such that $\mathfrak{p}_i$ "lies over" $\mathfrak{q}_i$ (and $m < n$), then the second chain can be extended to $\mathfrak{p}_1 \subset \cdots \subset \mathfrak{p}_n$ so that this remains true.

The going-up theorem has an important consequence.

1.8. Important exercise. Show that if $f : \operatorname{Spec} A \to \operatorname{Spec} B$ corresponds to an integral *extension of rings*, then $\dim \operatorname{Spec} A = \dim \operatorname{Spec} B$.

I'd like to walk you through much of this exercise. You can show that a chain downstairs gives a chain upstairs, by the going up theorem, of the same length. Conversely, a chain upstairs gives a chain downstairs. We need to check that no two elements of the chain upstairs goes to the same element of the chain downstairs. That boils down to this: If $\phi : k \to A$ is an integral extension, then $\dim A = 0$. *Proof.* Suppose $\mathfrak{p} \subset \mathfrak{m}$ are two prime ideals of $\mathfrak{p}$. Mod out by $\mathfrak{p}$, so we can assume that A is a domain. I claim that any non-zero element is invertible. Here's why. Say $x \in A$, and $x \neq 0$. Then the minimal monic polynomial for x has non-zero constant term. But then x is invertible (recall coefficients are in a field).

We now introduce another important and ancient result, Noether's Normalization Lemma.

1.9. *Noether Normalization Lemma. — Suppose* A *is an integral domain, finitely generated over a field* k*. If* $\mathrm{tr.deg.}_k A = n$*, then there are elements* $x_1, \dots, x_n \in A$*, algebraically independent over* k*, such that* A *is a finite (hence integral by Proposition 1.3) extension of* $k[x_1, \dots, x_n]$*.*

The geometric content behind this result is that given any integral affine k-scheme X, we can find a surjective finite morphism $X \to \mathbb{A}^n_k$, where n is the transcendence degree of the function field of X (over k).

Proof of Noether normalization. We give Nagata's proof, following Mumford's Red Book (§1.1). Suppose we can write $A = k[y_1, \dots, y_m]/\mathfrak{p}$, i.e. that A can be chosen to have m generators. Note that $m \geq n$. We show the result by induction on m. The base case $m = n$ is immediate.

Assume now that $m > n$, and that we have proved the result for smaller m. We will find $m-1$ elements $z_1, \dots, z_{m-1}$ of A such that A is finite over $A' := k[z_1, \dots, z_{m-1}]$ (by which we mean the subring of A generated by $z_1, \dots, z_{m-1}$). Then by the inductive hypothesis, A' is finite over some $k[x_1, \dots, x_n]$, and A is finite over A, so by Exercise 1.4 A is finite over $k[x_1, \dots, x_n]$.

As $y_1, \dots, y_m$ are algebraically dependent, there is some non-zero algebraic relation $f(y_1, \dots, y_m) = 0$ among them (where f is a polynomial in m variables).

Let $z_1 = y_1 - y_m^{r_1}, z_2 = y_2 - y_m^{r_2}, \dots, z_{m-1} = y_{m-1} - y_m^{r_{m-1}}$, where $r_1, \dots, r_{m-1}$ are positive integers to be chosen shortly. Then

$$f(z_1 + y_m^{r_1}, z_2 + y_m^{r_2}, \dots, z_{m-1} + y_m^{r_{m-1}}, y_m) = 0.$$

Then upon expanding this out, each monomial in f (as a polynomial in m variables) will yield a single term in that is a constant times a power of y_m (with no z_i factors). By choosing the r_i so that $0 \ll r_1 \ll r_2 \ll \cdots \ll r_{m-1}$, we can ensure that the powers of y_m appearing are all distinct, and so that in particular there is a leading term y_m^N, and all other terms (including those with z_i-factors) are of smaller degree in y_m. Thus we have described an integral dependence of y_m on $z_1, \dots, z_{m-1}$ as desired. □

Now we can give a proof of something we used a lot last quarter:

1.10. *Important Theorem about Dimension. — Suppose* R *is a finitely-generated domain over a field* k*. Then* $\dim \operatorname{Spec} R$ *is the transcendence degree of the fraction field* $\operatorname{Frac}(R)$ *over* k*.*

We proved this in class 9, but I think this proof is much slicker.

Proof. Suppose X is an integral affine k-scheme. We show that $\dim X$ equals the transcendence degree n of its function field, by induction on n. Fix X, and assume the result is known for all transcendence degrees less than n. The base case $n = -1$ is vacuous.

By Exercise 1.8, $\dim X = \dim \mathbb{A}^n_k$. If $n = 0$, we are done.

We now show that $\dim \mathbb{A}^n_k = n$ for $n > 0$. Clearly $\dim \mathbb{A}^n_k \geq n$, as we can describe a chain of irreducible subsets of length $n+1$: if $x_1, \ldots, x_n$ are coordinates on $\mathbb{A}^n$, consider the chain of ideals

$$(0) \subset (x_1) \subset \cdots \subset (x_1, \ldots, x_n)$$

in $k[x_1, \ldots, x_n]$. Suppose we have a chain of prime ideals of length at least n:

$$(0) = \mathfrak{p}_0 \subset \cdots \subset \mathfrak{p}_m.$$

where $\mathfrak{p}_1$ is a height 1 prime ideal. Then $\mathfrak{p}_1$ is principal (as $k[x_1, \ldots, x_n]$ is a unique factorization domain, cf. Exercises 1 and 4 on problem set 6); say $\mathfrak{p}_1 = (f(x_1, \ldots, x_n))$, where f is an irreducible polynomial. Then $k[x_1, \ldots, x_n]/(f(x_1, \ldots, x_n))$ has transcendence degree $n-1$, so by induction,

$$\dim k[x_1, \ldots, x_n]/(f) = n - 1.$$

□

2. Images of morphisms

Here are two applications of the going-up theorem, which are quite similar to each other.

2.1. *Exercise.* Show that finite morphisms are *closed*, i.e. the image of any closed subset is closed.

2.2. *Exercise.* Show that integral ring extensions induce a surjective map of spectra.

I now want to use the Noether normalization lemma to prove Chevalley's theorem. Recall that we define a *constructable subset* of a scheme to be a subset which belongs to the smallest family of subsets such that (i) every open set is in the family, (ii) a finite intersection of family members is in the family, and (iii) the complement of a family member is also in the family. So for example the image of $(x, y) \mapsto (x, xy)$ is constructable.

2.3. Exercise. Suppose X is a Noetherian scheme. Show that a subset of X is constructable if and only if it is the finite disjoint union of locally closed subsets.

Last quarter we stated the following.

2.4. *Chevalley's Theorem. — Suppose $f : X \to Y$ is a morphism of finite type of Noetherian schemes. Then the image of any constructable set is constructable.*

We'll now prove this using Noether normalization. (This is remarkable: Noether normalization is about finitely generated algebras over a field. There is no field in the statement of Chevalley's theorem. Hence if you prefer to work over arbitrary rings (or schemes), this shows that you still care about facts about finite type schemes over a field. Also, even if you are interested in finite type schemes over a given field (like $\mathbb{C}$), the field that comes up in the proof of Chevalley's theorem is *not* that field, so even if you prefer to work over $\mathbb{C}$, this argument shows that you still care about working over arbitrary fields, not necessarily algebraically closed.)

We say a morphism $f : X \to Y$ is *dominant* if the image of f meets every dense open subset of Y. (This is sometimes called *dominating*, but we will not use this notation.)

2.5. Exercise. Show that a dominant morphism of integral schemes $X \to Y$ induces an inclusion of function fields in the other direction.

2.6. *Exercise.* If $\phi : A \to B$ is a ring morphism, show that the corresponding morphism of affine schemes $\operatorname{Spec} B \to \operatorname{Spec} A$ is dominant iff ϕ has nilpotent kernel.

2.7. Exercise. Reduce the proof of the Going-up theorem to the following case: suppose $f : X = \operatorname{Spec} A \to Y = \operatorname{Spec} B$ is a dominant morphism, where A and B are domains, and f corresponds to $\phi : B \to B[x_1, \dots, x_n]/I \cong A$. Show that the image of f contains a dense open subset of $\operatorname{Spec} B$.

Proof. We prove the problem posed in the previous exercise. This argument uses Noether normalization 1.9 in an interesting context — even if we are interested in schemes over a field k, this argument will use a larger field, the field $K := \operatorname{Frac}(B)$. Now $A \otimes_B K$ is a localization of A with respect to B^*, so it is a domain, and it is finitely generated over K (by $x_1, \dots, x_n$), so it has finite transcendence degree r over K. Thus by Noether normalization, we can find a subring $K[y_1, \dots, y_r] \subset A \otimes_B K$, so that $A \otimes_B K$ is integrally dependent on $K[y_1, \dots, y_r]$. We can choose the y_i to be in A: each is in $(B^*)^{-1}A$ to begin with, so we can replace each y_i by a suitable K-multiple.

Sadly A is not necessarily integrally dependent on $K[y_1, \dots, y_r]$ (as this would imply that $\operatorname{Spec} A \to \operatorname{Spec} B$ is surjective). However, each x_i satisfies some integral equation

$$x_i^n + f_1(y_1, \dots, y_r)x_i^{n-1} + \cdots + f_n(y_1, \dots, y_r) = 0$$

where f_j are polynomials with coefficients in $K = \operatorname{Frac}(B)$. Let g be the product of the denominators of all the coefficients of all these polynomials (a finite set). Then A_g is integral over B_g, and hence $\operatorname{Spec} A_g \to \operatorname{Spec} B_g$ is surjective; $\operatorname{Spec} B_g$ is our open subset. $\square$

3. IMPORTANT EXAMPLE: MORPHISMS TO PROJECTIVE (AND QUASIPROJECTIVE) SCHEMES, AND INVERTIBLE SHEAVES

This will tell us why invertible sheaves are crucially important: they tell us about maps to projective space, or more generally to quasiprojective schemes. (And given that we have had a hard time naming any non-quasiprojective schemes, they tell us about maps to essentially all schemes that are interesting to us.)

3.1. *Important theorem. — Maps to $\mathbb{P}^n$ correspond to $n+1$ sections of a line bundle, not all vanishing at any point (= generated by global sections, by an earlier exercise, Class 16 Exercise 4.2, = Problem Set 7, Exercise 28), modulo sections of $\mathcal{O}_X^*$.*

The explanation and proof of the correspondence is in the notes for next day.

Here are some examples.

Example 1. Consider the $n+1$ functions $x_0, \dots, x_n$ on $\mathbb{A}^{n+1}$ (otherwise known as $n+1$ sections of the trivial bundle). They have no common zeros on $\mathbb{A}^n - 0$. Hence they determine a morphism $\mathbb{A}^{n+1} - 0 \to \mathbb{P}^n$. (We've talked about this morphism before. But now we don't have to worry about gluing.)

Example 2: the Veronese morphism. Consider the line bundle $\mathcal{O}_{\mathbb{P}^n}(m)$ on $\mathbb{P}^n$. We've checked that the number of sections of this line bundle are $\binom{n+m}{m}$, and they correspond to homogeneous degree m polynomials in the projective coordinates for $\mathbb{P}^n$. Also, they have no common zeros (as for example the subset of sections $x_0^m, x_1^m, \dots, x_n^m$ have no common zeros). Thus these determine a morphism $\mathbb{P}^n \to \mathbb{P}^{\binom{n+m}{m}-1}$. This is called the *Veronese morphism*. For example, if $n=2$ and $m=2$, we get a map $\mathbb{P}^2 \to \mathbb{P}^5$.

This is in fact a closed immersion. Reason: This map corresponds to a surjective map of graded rings. The first ring R_1 has one generator for each of degree m monomial in the x_i. The second ring is not $k[x_0, \dots, x_n]$, as R_1 does not surject onto it. Instead, we take $R_2 = k[x_0, \dots, x_n]_{(m)}$, i.e. we consider only those polynomials all of whose terms have degree divisible by m. Then the natural map $R_1 \to R_2$ is fairly clearly a surjection. Thus the corresponding map of projective schemes is a closed immersion by an earlier exercise.

How can you tell in general if something is a closed immersion, and not just a map? Here is one way.

3.2. *Exercise.* Let $f : X \to \mathbb{P}^n_A$ be a morphism of A-schemes, corresponding to an invertible sheaf $\mathcal{L}$ on X and sections $s_0, \dots, s_n \in \Gamma(X, \mathcal{L})$ as above. Then ϕ is a closed immersion iff (1) each open set $X_i = X_{s_i}$ is affine, and (2) for each i, the map of rings $A[y_0, \dots, y_n] \to \Gamma(X_i, \mathcal{O}_{X_i})$ given by $y_j \mapsto s_j/s_i$ is surjective.

We'll give another method of detecting closed immersions later. The intuition for this will come from differential geometry: the morphism should separate points, and also separate tangent vectors.

Example 3. The rational normal curve. The image of the Veronese morphism when $n = 1$ is called a *rational normal curve of degree* m. Our map is $\mathbb{P}^1 \to \mathbb{P}^m$ given by $[x; y] \to [x^m; x^{m-1}y; \cdots; xy^{m-1}; y^m]$. When $m = 3$, we get our old friend the *twisted cubic*. When $m = 2$, we get a smooth conic. What happens when $m = 1$?

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FOUNDATIONS OF ALGEBRAIC GEOMETRY CLASS 22

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CONTENTS

Last day: integral extensions, Going-up theorem, Noether Normalization, proof that transcendence degree = Krull dimension, proof of Chevalley's theorem.

Today: Morphisms to (quasi)projective schemes, and invertible sheaves; fibered products; fibers.

1. IMPORTANT EXAMPLE: MORPHISMS TO PROJECTIVE (AND QUASIPROJECTIVE) SCHEMES, AND INVERTIBLE SHEAVES

1.1. *Important theorem. — Maps to* $\mathbb{P}^n$ *correspond to* $n+1$ *sections of an invertible sheaf, not all vanishing at any point (= generated by global sections), modulo sections of* $\mathcal{O}_X^*$.

Here more precisely is the correspondence. If you have $n+1$ sections, then away from the intersection of their zero-sets, we have a morphism. Conversely, if you have a map to projective space $f : X \to \mathbb{P}^n$, then we have $n+1$ sections of $\mathcal{O}_{\mathbb{P}^n}(1)$, corresponding to the hyperplane sections, $x_0, \ldots, x_{n+1}$. then $f^*x_0, \ldots, f^*x_{n+1}$ are sections of $f^*\mathcal{O}_{\mathbb{P}^n}(1)$, and they have no common zero.

So to prove this, we just need to show that these two constructions compose to give the identity in either direction.

Given $n+1$ sections $s_0, \ldots, s_n$ of an invertible sheaf. We get trivializations on the open sets where each one vanishes. The transition functions are precisely s_i/s_j on $U_i \cap U_j$. We pull back $\mathcal{O}(1)$ by this map to projective space, This is trivial on the distinguished open sets. Furthermore, $f^*D(x_i) = D(s_i)$. Moreover, $s_i/s_j = f^*x_i/x_j$. Thus starting with the

Date: Thursday, January 12, 2006.

$n+1$ sections, taking the map to the projective space, and pulling back $\mathcal{O}(1)$ and taking the sections $x_0, \dots, x_n$, we recover the s_i's. That's one of the two directions.

Correspondingly, given a map $f : X \to \mathbb{P}^n$, let $s_i = f^* x_i$. The map $[s_0; \cdots; s_n]$ is precisely the map f. We see this as follows. The preimage of U_i is $D(s_i) = D(f^* x_i) = f^* D(x_i)$. So the right open sets go to the right open sets. And $D(s_i) \to D(x_i)$ is precisely by $s_j/s_i = f^*(x_j/x_i)$. □

1.2. *Exercise (Automorphisms of projective space).* Show that all the automorphisms of projective space $\mathbb{P}^n_k$ correspond to $(n+1) \times (n+1)$ invertible matrices over k, modulo scalars (also known as $PGL_{n+1}(k)$). (Hint: Suppose $f : \mathbb{P}^n_k \to \mathbb{P}^n_k$ is an automorphism. Show that $f^*\mathcal{O}(1) \cong \mathcal{O}(1)$. Show that $f^* : \Gamma(\mathbb{P}^n, \mathcal{O}(1)) \to \Gamma(\mathbb{P}^n, \mathcal{O}(1))$ is an isomorphism.)

This exercise will be useful later, especially for the case $n = 1$.

(A question for experts: why did I not state that previous exercise over an arbitrary base ring A? Where does the argument go wrong in that case?)

1.3. *Neat Exercise.* Show that any map from projective space to a smaller projective space is constant.

Here are some useful phrases to know.

A *linear series* on a scheme X over a field k is an invertible sheaf $\mathcal{L}$ and a finite-dimensional k-vector space V of sections. (We will not require that this vector space be a subspace of $\Gamma(X, \mathcal{L})$; in general, we just have a map $V \to \Gamma(X, \mathcal{L})$.) If the linear series is $\Gamma(X, \mathcal{L})$, we call it a *complete linear series*, and is often written $|\mathcal{L}|$. Given a linear series, any point $x \in X$ on which all elements of the linear series V vanish, we say that x is a *base-point* of V. If V has no base-points, we say that it is *base-point-free*. The union of base-points is called the *base locus*. In fact, it naturally has a scheme-structure — it is the (scheme-theoretic) intersection of the vanishing loci of the elements of V (or equivalently, of a basis of V). In this incarnation, it is called the *base scheme* of the linear series.

Then Theorem 1.1 says that each base-point-free linear series gives a morphism to projective space $X \to \mathbb{P}V^* = \operatorname{Proj} \oplus_n \mathcal{L}^{\otimes n}$. The resulting morphism is often written $X \xrightarrow{|V|} \mathbb{P}^n$. (I may not have this notation quite standard; I should check with someone. I always forget whether I should use "linear system" or "linear series".)

1.4. *Exercise.* If the image scheme-theoretically lies in a hyperplane of projective space, we say that it is *degenerate* (and otherwise, *non-degenerate*). Show that a base-point-free linear series V with invertible sheaf $\mathcal{L}$ is non-degenerate if and only if the map $V \to \Gamma(X, \mathcal{L})$ is an inclusion. Hence in particular a complete linear series is always non-degenerate.

Example: The Veronese and Segre morphisms. *Whoops! We don't know much about fibered products yet, so the Segre discussion may be a bit confusing. But fibered products are* *coming very very shortly...* The Veronese morphism can be interpreted in this way. The dth Veronese morphism on $\mathbb{P}^n$ corresponds to the complete linear series $|\mathcal{O}_{\mathbb{P}^n}(d)|$.

The Segre morphism can also be interpreted in this way. In case I haven't defined it yet, suppose $\mathcal{F}$ is a quasicoherent sheaf on a Z-scheme X, and $\mathcal{G}$ is a quasicoherent sheaf on a Z-scheme Y. Let π_X, π_Y be the projections from $X \times_Z Y$ to X and Y respectively. Then $\mathcal{F} \boxtimes \mathcal{G}$ is defined to be $\pi_X^* \mathcal{F} \otimes \pi_Y^* \mathcal{G}$. In particular, $\mathcal{O}_{\mathbb{P}^m \times \mathbb{P}^n}(a, b)$ is defined to be $\mathcal{O}_{\mathbb{P}^m}(a) \boxtimes \mathcal{O}_{\mathbb{P}^n}(b)$ (over any base Z). The Segre morphism $\mathbb{P}^m \times \mathbb{P}^n \to \mathbb{P}^{mn+m+n}$ corresponds to the complete linear system for the invertible sheaf $\mathcal{O}(1,1)$.

Both of these complete linear systems are easily seen to be base-point-free (*exercise*). We still have to check by hand that they are closed immersions. (We will later see, in class 34, a criterion for linear series to be a closed immersion, at least in the special case where we are working over an algebraically closed field.)

2. Fibered products

We will now construct the fibered product in the category of schemes. In other words, given $X, Y \to Z$, we will show that $X \times_Z Y$ exists. (Recall that the *absolute product* in a category is the fibered product over the final object, so $X \times Y = X \times_{\mathbb{Z}} Y$ in the category of schemes, and $X \times Y = X \times_S Y$ if we are implicitly working in the category of S-schemes, for example if S is the spectrum of a field.)

Here is a notation warning: in the literature (and indeed in this class) lazy people wanting to save chalk and ink will write $\times_k$ for $\times_{\operatorname{Spec} k}$, and similarly for $\times_{\mathbb{Z}}$. In fact it already happened in the paragraph above!

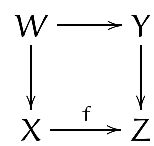
As always when showing that certain objects defined by universal properties exist, we have two ways of looking at the objects in practice: by using the universal property, or by using the details of the construction.

The key idea, roughly, is this: we cut everything up into affine open sets, do fibered products in that category (where it turns out we have seen the concept before in a different guise), and show that everything glues nicely. We can't do this too naively (e.g. by induction), as in general we won't be able to cut things into a finite number of affine open sets, so there will be a tiny bit of cleverness.

The argument will be an inspired bit of abstract nonsense, where we'll have to check almost nothing. This sort of argument is very powerful, and we will use it immediately after to construct lots of other interesting notions, so please pay attention!

Before we get started, here is a sign that something interesting happens for fibered products of schemes. Certainly you should believe that if we take the product of two affine lines (over your favorite algebraically field k, say), you should get the affine plane: $\mathbb{A}^1_k \times_k \mathbb{A}^1_k$ should be $\mathbb{A}^2_k$. But the underlying set of the latter is *not* the underlying set of the former —- we get additional points! I'll give an exercise later for you to verify this.

Let's take a break to introduce some language. Say

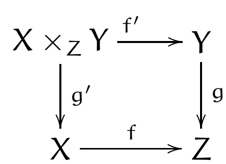


is a *fiber diagram* or *Cartesian diagram* or *base change diagram*. It is often called a *pullback diagram*, and $W \to X$ is called the *pullback* of $Y \to Z$ by f, and W is called the *pullback* of Y by f.

At this point, I drew some pictures on the blackboard giving some intuitive idea of what a pullback does. If $Y \to Z$ is a "family of schemes", then $W \to Z$ is the "pulled back family". To make this more explicit or precise, I need to tell you about fibers of a morphism. I also want to give you a bunch of examples. But before doing either of these things, I want to tell you how to compute fibered products in practice.

Okay, lets get to work.

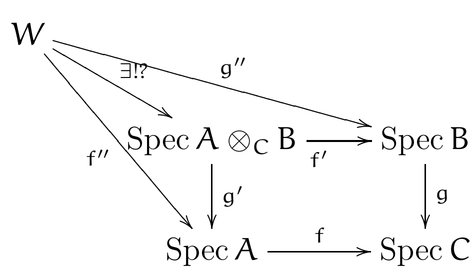
2.1. *Theorem (fibered products always exist). — Suppose $f : X \to Z$ and $g : Y \to Z$ are morphisms of schemes. Then the fibered product*



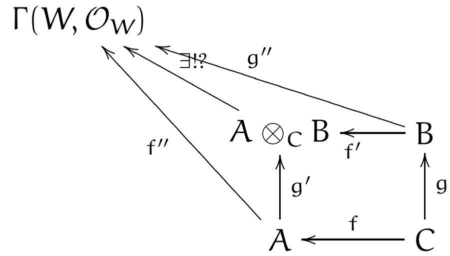
exists in the category of schemes.

We have an extended proof by universal property.

First, if X, Y, Z are affine schemes, say $X = \operatorname{Spec} A$, $Y = \operatorname{Spec} B$, $Z = \operatorname{Spec} C$, the fibered product exists, and is $\operatorname{Spec} A \otimes_C B$. Here's why. Suppose W is any scheme, along with morphisms $f'' : W \to X$ and $g'' : W \to Y$ such that $f \circ f'' = g \circ g''$ as morphisms $W \to Z$. We hope that there exists a unique $h : W \to \operatorname{Spec} A \otimes_C B$ such that $f'' = g' \circ h$ and $g'' = f' \circ h$.



But maps to affine schemes correspond precisely to maps of global sections in the other direction (class 19 exercise 0.1):



But this is precisely the universal property for tensor product! (The tensor product is the cofibered product in the category of rings.)

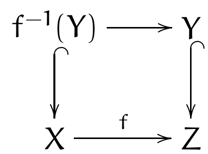
Thus indeed $\mathbb{A}^1 \times \mathbb{A}^1 \cong \mathbb{A}^2$, and more generally $(\mathbb{A}^1)^n \cong \mathbb{A}^n$.

Exercise. Show that the fibered product does not induce a bijection of points

$$\text{points}(\mathbb{A}^1_k) \times \text{points}(\mathbb{A}^1_k) \longrightarrow \text{points}(\mathbb{A}^2_k).$$

Thus products of schemes do something a little subtle on the level of sets.

Second, we note that the fibered product with open immersions always exists: if $Y \hookrightarrow Z$ an open immersion, then for any $f : X \to Z$, $X \times_Z Y$ is the open subset $f^{-1}(Y)$. (More precisely, this open subset satisfies the universal property.) We proved this in class 19 (exercise 1.2).



(An exercise to give you practice with this concept: show that the fibered product of two open immersions is their intersection.)

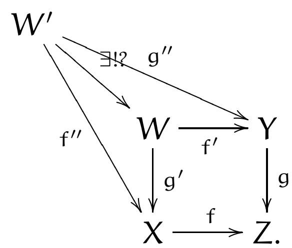
Hence the fibered product of a *quasiaffine* scheme (defined to be an open subscheme of an affine scheme) with an affine scheme over an affine scheme exists. *This isn't quite right; what we've shown, and what we'll use, is that the fibered product of a quasi-affine scheme with an affine scheme over an affine scheme Z exists so long as that quasi-affine scheme is an open subscheme of an affine scheme that also admits a map to Z extending the map from the quasiaffine. At some point I'll retype this to say this better. This sloppiness continues in later lectures, but the argument remains correct.*

Third, we show that $X \times_Z Y$ exists if Y and Z are affine and X is general. Before we show this, we remark that one special case of it is called "extension of scalars": if X is a k-scheme, and k' is a field extension (often k' is the algebraic closure of k), then $X \times_{\operatorname{Spec} k} \operatorname{Spec} k'$ (sometimes informally written $X \times_k k'$ or $X_{k'}$) is a k'-scheme. Often properties of X can be checked by verifying them instead on $X_{k'}$. This is the subject of *descent* — certain properties "descend" from $X_{k'}$ to X.

Let's verify this. It will follow from abstract nonsense and the gluing lemma. Recall the *gluing lemma* (a homework problem): assume we are given a bunch of schemes X_i indexed by some index set I, along with open subschemes $U_{ij} \subset X_i$ indexed by $I \times I$, and isomorphisms $f_{ij} : U_{ij} \xrightarrow{\sim} U_{ji}$, satisfying the cocycle condition: $f_{ij}(U_{ij} \cap U_{ik}) = U_{ji} \cap U_{jk}$, and $(f_{jk} \circ f_{ij})|_{U_{ij} \cap U_{ik}} = f_{ik}|_{U_{ij} \cap U_{ik}}$. Then they glue together to a unique scheme. (This was a homework problem long ago; I'll add a reference when I dig it up.)

We'll now apply this in our case. Cover X with affine open sets V_i. Let $V_{ij} = V_i \cap V_j$. Then for each of these, $X_i := V_i \times_Z Y$ exists, and each of them has open subsets $U_{ij} := V_{ij} \times_Z Y$, and isomorphisms satisfying the cocycle condition (because the V_i's and V_{ij}'s could be glued together via g_{ij} which satisfy the cocycle condition).

Call this glued-together scheme W. It comes with morphisms to X and Y (and their compositions to Z are the same). I claim that this satisfies the universal property for $X \times_Z Y$, basically because "morphisms glue" (yet another ancient exercise). Here's why. Suppose W' is any scheme, along with maps to X and Y that agree when they are composed to Z. We need to show that there is a unique morphism $W' \to W$ completing the diagram

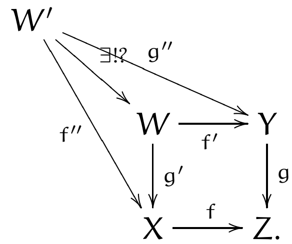


Now break W' up into open sets $W'_i = g''^{-1}(U_i)$. Then by the universal property for $V_i = U_i \times_Z Y$, there is a unique map $W'_i \to V_i$ (which we can interpret as $W'_i \to W$). (Thus we have already shown uniqueness of $W' \to W$.) These must agree on $W'_i \cap W'_j$, because there is only one map $W'_i \cap W'_j$ to W making the diagram commute (because of the second step — $(U_i \cap U_j) \times_Z Y$ exists). Thus all of these morphisms $W'_i \to W$ glue together; we have shown existence.

Fourth, we show that if Z is affine, and X and Y are arbitrary schemes, then $X \times_Z Y$ exists. We just repeat the process of the previous step, with the roles of X and Y repeated, using the fact that by the previous step, we can assume that the fibered product with an affine scheme with an arbitrary scheme over an affine scheme exists.

Fifth, we show that the fibered product of any two schemes over a *quasiaffine* scheme exists. Here is why: if $Z \hookrightarrow Z'$ is an open immersion into an affine scheme, then $X \times_Z Y = X \times_{Z'} Y$ are the same. (You can check this directly. But this is yet again an old exercise — problem set 1 problem A4 — following from the fact that $Z \hookrightarrow Z'$ is a monomorphism.)

Finally, we show that the fibered product of any scheme with any other scheme over any third scheme always exists. We do this in essentially the same way as the third step, using the gluing lemma and abstract nonsense. Say $f : X \to Z$, $g : Y \to Z$ are two morphisms of schemes. Cover Z with affine open subsets Z_i. Let $X_i = f^{-1}X_i$ and $Y_i = g^{-1}Y_i$. Define $Z_{ij} = Z_i \cap Z_j$, and X_{ij} and Y_{ij} analogously. Then $W_i := X_i \times_{Z_i} Y_i$ exists for all i, and has as open sets $W_{ij} := X_{ij} \times_{Z_{ij}} Y_{ij}$ along with gluing information satisfying the cocycle condition (arising from the gluing information for Z from the Z_i and Z_{ij}). Once again, we show that this satisfies the universal property. Suppose W' is any scheme, along with maps to X and Y that agree when they are composed to Z. We need to show that there is a unique morphism $W' \to W$ completing the diagram



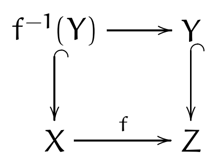
Now break W' up into open sets $W'_i = g'' \circ f^{-1}(Z_i)$. Then by the universal property for W_i, there is a unique map $W'_i \to W_i$ (which we can interpret as $W'_i \to W$). Thus we have already shown uniqueness of $W' \to W$. These must agree on $W'_i \cap W'_j$, because there is only one map $W'_i \cap W'_j$ to W making the diagram commute. Thus all of these morphisms $W'_i \to W$ glue together; we have shown existence. □

3. Computing fibered products in practice

There are four types of morphisms that it is particularly easy to take fibered products with, and all morphisms can be built from these four atomic components.

(1) *base change by open immersions*

We've already done the work for this one, and we used it above.

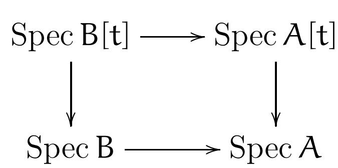


I'll describe the remaining three on the level of affine sets, because we obtain general fibered products by gluing.

(2) *adding an extra variable*

Exercise. Show that $B \otimes_A A[t] \cong B[t]$.

Hence the following is a fibered diagram.



(3) *base change by closed immersions*

If the right column is obtained by modding out by a certain ideal (i.e. if the morphism is a closed immersion, i.e. if the map of rings in the other direction is surjective), then the left column is obtained by modding out by the pulled back elements of that ideal. In other words, if $T \to R, S$ are two ring morphisms, and I is an ideal of R, and I^e is the extension of I to $R \otimes_T S$ (the elements $\sum_j i_j \otimes s_j$, where $i_j \in I$ and $s_j \in S$, then there is a natural isomorphism

$$R/I \otimes_T S \cong (R \otimes_T S)/I^e.$$

(This is precisely problem B3 on problem set 1.) Thus the natural morphism $R \otimes_T S \to R/I \otimes_T S$ is a surjection, and we have a base change diagram:

$$\begin{array}{ccc}
\operatorname{Spec}(R \otimes_T S)/I^e & \longrightarrow & \operatorname{Spec} R/I \\
\big\downarrow & & \big\downarrow \\
\operatorname{Spec} R \otimes_T S & \longrightarrow & \operatorname{Spec} R \\
\big\downarrow & & \big\downarrow \\
\operatorname{Spec} S & \longrightarrow & \operatorname{Spec} T
\end{array}$$

(where each rectangle is a fiber diagram).

Translation: the fibered product with a subscheme is the subscheme of the fibered product in the obvious way. We say that "closed immersions are preserved by base change".

(4) *base change by localization*

Exercise. Suppose $C \to B, A$ are two morphisms of rings. Suppose S is a multiplicative set of A. Then $(S \otimes 1)$ is a multiplicative set of $A \otimes_C B$. Show that there is a natural morphism $(S^{-1}A) \otimes_C B \cong (S \otimes 1)^{-1}(A \otimes_C B)$.

Hence we have a fiber diagram:

$$\begin{array}{ccc}
\operatorname{Spec}(S \otimes 1)^{-1}(A \otimes_C B) & \longrightarrow & \operatorname{Spec} S^{-1}A \\
\big\downarrow & & \big\downarrow \\
\operatorname{Spec} A \otimes_C B & \longrightarrow & \operatorname{Spec} A \\
\big\downarrow & & \big\downarrow \\
\operatorname{Spec} B & \longrightarrow & \operatorname{Spec} C
\end{array}$$

(where each rectangle is a fiber diagram).

Translation: the fibered product with a localization is the localization of the fibered product in the obvious way. We say that "localizations are preserved by base change". This is handy if the localization is of the form $A \hookrightarrow A_f$ (corresponding to taking distinguished open sets) or $A \hookrightarrow FF(A)$ (from A to the fraction field of A, corresponding to taking generic points), and various things in between.

These four tricks let you calculate lots of things in practice. For example,

$$\begin{gathered}\operatorname{Spec} k[x_1,\dots,x_m]/(f_1(x_1,\dots,x_m),\dots,f_r(x_1,\dots,x_m))\otimes_k \\ \operatorname{Spec} k[y_1,\dots,y_n]/(g_1(y_1,\dots,y_n),\dots,g_s(y_1,\dots,y_n)) \\ \cong \operatorname{Spec} k[x_1,\dots,x_m,y_1,\dots,y_n]/(f_1(x_1,\dots,x_m),\dots,f_r(x_1,\dots,x_m), \\ g_1(y_1,\dots,y_n),\dots,g_s(y_1,\dots,y_n)).\end{gathered}$$

Here are many more examples.

4. Examples

One important example is of *fibers* of morphisms. Suppose $p \to Z$ is the inclusion of a point (not necessarily closed). Then if $g : Y \to Z$ is any morphism, the base change with $p \to Z$ is called the *fiber of* g *above* p or the *preimage of* p, and is denoted $g^{-1}(p)$. If Z is irreducible, the fiber above the generic point is called the *generic fiber*. In an affine open subscheme $\operatorname{Spec} A$ containing p, p corresponds to some prime ideal $\mathfrak{p}$, and the morphism corresponds to the ring map $A \to A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}$. this is the composition if localization and closed immersion, and thus can be computed by the tricks above.

Here is an interesting example, that we will consider multiple times during this course. Consider the projection of the parabola $y^2 = x$ to the x axis, corresponding to the map of rings $\mathbb{Q}[x] \to \mathbb{Q}[y]$, with $x \mapsto y^2$. (If $\mathbb{Q}$ alarms you, replace it with your favorite field and see what happens.)

Then the preimage of 1 is 2 points:

$$\begin{aligned}\operatorname{Spec}\mathbb{Q}[x,y]/(y^2-x)\otimes_{\mathbb{Q}}\operatorname{Spec}\mathbb{Q}[x]/(x-1) &\cong \operatorname{Spec}\mathbb{Q}[x,y]/(y^2-x,x-1) \\ &\cong \operatorname{Spec}\mathbb{Q}[y]/(y^2-1) \\ &\cong \operatorname{Spec}\mathbb{Q}[y]/(y-1)\coprod\operatorname{Spec}\mathbb{Q}[y]/(y+1).\end{aligned}$$

The preimage of 0 is 1 nonreduced point:

$$\operatorname{Spec}\mathbb{Q}[x,y]/(y^2-x,x)\cong\operatorname{Spec}\mathbb{Q}[y]/(y^2).$$

The preimage of -1 is 1 reduced point, but of "size 2 over the base field".

$$\operatorname{Spec}\mathbb{Q}[x,y]/(y^2-x,x+1)\cong\operatorname{Spec}\mathbb{Q}[y]/(y^2+1)\cong\operatorname{Spec}\mathbb{Q}[i].$$

The preimage of the generic fiber is again 1 reduced point, but of "size 2 over the residue field".

$$\operatorname{Spec}\mathbb{Q}[x,y]/(y^2-x)\otimes\mathbb{Q}(x)\cong\operatorname{Spec}\mathbb{Q}[y]\otimes\mathbb{Q}(y^2)$$

i.e. you take elements polynomials in y, and you are allowed to invert polynomials in y^2. A little thought shows you that you are then allowed to invert polynomials in y, as if $f(y)$ is any polynomial in y, then

$$\frac{1}{f(y)}=\frac{f(-y)}{f(y)f(-y)},$$

and the latter denominator is a polynomial in y^2. Thus

$$\operatorname{Spec} \mathbb{Q}[x,y]/(y^2 - x) \otimes \mathbb{Q}(x) \cong \mathbb{Q}(y)$$

which is a degree 2 field extension of $\mathbb{Q}(x)$.

For future reference notice the following interesting fact: in each case, the number of preimages can be interpreted as 2, where you count to two in several ways: you can count points; you can get non-reduced behavior; or you can have field extensions. This is going to be symptomatic of a very special and important kind of morphism (a finite flat morphism).

Here are some other examples.

4.1. *Exercise.* Prove that $\mathbb{A}^n_R \cong \mathbb{A}^n_{\mathbb{Z}} \times_{\operatorname{Spec} \mathbb{Z}} \operatorname{Spec} R$. Prove that $\mathbb{P}^n_R \cong \mathbb{P}^n_{\mathbb{Z}} \times_{\operatorname{Spec} \mathbb{Z}} \operatorname{Spec} R$.

4.2. *Exercise.* Show that for finite-type schemes over $\mathbb{C}$, the complex-valued points of the fibered product correspond to the fibered product of the complex-valued points. (You will just use the fact that $\mathbb{C}$ is algebraically closed.)

Here is a definition in common use. The terminology is a bit unfortunate, because it is a second (different) meaning of "points of a scheme". If T is a scheme, the T-*valued points of a scheme* X are defined to be the morphism $T \to X$. They are sometimes denoted $X(T)$. If R is a ring (most commonly in this context a field), the R-*valued points of a scheme* X are defined to be the morphism $\operatorname{Spec} R \to X$. They are sometimes denoted $X(R)$. For example, if k is an algebraically closed field, then the k-valued points of a finite type scheme are just the closed points; but in general, things can be weirder. (When we say "points of a scheme", and not T-valued points, we will always mean the usual meaning, not this meaning.)

Exercise. Describe a natural bijection $(X \times_Z Y)(T) \cong X(T) \times_{Z(T)} Y(T)$. (The right side is a fibered product of sets.) In other words, fibered products behaves well with respect to T-valued points. This is one of the motivations for this notion.

4.3. *Exercise.* Describe $\operatorname{Spec} \mathbb{C} \times_{\operatorname{Spec} \mathbb{R}} \operatorname{Spec} \mathbb{C}$. This small example is the first case of something incredibly important.

4.4. *Exercise.* Consider the morphism of schemes $X = \operatorname{Spec} k[t] \to Y = \operatorname{Spec} k[u]$ corresponding to $k[u] \to k[t]$, $t = u^2$. Show that $X \times_Y X$ has 2 irreducible components. Compare what is happening above the generic point of Y to the previous exercise.

4.5. *A little too vague to be an exercise.* More generally, suppose $K/\mathbb{Q}$ is a finite Galois field extension. Investigate the analogue of the previous two exercises. Try degree 2. Try degree 3.

4.6. *Hard but fascinating exercise for those familiar with the Galois group of $\overline{\mathbb{Q}}$ over $\mathbb{Q}$.* Show that the points of $\operatorname{Spec} \overline{\mathbb{Q}} \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}$ are in natural bijection with $\operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, and the Zariski topology on the former agrees with the profinite topology on the latter.

4.7. *Exercise (A weird scheme).* Show that $\operatorname{Spec} \mathbb{Q}(t) \otimes_{\mathbb{Q}} \mathbb{C}$ is an integral dimension one scheme, with closed points in natural correspondence with the transcendental complex numbers. (If the description $\operatorname{Spec} \mathbb{C}[t] \otimes_{\mathbb{Q}[t]} \mathbb{Q}(t)$ is more striking, you can use that instead.) This scheme doesn't come up in nature, but it is certainly neat!

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FOUNDATIONS OF ALGEBRAIC GEOMETRY CLASS 23

RAVI VAKIL

CONTENTS

Last day: Morphisms to (quasi)projective schemes, and invertible sheaves; fibered products.

Today: Fibers of morphisms. Properties preserved by base change: open immersions, closed immersions, Segre embedding. Other schemes defined by universal property: reduction, normalization.

Last day, I showed you that fibered products exist, and I gave an argument that had fairly few moving parts: fibered products exist when the schemes in question are affine schemes; the universal property; and the fact that morphisms glue. I'll give you an exercise later today to give you a chance to make a similar argument, when I give the universal property for reducedness.

1. FIBERS OF MORPHISMS

We can informally interpret fibered product in the following geometric way. Suppose $Y \to Z$ is a morphism. We interpret this as a "family of schemes parametrized by a base scheme (or just plain *base*) Z." Then if we have another morphism $X \to Z$, we interpret the induced map $X \times_Z Y \to X$ as the "pulled back family". I drew a picture of this on the blackboard. I discussed the example: the family $y^2z = x^3 + txz^2$ of cubics in $\mathbb{P}^2$ parametrized by the affine line, and what happens if you pull back to the affine plane via $t = uv$, to get the family $y^2z = x^3 + uvxz^2$.

For this reason, fibered product is often called *base change* or *change of base* or *pullback*.

For instance, if X is a closed point of Z, then we will get the fiber over Z. As an example, consider the map of schemes $f : Y = \operatorname{Spec} \mathbb{Q}[t] \to Z = \operatorname{Spec} \mathbb{Q}[u]$ given by $u \mapsto t^2$ (or

Date: Tuesday, January 17, 2006. Trivial update October 26, 2006.

$u = t^2$). (I drew a picture on the blackboard. It looked like a parabola with horizontal axis of symmetry, projecting to the x-axis.) The fiber above $u = 1$ corresponds to the base change $X = \operatorname{Spec} \mathbb{Q}[u]/(u-1) \to \operatorname{Spec} \mathbb{Q}[u]$. Let's do the algebra: $X \times_Z Y = \operatorname{Spec} \mathbb{Q}[t, u]/(u-1, u-t^2) \cong \operatorname{Spec} \mathbb{Q}[t]/(t^2-1) \cong \operatorname{Spec} \mathbb{Q}[t]/(t-1) \times \mathbb{Q}[t]/(t+1)$. We see two reduced points (at "$u = 1, t = 1$ and $u = 1, t = -1$").

Next let's examine the fiber above $u = 0$. We get $\operatorname{Spec} \mathbb{Q}[t]/(t^2)$ — a point with non-reduced structure!

Finally, let's consider $u = -1$. We get $\operatorname{Spec} \mathbb{Q}[t]/(t^2 + 1)$. We get a single reduced point. The residue field $\mathbb{Q}(i)$ is a degree 2 field extension over $\mathbb{Q}$.

(Notice that in each case, we get something of "size two", informally speaking. One way of making this precise is that the rank of the sheaf $f_* \mathcal{O}_Y$ is rank 2 everywhere. In the first case, we see it as getting two different points. In the second, we get one point, with non-reduced behavior. In the last case, we get one point, of "size two". We will later see this "constant rank of $f_* \mathcal{O}_Y$" as symptomatic of the fact that this morphism is "particularly nice", i.e. finite and flat.)

We needn't look at fibers over just closed points; we can consider fibers over any points. More precisely, if p is a point of Z with residue field K, then we get a map $\operatorname{Spec} K \to Z$, and we can base change with respect to this morphism.

In the case of the generic point of $\operatorname{Spec} \mathbb{Q}[u]$ in the above example, we have $K = \mathbb{Q}(u)$, and $\mathbb{Q}[u] \to \mathbb{Q}(u)$ is the inclusion of the generic point. Let $X = \operatorname{Spec} \mathbb{Q}(u)$. Then you can verify that $X \times_Z Y = \operatorname{Spec} \mathbb{Q}[t, u]/(u - t^2) \otimes \mathbb{Q}(u) \cong \operatorname{Spec} \mathbb{Q}(t)$. We get the morphism $\mathbb{Q}(u) \to \mathbb{Q}(t)$ given by $u = t^2$ — a quadratic field extension.

Implicit here is a notion I should make explicit, about how you base change with respect to localization. Given $A \to B$, and a multiplicative set S of A, we have $(S^{-1}A) \otimes_A B \cong S^{-1}B$, where $S^{-1}B$ has the obvious interpretation. In other words,

$$\begin{array}{ccc} S^{-1}B & \longleftarrow & B \\ \uparrow & & \uparrow \\ S^{-1}A & \longleftarrow & A \end{array}$$

is "cofiber square" (or "pushout diagram").

1.1. *Remark: Geometric points.* We have already given two meanings for the "points of a scheme". We used one to define the notion of a scheme. Secondly, if T is a scheme, people sometimes say that $\operatorname{Hom}(T, X)$ are the "T-valued points of X". That's already confusing. But also, people say that the geometric points correspond to $\operatorname{Hom}(T, X)$ where T is the Spec of an algebraically closed field. Then for example the *geometric fibers* are the fibers over geometric points. In the example above, here is a geometric point: $\operatorname{Spec} \overline{\mathbb{Q}}[u]/(u-1) \to \operatorname{Spec} \mathbb{Q}[u]$. And here is a geometric fiber: $\operatorname{Spec} \overline{\mathbb{Q}}[t]/(t^2-1)$. Notice that the geometric fiber above $u = -1$ also consists of two points, unlike the "usual" fiber.

(I should check: possibly the definition should just be for T the algebraic closure of the residue field of a not-necessarily-closed point.)

1.2. *Exercise for the arithmetically-minded.* Show that for the morphism $\operatorname{Spec} \mathbb{C} \to \operatorname{Spec} \mathbb{R}$, all geometric fibers consist of two reduced points. This exercise should be removed if I have the wrong definition of geometric point!

We will discuss more about geometric points and properties of geometric fibers shortly.

2. Properties preserved by base change

We now discuss a number of properties that behave well under base change.

We've already shown that the notion of "open immersion" is preserved by base change (problem 6 on problem set 9, see class 19). We did this by explicitly describing what the fibered product of an open immersion is: if $Y \hookrightarrow Z$ is an open immersion, and $f : X \to Z$ is any morphism, then we checked that the open subscheme $f^{-1}(Y)$ of X satisfies the universal property of fibered products.

2.1. *Important exercise (problem 8+ on the last problem set).* Show that the notion of "closed immersion" is preserved by base change. (This was stated in class 19.) Somewhat more precisely, given a fiber diagram

$$\begin{array}{ccc} W & \longrightarrow & X \\ \downarrow & & \downarrow \\ Y & \longrightarrow & Z \end{array}$$

where $Y \hookrightarrow Z$ is a closed immersion, then $W \hookrightarrow X$ is as well. (Hint: in the case of affine schemes, you have done this before in a different guise — see problem B3 on problem set 1!) In the course of the proof, you will show that W is cut out by the same equations in X as Y is in Z, or more precisely by pullback of those equations. Hence fibered products (over k) of schemes of finite type over k may be computed easily:

$$\operatorname{Spec} k[x_1, \dots, x_m]/(f_1(x_1, \dots, x_m), \dots, f_r(x_1, \dots, x_m)) \times_{\operatorname{Spec} k}$$

$$\operatorname{Spec} k[y_1, \dots, y_m]/(g_1(y_1, \dots, y_m), \dots, g_s(y_1, \dots, y_m))$$

$$\cong \operatorname{Spec} k[x_1, \dots, x_m, y_1, \dots, y_m]/(f_1(x_1, \dots, x_m), \dots, f_r(x_1, \dots, x_m),$$

$$g_1(y_1, \dots, y_m), \dots, g_s(y_1, \dots, y_m)).$$

We sometimes say that W is the *scheme-theoretic pullback of* Y, *scheme-theoretic inverse image*, or *inverse image scheme of* Y. The ideal sheaf of W is sometimes called the *inverse image (quasicoherent) ideal sheaf*.

Note for experts: It is not necessarily the quasicoherent pullback (f^*) of the ideal sheaf, as the following example shows. (Thanks Joe!)

$$\begin{array}{ccc} \operatorname{Spec} k[x]/(x) & \longrightarrow & \operatorname{Spec} k[x]/(x) \\ \downarrow & & \downarrow \\ \operatorname{Spec} k[x]/(x) & \longrightarrow & \operatorname{Spec} k[x] \end{array}$$

Instead, the correct thing to pullback (the thing that "pulls back well") is the surjection $\mathcal{O}_Z \to \mathcal{O}_Y \to 0$, which pulls back to $\mathcal{O}_X \to \mathcal{O}_W \to 0$. The key issue is that pullback of quasicoherent sheaves is right-exact, so we shouldn't expect the pullback of $0 \to \mathcal{I}_{Y/Z} \to \mathcal{O}_Z \to \mathcal{O}_Y \to 0$ to be exact, only right-exact. (Thus for example we get a natural map $f^*\mathcal{I}_{Y/Z} \to \mathcal{I}_{W/X}$.)

Similarly, other important properties are preserved by base change.

2.2. *Exercise.* Show that the notion of "morphism locally of finite type" is preserved by base change. Show that the notion of "affine morphism" is preserved by base change. Show that the notion of "finite morphism" is preserved by base change.

2.3. *Exercise.* Show that the notion of "quasicompact morphism" is preserved by base change.

2.4. *Exercise.* Show that the notion of "morphism of finite type" is preserved by base change.

2.5. *Exercise.* Show that the notion of "quasifinite morphism" (= finite type + finite fibers) is preserved by base change. (Note: the notion of "finite fibers" is not preserved by base change. $\operatorname{Spec} \overline{\mathbb{Q}} \to \operatorname{Spec} \mathbb{Q}$ has finite fibers, but $\operatorname{Spec} \overline{\mathbb{Q}} \otimes_{\mathbb{Q}} \overline{\mathbb{Q}} \to \operatorname{Spec} \overline{\mathbb{Q}}$ has one point for each element of $\operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$.)

2.6. *Exercise.* Show that surjectivity is preserved by base change (or fibered product). In other words, if $X \to Y$ is a surjective morphism, then for any $Z \to Y$, $X \times_Y Z \to Z$ is surjective. (You may end up using the fact that for any fields k_1 and k_2 containing k_3, $k_1 \otimes_{k_3} k_2$ is non-zero, and also the axiom of choice.)

2.7. *Exercise.* Show that the notion of "irreducible" is not necessarily preserved by base change. Show that the notion of "connected" is not necessarily preserved by base change. (Hint: $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$, $\mathbb{Q}[i] \otimes_{\mathbb{Q}} \mathbb{Q}[i]$.)

If X is a scheme over a field k, it is said to be *geometrically irreducible* if its base change to $\overline{k}$ (i.e. $X \times_{\operatorname{Spec} k} \operatorname{Spec} \overline{k}$) is irreducible. Similarly, it is *geometrically connected* if its base change to $\overline{k}$ (i.e. $X \times_{\operatorname{Spec} k} \operatorname{Spec} \overline{k}$) is connected. Similarly also for *geometrically reduced* and *geometrically integral*. We say that $f : X \to Y$ has *geometrically irreducible (resp. connected, reduced, integral) fibers* if the geometric fibers are geometrically irreducible (resp. connected, reduced, integral).

If you care about such notions, see Hartshorne Exercise II.3.15 for some facts (stated in a special case). In particular, to check geometric irreducibility, it suffices to check over *separably closed* (not necessarily algebraically closed) fields. To check geometric reducedness, it suffices to check over *perfect* fields.

2.8. *Exercise.* Show that $\operatorname{Spec} \mathbb{C}$ is not a geometrically irreducible $\mathbb{R}$-scheme. If $\operatorname{char} k = p$, show that $\operatorname{Spec} k(u)$ is not a geometrically reduced $\operatorname{Spec} k(u^p)$-scheme.

2.9. *Exercise.* Show that the notion of geometrically irreducible (resp. connected, reduced, integral) fibers behaves well with respect to base change.

On a related note:

2.10. *Exercise (less important).* Suppose that l/k is a finite field extension. Show that a k-scheme X is normal if and only if $X \times_{\operatorname{Spec} k} \operatorname{Spec} l$ is normal. Hence deduce that if k is any field, then $\operatorname{Spec} k[w, x, y, z]/(wz - xy)$ is normal. (I think this was promised earlier.) Hint: we showed earlier (Problem B4 on set 4) that $\operatorname{Spec} k[a, b, c, d]/(a^2 + b^2 + c^2 + d^2)$ is normal.

3. Products of projective schemes: The Segre embedding

I will next describe products of projective A-schemes over A. The case of greatest initial interest is if $A = k$. (A reminder of why we like projective schemes. (i) it is an easy way of getting interesting non-affine schemes. (ii) we get lots of schemes of classical interest. (iii) we have a hard time thinking of anything that isn't projective or an open subset of a projective. (iv) a k-scheme is a first approximation of what we mean by compact.)

In order to do this, I need only describe $\mathbb{P}^m_A \times_A \mathbb{P}^n_A$, because any projective scheme has a closed immersion in some $\mathbb{P}^m_A$, and closed immersions behave well under base change: so if $X \hookrightarrow \mathbb{P}^m_A$ and $Y \hookrightarrow \mathbb{P}^n_A$ are closed immersions, then $X \times_A Y \hookrightarrow \mathbb{P}^m_A \times_A \mathbb{P}^n_A$ is also a closed immersion, cut out by the equations of X and Y.

We'll describe $\mathbb{P}^m_A \times_A \mathbb{P}^n_A$, and see that it too is a projective A-scheme. Consider the map $\mathbb{P}^m_A \times_A \mathbb{P}^n_A \to \mathbb{P}^{mn+m+n}_A$ given by

$$([x_0; \dots; x_m], [y_0; \dots; y_n]) \to [z_{00}; z_{01}; \cdots; z_{ij}; \cdots; z_{mn}] = [x_0 y_0; x_0 y_1; \cdots; x_i y_j; \cdots x_m y_n].$$

First, you should verify that this is a well-defined morphism! On the open chart $U_i \times V_j$, this gives a map $(x_{0/i}, \dots, x_{m/i}, y_{0/j}, \dots, y_{n/j}) \mapsto [x_{0/i} y_{0/j}; \dots; x_{i/i} y_{j/j}; \dots; x_{m/i} y_{n/j}]$. Note that this gives an honest map to projective space — not all the entries on the right are zero, as one of the entries $(x_{i/i} y_{j/j})$ is 1.

(Aside: we now well know that a map to projective space corresponds to an invertible sheaf with a bunch of sections. The invertible sheaf on this case is $\pi_1^* \mathcal{O}_{\mathbb{P}^m_A}(1) \otimes \pi_2^* \mathcal{O}_{\mathbb{P}^n_A}(1)$, where π_i are the projections of the product onto the two factors. The notion $\boxtimes$ is often used for this notion, when you pull back sheaves from each factor of a product, and tensor. For example, this invertible sheaf could be written $\mathcal{O}(1) \boxtimes \mathcal{O}(1)$. People often write $\mathcal{O}(a) \boxtimes \mathcal{O}(b)$ for $\mathcal{O}(a, b)$.)

I claim this morphism is a closed immersion. (We are essentially using Exercise 3.2 in the class 21 notes, problem 40 in problem set 9. But don't waste your time by looking back at it.) Let's check this on the open set where $z_{ab} \neq 0$. Without loss of generality, I'll take $a = b = 0$, to make notation simpler. Then the preimage of this open set in $\mathbb{P}^m_A \times \mathbb{P}^n_A$ is the locus where $x_0 \neq 0$ and $y_0 \neq 0$, i.e. $U_0 \times V_0$, U_0 and V_0 are the usual distinguished open sets of $\mathbb{P}^m_A$ and $\mathbb{P}^n_A$ respectively. The coordinates here are $x_{1/0}, \ldots, x_{m/0}, y_{1/0}, \ldots, y_{n/0}$. Thus the map corresponds to $z_{ab/00} \to x_{a/0} y_{b/0}$, which clearly induces a surjection of rings

$$A[z_{00/00}, \ldots, z_{mn/00}] \to A[x_{1/0}, \ldots, x_{m/0}, y_{1/0}, \ldots, y_{n/0}].$$

(Recall that $z_{a0/00} \mapsto x_{a/0}$ and $z_{0b/00} \mapsto y_{b/0}$.)

Hence we are done! This map is called the *Segre morphism* or *Segre embedding*. If A is a field, the image is called the *Segre variety* — although we don't yet know what a variety is!

Here are some useful comments.

3.1. *Exercise.* Show that the Segre scheme (the image of the Segre morphism) is cut out by the equations corresponding to

$$\operatorname{rank} \begin{pmatrix} a_{00} & \cdots & a_{0n} \\ \vdots & \ddots & \vdots \\ a_{m0} & \cdots & a_{mn} \end{pmatrix} = 1,$$

i.e. that all 2×2 minors vanish. (Hint: suppose you have a polynomial in the a_{ij} that becomes zero upon the substitution $a_{ij} = x_i y_j$. Give a recipe for subtracting polynomials of the form monomial times 2×2 minor so that the end result is 0.)

3.2. *Example.* Let's consider the first non-trivial example, when $m = n = 1$. We get $\mathbb{P}^1 \times \mathbb{P}^1 \hookrightarrow \mathbb{P}^3$. We get a single equation

$$\operatorname{rank} \begin{pmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{pmatrix} = 1,$$

i.e. $a_{00} a_{11} - a_{01} a_{10} = 0$. We get our old friend, the quadric surface! Hence: the nonsingular quadric surface $wz - xy = 0$ is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. Note that we can reinterpret the rulings; I pointed this out on the model. Since (by diagonalizability of quadratics) all nonsingular quadratics over an algebraically closed field are isomorphic, we have that all nonsingular quadric surfaces over an algebraically closed field are isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$.

Note that this is not true even over a field that is not algebraically closed. For example, over $\mathbb{R}$, $w^2 + x^2 + y^2 + z^2 = 0$ is not isomorphic to $\mathbb{P}^1_{\mathbb{R}} \times_{\mathbb{R}} \mathbb{P}^1_{\mathbb{R}}$. Reason: the former has no real points, while the latter has lots of real points.

3.3. Let's return to the general Segre situation. We can describe the closed subscheme alternatively the Proj of the subring R of

$$A[x_0, \ldots, x_m, y_0, \ldots, y_n]$$

generated by monomials of equal degree in the x's and the y's. Using this, you can give a co-ordinate free description of this product (i.e. without using the co-ordinates x_i and y_j): $\mathbb{P}^m_A \times_A \mathbb{P}^n_A = \operatorname{Proj} R$ where

$$R = \oplus_{i=0}^{\infty} \operatorname{Sym}^i H^0(\mathbb{P}^m_A, \mathcal{O}(1)) \otimes \operatorname{Sym}^i H^0(\mathbb{P}^n_A, \mathcal{O}(1)).$$

Kirsten asks an interesting question: show that $\mathcal{O}(a, b)$ gives a closed immersion to projective space if $a, b > 0$.

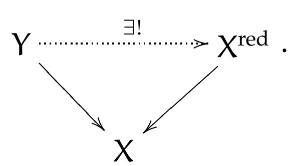
You may want to ponder how to think of products of three projective spaces.

4. Other schemes defined by universal property: reduction, normalization

I now want to define other schemes using universal properties, in ways that are vaguely analogous to fibered product.

As a warm-up, I'd like to revisit an earlier topic: reduction of a scheme. Recall that if X is a scheme, we defined a closed immersion $X^{red} \hookrightarrow X$. (See the comment just before §1.4 in class 19.) I'd like to revisit this.

4.1. *Potentially enlightening exercise.* Show that $X^{red} \to X$ satisfies the following universal property: any morphism from a reduced scheme Y to X factors uniquely through X^{red}.



You can use this as a definition for $X^{red} \to X$. Let me walk you through part of this. First, prove this for X affine. (Here you use the fact that we know that maps to an affine scheme correspond to a maps of global sections in the other direction.) Then use the universal property to show the result for quasiaffine X. Then use the universal property to show it in general. **Oops! I don't think I've defined quasiaffine before. It is any scheme that can be expressed as an open subset of an affine scheme. I should eventually put this definition earlier in the course notes, but may not get a chance to. It may appear in the class 22 notes, which are yet to be written up. The concept is reintroduced yet again in Exercise 4.4 below.**

4.2. Normalization.

I now want to tell you how to normalize a reduced Noetherian scheme. A normalization of a scheme X is a morphism $\nu : \tilde{X} \to X$ from a normal scheme, where ν induces a bijection of components of $\tilde{X}$ and X, and ν gives a birational morphism on each of the components; it will be nicer still, as it will satisfy a universal property. (I drew a picture of a normalization of a curve.) **Oops! I didn't define *birational* until class 27. Please just plow ahead! I may later patch this anachronism, but most likely I won't get the chance.**

I'll begin by dealing with the case where X is irreducible, and hence integral. (I'll then deal with the more general case, and also discuss normalization in a function field extension.)

In this case of X irreducible, the normalization satisfies dominant morphism from an irreducible normal scheme to X, then this morphism factors uniquely through ν:

$$\begin{array}{ccc} Y & \overset{\exists!}{\dashrightarrow} & \tilde{X} \\ & \searrow \qquad \swarrow_{\nu} & \\ & X & \end{array}$$

Thus if it exists, then it is unique up to unique isomorphism. We now have to show that it exists, and we do this in the usual way. We deal first with the case where X is affine, say $X = \operatorname{Spec} R$, where R is an integral domain. Then let $\tilde{R}$ be the integral closure of R in its fraction field $\operatorname{Frac}(R)$.

4.3. *Exercise.* Show that $\nu : \operatorname{Spec} \tilde{R} \to \operatorname{Spec} R$ satisfies the universal property.

4.4. *Exercise.* Show that normalizations exist for any quasiaffine X (i.e. any X that can be expressed as an open subset of an affine scheme).

4.5. *Exercise.* Show that normalizations exist in general.

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FOUNDATIONS OF ALGEBRAIC GEOMETRY CLASS 24

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CONTENTS

Last day: Fibers of morphisms. Properties preserved by base change: open immersions, closed immersions, Segre embedding. Other schemes defined by universal property: reduction, normalization.

Today: normalization (in a field extension), "sheaf Spec", "sheaf Proj", projective morphism.

1. NORMALIZATION, CONTINUED

Last day, I defined the normalization of a reduced scheme. I have an interesting question for experts: there is a reasonable extension to schemes in general; does anything go wrong? I haven't yet given this much thought, but it seems worth exploring.

I described normalization last day in the case when X is irreducible, and hence integral. In this case of X irreducible, the normalization satisfies the universal property, that if $Y \to X$ is any other dominant morphism from a normal scheme to X, then this morphism factors uniquely through ν:

$$\begin{array}{ccc} Y & \overset{\exists!}{\dashrightarrow} & \tilde{X} \\ & \searrow & \swarrow \nu \\ & X & \end{array} .$$

Thus if it exists, then it is unique up to unique isomorphism. We then showed that it exists, using an argument we saw for the third time. (The first time was in the existence of the fibered product. The second was an argument for the existence of the reduction morphism.) The ring-theoretic case got us started: if $X = \operatorname{Spec} R$, then and $\tilde{R}$ is the integral closure of R in its fraction field $\operatorname{Frac}(R)$, then I gave as an exercise that $\nu : \operatorname{Spec} \tilde{R} \to \operatorname{Spec} R$ satisfies the universal property.

Date: Thursday, January 19, 2006. Last minor update June 28, 2007.

1.1. *Exercise.* Show that the normalization morphism is surjective. (Hint: Going-up!)

We now mention some bells and whistles. The following fact is handy.

1.2. *Theorem (finiteness of integral closure). — Suppose* A *is a domain,* $K = \mathrm{Frac}(A)$, L/K *is a finite field extension, and* B *is the integral closure of* A *in* L *("the integral closure of* A *in the field extension* L/K*", i.e. those elements of* L *integral over* A*).*
(a) if A *is integrally closed, then* B *is a finitely generated* A*-module.*
(b) if A *is a finitely generated* k*-algebra, then* B *(the integral closure of* A *in its fraction field) is a finitely generated* A*-module.*

I hope to type up a proof of these facts at some point to show you that they are not that bad. Much of part (a) was proved by Greg Brumfiel in 210B last year.

Warning: (b) does *not* hold for Noetherian A in general. I find this very alarming. I don't know an example offhand, but one is given in Eisenbud's book.

1.3. *Exercise.* Show that $\dim \tilde{X} = \dim X$ (hint: see our going-up discussion).

1.4. *Exercise.* Show that if X is an integral finite-type k-scheme, then its normalization $\nu : \tilde{X} \to X$ is a finite morphism.

1.5. *Exercise.* Explain how to generalize the notion of normalization to the case where X is a reduced Noetherian scheme (with possibly more than one component). This basically requires defining a universal property. I'm not sure what the "perfect" definition, but all reasonable universal properties should lead to the same space.

1.6. *Exercise.* Show that if X is an integral finite type k-scheme, then its non-normal points form a closed subset. (This is a bit trickier. Hint: consider where $\nu_* \mathcal{O}_{\tilde{X}}$ has rank 1.) I haven't thought through all the details recently, so I hope I've stated this correctly.

Here is an explicit example to think through some of these ideas.

1.7. Exercise. Suppose $X = \operatorname{Spec} \mathbb{Z}[15i]$. Describe the normalization $\tilde{X} \to X$. (Hint: it isn't hard to find an integral extension of $\mathbb{Z}[15i]$ that is integrally closed. By the above discussion, you've then found the normalization!) Over what points of X is the normalization not an isomorphism?

1.8. *Exercise.* (This is an important generalization!) Suppose X is an integral scheme. Define the *normalization of* X, $\nu : \tilde{X} \to X$, *in a given finite field extension of the function field of* X. Show that $\tilde{X}$ is normal. (This will be hard-wired into your definition.) Show that if either X is itself normal, or X is finite type over a field k, then the normalization in a finite field extension is a finite morphism.

Let's try this in a few cases.

1.9. Exercise. Suppose $X = \operatorname{Spec} \mathbb{Z}$ (with function field $\mathbb{Q}$). Find its integral closure in the field extension $\mathbb{Q}(i)$.

A finite extension K of $\mathbb{Q}$ is called a *number field*, and the integral closure of $\mathbb{Z}$ in K the *ring of integers of* K, denoted $\mathcal{O}_K$. (This notation is a little awkward given our other use of the symbol $\mathcal{O}$.) By the previous exercises, $\operatorname{Spec} \mathcal{O}_K$ is a Noetherian normal domain of dimension 1 (hence regular). This is called a *Dedekind domain*. We think of it as a smooth curve.

1.10. Exercise. (a) Suppose $X = \operatorname{Spec} k[x]$ (with function field $k(x)$). Find its integral closure in the field extension $k(y)$, where $y^2 = x^2 + x$. (Again we get a Dedekind domain.)
(b) Suppose $X = \mathbb{P}^1$, with distinguished open $\operatorname{Spec} k[x]$. Find its integral closure in the field extension $k(y)$, where $y^2 = x^2 + x$. (Part (a) involves computing the normalization over one affine open set; now figure out what happens over the "other".)

2. Sheaf Spec

Given an A-algebra, B, we can take its Spec to get an affine scheme over $\operatorname{Spec} A$: $\operatorname{Spec} B \to \operatorname{Spec} A$. I'll now give a universal property description of a globalization of that notation. We will take an arbitrary scheme X, and a quasicoherent sheaf of algebras $\mathcal{A}$ on it. We will define how to take Spec of this sheaf of algebras, and we will get a scheme $\underline{\operatorname{Spec}}\, \mathcal{A} \to X$ that is "affine over X", i.e. the structure morphism is an affine morphism.

We will do this as you might by now expect: for each affine on X, we use our affine construction, and show that everything glues together nicely. We do this instead by describing $\underline{\operatorname{Spec}}\, \mathcal{A} \to X$ in terms of a good universal property: given any morphism $\pi : Y \to X$ along with a morphism of $\mathcal{O}_X$-modules

$$\alpha : \mathcal{A} \to \pi_* \mathcal{O}_Y,$$

there is a unique map $Y \to \underline{\operatorname{Spec}}\, \mathcal{A}$ factoring π, i.e. so that the following diagram commutes,

$$\begin{array}{ccc} Y & \overset{\exists !}{\dashrightarrow} & \underline{\operatorname{Spec}}\, \mathcal{A} \\ & {\scriptstyle \pi} \searrow \quad \swarrow {\scriptstyle \beta} & \\ & X & \end{array}$$

and an isomorphism $\phi : \mathcal{A} \to \beta_* \mathcal{O}_{\underline{\operatorname{Spec}}\, \mathcal{A}}$ inducing α.

(For experts: we need $\mathcal{O}_X$-modules, and to leave our category of quasicoherent sheaves on X, because we only showed that the pushforward of quasicoherent sheaves are quasicoherent for certain morphisms, where the preimage of each affine was a finite union of affines, the pairwise intersection of which were also finite unions. This notion will soon be formalized as quasicompact and quasiseparated.)

At this point we're getting to be experts on this, so let's show that this $\underline{\mathrm{Spec}}\,\mathcal{A}$ exists. In the case where X is affine, we are done by our affine discussion. In the case where X is quasiaffine, we are done for the same reason as before. And finally, in the case where X is general, we are done once again!

In particular, note that $\underline{\mathrm{Spec}}\,\mathcal{A} \to X$ is an affine morphism.

2.1. *Exercise.* Show that if $f : Z \to X$ is an affine morphism, then we have a natural isomorphism $Z \cong \underline{\mathrm{Spec}}\, f_*\mathcal{O}_Z$ of X-schemes.

Hence we can recover any affine morphism in this way. More precisely, a morphism is affine if and only if it is of the form $\underline{\mathrm{Spec}}\,\mathcal{A} \to X$.

2.2. *Exercise ($\underline{\mathrm{Spec}}$ behaves well with respect to base change).* Suppose $f : Z \to X$ is any morphism, and $\mathcal{A}$ is a quasicoherent sheaf of algebras on X. Show that there is a natural isomorphism $Z \times_X \underline{\mathrm{Spec}}\,\mathcal{A} \cong \underline{\mathrm{Spec}}\, f^*\mathcal{A}$.

An important example of this $\underline{\mathrm{Spec}}$ construction is the total space of a finite rank locally free sheaf $\mathcal{F}$, which is a *vector bundle*. It is $\underline{\mathrm{Spec}}\,\mathrm{Sym}^* \mathcal{F}^\vee$.

2.3. *Exercise.* Show that this is a vector bundle, i.e. that given any point $p \in X$, there is a neighborhood $p \in U \subset X$ such that $\underline{\mathrm{Spec}}\,\mathrm{Sym}^* \mathcal{F}^\vee|_U \cong \mathbb{A}^n_U$. Show that $\mathcal{F}$ is isomorphic to the sheaf of sections of it.

As an easy example: if $\mathcal{F}$ is a *free* sheaf of rank n, then $\underline{\mathrm{Spec}}\,\mathrm{Sym}^* \mathcal{F}^\vee$ is called $\mathbb{A}^n_X$, generalizing our earlier notions of $\mathbb{A}^n_A$. As the notion of a free sheave behaves well with respect to base change, so does the notion of $\mathbb{A}^n_X$, i.e. given $X \to Y$, $\mathbb{A}^n_Y \times_Y X \cong \mathbb{A}^n_X$.

Here is one last fact that might come in handy.

2.4. *Exercise.* Suppose $f : \underline{\mathrm{Spec}}\,\mathcal{A} \to X$ is a morphism. Show that the category of quasicoherent sheaves on $\underline{\mathrm{Spec}}\,A$ is "essentially the same as" (=equivalent to) the category of quasicoherent sheaves on X with the structure of $\mathcal{A}$-modules (quasicoherent $\mathcal{A}$-modules on X).

The reason you could imagine caring is when X is quite simple, and $\underline{\mathrm{Spec}}\,\mathcal{A}$ is complicated. We'll use this before long when $X \cong \mathbb{P}^1$, and $\underline{\mathrm{Spec}}\,\mathcal{A}$ is a more complicated curve. (I drew a picture of this.)

3. Sheaf Proj

We'll now do a global (or "sheafy") version of Proj, which we'll denote $\underline{\mathrm{Proj}}$.

Suppose now that $\mathcal{S}_*$ is a quasicoherent sheaf of graded algebras of X. To be safe, let me assume that $\mathcal{S}_*$ is locally generated in degree 1 (i.e. there is a cover by small affine open sets, where for each affine open set, the corresponding algebra is generated in degree 1), and $\mathcal{S}_1$ is finite type. We will define $\underline{\operatorname{Proj}}\, \mathcal{S}_*$.

The essential ideal is that we do this affine by affine, and then glue the result together. But as before, this is tricky to do, but easier if you state the right universal property.

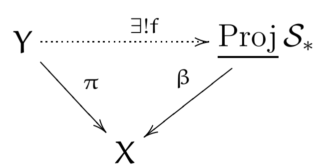
As a preliminary, let me re-examine our earlier theorem, that "Maps to $\mathbb{P}^n$ correspond to $n+1$ sections of an invertible sheaf, not all vanishing at any point (= generated by global sections), modulo sections of $\mathcal{O}_X^*$."

I will now describe this in a more "relative" setting, where relative means that we do this with morphisms of schemes. We begin with a relative notion of base-point free. Suppose $f : Y \to X$ is a morphism, and $\mathcal{L}$ is an invertible sheaf on Y. We say that $\mathcal{L}$ is *relatively base point free* if for every point $p \in X$, $q \in Y$, with $f(q) = p$, there is a neighborhood U for which there is a section of $\mathcal{L}$ over $f^{-1}(U)$ not vanishing at q. Similarly, we define *relatively generated by global sections* if there is a neighborhood U for which there are sections of $\mathcal{L}$ over $f^{-1}(U)$ generating *every* stalk of $f^{-1}(U)$. This is admittedly hideous terminology. (One can also define *relatively generated by global sections at a point* $p \in Y$. See class 16 where we defined these notions in a non-relative setting. In class 32, this will come up again.) More generally, we can define the notion of "relatively generated by global sections by a subsheaf of $f_*\mathcal{L}$".

Definition. $(\underline{\operatorname{Proj}}\, \mathcal{S}_*, \mathcal{O}_{\underline{\operatorname{Proj}}\, \mathcal{S}_*}(1)) \to X$ satisfies the following universal property. Given any other X-scheme Y with an invertible sheaf $\mathcal{L}$, and a map of graded $\mathcal{O}_X$-algebras

$$\alpha : \mathcal{S}_* \to \oplus_{n=0} \pi_* \mathcal{L}^{\otimes n},$$

such that $\mathcal{L}$ is relatively generated by the global sections of $\alpha(S_1)$, there is a unique factorization



and a canonical isomorphism $\mathcal{L} \cong f^*\mathcal{O}_{\underline{\operatorname{Proj}}\, \mathcal{S}_*}(1)$ and a morphism $\mathcal{S}_* \to \oplus_n \beta_* \mathcal{O}(n)$ inducing α.

In particular, $\underline{\operatorname{Proj}}\, \mathcal{S}_*$ comes with an invertible sheaf $\mathcal{O}_{\underline{\operatorname{Proj}}\, \mathcal{S}_*}(1)$, and this $\mathcal{O}(1)$ should be seen as part of the data.

This definition takes some getting used to.

But we prove this as usual!

We first deal with the case where X is affine, say $X = \operatorname{Spec} A$, $\mathcal{S}_* = \tilde{S}_*$. You won't be surprised to hear that in this case, $(\operatorname{Proj} S_*, \mathcal{O}(1))$ satisfies the universal property.

We outline why. Clearly, given a map $Y \to \operatorname{Proj} S_*$, we get a pullback map α. Conversely, given such a pullback map, we want to show that this induces a (unique) map $Y \to \operatorname{Proj} S_*$. Now because S_* is generated in degree 1, we have a closed immersion

$\mathrm{Proj}\, S_* \hookrightarrow \mathrm{Proj}\,\mathrm{Sym}^* S_1$. The map in degree 1, $S_1 \to \pi_* \mathcal{L}$, gives a map $Y \to \mathrm{Proj}\,\mathrm{Sym}^* S_1$ by our magic theorem "Maps to $\mathbb{P}^n$ correspond to $n+1$ sections of an invertible sheaf, not all vanishing at any point (= generated by global sections), modulo sections of $\mathcal{O}_X^*$."

3.1. Exercise. Complete this argument that if $X = \mathrm{Spec}\, A$, then $(\mathrm{Proj}\,\mathcal{S}_*, \mathcal{O}(1))$ satisfies the universal property.

3.2. Exercise. Show that $(\mathrm{Proj}\,\mathcal{S}_*, \mathcal{O}(1))$ exists in general, by following the analogous universal property argument: show that it exists for X quasiaffine, then in general.

3.3. *Exercise ($\underline{\mathrm{Proj}}$ behaves well with respect to base change).* Suppose $\mathcal{S}_*$ is a quasicoherent sheaf of graded algebras on X satisfying the required hypotheses above for $\underline{\mathrm{Proj}}\,\mathcal{S}_*$ to exist. Let $f : Y \to X$ be any morphism. Give a natural isomorphism

$$(\underline{\mathrm{Proj}}\, f^*\mathcal{S}_*, \mathcal{O}_{\underline{\mathrm{Proj}}\, f^*\mathcal{S}_*}(1)) \cong (Y \times_X \underline{\mathrm{Proj}}\,\mathcal{S}_*, g^*\mathcal{O}_{\underline{\mathrm{Proj}}\,\mathcal{S}_*}(1)) \cong$$

where g is the natural morphism in the base change diagram

$$\begin{array}{ccc} Y \times_X \underline{\mathrm{Proj}}\,\mathcal{S}_* & \xrightarrow{g} & \underline{\mathrm{Proj}}\,\mathcal{S}_* \\ \downarrow & & \downarrow \\ Y & \longrightarrow & X. \end{array}$$

3.4. *Definition.* If $\mathcal{F}$ is a finite rank locally free sheaf on X. Then $\underline{\mathrm{Proj}}\,\mathrm{Sym}^*\mathcal{F}$ is called its *projectivization*. If $\mathcal{F}$ is a free sheaf of rank $n+1$, then we define $\mathbb{P}^n_X := \underline{\mathrm{Proj}}\,\mathrm{Sym}^*\mathcal{F}$. (Then $\mathbb{P}^n_{\mathrm{Spec}\, A}$ agrees with our earlier definition of $\mathbb{P}^n_A$.) Clearly this notion behaves well with respect to base change.

This "relative $\mathcal{O}(1)$" we have constructed is a little subtle. Here are couple of exercises to give you practice with the concept.

3.5. *Exercise.* $\underline{\mathrm{Proj}}(\mathcal{S}_*[t]) \cong \underline{\mathrm{Spec}}\,\mathcal{S}_* \coprod \underline{\mathrm{Proj}}\,\mathcal{S}_*$, where $\underline{\mathrm{Spec}}\,\mathcal{S}_*$ is an open subscheme, and $\underline{\mathrm{Proj}}\,\mathcal{S}_*$ is a closed subscheme. Show that $\underline{\mathrm{Proj}}\,\mathcal{S}*$ is an effective Cartier divisor, corresponding to the invertible sheaf $\mathcal{O}_{\underline{\mathrm{Proj}}\, N}(1)$. (This is the generalization of the projective and affine cone. At some point I should give an explicit reference to our earlier exercise on this.)

3.6. *Exercise.* Suppose $\mathcal{L}$ is an invertible sheaf on X, and $\mathcal{S}_*$ is a quasicoherent sheaf of graded algebras on X satisfying the required hypotheses above for $\underline{\mathrm{Proj}}\,\mathcal{S}_*$ to exist. Define $\mathcal{S}'_* = \oplus_{n=0}\mathcal{S}_n \otimes \mathcal{L}_n$. Give a natural isomorphism of X-schemes

$$(\underline{\mathrm{Proj}}\,\mathcal{S}'_*, \mathcal{O}_{\underline{\mathrm{Proj}}\,\mathcal{S}'_*}(1)) \cong (\underline{\mathrm{Proj}}\,\mathcal{S}_*, \mathcal{O}_{\underline{\mathrm{Proj}}\,\mathcal{S}_*}(1) \otimes \pi^*\mathcal{L}),$$

where $\pi : \underline{\mathrm{Proj}}\,\mathcal{S}_* \to X$ is the structure morphism. In other words, informally speaking, the $\underline{\mathrm{Proj}}$ is the same, but the $\mathcal{O}(1)$ is twisted by $\mathcal{L}$.

3.7. Projective morphisms.

If you are tuning out because of these technicalities, please tune back in! I now want to define an essential notion.

Recall that we have recast affine morphisms in the following way: $X \to Y$ is an affine morphism if $X \cong \underline{\mathrm{Spec}}\, \mathcal{A}$ for some quasicoherent sheaf of algebras $\mathcal{A}$ on Y.

I will now *define* the notion of a projective morphism similarly.

3.8. *Definition.* A morphism $X \to Y$ is *projective* if there is an isomorphism

$$\begin{array}{ccc} X & \xrightarrow{\sim} & \underline{\mathrm{Proj}}\, \mathcal{S}_* \\ & \searrow \quad \swarrow & \\ & Y & \end{array}$$

for a quasicoherent sheaf of algebras $\mathcal{S}_*$ on Y satisfying the required hypothesis for $\underline{\mathrm{Proj}}$ to exist.

Two warnings! 1. Notice that I didn't say anything about the $\mathcal{O}(1)$, which is an important definition. The notion of affine morphism is affine-local on the target, but this notion is not affine-local on the target! (In nice circumstances it is, as we'll see later. We'll also see an example where this is not.) 2. Hartshorne gives a different definition; I'm following the more general definition of Grothendieck. But again, these definitions turn out to be the same in nice circumstances.

This is the "relative version" of $\mathrm{Proj}\, S_* \to \mathrm{Spec}\, A$.

3.9. *Exercise.* Show that closed immersions are projective morphisms. (Hint: Suppose the closed immersion $X \to Y$ corresponds to $\mathcal{O}_Y \to \mathcal{O}_X$. Consider $\mathcal{S}_0 = \mathcal{O}_X$, $\mathcal{S}_i = \mathcal{O}_Y$ for $i > 1$.)

3.10. *Exercise (suggested by Kirsten).* Suppose $f : X \hookrightarrow \mathbb{P}^n_S$ where S is some scheme. Show that the structure morphism $\pi : X \to S$ is a projective morphism as follows: let $\mathcal{L} = f^* \mathcal{O}_{\mathbb{P}^n_S}(1)$, and show that $X = \underline{\mathrm{Proj}}\, \pi_* \mathcal{L}^{\otimes n}$.

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FOUNDATIONS OF ALGEBRAIC GEOMETRY CLASS 25

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CONTENTS

Last day: Normalization (in a finite field extension), "sheaf Spec", "sheaf Proj", projective morphisms.

Today: separatedness, definition of variety.

0.1. Here is a notion I should have introduced earlier: *induced reduced subscheme structure*. Suppose X is a scheme, and S is a *closed subset* of X. Then there is a unique reduced closed subscheme Z of X "supported on S". More precisely, it can be defined by the following universal property: for any morphism from a *reduced* scheme Y to X, whose image lies in S (as a set), this morphism factors through Z uniquely. Over an affine $X = \mathrm{Spec}\, R$, we get $\mathrm{Spec}\, R/I(S)$. (Exercise: verify this.) For example, if S is the entire underlying set of X, we get X^{red}.

1. SEPARATED MORPHISMS

We will now describe a very useful notion, that of morphisms being *separated*. Separatedness is one of the definitions in algebraic geometry (like flatness) that seems initially unmotivated, but later turns out to be the answer to a large number of desiderata.

Here are some initial reasons. First, in some sense it is the analogue of Hausdorff. A better description is the following: if you take the definition I'm about to give you and apply it to the "usual" topology, you'll get a correct (if unusual) definition of Hausdorffness. The reason this doesn't give Hausdorffness in the category of schemes is because the topology on the product is not the product topology. (An earlier exercise was to show that $\mathbb{A}^2_k$ does not have the product topology on $\mathbb{A}^1_k \times_k \mathbb{A}^1_k$.) One benefit of this definition is that we will be finally ready to define a *variety*, in a way that corresponds to the classical definition.

Date: Tuesday, January 24, 2006. Small update June 20.

Second, a separated morphism has the property that the intersection of a two affine open sets is affine, which is precisely the odd hypothesis needed to make Cech cohomology work.

A third motivation is that nasty line with doubled origin, which is a counterexample to many statements one might hope are true. The line with double origin is not separated, and by adding a separatedness hypothesis, the desired statements turn out to be true.

A fourth motivation is to give a good foundation for the notion of rational maps, which we will discuss shortly.

A lesson arising from the construction is the importance of the diagonal morphism. More precisely given a morphism $X \to Y$, nice consequences can be leveraged from good behavior of the diagonal morphism $\delta : X \to X \times_Y X$, usually through fun diagram chases. This is a lesson that applies across many fields of mathematics. (Another nice gift the diagonal morphism: it will soon give us a good algebraic definition of differentials.)

1.1. *Proposition. — Let* $X \to Y$ *be a morphism of schemes. Then the diagonal morphism* $\delta : X \to X \times_Y X$ *is a locally closed immersion.*

This locally closed subscheme of $X \times_Y X$ (the diagonal) will be denoted Δ.

Proof. We will describe a union of open subsets of $X \times_Y X$ covering the image of X, such that the image of X is a closed immersion in this union.

1.2. Say Y is covered with affine opens V_i and X is covered with affine opens U_{ij}, with $\pi : U_{ij} \to V_i$. Then the diagonal is covered by $U_{ij} \times_{V_i} U_{ij}$. (Any point $p \in X$ lies in some U_{ij}; then $\delta(p) \in U_{ij} \times_{V_i} U_{ij}$.) Note that $\delta^{-1}(U_{ij} \times_{V_i} U_{ij}) = U_{ij}$: $U_{ij} \times_{V_i} U_{ij} \cong U_{ij} \times_Y U_{ij}$ because $V_i \hookrightarrow Y$ is a monomorphism. Then because open immersions behave well with respect to base change, we have the fiber diagram

$$\begin{array}{ccc} U_{ij} & \longrightarrow & X \\ \downarrow & & \downarrow \\ U_{ij} \times_Y X & \longrightarrow & X \times_Y X \end{array}$$

from which $\delta^{-1}(U_{ij} \times_Y X) = U_{ij}$. As $\delta^{-1}(U_{ij} \times_Y U_{ij})$ contains U_{ij}, we must have $\delta^{-1}(U_{ij} \times_Y U_{ij}) = U_{ij}$.

Finally, we'll check that $U_{ij} \to U_{ij} \times_{V_i} U_{ij}$ is a closed immersion. Say $V_i = \operatorname{Spec} S$ and $U_{ij} = \operatorname{Spec} R$. Then this corresponds to the natural ring map $R \times_S R \to R$, which is obviously surjective. ☐

(A picture is helpful here.)

Note that the open subsets we described may not cover $X \times_Y X$, so we have not shown that δ is a closed immersion.

1.3. *Definition.* A morphism $X \to Y$ is said to be **separated** if the diagonal morphism $\delta : X \to X \times_Y X$ is a closed immersion. If R is a ring, an R-scheme X is said to be *separated over* R if the structure morphism $X \to \operatorname{Spec} R$ is separated. When people say that a scheme (rather than a morphism) X is separated, they mean implicitly that some morphism is separated. For example, if they are talking about R-schemes, they mean that X is separated over R.

Thanks to Proposition 1.1, a morphism is separated if and only if the image of the diagonal morphism is closed.

1.4. *Important easy exercise.* Show that open immersions and closed immersions are separated. (Answer: Show that monomorphisms are separated. Open and closed immersions are monomorphisms, by earlier exercises. Alternatively, show this by hand.)

1.5. *Important easy exercise.* Show that every morphism of affine schemes is separated. (Hint: this was essentially done in Proposition 1.1.)

I'll now give you an example of something separated that is not affine. The following single calculation will eventually easily imply that all quasiprojective morphisms are separated.

1.6. *Proposition.* — $\mathbb{P}^n_{\mathbb{Z}} \to \operatorname{Spec} \mathbb{Z}$ *is separated.*

(The identical argument holds with $\mathbb{Z}$ replaced by any ring.)

Proof. We cover $\mathbb{P}^n_{\mathbb{Z}} \times_{\mathbb{Z}} \mathbb{P}^n_{\mathbb{Z}}$ with open sets of the form $U_i \times U_j$, where $U_0, \ldots, U_n$ form the "usual" affine open cover. The case $i = j$ was taken care of before, in the proof of Proposition 1.1. For $i \neq j$, we may take $i = 0$, $j = n$. Then

$$U_0 \times_{\mathbb{Z}} U_n \cong \operatorname{Spec} \mathbb{Z}[x_{1/0}, \ldots, x_{n/0}, y_{0/n}, \ldots, y_{n-1/n}],$$

and the image of the diagonal morphism meets this open set in the closed subscheme $y_{0/n} x_{n/0} = 1$, $x_{i/0} = x_{n/0} y_{i/n}$, $y_{j/n} = y_{0/n} x_{j/0}$. □

1.7. *Exercise.* Verify the last sentence of the proof. Note that you should check that the diagonal morphism restricted to this open set has source $U_0 \cap U_n$; see §1.2.

1.8. *Exercise.* Show that the line with doubled origin X is not separated, by verifying that the image of the diagonal morphism is not closed. (Another argument is given below, in Exercise 1.28.)

We finally define then notion of variety!

1.9. *Definition.* A **variety** over a field k is defined to be a reduced, separated scheme of finite type over k. We may use the language k-variety.

Example: a reduced finite type affine k-scheme is a variety. In other words, to check if $\operatorname{Spec} k[x_1, \ldots, x_n]/(f_1, \ldots, f_r)$ is a variety, you need only check reducedness.

Notational caution: In some sources (including, I think, Mumford), the additional condition of irreducibility is imposed. We will not do this. Also, it is often assumed that k is algebraically closed. We will not do this either.

Here is a very handy consequence of separatedness!

1.10. *Proposition. — Suppose* $X \to \operatorname{Spec} R$ *is a separated morphism to an affine scheme, and* U *and* V *are affine open sets of* X*. Then* $U \cap V$ *is an affine open subset of* X*.*

We'll prove this shortly.

Consequence: if $X = \operatorname{Spec} A$, then the intersection of any two affine opens is open (just take $R = \mathbb{Z}$ in the above proposition). This is certainly not an obvious fact! We know that the intersection of any two distinguished affine open sets is affine (from $D(f) \cap D(g) = D(fg)$), but we have very little handle on affine open sets in general.

Warning: this property does not characterize separatedness. For example, if $R = \operatorname{Spec} k$ and X is the line with doubled origin over k, then X also has this property. This will be generalized slightly in Exercise 1.31.

Proof. Note that $(U \times_{\operatorname{Spec} R} V) \cap \Delta = U \cap V$, where Δ is the diagonal. (This is clearest with a figure. See also §1.2.)

$U \times_{\operatorname{Spec} R} V$ is affine ($\operatorname{Spec} S \times_{\operatorname{Spec} R} \operatorname{Spec} T = \operatorname{Spec} S \otimes_R T$), and Δ is a closed subscheme of an affine scheme, and hence affine. □

1.11. Sample application: The graph morphism.

1.12. *Definition.* Suppose $f : X \to Y$ is a morphism of Z-schemes. The morphism $\Gamma : X \to X \times_Z Y$ given by $\Gamma = (\mathrm{id}, f)$ is called the **graph morphism**.

1.13. *Proposition. — Show that* Γ *is a locally closed immersion. Show that if* Y *is a separated* Z*-scheme (i.e. the structure morphism* $Y \to Z$ *is separated), then* Γ *is a closed immersion.*

This will be generalized in Exercise 1.29.

Proof by diagram.

$$\begin{array}{ccc} X & \longrightarrow & X \times_Z Y \\ \downarrow & & \downarrow \\ Y & \xrightarrow{\delta} & Y \times_Z Y \end{array}$$

□

1.14. Quasiseparated morphisms.

We now define a handy relative of separatedness, that is also given in terms of a property of the diagonal morphism, and has similar properties. The reason it is less famous is because it automatically holds for the sorts of schemes that people usually deal with. We say a morphism $f : X \to Y$ is **quasiseparated** if the diagonal morphism $\delta : X \to X \times_Y X$ is quasicompact. I'll give a more insightful translation shortly, in Exercise 1.15.

Most algebraic geometers will only see quasiseparated morphisms, so this may be considered a very weak assumption. Here are two large classes of morphisms that are quasiseparated. (a) As closed immersions are quasicompact (not hard), separated implies quasiseparated. (b) If X is a Noetherian scheme, then any morphism to another scheme is quasicompact (not hard; *Exercise*), so any $X \to Y$ is quasiseparated. Hence those working in the category of Noetherian schemes need never worry about this issue.

It is the following characterization which makes quasiseparatedness a useful hypothesis in proving theorems.

1.15. *Exercise.* Show that $f : X \to Y$ is quasiseparated if and only if for any affine open $\operatorname{Spec} R$ of Y, and two affine open subsets U and V of X mapping to $\operatorname{Spec} R$, $U \cap V$ is a *finite* union of affine open sets.

1.16. *Exercise.* Here is an example of a nonquasiseparated scheme. Let $X = \operatorname{Spec} k[x_1, x_2, \dots]$, and let U be $X - \mathfrak{m}$ where $\mathfrak{m}$ is the maximal ideal $(x_1, x_2, \dots)$. Take two copies of X, glued along U. Show that the result is not quasiseparated.

In particular, the condition of quasiseparatedness is often paired with quasicompactness in hypotheses of theorems. A morphism $f : X \to Y$ is quasicompact and quasiseparated if and only if the preimage of any affine open subset of Y is a *finite* union of affine open sets in X, whose pairwise intersections are all *also* finite unions of affine open sets.

This strong finiteness assumption can be very useful, as the following result shows:

1.17. *Proposition. — If $X \to Y$ is a quasicompact, quasiseparated morphism, and $\mathcal{F}$ is quasicoherent sheaf on X, show that $f_*\mathcal{F}$ is a quasicoherent sheaf on Y.*

Proof. The proof we gave earlier (Theorem 2.2 of Class 20) applies without change. We just didn't have the name "quasiseparated" to attach to these hypothesis. □

1.18. *Theorem. — Both separatedness and quasiseparatedness are preserved by base change.*

Proof. Suppose

$$\begin{array}{ccc} W & \longrightarrow & X \\ \downarrow & & \downarrow \\ Y & \longrightarrow & Z \end{array}$$

is a fiber square. We will show that if $Y \to Z$ is separated or quasiseparated, then so is $W \to X$. The reader should verify (using only category theory!) that

$$\begin{CD} W @>{\delta_W}>> W \times_X W \\ @VVV @VVV \\ Y @>{\delta_Y}>> Y \times_Z Y \end{CD}$$

is a fiber diagram. As the property of being a closed immersion is preserved by base change (shown earlier when we showed many properties are well behaved under base change) , if δ_Y is a closed immersion, so is δ_X.

Quasiseparatedness follows in the identical manner, as quasicompactness is also preserved by base change. □

1.19. *Proposition. — The condition of being separated is local on the target. Precisely, a morphism* $f : X \to Y$ *is separated if and only if for any cover of* Y *by open subsets* U_i, $f^{-1}(U_i) \to U_i$ *is separated for each* i.

Hence affine morphisms are separated, by Exercise 1.5. (Thus finite morphisms are separated.)

Proof. If $X \to Y$ is separated, then for any $U_i \hookrightarrow Y$, $f^{-1}(U_i) \to U_i$ is separated by Theorem 1.18. Conversely, to check if $\Delta \hookrightarrow X \times_Y X$ is a closed subset, it suffices to check this on an open cover. If $g : X \times_Y X \to Y$ is the natural morphism, our open cover U_i of Y induces an open cover $g^{-1}(U_i)$ of $X \times_Y X$. □

1.20. *Exercise.* Prove that the condition of being quasiseparated is local on the target. (Hint: the condition of being quasicompact is local on the target by an earlier exercise; use a similar argument.)

1.21. *Proposition. — The condition of being separated is closed under composition. In other words, if* $f : X \to Y$ *is separated and* $g : Y \to Z$ *is separated, then* $g \circ f : X \to Z$ *is separated.*

Proof. This is a good excuse to show you a very useful fiber diagram:

$$\begin{CD} U \times_X V @>>> U \times_S V \\ @VVV @VVV \\ X @>>> X \times_S X \end{CD}$$

We are given that $a : X \hookrightarrow X \times_Y X$ and $b : Y \to Y \times_Z Y$ are closed immersions, and we wish to show that $X \to X \times_Z X$ is a closed immersion. Consider the diagram

$$\begin{array}{ccccc} X & \xrightarrow{a} & X \times_Y X & \xrightarrow{c} & X \times_Z X \\ & & \downarrow & & \downarrow \\ & & Y & \xrightarrow{b} & Y \times_Z Y. \end{array}$$

The square on the right is a fiber diagram (see the very useful diagram above). As b is a closed immersion, c is too (closed immersions behave well under fiber diagrams). Thus $c \circ a$ is a closed immersion (the composition of two closed immersions is also a closed immersion). □

The identical argument (with "closed immersion" replaced by "quasicompact") shows that the condition of being quasiseparated is closed under composition.

1.22. *Proposition. — Any quasiprojective morphism is separated.*

As a corollary, any reduced quasiprojective k-scheme is a k-variety.

Proof. Open immersions are separated by Exercise 1.4. Hence by Proposition 1.21, it suffices to check that projective morphisms are separated. We can check that this locally on the target by Proposition 1.19, so it suffices to check that $f : X \to Z$ where f factors through $\mathbb{P}^n_Z$, and $X \hookrightarrow \mathbb{P}^n_Z$ is a closed immersion. But closed immersions are separated, so $X \hookrightarrow \mathbb{P}^n_Z$ is separated, so it suffices to check $\mathbb{P}^n_Z \to Z$ is separated. But this is obtained by base change from $\mathbb{P}^n_{\mathbb{Z}} \to \operatorname{Spec} \mathbb{Z}$, so we are done (as this latter morphism is separated by the previous proposition, and separatedness is preserved by base change by Proposition 1.18). □

1.23. *Proposition. — Suppose $f : X \to Y$ and $f' : X' \to Y'$ are separated morphisms of S-schemes. Then the product morphism $f \times f' : X \times_S X' \to Y \times_S Y'$ is separated.*

Proof. Consider the following diagram, and use the fact that separatedness is preserved under base change and composition.

$$\begin{array}{ccccccc} & X \times_S X' & \longrightarrow & X \times_S Y' & \longrightarrow & Y \times_S Y' & \\ \swarrow & & \swarrow & & \searrow & & \searrow \\ X' & \longrightarrow & Y' & & X & \longrightarrow & Y \end{array}$$

□

1.24. A very fun result.

We now come to a very useful, but bizarre-looking, result.

1.25. *Proposition. — Let* P *be a class of morphisms that is preserved by base change and composition. Suppose*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & {\scriptstyle h}\searrow \quad \swarrow{\scriptstyle g} & \\ & Z & \end{array}$$

is a commuting diagram of schemes.

(a) *Suppose that the diagonal morphism* $\delta_g : Y \to Y \times_Z Y$ *is in* P *and* $h : X \to Z$ *is in* P*. The* $f : X \to Y$ *is in* P*.*
(b) *In particular, if closed immersions are in* P*, then if* h *is in* P *and* g *is separated, then* f *is in* P*.*

I like this because when you plug in different P, you get very different-looking (and non-obvious) consequences.

Here are some examples.

Locally closed immersions are separated, so part (a) applies, and the first clause always applies. In other words, if you factor a locally closed immersion $X \to Z$ into $X \to Y \to Z$, then $X \to Y$ must be a locally closed immersion.

A morphism (over $\operatorname{Spec} k$) from a projective k-scheme to a separated k-scheme is always projective.

Possibilities for P in case (b) include: finite morphisms, morphisms of finite type, projective morphisms (needed exercise: closed immersions are projective), closed immersions, affine morphisms.

Proof. By the fibered square

$$\begin{array}{ccc} X & \xrightarrow{\Gamma} & X \times_Z Y \\ \downarrow & & \downarrow \\ Y & \xrightarrow{\delta_Y} & Y \times_Z Y \end{array}$$

we see that the graph morphism $\Gamma : X \to X \times_Z Y$ is in P (Definition 1.12), as P is closed under base change. By the fibered square

$$\begin{array}{ccc} X \times_Z Y & \xrightarrow{h'} & Y \\ \downarrow & & \downarrow{\scriptstyle g} \\ X & \xrightarrow{h} & Z \end{array}$$

the projection $h' : X \times_Z Y \to Y$ is in P as well. Thus $f = h' \circ \Gamma$ is in P □

1.26. *Exercise.* Show that a k-scheme is separated (over k) iff it is separated over $\mathbb{Z}$.

Here now are some fun and useful exercises.

1.27. *Useful exercise: The locus where two morphisms agree.* We can now make sense of the following statement. Suppose

$$f, g : \quad \begin{array}{ccc} X & \longrightarrow & Y \\ & \searrow \quad \swarrow & \\ & Z & \end{array}$$

are two morphisms over Z. Then the locus on X where f and g agree is a locally closed subscheme of X. If $Y \to Z$ is separated, then the locus is a closed subscheme of X. More precisely, define V to be the following fibered product:

$$\begin{array}{ccc} V & \longrightarrow & Y \\ \downarrow & & \downarrow \delta \\ X & \xrightarrow{(f,g)} & Y \times_Z Y. \end{array}$$

As δ is a locally closed immersion, $V \to X$ is too. Then if $h : W \to X$ is any scheme such that $g \circ h = f \circ h$, then h factors through V. (Put differently: we are describing $V \hookrightarrow X$ by way of a universal property. Taking this as the definition, it is not a priori clear that V is a locally closed subscheme of X, or even that it exists.) Now we come to the exercise: prove this (the sentence before the parentheses). (Hint: we get a map $g \circ h = f \circ h : W \to Y$. Use the definition of fibered product to get $W \to V$.)

1.28. *Exercise.* Show that the line with doubled origin X is not separated, by finding two morphisms $f_1, f_2 : W \to X$ whose domain of agreement is not a closed subscheme (cf. Proposition 1.1). (Another argument was given above, in Exercise 1.8.)

1.29. *Exercise.* Suppose $\pi : Y \to X$ is a morphism, and $s : X \to Y$ is a *section* of a morphism, i.e. $\pi \circ s$ is the identity on X. Show that s is a locally closed immersion. Show that if π is separated, then s is a closed immersion. (This generalizes Proposition 1.13.)

1.30. *Less important exercise.* Suppose P is a class of morphisms such that closed immersions are in P, and P is closed under fibered product and composition. Show that if $X \to Y$ is in P then $X^{red} \to Y^{red}$ is in P. (Two examples are the classes of separated morphisms and quasiseparated morphisms.) (Hint:

$$\begin{array}{ccccc} X^{red} & \longrightarrow & X \times_Y Y^{red} & \longrightarrow & Y^{red} \\ & \searrow & \downarrow & & \downarrow \\ & & X & \longrightarrow & Y \end{array}$$

)

1.31. *Exercise.* Suppose $\pi : X \to Y$ is a morphism or a ring R, Y is a separated R-scheme, U is an affine open subset of X, and V is an affine open subset of Y. Show that $U \cap \pi^{-1} V$ is an affine open subset of X. (Hint: this generalizes Proposition 1.9 of the Class 25 notes. Use Proposition 1.12 or 1.13.) This will be used in the proof of the Leray spectral sequence.

2. Valuative criteria for separatedness

Describe fact that some people love. It can be useful. I've never used it. But it gives good intuition.

It is possible to verify separatedness by checking only maps from valuations rings.

We begin with a valuative criterion that applies in a case that will suffice for the interests of most people, that of finite type morphisms of Noetherian schemes. We'll then give a more general version for more general readers.

2.1. *Theorem (Valuative criterion for separatedness for morphisms of finite type of Noetherian schemes). — Suppose* $f : X \to Y$ *is a morphism of finite type of Noetherian schemes. Then* f *is separated if and only if the following condition holds. For any discrete valuation ring* R *with function field* K*, and any diagram of the form*

(1)
$$\begin{array}{ccc} \operatorname{Spec} K & \longrightarrow & X \\ \downarrow & & \downarrow f \\ \operatorname{Spec} R & \longrightarrow & Y \end{array}$$

(where the vertical morphism on the left corresponds to the inclusion $R \hookrightarrow K$*), there is at most one morphism* $\operatorname{Spec} R \to X$ *such that the diagram*

(2)
$$\begin{array}{ccc} \operatorname{Spec} K & \longrightarrow & X \\ \downarrow & \nearrow & \downarrow f \\ \operatorname{Spec} R & \longrightarrow & Y \end{array}$$

commutes.

A useful thing to take away from this statement is the intuition behind it. We think of $\operatorname{Spec} R$ as a "germ of a curve", and $\operatorname{Spec} K$ as the "germ minus the origin". Then this says that if we have a map from a germ of a curve to Y, and have a lift of the map away from the origin to X, then there is at most one way to lift the map from the entire germ. (A picture is helpful here.)

For example, this captures the idea of what is wrong with the map of the line with the doubled origin over k: we take $\operatorname{Spec} R$ to be the germ of the affine line at the origin, and consider the map of the germ minus the origin to the line with doubled origin. Then we have two choices for how the map can extend over the origin.

2.2. *Exercise.* Make this precise: show that the line with the doubled origin fails the valuative criterion for separatedness.

Proof. (This proof is more telegraphic than I'd like. I may fill it out more later. Because we won't be using this result later in the course, you should feel free to skip it, but you may want to skim it.) One direction is fairly straightforward. Suppose $f : X \to Y$ is separated, and such a diagram (1) were given. suppose g_1 and g_2 were two morphisms

$\operatorname{Spec} R \to X$ making (2) commute. Then $g = (g_1, g_2) : \operatorname{Spec} R \to X \times_Y X$ is a morphism, with $g(\operatorname{Spec} K)$ contained in the diagonal. Hence as $\operatorname{Spec} K$ is dense in $\operatorname{Spec} R$, and g is continuous, $g(\operatorname{Spec} R)$ is contained in the closure of the diagonal. As the diagonal is closed (the separated hypotheses), $g(\operatorname{Spec} R)$ is also contained *set-theoretically* in the diagonal. As $\operatorname{Spec} R$ is reduced, g factors through the reduced induced subscheme structure (§0.1) of the diagonal. Hence g factors through the diagonal:

$$\operatorname{Spec} R \longrightarrow X \xrightarrow{\delta} X \times_Y X,$$

which means $g_1 = g_2$ by Exercise 1.27.

Suppose conversely that f is not separated, i.e. that the diagonal $\Delta \subset X \times_Y X$ is not closed. As $X \times_Y X$ is Noetherian (X is Noetherian, and $X \times_Y X \to X$ is finite type as it is obtained by base change from the finite type $X \to Y$) we have a well-defined notion of dimension of all irreducible closed subsets, and it is bounded. Let P be a point in $\overline{\Delta} - \Delta$ of largest dimension. Let Q be a point in Δ such that $P \in \overline{Q}$. (A picture is handy here.) Let Z be the scheme obtained by giving the reduced induced subscheme structure to $\overline{Q}$. Then P is a codimension 1 point on Z; let $R' = \mathcal{O}_{Z,P}$ be the local ring of Z at P. Then R' is a Noetherian local domain of dimension 1. Let R'' be the normalization of R. Choose any point P'' of $\operatorname{Spec} R''$ mapping to P; such a point exists because the normalization morphism $\operatorname{Spec} R \to \operatorname{Spec} R'$ is surjective (normalization is an integral extension, hence surjective by the Going-up theorem, lecture 21 theorem 1.5). Let R be the localization of R'' at P''. Then R is a normal Noetherian local domain of dimension 1, and hence a discrete valuation ring. Let K be its fraction field. Then $\operatorname{Spec} R \to X \times_Y X$ does not factor through the diagonal, but $\operatorname{Spec} K \to X \times_Y X$ does, and we are done. □

Here is a more general statement. I won't give a proof here, but I think the proof given in Hartshorne Theorem II.4.3 applies (even though the hypotheses are more restrictive).

2.3. *Theorem (Valuative criterion of separatedness). — Suppose $f : X \to Y$ is a quasicompact, quasiseparated morphism. Then f is separated if and only if the following condition holds. For any valuation ring R with function field K, and or any diagram of the form* (1), *there is at most one morphism* $\operatorname{Spec} R \to X$ *such that the diagram* (2) *commutes.*

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FOUNDATIONS OF ALGEBRAIC GEOMETRY CLASS 26

RAVI VAKIL

CONTENTS

Last day: separatedness, definition of variety.

Today: proper morphisms.

I said a little more about separatedness of moduli spaces, for those familiar such objects. Suppose we are interested in a moduli space of a certain kind of object. That means that there is a scheme M with a "universal family" of such objects over M, such that there is a bijection between families of such objects over an arbitrary scheme S, and morphisms $S \to B$. (One direction of this map is as follows: given a morphism $S \to B$, we get a family of objects over S by pulling back the universal family over B.) The separatedness of the moduli space (over the base field, for example, if there is one) can be interpreted as follows. Fix a valuation ring A (or even discrete valuation ring, if our moduli space of of finite type) with fraction field K. We interpret Spec intuitively as a germ of a curve, and we interpret $\operatorname{Spec} K$ as the germ minus the "origin" (an analogue of a small punctured disk). Then we have a family of objects over $\operatorname{Spec} K$ (or over the punctured disk), or equivalently a map $\operatorname{Spec} K \to M$, and the moduli space is separated if there is *at most one way* to fill in the family over the origin, i.e. a family over $\operatorname{Spec} A$.

1. PROPER MORPHISMS

I'll now tell you about a new property of morphisms, the notion of *properness*. You can think about this in several ways.

Recall that a map of topological spaces (also known as a continuous map!) is said to be proper if the preimage of compact sets is compact. Properness of morphisms is an analogous property. For example, proper varieties over $\mathbb{C}$ will be the same as compact in the "usual" topology.

Alternatively, we will see that projective morphisms are proper — this is the hardest thing we will prove — so you can see this as nice property satisfied by projective morphisms, and hence as a generalization of projective morphisms. Indeed, in some sense,

Date: Thursday, January 26, 2006. Minor update May 28.

essentially all interesting properties of projective morphisms that don't explicitly involve $\mathcal{O}(1)$ turn out to be properties of proper morphisms. The key tool in showing such results is Chow's Lemma, which I will state but not prove. Like separatedness, there is a valuative criterion for properness.

Definition. We say a map of topological spaces (i.e. a continuous map) $f : X \to Y$ is *closed* if for each closed subset $S \subset X$, $f(S)$ is also closed. (This is the definition used elsewhere in mathematics.) We say a morphism of schemes is closed if the underlying continuous map is closed. We say that a morphism of schemes $f : X \to Y$ is *universally closed* if for every morphism $g : Z \to Y$, the induced $Z \times_Y X \to Z$ is closed. In other words, a morphism is universally closed if it remains closed under any base change. (A note on terminology: if P is some property of schemes, then a morphism of schemes is said to be "universally P" if it remains P under any base change.)

A morphism $f : X \to Y$ *is* **proper** *if it is separated, finite type, and universally closed.*

As an example: we expect that $\mathbb{A}^1_{\mathbb{C}} \to \operatorname{Spec} \mathbb{C}$ is not proper, because the complex manifold corresponding to $\mathbb{A}^1_{\mathbb{C}}$ is not compact. However, note that this map is separated (it is a map of affine schemes), finite type, and closed. So the "universally" is what matters here. What's the base change that turns this into a non-closed map? Consider $\mathbb{A}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}} \to \mathbb{P}^1_{\mathbb{C}}$.

1.1. *Exercise.* Show that $\mathbb{A}^1_{\mathbb{C}} \to \operatorname{Spec} \mathbb{C}$ is not proper.

Here are some examples of proper maps.

1.2. Closed immersions are proper: they are clearly separated (as affine morphisms are separated). They are finite type. After base change, they remain closed immersions, and closed immersions are always closed.

More generally, finite morphisms are proper: they are separated (as affine), and finite type. The notion of "finite morphism" behaves well under base change, and we have checked that finite morphisms are always closed (I believe in class 21, using the Going-up theorem).

I mentioned that we are going to show that projective morphisms are proper. In fact, finite morphisms are projective (and closed immersions are finite), so the previous two facts will follow from our fancier fact. I should have explained earlier why finite morphisms are projective, but I'll do so now. Suppose $X \to Y$ is a finite morphism, i.e. $X = \underline{\operatorname{Spec}} \mathcal{A}$ where $\mathcal{A}$ is a finite type sheaf of algebras. I will now show that $X = \underline{\operatorname{Proj}} \mathcal{S}_*$, where $\mathcal{S}_*$ is a sheaf of graded algebras, satisfying all of our various conditions: $\mathcal{S}_0 = \mathcal{O}_Y$, $\mathcal{S}_*$ is "locally generated" by $\mathcal{S}_1$ as a $\mathcal{S}_0$-algebra (i.e. this is true over every open affine subset of Y). Given the statement, you might be able to guess what $\mathcal{S}_*$ should be. I must tell you what $\mathcal{S}_n$ is, and how to multiply. Take $\mathcal{S}_n = \mathcal{A}$ for $n > 0$, with the "obvious" map.

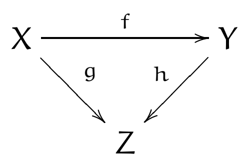
1.3. *Exercise.* Verify that $X = \underline{\operatorname{Proj}} \mathcal{S}_*$. What is $\mathcal{O}_{\underline{\operatorname{Proj}} \mathcal{S}_*}(1)$?

1.4. Properties of proper morphisms.

1.5. *Proposition. —*

(a) *The notion of "proper morphism" is stable under base change.*
(b) *The notion of "proper morphism" is local on the target (i.e.* $f : X \to Y$ *is proper if and only if for any affine open cover* $U_i \to Y$*,* $f^{-1}(U_i) \to U_i$ *is proper). Note that the "only if" direction follows from (a) — consider base change by* $U_i \hookrightarrow Y$*.*
(c) *The notion of "proper morphism" is closed under composition.*
(d) *The product of two proper morphisms is proper (i.e. if* $f : X \to Y$ *and* $g : X' \to Y'$ *are proper, where all morphisms are morphisms of* Z*-schemes) then* $f \times g : X \times_Z X' \to Y \times_Z Y'$ *is proper.*
(e) *Suppose*

(1)



is a commutative diagram, and g *is proper, and* h *is separated. Then* f *is proper.*
(f) *(I don't know if this is useful, but I may as well say it anyway.) Suppose* (1) *is a commutative diagram, and* f *is surjective,* g *is proper, and* h *is separated and finite type. Then* h *is proper.*

Proof. (a) We have already shown that the notions of separatedness and finite type are local on the target. The notion of closedness is local on the target, and hence so is the notion of universal closedness.

(b) The notions of separatedness, finite type, and universal closedness are all preserved by fiber product. (Notice that this is why universal closedness is better than closedness — it is automatically preserved by base change!)

(c) The notions of separatedness, finite type, and universal closedness are all preserved by composition.

(d) Both $X \times_Z Y \to X' \times_Z Y$ and $X' \times_Z Y \to X' \times_Z Y'$ are proper, because the notion is preserved by base change (part (b)). Then their composition is also proper (part (c)).

(e) Closed immersions are proper, so we invoke our magic and weird "property P fact" from last day.

(f) *Exercise.* □

We come to the hardest thing I will prove today.

1.6. *Theorem. — Projective morphisms are proper.*

It is not easy to come up with an example of a morphism that is proper but not projective! I'll give an simple example before long of a proper but not projective surface (over a field), once we have the notion of the fact that line bundles on nice families of curves have constant degree. Once we discuss blow-ups, I'll give Hironaka's example of a proper but not projective *nonsingular* threefold over $\mathbb{C}$.

I'll give part of the proof today, and the rest next day (because I thought I had a simplification that I realized this morning didn't work out).

Proof. Suppose $f : X \to Y$ is projective. Because the notion of properness is local on the base, we may assume that Y is affine, say $\operatorname{Spec} A$. Then $X \hookrightarrow \mathbb{P}^n_A$ for some n. As closed immersions are proper (§1.2), and the composition of two proper morphisms is proper, it suffices to prove that $\mathbb{P}^n_A \to \operatorname{Spec} A$ is proper. However, we have shown that projective morphisms are separated (last day), and finite type, so it suffices to show that $\mathbb{P}^n_A \to \operatorname{Spec} A$ is universally closed.

We will next show that it suffices to show that $\mathbb{P}^n_R \to \operatorname{Spec} R$ is closed for all rings R. Indeed, we need to show that given any base change $X \to \operatorname{Spec} A$, the resulting base changed morphisms $\mathbb{P}^n_X \to X$ is closed. But the notion of being "closed" is local on the base, so we can replace X by an affine cover.

Next day I will complete the proof by showing that $\mathbb{P}^n_A \to \operatorname{Spec} A$ is closed. This is sometimes called the fundamental theorem of elimination theory. Here are some examples to show you that this is a bit subtle.

First, let $A = k[a, b, c, \dots, i]$, and consider the closed subscheme of $\mathbb{P}^2_A$ (taken with coordinates x, y, z) corresponding to $ax + by + cz = 0$, $dx + ey + fz = 0$, $gx + hy + iz = 0$. Then we are looking for the locus in $\operatorname{Spec} A$ where these equations have a non-trivial solution. This indeed corresponds to a Zariski-closed set — where

$$\det \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = 0.$$

As a second example, let $A = k[a_0, a_1, \dots, a_m, b_0, b_1, \dots, b_n]$. Now consider the closed subscheme of $\mathbb{P}^1_A$ (taken with coordinates x and y) corresponding to $a_0 x^m + a_1 x^{m-1} y + \cdots + a_m y^m = 0$ and $b_0 x^n + b_1 x^{m-1} y + \cdots + b_n y^n = 0$. Then we are looking at the locus in $\operatorname{Spec} A$ where these two polynomials have a common root — this is known as the *resultant*. □

I'll end my discussion of properness with some results that I'll not prove and not use.

1.7. Miscellaneous facts.

Here are some enlightening facts.

(a) Proper and affine = finite. (b) Proper and quasifinite = finite.

(We'll show all three of this in the case of projective morphisms.)

As an application: quasifinite morphisms from proper schemes to separated schemes are finite. Here is why: suppose $X \to Y$ is a quasifinite morphism over Z, where X is proper over Z. Then by one of our weird "property P" facts (Proposition 1.24(b) in class 25), $X \to Y$ is proper. Hence by (b) above, it is finite.

Here is an explicit example: consider a morphism $\mathbb{P}^1 \to \mathbb{P}^1$ given by two distinct sections of $\mathcal{O}_{\mathbb{P}^1}(2)$. The fibers are finite, hence this is a finite morphism. (This could also be checked directly.)

Here is a third fact: If $\pi : X \to Y$ is proper, and $\mathcal{F}$ is a coherent sheaf on X, then $\pi_* \mathcal{F}$ is coherent.

In particular, if X is proper over k, $H^0(X, \mathcal{F})$ is finite-dimensional. (This is just the special case of the morphism $X \to k$.)

1.8. Valuative criterion.

There is a valuative criterion for properness too. I've never used it personally, but it *is* useful, both directly, and also philosophically. I'll make statements, and then discuss some philosophy.

1.9. *Theorem (Valuative criterion for properness for morphisms of finite type of Noetherian schemes). — Suppose* $f : X \to Y$ *is a morphism of finite type of locally Noetherian schemes. Then* f *is proper if and only if the following condition holds. For any discrete valuation ring* R *with function field* K*, and or any diagram of the form*

(2)
$$\begin{array}{ccc} \operatorname{Spec} K & \longrightarrow & X \\ \big\downarrow & & \big\downarrow f \\ \operatorname{Spec} R & \longrightarrow & Y \end{array}$$

(where the vertical morphism on the left corresponds to the inclusion $R \hookrightarrow K$*), there is* exactly *one morphism* $\operatorname{Spec} R \to X$ *such that the diagram*

(3)
$$\begin{array}{ccc} \operatorname{Spec} K & \longrightarrow & X \\ \big\downarrow & \nearrow & \big\downarrow f \\ \operatorname{Spec} R & \longrightarrow & Y \end{array}$$

commutes.

Recall that the valuative criterion for properness was the same, except that *exact* was replaced by *at most*.

In the case where Y is a field, you can think of this as saying that limits of one-parameters always exist, and are unique.

1.10. *Theorem (Valuative criterion of properness). — Suppose* $f : X \to Y$ *is a quasiseparated, finite type (hence quasicompact) morphism. Then* f *is proper if and only if the following condition* *holds. For any valuation ring* R *with function field* K*, and or any diagram of the form* (2)*, there is* exactly *one morphism* $\operatorname{Spec} R \to X$ *such that the diagram* (3) *commutes.*

Uses: (1) intuition. (2) moduli idea: exactly one way to fill it in (stable curves). (3) motivates the definition of properness for stacks.

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FOUNDATIONS OF ALGEBRAIC GEOMETRY CLASS 27

RAVI VAKIL

CONTENTS

Last day: proper morphisms.

Today: a little more propriety. Rational maps. Curves.

(These notes include some facts discussed in class 28, for the sake of continuity.)

1. PROPER MORPHISMS

Last day we mostly proved:

1.1. *Theorem. — Projective morphisms are proper.*

We had reduced it to the following fact:

1.2. *Proposition. —* $\pi : \mathbb{P}^n_A \to \operatorname{Spec} A$ *is a closed morphism.*

Proof. Suppose $Z \hookrightarrow \mathbb{P}^n_A$ is a closed *subset*. We wish to show that $\pi(Z)$ is closed.

Suppose $y \notin \pi(Z)$ is a *closed* point of $\operatorname{Spec} A$. We'll check that there is a distinguished open neighborhood $D(f)$ of y in $\operatorname{Spec} A$ such that $D(f)$ doesn't meet $\pi(Z)$. (If we could show this for *all* points of $\pi(Z)$, we would be done. But I prefer to concentrate on closed points for now.) Suppose y corresponds to the maximal ideal $\mathfrak{m}$ of A. We seek $f \in A - \mathfrak{m}$ such that $\pi^* f$ vanishes on Z.

A picture helps here, but I haven't put it in the notes.

Date: Tuesday, January 31, 2006. Updated March 8, 2006.

Let $U_0, \dots, U_n$ be the usual affine open cover of $\mathbb{P}^n_A$. The closed subsets $\pi^{-1}y$ and Z do not intersect. On the affine open set U_i, we have two closed subsets that do not intersect, which means that the ideals corresponding to the two open sets generate the unit ideal, so in the ring of functions on U_i, we can write

$$1 = a_i + \sum m_{ij} g_{ij}$$

where $m_{ij} \in \mathfrak{m}$, and a_i vanishes on Z. Note that $a_i, g_{ij} \in A[x_{0/i}, x_{1/i}, \dots, x_{n/i}]$. So by multiplying by a sufficiently high power x_i^n of x_i, we have an equality

$$x_i^N = a_i' + \sum m_{ij} g_{ij}'$$

on U_i, where both sides are expressions in $A[x_0, \dots, x_n]$. We may take N large enough so that it works for all i. Thus for N' sufficiently large, we can write any monomial in $x_1, \dots, x_n$ of degree N' as something vanishing on Z plus a linear combination of elements of $\mathfrak{m}$ times other polynomials. Hence if $S_* = A[x_0, \dots, x_n]$,

$$S_{N'} = I(Z)_{N'} + \mathfrak{m} S_{N'}$$

where $I(Z)_*$ is the graded ideal of functions vanishing on Z. Hence by Nakayama's lemma, there exists $f \in A - \mathfrak{m}$ such that

$$f S_{N'} \subset I(Z)_{N'}.$$

Thus we have found our desired f!

We are now ready to tackle the proposition in general. Suppose $y \in \operatorname{Spec} A$ is no longer necessarily a closed point, and say $y = [\mathfrak{p}]$. Then we apply the same argument in $\operatorname{Spec} A_\mathfrak{p}$. We get $S_{N'} \otimes A_\mathfrak{p} = I(Z)_{N'} \otimes A_\mathfrak{p} + \mathfrak{m} S_{N'} \otimes A_\mathfrak{p}$, from which $g(S_{N'}/I(Z)_{N'}) \otimes A_\mathfrak{p} = 0$ for some $g \in A_\mathfrak{p} - \mathfrak{p}A_\mathfrak{p}$, from which $(S_{N'}/I(Z)_{N'}) \otimes A_\mathfrak{p} = 0$. Now $S_{N'}$ is a finitely generated A-module, so there is some $f \in R - \mathfrak{p}$ with $fS_N \subset I(Z)$ (if the module-generators of $S_{N'}$, and $f_1, \dots, f_a$ are annihilate the generators respectively, then take $f = \prod f_i$), so once again we have found $D(f)$ containing $\mathfrak{p}$, with (the pullback of) f vanishing on Z. □

2. Scheme-theoretic closure, and scheme-theoretic image

Have I defined scheme-theoretic closure of a locally closed subscheme $W \hookrightarrow Y$? I think I have neglected to. It is the smallest closed subscheme of Y containing W. *Exercise.* Show that this notion is well-defined. More generally, if $f : W \to Y$ is any morphism, define the scheme-theoretic image as the smallest closed subscheme $Z \to Y$ so that f factors through $Z \hookrightarrow Y$. *Exercise.* Show that this is well-defined. (One possible hint: use a universal property argument.) If Y is affine, the ideal sheaf corresponds to the functions on Y that are zero when pulled back to Z. Show that the closed set underlying the image subscheme may be strictly larger than the closure of the set-theoretic image: consider $\coprod_{n \geq 0} \operatorname{Spec} k[t]/t^n \to \operatorname{Spec} k[t]$. (I suspect that such a pathology cannot occur for finite type morphisms of Noetherian schemes, but I haven't investigated.)

3. Rational maps

This is a very old topic, near the beginning of any discussion of varieties. It has appeared late for us because we have just learned about separatedness.

For this section, I will suppose that X and Y are integral and separated, although these notions are often useful in more general circumstances. The interested reader should consider the first the case where the schemes in question are reduced and separated (but not necessarily irreducible). Many notions can make sense in more generality (without reducedness hypotheses for example), but I'm not sure if there is a widely accepted definition.

A key example will be irreducible varieties, and the language of rational maps is most often used in this case.

A **rational map** $X \dashrightarrow Y$ is a morphism on a dense open set, with the equivalence relation: $(f : U \to Y) \sim (g : V \to Y)$ if there is a dense open set $Z \subset U \cap V$ such that $f|_Z = g|_Z$. (We will soon see that we can add: if $f|_{U \cap V} = g|_{U \cap V}$.)

An obvious example of a rational map is a morphism. Another example is a rational function, which is a rational map to $\mathbb{A}^1_{\mathbb{Z}}$ (*easy exercise*).

3.1. *Exercise.* Show that you can compose two rational maps $f : X \dashrightarrow Y$, $g : Y \dashrightarrow Z$ if f is dominant.

3.2. *Easy exercise.* Show that dominant rational maps give morphisms of function fields in the opposite direction. (This was problem 37 on problem set 9.)

It is not true that morphisms of function fields give dominant rational maps, or even rational maps. For example, $k[x]$ and $\operatorname{Spec} k(x)$ have the same function field ($k(x)$), but there is no rational map $\operatorname{Spec} k[x] \dashrightarrow k(x)$. Reason: that would correspond to a morphism from an open subset U of $\operatorname{Spec} k[x]$, say $k[x, 1/f(x)]$, to $k(x)$. But there is no map of rings $k(x) \to k[x, 1/f(x)]$ for any one $f(x)$.

However, this is true in the case of varieties (see Proposition 3.4 below).

A rational map $f : X \to Y$ is said to be *birational* if it is dominant, and there is another morphism (a "rational inverse") that is also dominant, such that $f \circ g$ is (in the same equivalence class as) the identity on Y, and $g \circ f$ is (in the same equivalence class as) the identity on X.

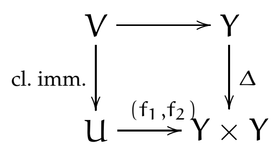
A morphism is **birational** if it is birational as a rational map. We say X and Y are *birational* to each other if there exists a birational map $X \dashrightarrow Y$. This is the same as our definition before. Birational maps induce isomorphisms of function fields.

3.3. *Important Theorem. — Two S-morphisms $f_1, f_2 : U \to Z$ from a reduced scheme to a separated S-scheme agreeing on a dense open subset of U are the same.*

Note that this generalizes the easy direction of the valuative criterion of separatedness (which is the special case where U is Spec of a discrete valuation ring — which consists of two points — and the dense open set is the generic point).

It is useful to see how this breaks down when we give up reducedness of the base or separatedness of the target. For the first, consider the two maps $\mathrm{Spec}\, k[x, y]/(x^2, xy) \to \mathrm{Spec}\, k[t]$, where we take f_1 given by $t \mapsto y$ and f_2 given by $t \mapsto y + x$; f_1 and f_2 agree on the distinguished open set $D(y)$. (A picture helps here!) For the second, consider the two maps from $\mathrm{Spec}\, k[t]$ to the line with the doubled origin, one of which maps to the "upper half", and one of which maps to the "lower half". these to morphisms agree on the dense open set $D(f)$.

Proof.



We have a closed subscheme of U containing the generic point. It must be all of U. □

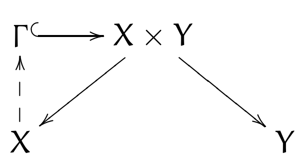
Consequence 1. Hence (as X is reduced and Y is separated) if we have two morphisms from open subsets of X to Y, say $f : U \to Y$ and $g : V \to Y$, and they agree on a dense open subset $Z \subset U \cap V$, then they necessarily agree on $U \cap V$.

Consequence 2. Also: a rational map has a largest *domain of definition* on which $f : U \dashrightarrow Y$ is a morphism, which is the union of all the domains of definition.

In particular, a rational function from a reduced scheme has a largest *domain of definition*.

We define the *graph* of a rational map $f : X \dashrightarrow Y$ as follows: let (U, f') be any representative of this rational map (so $f' : U \to Y$ is a morphism). Let Γ_f be the scheme-theoretic closure of $\Gamma_{f'} \hookrightarrow U \times Y \hookrightarrow X \times Y$, where the first map is a closed immersion, and the second is an open immersion. *Exercise.* Show that this is independent of the choice of U.

Here is a handy diagram involving the graph of a rational map:



(that "up arrow" should be dashed).

We now prove a Proposition promised earlier.

3.4. *Proposition. — Suppose* X, Y *are irreducible varieties, and we are given* $f^\# : FF(Y) \hookrightarrow FF(Y)$. *Then there exists a dominant rational map* $f : X \dashrightarrow Y$ *inducing* $f^\#$.

Proof. By replacing Y with an affine open set, we may assume Y is affine, say $Y = \operatorname{Spec} k[x_1, \dots, x_n]/(f_1, \dots, f_r)$. Then we have $x_1, \dots, x_n \in K(X)$. Let U be an open subset of the domains of definition of these rational functions. Then we get a morphism $U \to \mathbb{A}^n_k$. But this morphism factors through $Y \subset \mathbb{A}^n$, as $x_1, \dots, x_n$ satisfy all the relations $f_1, \dots, f_r$. $\square$

3.5. *Exercise.* Let K be a finitely generated field extension of transcendence degree m over k. Show there exists an irreducible k-variety W with function field K. (Hint: let $x_1, \dots, x_n$ be generators for K over k. Consider the map $\operatorname{Spec} K \to \operatorname{Spec} k[t_1, \dots, t_n]$ given by the ring map $t_i \mapsto x_i$. Take the scheme-theoretic closure of the image.)

3.6. *Proposition. — Suppose X and Y are integral and separated (our standard hypotheses from last day). Then X and Y are birational if and only if there is a dense=non-empty open subscheme U of X and a dense=non-empty open subscheme V of Y such that $U \cong Y$.*

This gives you a good idea of how to think of birational maps.

3.7. *Exercise.* Prove this. (Feel free to consult Iitaka or Hartshorne (Corollary I.4.5).)

4. Examples of rational maps

We now give a bunch of examples Here are some examples of rational maps, and birational maps. A recurring theme is that domains of definition of rational maps to projective schemes extend over nonsingular codimension one points. We'll make this precise when we discuss curves shortly.

(A picture is helpful here.) The first example is how you find a formula for Pythagorean triples. Suppose you are looking for rational points on the circle C given by $x^2 + y^2 = 1$. One rational point is $p = (1, 0)$. If q is another rational point, then pq is a line of rational (non-infinite) slope. This gives a rational map from the conic to $\mathbb{A}^1$. Conversely, given a line of slope m through p, where m is rational, we can recover q as follows: $y = m(x-1)$, $x^2 + y^2 = 1$. We substitute the first equation into the second, to get a quadratic equation in x. We know that we will have a solution $x = 1$ (because the line meets the circle at $(x, y) = (1, 0)$), so we expect to be able to factor this out, and find the other factor. This indeed works:

$$\begin{aligned} x^2 + (m(x-1))^2 &= 1 \\ (m^2+1)x^2 + (-2)x + (m^2 - 1) &= 0 \\ (x-1)((m^2+1)x - (m^2-1)) &= 0 \end{aligned}$$

The other solution is $x = (m^2 - 1)/(m^2 + 1)$, which gives $y = 2m/(m^2 + 1)$. Thus we get a birational map between the conic C and $\mathbb{A}^1$ with coordinate m, given by $f : (x, y) \mapsto y/(x - 1)$ (which is defined for $x \neq 1$), and with inverse rational map given by $m \mapsto ((m^2 - 1)/(m^2 + 1), 2m/(m^2 + 1))$ (which is defined away from $m^2 + 1 = 0$).

We can extend this to a rational map $C \dashrightarrow \mathbb{P}^1$ via the inclusion $\mathbb{A}^1 \to \mathbb{P}^1$. Then f is given by $(x, y) \mapsto [y; x-1]$. (Remember that we give maps to projective space by giving sections of line bundles — in this case, we are using the structure sheaf.) We then have an interesting question: what is the domain of definition of f? It appears to be defined everywhere except for where $y = x - 1 = 0$, i.e. everywhere but p. But in fact it can be extended over p! Note that $(x, y) \mapsto [x+1; -y]$ (where $(x, y) \neq (-1, y)$) agrees with f on their common domains of definition, as $[x+1; -y] = [y; x-1]$. Hence this rational map can be extended farther than we at first thought. This will be a special case of a result we'll see later today.

(For the curious: we are working with schemes over $\mathbb{Q}$. But this works for any scheme over a field of characteristic not 2. What goes wrong in characteristic 2?)

4.1. *Exercise.* Use the above to find a "formula" for all Pythagorean triples.

4.2. *Exercise.* Show that the conic $x^2 + y^2 = z^2$ in $\mathbb{P}^2_k$ is isomorphic to $\mathbb{P}^1_k$ for any field k of characteristic not 2. (Presumably this is true for any ring in which 2 is invertible too...)

In fact, any conic in $\mathbb{P}^2_k$ with a k-valued point (i.e. a point with residue field k) is isomorphic to $\mathbb{P}^1_k$. (This hypothesis is certainly necessary, as $\mathbb{P}^1_k$ certainly has k-valued points. $x^2 + y^2 + z^2 = 0$ over $k = \mathbb{R}$ gives an example of a conic that is not isomorphic to $\mathbb{P}^1_k$.)

4.3. *Exercise.* Find all rational solutions to $y^2 = x^3 + x^2$, by finding a birational map to $\mathbb{A}^1$, mimicking what worked with the conic.

You will obtain a rational map to $\mathbb{P}^1$ that is not defined over the node $x = y = 0$, and *can't* be extended over this codimension 1 set. This is an example of the limits of our future result showing how to extend rational maps to projective space over codimension 1 sets: the codimension 1 sets have to be nonsingular. More on this soon!

4.4. *Exercise.* Use something similar to find a birational map from the quadric $Q = \{x^2 + y^2 = w^2 + z^2\}$ to $\mathbb{P}^2$. Use this to find all rational points on Q. (This illustrates a good way of solving Diophantine equations. You will find a dense open subset of Q that is isomorphic to a dense open subset of $\mathbb{P}^2$, where you can easily find all the rational points. There will be a closed subset of Q where the rational map is not defined, or not an isomorphism, but you can deal with this subset in an ad hoc fashion.)

4.5. *Exercise (a first view of a blow-up).* Let k be an algebraically closed field. (We make this hypothesis in order to not need any fancy facts on nonsingularity.) Consider the rational map $\mathbb{A}^2_k \dashrightarrow \mathbb{P}^1_k$ given by $(x, y) \mapsto [x; y]$. I think you have shown earlier that this rational map cannot be extended over the origin. Consider the graph of the birational map, which we denote $\mathrm{Bl}_{(0,0)} \mathbb{A}^2_k$. It is a subscheme of $\mathbb{A}^2_k \times \mathbb{P}^1_k$. Show that if the coordinates on $\mathbb{A}^2$ are x, y, and the coordinates on $\mathbb{P}^1$ are u, v, this subscheme is cut out in $\mathbb{A}^2 \times \mathbb{P}^1$ by the single equation $xv = yu$. Show that $\mathrm{Bl}_{(0,0)} \mathbb{A}^2_k$ is nonsingular. Describe the fiber of the morphism $\mathrm{Bl}_{(0,0)} \mathbb{A}^2_k \to \mathbb{P}^1_k$ over each closed point of $\mathbb{P}^1_k$. Describe the fiber of the morphism

$\mathrm{Bl}_{(0,0)} \mathbb{A}^2_k \to \mathbb{A}^2_k$. Show that the fiber over $(0,0)$ is an effective Cartier divisor. It is called the *exceptional divisor*.

4.6. *Exercise (the Cremona transformation, a useful classical construction).* Consider the rational map $\mathbb{P}^2 \dashrightarrow \mathbb{P}^2$, given by $[x;y;z] \to [1/x;1/y;1/z]$. What is the the domain of definition? (It is bigger than the locus where $xyz \neq 0$!) You will observe that you can extend it over codimension 1 sets. This will again foreshadow a result we will soon prove.

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FOUNDATIONS OF ALGEBRAIC GEOMETRY CLASS 28

RAVI VAKIL

CONTENTS

Last day: More on properness. Rational maps.

Today: Curves.

(I also discussed rational maps a touch more, but I've included that in the class 27 notes for the sake of continuity.)

1. CURVES

Let's now use our technology to study something explicit! For our discussion here, we will temporarily define a *curve* to be an integral variety over k of dimension 1. (In particular, curves are reduced, irreducible, separated, and finite type over k.)

I gave an incomplete proof to the following proposition. Because I don't think I'll use it, I haven't tried to patch it. But if there is interest, I'll include the proof with the hole, in case one of you can figure out how to make it work. (We showed that each closed point gives a discrete valuation, and we showed that each discrete valuation gives a morphism from the Spec corresponding discrete valuation ring to the curve, but we didn't show that it was the local ring of the corresponding closed point. I would like to do this without invoking any algebra that we haven't yet proved.)

1.1. *Proposition. — Suppose C is a projective nonsingular curve. Then each closed point of C yields a discrete valuation ring, and hence a discrete valuation on $FF(C)$. This gives a bijection from closed points of C and discrete valuations on $FF(C)$.*

Thus a projective nonsingular curve is a convenient way of seeing all the discrete valuations at once, in a nice geometric package.

I had wanted to ask you the following exercise (for those with arithmetic proclivities), but I won't now: Suppose A is the ring of integers in a number field (i.e. the integral

Date: Thursday, February 2, 2006.

closure of $\mathbb{Z}$ in a finite field extension $K/\mathbb{Q}$ — $K = FF(A)$). Show that there is a natural bijection between discrete valuations on K are in bijection with the maximal ideals of A.

1.2. *Key Proposition.* — *Suppose C is a dimension 1 finite type k-scheme, and p is a nonsingular point of it. Suppose Y is a projective k-scheme. Then any morphism $C - p \to Y$ extends to $C \to Y$.*

Note: if such an extension exists, then it is unique: The non-reduced locus of C is a closed subset (we checked this earlier for any Noetherian scheme), not including p, so by replacing C by an open neighborhood of p that is reduced, we can use our recently-proved theorem that maps from reduced schemes to separated schemes are determined by their behavior on a dense open set (Important Theorem 3.3 in last day's notes).

I'd like to give two proofs, which are enlightening in different ways.

Proof 1. By restricting to an affine neighborhood of C, we can reduce to the case where C is affine.

We next reduce to the case where $Y = \mathbb{P}^n_k$. Here is how. Choose a closed immersion $Y \to \mathbb{P}^n_k$. If the result holds for $\mathbb{P}^n$, and we have a morphism $C \to \mathbb{P}^n$ with $C - p$ mapping to Y, then C must map to Y as well. Reason: we can reduce to the case where the source is an affine open subset, and the target is $\mathbb{A}^n_k \subset \mathbb{P}^n_k$ (and hence affine). Then the functions vanishing on $Y \cap \mathbb{A}^n_k$ pull back to functions that vanish at the generic point of C and hence vanish everywhere on C, i.e. C maps to Y.

Choose a uniformizer $t \in \mathfrak{m} - \mathfrak{m}^2$ in the local ring. By discarding the points of the vanishing set $V(t)$ aside from p, and taking an affine open subset of p in the remainder we reduce to the case where t cuts out precisely $\mathfrak{m}$ (i.e. $\mathfrak{m} = (y)$). Choose a dense open subset U of $C - p$ where the pullback of $\mathcal{O}(1)$ is trivial. Take an affine open neighborhood $\operatorname{Spec} A$ of p in $U \cup \{p\}$. Then the map $\operatorname{Spec} A - p \to \mathbb{P}^n$ corresponds to $n+1$ functions, say $f_0, \dots, f_n \in A_\mathfrak{m}$, not all zero. Let m be the smallest valuation of all the f_i. Then $[t^{-m} f_0; \dots; t^{-m} f_n]$ has all entries in A, and not all in the maximal ideal, and thus is defined at p as well. □

Proof 2. We extend the map $\operatorname{Spec} FF(C) \to Y$ to $\operatorname{Spec} \mathcal{O}_{C,p} \to Y$ as follows. Note that $\mathcal{O}_{C,p}$ is a discrete valuation ring. We show first that there is a morphism $\operatorname{Spec} \mathcal{O}_{C,p} \to \mathbb{P}^n$. The rational map can be described as $[a_0; a_1; \cdots; a_n]$ where $a_i \in \mathcal{O}_{C,p}$. Let m be the minimum valuation of the a_i, and let t be a uniformizer of $\mathcal{O}_{C,p}$ (an element of valuation 1). Then $[t^{-m} a_0; t^{-m} a_1; \dots t^{-m} a_n]$ is another description of the morphism $\operatorname{Spec} FF(\mathcal{O}_{C,p}) \to \mathbb{P}^n$, and each of the entries lie in $\mathcal{O}_{C,p}$, and not all entries lie in $\mathfrak{m}$ (as one of the entries has valuation 0). This same expression gives a morphism $\operatorname{Spec} \mathcal{O}_{C,p} \to \mathbb{P}^n$.

Our intuition now is that we want to glue the maps $\operatorname{Spec} \mathcal{O}_{C,p} \to Y$ (which we picture as a map from a germ of a curve) and $C - p \to Y$ (which we picture as the rest of the curve). Let $\operatorname{Spec} R \subset Y$ be an affine open subset of Y containing the image of $\operatorname{Spec} \mathcal{O}_{C,p}$. Let $\operatorname{Spec} A \subset C$ be an affine open of C containing p, and such that the image of $\operatorname{Spec} A - p$ in Y lies in $\operatorname{Spec} R$, and such that p is cut out scheme-theoretically by a single equation (i.e.

there is an element $t \in A$ such that (t) is the maximal ideal corresponding to p. Then R and A are domains, and we have two maps $R \to A_{(t)}$ (corresponding to $\operatorname{Spec} \mathcal{O}_{C,p} \to \operatorname{Spec} R$) and $R \to A_t$ (corresponding to $\operatorname{Spec} A - p \to \operatorname{Spec} R$) that agree "at the generic point", i.e. that give the same map $R \to FF(A)$. But $A_t \cap A_{(t)} = A$ in $FF(A)$ (e.g. by Hartogs' theorem — elements of the fraction field of A that don't have any poles away from t, nor at t, must lie in A), so we indeed have a map $R \to A$ agreeing with both morphisms. □

1.3. *Exercise (Useful practice!).* Suppose X is a Noetherian k-scheme, and Z is an irreducible codimension 1 subvariety whose generic point is a nonsingular point of X (so the local ring $\mathcal{O}_{X,Z}$ is a discrete valuation ring). Suppose $X \dashrightarrow Y$ is a rational map to a projective k-scheme. Show that the domain of definition of the rational map includes a dense open subset of Z. In other words, rational maps from Noetherian k-schemes to projective k-schemes can be extended over nonsingular codimension 1 sets. We saw this principle in action with the Cremona transformation, in Class 27 Exercise 4.6. (By the easy direction of the valuative criterion of separatedness, or the theorem of uniqueness of extensions of maps from reduced schemes to separated schemes — Theorem 3.3 of Class 27 — this map is unique.)

1.4. *Theorem. — If C is a nonsingular curve, then there is some projective nonsingular curve C' and an open immersion $C \hookrightarrow C'$.*

This proof has a bit of a different flavor than proofs we've seen before. We'll use make particular use of the fact that one-dimensional Noetherian schemes have a boring topology.

Proof. Given a nonsingular curve C, take a non-empty=dense affine open set, and take any non-constant function f on that affine open set to get a rational map $C \dashrightarrow \mathbb{P}^1$ given by $[1; f]$. As a dense open set of a dimension 1 scheme consists of everything but a finite number of points, by Proposition 1.2, this extends to a morphism $C \to \mathbb{P}^1$.

We now take the normalization of $\mathbb{P}^1$ in the function field $FF(C)$ of C (a finite extension of $FF(\mathbb{P}^1)$), to obtain $C' \to \mathbb{P}^1$. Now C' is normal, hence nonsingular (as nonsingular = normal in dimension 1). By the finiteness of integral closure, $C' \to \mathbb{P}^1$ is a finite morphism. Moreover, finite morphisms are projective, so by considering the composition of projective morphisms $C' \to \mathbb{P}^1 \to \operatorname{Spec} k$, we see that C' is projective over k. Thus we have an isomorphism $FF(C) \to FF(C')$, hence a rational map $C \dashrightarrow C'$, which extends to a morphism $C \to C'$ by Key Proposition 1.2.

Finally, I claim that $C \to C'$ is an open immersion. If we can prove this, then we are done. I note first that this is an injection of sets:

- the generic point goes to the generic point
- the closed points of C correspond to distinct valuations on $FF(C)$ (as C is separated, by the easy direction of the valuative criterion of separatedness)

Thus as sets, C is C' minus a finite number of points. As the topology on C and C' is the "cofinite topology" (i.e. the open sets include the empty set, plus everything minus a finite number of closed points), the map $C \to C'$ of topological spaces expresses C as a homeomorphism of C onto its image $\mathrm{im}(C)$. Let $f : C \to \mathrm{im}(C)$ be this morphism of schemes. Then the morphism $\mathcal{O}_{\mathrm{im}(C)} \to f_* \mathcal{O}_C$ can be interpreted as $\mathcal{O}_{\mathrm{im}(C)} \to \mathcal{O}_C$ (where we are identifying C and $\mathrm{im}(C)$ via the homeomorphism f). This morphism of sheaves is an isomorphism of stalks at all points $p \in \mathrm{im}(C)$ (it is the isomorphism the discrete valuation ring corresponding to $p \in C'$), and is hence an isomorphism. Thus $C \to \mathrm{im}(C)$ is an isomorphism of schemes, and thus $C \to C'$ is an open immersion. □

We now come to the big theorem of today (although the Key Proposition 1.2 above was also pretty big).

1.5. *Theorem. — The following categories are equivalent.*

(i) *nonsingular projective curves, and surjective morphisms.*
(ii) *nonsingular projective curves, and dominant morphisms.*
(iii) *nonsingular projective curves, and dominant rational maps*
(iv) *quasiprojective reduced curves, and dominant rational maps*
(v) *function fields of dimension* 1 *over* k*, and* k*-homomorphisms.*

(All morphisms and maps are assumed to be k-morphisms and k-rational maps, i.e. they are all over k. Remember that today we are working in the category of k-schemes.)

This has a lot of implications. For example, each quasiprojective reduced curve is isomorphic to precisely one projective nonsingular curve.

This leads to a motivating question that I mentioned informally last day (and that isn't in the notes). Suppose k is algebraically closed (such as $\mathbb{C}$). Is it true that all nonsingular projective curves are isomorphic to $\mathbb{P}^1_k$? Equivalently, are all quasiprojective reduced curves birational to $\mathbb{A}^1_k$? Equivalently, are all transcendence degree 1 extensions of k generated (as a field) by a single element? The answer (as most of you know) is *no*, but we can't yet see that.

1.6. *Exercise.* Show that all nonsingular proper curves are projective.

(We may eventually see that all reduced proper curves over k are projective, but I'm not sure; this will use the Riemann-Roch theorem, and I may just prove it for projective curves.)

Before we get to the proof, I want to mention a sticky point that came up in class. If $k = \mathbb{R}$, then we are allowing curves such as $\mathbb{P}^1_{\mathbb{C}}$ that "we don't want". One way of making this precise is noting that they are not geometrically irreducible (as $\mathbb{C}(t)_{\otimes_{\mathbb{R}}}\mathbb{C} \cong \mathbb{C}(t) \oplus \mathbb{C}(t)$). Another way is to note that this function field K does not satisfy $\overline{k} \cap K = k$ in $\overline{K}$. If this bothers you, then add it to each of the 5 categories. (For example, in (i)–(iii), we consider only nonsingular projective curves whose function field K satisfies $\overline{k} \cap K = k$ in $\overline{K}$.) If this doesn't bother you, please ignore this paragraph!

Proof. Any surjective morphism is a dominant morphism, and any dominant morphism is a dominant rational map, and each nonsingular projective curve is a quasiprojective curve, so we've shown (i) $\rightarrow$ (ii) $\rightarrow$ (iii) $\rightarrow$ (iv). To get from (iv) to (i), we first note that the nonsingular points on a quasiprojective reduced curve are dense. (One method, suggested by Joe: we know that normalization is an isomorphism away from a closed subset.) Given a dominant rational map between quasiprojective reduced curves $C \rightarrow C'$, we get a dominant rational map between their normalizations, which in turn gives a dominant rational map between their projective models $D \dashrightarrow D'$. The the dominant rational map is necessarily a morphism by Proposition 1.2, and then this morphism is necessarily projective and hence closed, and hence surjective (as the image contains the generic point of D', and hence its closure). Thus we have established (iv) $\rightarrow$ (i).

It remains to connect (i). Each dominant rational map of quasiprojective reduced curves indeed yields a map of function fields of dimension 1 (their fraction fields). Each function field of dimension 1 yields a reduced affine (hence quasiprojective) curve over k, and each map of two such yields a dominant rational map of the curves. □

1.7. Degree of a morphism between projective nonsingular curves.

We conclude with a useful fact: Any non-constant morphism from one projective nonsingular curve to another has a well-behaved degree, in a sense that we will now make precise. We will also show that any non-constant finite morphism from one nonsingular curve to another has a well-behaved degree in the same sense.

Suppose $f : C \rightarrow C'$ is a surjective (or equivalently, dominant) map of nonsingular projective curves.

It is a finite morphism. Here is why. (If we had already proved that quasifinite projective or proper morphisms with finite fibers were finite, we would know this. Once we *do* know this, the contents of this section would extend to the case where C is not necessarily non-singular.) Let C'' be the normalization of C' in the function field of C. Then we have an isomorphism $FF(C) \cong FF(C'')$ which leads to birational maps $C \dashleftarrow\dashrightarrow C''$ which extend to morphisms as both C and C'' are nonsingular and projective. Thus this yields an isomorphism of C and C''. But $C'' \rightarrow C$ is a finite morphism by the finiteness of integral closure.

We can then use the following proposition, which applies in more general situations.

1.8. *Proposition. — Suppose that $\pi : C \rightarrow C'$ is a surjective finite morphism, where C is an integral curve, and C' is an integral nonsingular curve. Then $\pi_* \mathcal{O}_C$ is locally free of finite rank.*

As π is finite, $\pi_* \mathcal{O}_C$ is a finite type sheaf on $\mathcal{O}'_C$. In case you care, the hypothesis "integral" on C' is redundant.

Before proving the proposition. I want to remind you what this means. Suppose d is the rank of this allegedly locally free sheaf. Then the fiber over any point of C with residue field K is the Spec of an algebra of dimension d over K. This means that the number of points in the fiber, counted with appropriate multiplicity, is always d.

As a motivating example, consider the map $\mathbb{Q}[y] \to \mathbb{Q}[x]$ given by $x \mapsto y^2$. (We've seen this example before.) I picture this as the projection of the parabola $x = y^2$ to the x-axis. (i) The fiber over $x = 1$ is $\mathbb{Q}[y]/(y^2 - 1)$, so we get 2 points. (ii) The fiber over $x = 0$ is $\mathbb{Q}[y]/(y^2)$ — we get one point, with multiplicity 2, arising because of the nonreducedness. (iii) The fiber over $x = -1$ is $\mathbb{Q}[y]/(y^2 + 1) \cong \mathbb{Q}[i]$ — we get one point, with multiplicity 2, arising because of the field extension. (iv) Finally, the fiber over the generic point $\operatorname{Spec} \mathbb{Q}(x)$ is $\operatorname{Spec} \mathbb{Q}(y)$, which is one point, with multiplicity 2, arising again because of the field extension (as $\mathbb{Q}(y)/\mathbb{Q}(x)$ is a degree 2 extension). We thus see three sorts of behaviors (as (iii) and (iv) are the same behavior). Note that even if you only work with algebraically closed fields, you will still be forced to this third type of behavior, because residue fields at generic points tend not to be algebraically closed (witness case (iv) above).

Note that we need C' to be nonsingular for this to be true. (I gave a picture of the normalization of a nodal curve as an example. A picture would help here.)

We will see the proof next day.

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FOUNDATIONS OF ALGEBRAIC GEOMETRY CLASS 29

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CONTENTS

Last day: One last bit of rational maps. Curves.

Today: A bit more curves. Introduction to cohomology.

1. SCHEME-THEORETIC CLOSURE, AND SCHEME-THEORETIC IMAGE

I discussed the scheme-theoretic closure of a locally closed scheme, and more generally, the scheme-theoretic image of a morphism. I've moved this discussion into the class 27 notes.

2. CURVES

Last day we proved a couple of important theorems:

2.1. *Key Proposition. — Suppose C is a dimension 1 finite type k-scheme, and p is a nonsingular point of it. Suppose Y is a projective k-scheme. Then any morphism $C - p \to Y$ extends to $C \to Y$.*

2.2. *Theorem. — If C is a nonsingular curve, then there is some projective nonsingular curve C' and an open immersion $C \hookrightarrow C'$.*

2.3. *Theorem. — The following categories are equivalent.*

Date: Tuesday, February 7, 2006. Updated June 25, 2007.

(i) *nonsingular projective curves, and surjective morphisms.*
(ii) *nonsingular projective curves, and dominant morphisms.*
(iii) *nonsingular projective curves, and dominant rational maps*
(iv) *quasiprojective reduced curves, and dominant rational maps*
(v) *fields of transcendence dimension* 1 *over* k*, and* k*-homomorphisms.*

We then discussed the degree of a morphism between projective nonsingular curves. In particular, we are in the midst of showing that any non-constant morphism from one projective nonsingular curve to another has a well-behaved degree. Suppose $f : C \to C'$ is a surjective (or equivalently, dominant) map of nonsingular projective curves. We showed that f is a finite morphism, by showing that f is the normalization of C' in the function field of C; hence the result follows by finiteness of integral closure.

2.4. *Proposition. — Suppose that* $\pi : C \to C'$ *is a surjective finite morphism, where* C *is an integral curve, and* C' *is an integral nonsingular curve. Then* $\pi_* \mathcal{O}_C$ *is locally free of finite rank.*

All we will really need is that C is reduced of pure dimension 1.

We are about to prove this.

Let's discuss again what this means. (I largely said this last day.) Suppose d is the rank of this allegedly locally free sheaf. Then the fiber over any point of C with residue field K is the Spec of an algebra of dimension d over K. This means that the number of points in the fiber, counted with appropriate multiplicity, is always d.

Proof. (For experts: we will later see that what matters here is that the morphism is finite and *flat*. But we don't yet know what flat is.)

The question is local on the target, so we may assume that C' is affine. Note that $\pi_* \mathcal{O}_C$ is torsion-free (as $\Gamma(C, \mathcal{O}_C)$ is an integral domain). Our plan is as follows: by an important exercise from last quarter (Exercise 5.2 of class 15; problem 10 on problem set 7), if the rank of the coherent sheaf $\pi_* \mathcal{O}_C$ is constant, then (as C' is reduced) $\pi_* \mathcal{O}_C$ is locally free. We'll show this by showing the rank at any closed point of C' is the same as the rank at the generic point.

The notion of "rank at a point" behaves well under base change, so we base change to the discrete valuation ring $\mathcal{O}_{C',p}$, where p is some closed point of C'. Then $\pi_* \mathcal{O}_C$ is a finitely generated module over a discrete valuation ring which is torsion-free. By the classification of finitely generated modules over a principal ideal domain, any finitely generate module over a principal ideal domain A is a direct sum of modules of the form $A/(d)$ for various $d \in A$. But if A is a discrete valuation ring, and $A/(d)$ is torsion-free, then $A/(d)$ is necessarily A (as for example all ideals of A are of the form 0 or a power of the maximal ideal). Thus we are done. $\square$

Remark. If we are working with complex curves, this notion of degree is the same as the notion of the topological degree.

Suppose C is a projective curve, and $\mathcal{L}$ is an invertible sheaf. We will define $\deg \mathcal{L}$.

Let s be a non-zero rational section of $\mathcal{L}$. For any $p \in C$, recall the valuation of s at p ($\nu_p(s) \in \mathbb{Z}$). (Pick any local section t of $\mathcal{L}$ not vanishing at p. Then $s/t \in FF(C)$. $\nu_p(s) := \nu_p(s/t)$. We can show that this is well-defined.)

Define $\deg(\mathcal{L}, s)$ (where s is a non-zero rational section of $\mathcal{L}$) to be the number of zeros minus the number of poles, counted with appropriate multiplicity. (In other words, each point contributes the valuation at that point times the degree of the field extension.) We'll show that this is independent of s. (Note that we need the projective hypothesis: the sections x and 1 of the structure sheaf on $\mathbb{A}^1$ have different degrees.)

Notice that $\deg(\mathcal{L}, s)$ is additive under products: $\deg(\mathcal{L}, s)+\deg(\mathcal{M}, t) = \deg(\mathcal{L}\otimes\mathcal{M}, s\otimes t)$. Thus to show that $\deg(\mathcal{L}, s) = \deg(\mathcal{L}, t)$, we need to show that $\deg(\mathcal{O}_C, s/t) = 0$. Hence it suffices to show that $\deg(\mathcal{O}_C, u) = 0$ for a non-zero rational function u on C. Then u gives a rational map $C \dashrightarrow \mathbb{P}^1$. By our recent work (Proposition 2.1 above), this can be extended to a morphism $C \to \mathbb{P}^1$. The preimage of 0 is the number of 0's, and the preimage of ∞ is the number of ∞'s. But these are the same by our previous discussion of degree of a morphism! Finally, suppose $p \mapsto 0$. I claim that the valuation of u at p times the degree of the field extension is precisely the contribution of p to $u^{-1}(0)$. (A similar computation for ∞ will complete the proof of the desired result.) This is because the contribution of p to $u^{-1}(0)$ is precisely

$$\dim_k \mathcal{O}_{C,p}/(u) = \dim_k \mathcal{O}_{C,p}/\mathfrak{m}^{\nu_p(u)} = \nu_p(u) \dim_k \mathcal{O}_{C,p}/\mathfrak{m}.$$

□

We can define the degree of an invertible sheaf $\mathcal{L}$ on an integral *singular* projective curve C as follows: if $\nu : \tilde{C} \to C$ be the normalization, let $\deg_C \mathcal{L} := \deg_{\tilde{C}} \nu^*\mathcal{L}$. Notice that if s is a meromorphic section that has neither zeros nor poles at the singular points of C, then $\deg_C \mathcal{L}$ is still the number of zeros minus the number of poles (suitably counted), because the zeros and poles of $\nu^*\mathcal{L}$ are just the same as those of $\mathcal{L}$.

3.1. *Exercise.* Suppose $f : C \to C'$ is a degree d morphism of integral projective nonsingular curves, and $\mathcal{L}$ is an invertible sheaf on C'. Show that $\deg_C f^*\mathcal{L} = d \deg_{C'} \mathcal{L}$.

3.2. Degree of a Cartier divisor on a curve.

I said the following in class 30. (I've repeated this in the class 30 notes.)

Suppose D is an effective Cartier divisor on a projective curve, or a Cartier divisor on a projective nonsingular curve (over a field k). (I should really say: suppose D is a Cartier divisor on a projective curve, but I don't think I defined Cartier divisors in that generality.) Then define the *degree* of D (denoted $\deg D$) to be the degree of the corresponding invertible sheaf.

Exercise. If D is an effective Cartier divisor on a projective nonsingular curve, say $D = \sum n_i p_i$, prove that $\deg D = \sum n_i \deg p_i$, where $\deg p_i$ is the degree of the field extension of the residue field at p_i over k.

4. Cech cohomology of quasicoherent sheaves

One idea behind the cohomology of quasicoherent sheaves is as follows. If $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$ is a short exact sequence of sheaves on X, we know that

$$0 \to \mathcal{F}(X) \to \mathcal{G}(X) \to \mathcal{H}(X).$$

In other words, $\Gamma(X, \cdot)$ is a left-exact functor. We dream that this is something called H^0, and that this sequence continues off to the right, giving a long exact sequence in cohomology. (In general, whenever we see a left-exact or right-exact functor, we should hope for this, and in most good cases our dreams are fulfilled. The machinery behind this is sometimes called *derived functor cohomology*, which we may discuss in the third quarter.)

We'll show that these cohomology groups exist. Before defining them explicitly, we first describe their important properties.

Suppose X is an R-scheme. Assume throughout that X is separated and quasicompact. Then for each quasicoherent sheaf $\mathcal{F}$ on X, we'll define R-modules $H^i(X, \mathcal{F})$. (In particular, if $R = k$, they are k-vector spaces.) First, $\boxed{H^0(X, \mathcal{F}) = \Gamma(X, \mathcal{F})}$. Each H^i will be a contravariant functor in the space X, and a $\boxed{\text{covariant functor in the sheaf } \mathcal{F}}$. The functor H^i behaves well under direct sums: $\boxed{H^i(X, \oplus_j \mathcal{F}_j) = \oplus_j H^i(X, \mathcal{F}_j)}$. (We will need infinite sums, not just finite sums.) If $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$ is a short exact sequence of quasicoherent sheaves on X, then we have a $\boxed{\text{long exact sequence}}$

$$\begin{aligned} 0 \to H^0(X, \mathcal{F}) \to H^0(X, \mathcal{G}) \to H^0(X, \mathcal{H}) \\ \to H^1(X, \mathcal{F}) \to H^1(X, \mathcal{G}) \to H^1(X, \mathcal{H}) \to \cdots . \end{aligned}$$

(The maps $H)(X, ?) \to H^i(X, ??)$ will be those coming from covariance; the *connecting homomorphisms* $H^i(x, \mathcal{H}) \to H^{i+1}(X, \mathcal{F})$ will have to be defined.) We'll see that if X can be covered by n affines, then $\boxed{H^i(X, \mathcal{F}) = 0}$ for $i \geq n$ for all $\mathcal{F}$, i. (In particular, all higher quasicoherent cohomology groups on affine schemes vanish.) If $X \hookrightarrow Y$ is a closed immersion, and $\mathcal{F}$ is a quasicoherent sheaf on X, then $\boxed{H^i(X, \mathcal{F}) = H^i(Y, f_* \mathcal{F})}$. (We'll care about this particularly in the case when $X \subset Y = \mathbb{P}^N_R$, which will let us reduce calculations on arbitrary projective R-schemes to calculations on $\mathbb{P}^n_R$.)

We will also identify the cohomology of all the invertible sheaves on $\mathbb{P}^n_R$:

4.1. *Proposition.* —

- $H^0(\mathbb{P}^n_R, \mathcal{O}_{\mathbb{P}^n_R}(m))$ *is a free* R*-module of rank* $\binom{n+m}{n}$ *if* $i = 0$ *and* $m \geq 0$*, and* 0 *otherwise.*
- $H^n(\mathbb{P}^n_R, \mathcal{O}_{\mathbb{P}^n_R}(m))$ *is a free* R*-module of rank* $\binom{-m-1}{-n-m-1}$ *if* $m \leq -n-1$*, and* 0 *otherwise.*
- $H^i(\mathbb{P}^n_R, \mathcal{O}_{\mathbb{P}^n_R}(m)) = 0$ *if* $0 < i < n$*.*

It is more helpful to say the following imprecise statement: $H^0(\mathbb{P}^n_R, \mathcal{O}_{\mathbb{P}^n_R}(m))$ should be interpreted as the homogeneous degree m polynomials in $x_0, \dots, x_n$ (with R-coefficients), and $H^n(\mathbb{P}^n_R, \mathcal{O}_{\mathbb{P}^n_R}(m))$ should be interpreted as the homogeneous degree m Laurent polynomials in $x_0, \dots, x_n$, where in each monomial, each x_i appears with degree at most -1.

We'll prove this next day.

Here are some features of this Proposition that I wish to point out, that will be the first appearances of things that we'll prove later.

- The cohomology of these bundles vanish above the dimension of the space *if* $R = k$; we'll generalize this for $\operatorname{Spec} R$, and even more, in before long.
- These cohomology groups are always finitely-generated R modules.
- The top cohomology group vanishes for $m > -n-1$. (This is a first appearance of "Kodaira vanishing".)
- The top cohomology group is "1-dimensional" for $m = -n-1$ if $R = k$. This is the first appearance of a dualizing sheaf.
- We have a natural duality $H^i(X, \mathcal{O}(m)) \times H^{n-i}(X, \mathcal{O}(-n-1-m)) \to H^n(X, \mathcal{O}(-n-1))$. This is the first appearance of Serre duality.

I'd like to use all these properties to prove things, so you'll see how handy they are. We'll worry later about defining cohomology, and proving these properties.

When we discussed global sections, we worked hard to show that for any coherent sheaf $\mathcal{F}$ on $\mathbb{P}^n_R$ we could find a surjection $\mathcal{O}(m)^{\oplus j} \to \mathcal{F}$, which yields the exact sequence

$$0 \to \mathcal{G} \to \mathcal{O}(m)^{\oplus j} \to \mathcal{F} \to 0 \tag{1}$$

for some coherent sheaf $\mathcal{G}$. We can use this to prove the following.

4.2. *Theorem. — (i) For any coherent sheaf $\mathcal{F}$ on a projective R-scheme where R is Noetherian, $h^i(X, \mathcal{F})$ is a finitely generated R-module. (ii) (Serre vanishing) Furthermore, for $m \gg 0$, $H^i(X, \mathcal{F}(m)) = 0$ for all i, even without Noetherian hypotheses.*

Proof. Because cohomology of a closed scheme can be computed on the ambient space, we may reduce to the case $X = \mathbb{P}^n_R$.

(i) Consider the long exact sequence:

$$0 \longrightarrow H^0(\mathbb{P}^n_R, \mathcal{G}) \longrightarrow H^0(\mathbb{P}^n_R, \mathcal{O}(m)^{\oplus j}) \longrightarrow H^0(\mathbb{P}^n_R, \mathcal{F}) \longrightarrow$$

$$H^1(\mathbb{P}^n_R, \mathcal{G}) \longrightarrow H^1(\mathbb{P}^n_R, \mathcal{O}(m)^{\oplus j}) \longrightarrow H^1(\mathbb{P}^n_R, \mathcal{F}) \longrightarrow \cdots$$

$$\cdots \longrightarrow H^{n-1}(\mathbb{P}^n_R, \mathcal{G}) \longrightarrow H^{n-1}(\mathbb{P}^n_R, \mathcal{O}(m)^{\oplus j}) \longrightarrow H^{n-1}(\mathbb{P}^n_R, \mathcal{F}) \longrightarrow$$

$$H^n(\mathbb{P}^n_R, \mathcal{G}) \longrightarrow H^n(\mathbb{P}^n_R, \mathcal{O}(m)^{\oplus j}) \longrightarrow H^n(\mathbb{P}^n_R, \mathcal{F}) \longrightarrow 0$$

The exact sequence ends here because $\mathbb{P}^n_R$ is covered by $n+1$ affines. Then $H^n(\mathbb{P}^n_R, \mathcal{O}(m)^{\oplus j})$ is finitely generated by Proposition 4.1, hence $H^n(\mathbb{P}^n_R, \mathcal{F})$ is finitely generated for all coherent sheaves $\mathcal{F}$. Hence in particular, $H^n(\mathbb{P}^n_R, \mathcal{G})$ is finitely generated. As $H^{n-1}(\mathbb{P}^n_R, \mathcal{O}(m)^{\oplus j})$ is finitely generated, and $H^n(\mathbb{P}^n_R, \mathcal{G})$ is too, we have that $H^{n-1}(\mathbb{P}^n_R, \mathcal{F})$ is finitely generated for all coherent sheaves $\mathcal{F}$. We continue inductively downwards.

(ii) Twist (4.1) by $\mathcal{O}(N)$ for $N \gg 0$. Then $H^n(\mathbb{P}^n_R, \mathcal{O}(m+N)^{\oplus j}) = 0$, so $H^n(\mathbb{P}^n_R, \mathcal{F}(N)) = 0$. Translation: for any coherent sheaf, its top cohomology vanishes once you twist by $\mathcal{O}(N)$ for N sufficiently large. Hence this is true for $\mathcal{G}$ as well. Hence from the long exact sequence, $H^{n-1}(\mathbb{P}^n_R, \mathcal{F}(N)) = 0$ for $N \gg 0$. As in (i), we induct downwards, until we get that $H^1(\mathbb{P}^n_R, \mathcal{F}(N)) = 0$. (The induction proceeds no further, as it is *not* true that $H^0(\mathbb{P}^n_R, \mathcal{O}(m+N)^{\oplus j}) = 0$ for large N — quite the opposite. □

Exercise for those who like working with non-Noetherian rings: Prove part (i) in the above result without the Noetherian hypotheses, assuming only that R is a coherent R-module (it is "coherent over itself"). (Hint: induct downwards as before. The order is as follows: $H^n(\mathbb{P}^n_R, \mathcal{F})$ finitely generated, $H^n(\mathbb{P}^n_R, \mathcal{G})$ finitely generated, $H^n(\mathbb{P}^n_R, \mathcal{F})$ coherent, $H^n(\mathbb{P}^n_R, \mathcal{G})$ coherent, $H^{n-1}(\mathbb{P}^n_R, \mathcal{F})$ finitely generated, $H^{n-1}(\mathbb{P}^n_R, \mathcal{G})$ finitely generated, etc.)

In particular, we have proved the following, that we would have cared about even before we knew about cohomology.

4.3. *Corollary. — Any projective* k*-scheme has a finite-dimensional space of global sections. More generally, if* $\mathcal{F}$ *is a coherent sheaf on a projective* R*-scheme, then* $h^0(X, \mathcal{F})$ *is a finitely generated* R*-module.*

This is true more generally for proper k-schemes, not just projective k-schemes, but I won't give the argument here.

Here is another a priori interesting consequence:

4.4. *Corollary. — If* $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$ *is an exact sequence of coherent sheaves on projective* X *with* $\mathcal{F}$ *coherent, then for* $n \gg 0$, $0 \to H^0(X, \mathcal{F}(n)) \to H^0(X, \mathcal{G}(n)) \to H^0(X, \mathcal{H}(n)) \to 0$ *is also exact.*

(Proof: for $n \gg 0$, $H^1(X, \mathcal{F}(n)) = 0$.)

This result can also be shown directly, without the use of cohomology.

5. Proving the things you need to know

As you read this, you should go back and check off all the facts, to make sure that I've shown all that I've promised.

5.1. Cech cohomology. Works nicely here. In general: take finer and finer covers. Here we take a single cover.

Suppose X is quasicompact and separated, e.g. X is quasiprojective over A. In particular, X may be covered by a finite number of affine open sets, and the intersection of any two affine open sets is also an affine open set; these are the properties we will use. Suppose $\mathcal{F}$ is a quasicoherent sheaf, and $\mathcal{U} = \{U_i\}_{i=1}^n$ is a *finite* set of affine open sets of X whose union is U. For $I \subset \{1, \ldots, n\}$ define $U_i = \cap_{i \in I} U_i$. It is affine by the separated hypothesis. **Define** $H^i_{\mathcal{U}}(U, \mathcal{F})$ to be the ith cohomology group of the complex

$$0 \to \bigoplus_{\substack{|I|=1 \\ I \subset \{1,\ldots,n\}}} \mathcal{F}(U_I) \to \cdots \to \bigoplus_{\substack{|I|=i \\ I \subset \{1,\ldots,n\}}} \mathcal{F}(U_I) \to \bigoplus_{\substack{|I|=i+1 \\ I \subset \{1,\ldots,n\}}} \mathcal{F}(U_I) \to \cdots . \tag{2}$$

Note that if X is an R-scheme, then $H^i_{\mathcal{U}}(X, \mathcal{F})$ is an R-module. Also $H^0_{\mathcal{U}}(X, \mathcal{F}) = \Gamma(X, \mathcal{F})$.

5.2. *Exercise.* Suppose $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ is a short exact sequence of sheaves on a topological space, and $\mathcal{U}$ is an open cover such that on any intersection the sections of $\mathcal{F}_2$ surject onto $\mathcal{F}_3$. Show that we get a long exact sequence of cohomology. (Note that this applies in our case!)

I ended by stating the following result, which we will prove next day.

5.3. *Theorem/Definition. — Recall that X is quasicompact and separated. $H^i_{\mathcal{U}}(U, \mathcal{F})$ is independent of the choice of (finite) cover $\{U_i\}$. More precisely,*

- (*) *for all k, for any two covers $\{U_i\} \subset \{V_i\}$ of size at most k, the maps $H^i_{\{V_i\}}(X, \mathcal{F}) \to H^i_{\{U_i\}}(X, \mathcal{F})$ induced by the natural maps of complex* (2) *are isomorphisms.*

Define the Cech cohomology group $H^i(X, \mathcal{F})$ to be this group.

I needn't have stated in terms of some k; I've stated it in this way so I can prove it by induction.

(For experts: we'll get natural quasiisomorphisms of Cech complexes for various $\mathcal{U}$.)

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FOUNDATIONS OF ALGEBRAIC GEOMETRY CLASS 30

RAVI VAKIL

CONTENTS

Last day: More curves. Cohomology take 1.

Today: Cohomology continued. Hilbert functions and Hilbert polynomials.

1. Left-over: Degree of a Cartier divisor on a projective curve

As always, there is something small that I should have said last day. Suppose D is an effective Cartier divisor on a projective curve, or a Cartier divisor on a projective nonsingular curve (over a field k). (I should really say: suppose D is a Cartier divisor on a projective curve, but I don't think I defined Cartier divisors in that generality.) Then define the *degree* of D (denoted $\deg D$) to be the degree of the corresponding invertible sheaf.

Exercise. If D is an effective Cartier divisor on a projective nonsingular curve, say $D = \sum n_i p_i$, prove that $\deg D = \sum n_i \deg p_i$, where $\deg p_i$ is the degree of the field extension of the residue field at p_i over k.

(This is also now in the class 29 notes, where it belongs.)

2. Cohomology continued

Last day, I gave you lots of facts that we wanted cohomology to satisfy. Suppose X is a separated and quasicompact R-scheme. In particular, X can be covered by a finite number of affine open sets, and the intersection of any two affine open sets is another affine open

Date: Thursday, February 9, 2006. Minor update March 8.

set. We are going to define $H^i(X, \mathcal{F})$ for any quasicoherent sheaf $\mathcal{F}$ on X, that satisfies the following properties.

- $H^0(X, \mathcal{F}) = \Gamma(X, \mathcal{F})$
- H^i is a contravariant functor in X and a covariant functor in $\mathcal{F}$.
- $H^i(X, \oplus_j \mathcal{F}_j) = \oplus_j H^i(X, \mathcal{F}_j)$: cohomology commutes with arbitrary direct sums.
- long exact sequences
- $H^i(\operatorname{Spec} R, \mathcal{F}) = 0$.
- If $X \hookrightarrow Y$ is a closed immersion, and $\mathcal{F}$ is a quasicoherent sheaf on X, then $H^i(X, \mathcal{F}) = H^i(Y, f_*\mathcal{F})$.
- $H^i(\mathbb{P}^n_R, \mathcal{O}_{\mathbb{P}^n_R}(r))$ is something nice (we described it in a statement last day that we will prove today)

Last day, we defined these cohomology groups given the additional data of an affine open cover $\mathcal{U}$; I used the notation $H^i_{\mathcal{U}}(X, \mathcal{F})$. We'll start today by showing that this is independent of $\mathcal{U}$.

2.1. *Theorem/Definition. — Recall that X is quasicompact and separated. $H^i_{\mathcal{U}}(U, \mathcal{F})$ is independent of the choice of (finite) cover $\{U_i\}$. More precisely,*

(*) *for all k, for any two covers $\{U_i\} \subset \{V_i\}$ of size at most k, the maps $H^i_{\{V_i\}}(X, \mathcal{F}) \to H^i_{\{U_i\}}(X, \mathcal{F})$ induced by the natural maps of complex (1) are isomorphisms.*

Define the Cech cohomology group $H^i(X, \mathcal{F})$ to be this group.

$$(1) \qquad 0 \to \bigoplus_{\substack{|I|=1 \\ I \subset \{1,\dots,n\}}} \mathcal{F}(U_I) \to \cdots \to \bigoplus_{\substack{|I|=i \\ I \subset \{1,\dots,n\}}} \mathcal{F}(U_I) \to \bigoplus_{\substack{|I|=i+1 \\ I \subset \{1,\dots,n\}}} \mathcal{F}(U_I) \to \cdots.$$

I needn't have stated in terms of some k; I've stated it in this way so I can prove it by induction.

(For experts: we'll get natural quasiisomorphisms of Cech complexes for various $\mathcal{U}$.)

Proof. We prove this by induction on k. The base case is trivial. We need only prove the result for $\{U_i\}_{i=1}^n \subset \{U_i\}_{i=0}^n$, where the case $k = n$ is known. Consider the exact sequence of complexes

$$
\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots \longrightarrow & & \bigoplus_{\substack{|I|=i-1 \\ 0\in I\subset\{0,\dots,n\}}} \mathcal{F}(U_I) & \longrightarrow & \bigoplus_{\substack{|I|=i \\ 0\in I\subset\{0,\dots,n\}}} \mathcal{F}(U_I) & \longrightarrow & \bigoplus_{\substack{|I|=i+1 \\ 0\in I\subset\{0,\dots,n\}}} \mathcal{F}(U_I) & \longrightarrow \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots \longrightarrow & & \bigoplus_{\substack{|I|=i-1 \\ I\subset\{0,\dots,n\}}} \mathcal{F}(U_I) & \longrightarrow & \bigoplus_{\substack{|I|=i \\ I\subset\{0,\dots,n\}}} \mathcal{F}(U_I) & \longrightarrow & \bigoplus_{\substack{|I|=i+1 \\ I\subset\{0,\dots,n\}}} \mathcal{F}(U_I) & \longrightarrow \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots \longrightarrow & & \bigoplus_{\substack{|I|=i-1 \\ I\subset\{1,\dots,n\}}} \mathcal{F}(U_I) & \longrightarrow & \bigoplus_{\substack{|I|=i \\ I\subset\{1,\dots,n\}}} \mathcal{F}(U_I) & \longrightarrow & \bigoplus_{\substack{|I|=i+1 \\ I\subset\{1,\dots,n\}}} \mathcal{F}(U_I) & \longrightarrow \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

We get a long exact sequence of cohomology from this. Thus by Exercise 5.2 of last day, we wish to show that the top row is exact. But the ith cohomology of the top row is precisely $H^i_{\{U_i \cap U_0\}_{i>0}}(U_i, \mathcal{F})$ except at step 0, where we get 0 (because the complex starts off $0 \to \mathcal{F}(U_0) \to \oplus_{j=1}^n \mathcal{F}(U_0 \cap U_j)$). So we just need to show that higher Cech groups of affine schemes are 0. Hence we are done by the following result. □

2.2. *Theorem. — The higher Cech cohomology $H^i_{\mathcal{U}}(X, \mathcal{F})$ of an affine R-scheme X vanishes (for any affine cover $\mathcal{U}$, $i > 0$, and quasicoherent $\mathcal{F}$).*

Serre describes this as a partition of unity argument.

A spectral sequence argument can make quick work of this, but I'd like to avoid introducing spectral sequences until I have to.

Proof. We want to show that the "extended" complex (where you tack on global sections to the front) has no cohomology, i.e. that

$$0 \to \mathcal{F}(X) \to \oplus_{|I|=1}\mathcal{F}(U_I) \to \oplus_{|I|=2}\mathcal{F}(U_I) \to \cdots \tag{2}$$

is exact. We do this with a trick.

Suppose first that some U_i (say U_0) is X. Then the complex can be described as the middle row of the following short exact sequence of complexes

$$\begin{array}{ccccccccc}
0 & \longrightarrow & 0 & \longrightarrow & \oplus_{|I|=1, 0 \in I} \mathcal{F}(U_I) & \longrightarrow & \oplus_{|I|=2, 0 \in I} \mathcal{F}(U_I) & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{F}(X) & \longrightarrow & \oplus_{|I|=1} \mathcal{F}(U_I) & \longrightarrow & \oplus_{|I|=2} \mathcal{F}(U_I) & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{F}(X) & \longrightarrow & \oplus_{|I|=1, 0 \notin I} \mathcal{F}(U_I) & \longrightarrow & \oplus_{|I|=2, 0 \notin I} \mathcal{F}(U_I) & \longrightarrow & \cdots
\end{array}$$

The top row is the same as the bottom row, slid over by 1. The corresponding long exact sequence of cohomology shows that the central row has vanishing cohomology. (Topological experts will recognize a *mapping cone* in the above construction.)

We next prove the general case by sleight of hand. Say $X = \operatorname{Spec} S$. We wish to show that the complex of R-modules (2) is exact. It is also a complex of S-modules, so we wish to show that the complex of S-modules (2) is exact. To show that it is exact, it suffices to show that for a cover of $\operatorname{Spec} S$ by distinguished opens $D(f_i)$ ($1 \leq i \leq s$) (i.e. $(f_1, \dots, f_s) = 1$ in S) the complex is exact. (Translation: exactness of a sequence of sheaves may be checked locally.) We choose a cover so that each $D(f_i)$ is contained in some $U_j = \operatorname{Spec} R_j$. Consider the complex localized at f_i. As

$$\Gamma(\operatorname{Spec} R, \mathcal{F})_f = \Gamma(\operatorname{Spec}(R_j)_f, \mathcal{F})$$

(as this is one of the definitions of a quasicoherent sheaf), as $U_j \cap D(f_i) = D(f_i)$, we are in the situation where one of the U_i's is X, so we are done. □

2.3. *Exercise.* Suppose $V \subset U$ are open subsets of X. Show that we have restriction morphisms $H^i(U, \mathcal{F}) \to H^i(V, \mathcal{F})$ (if U and V are quasicompact, and U hence V is separated). Show that restrictions commute. Hence if X is a Noetherian space, $H^i(;\mathcal{F})$ this is a contravariant functor from the category $\mathrm{Top}(X)$ to abelian groups. (For experts: this means that it is a presheaf. But this is not a good way to think about it, as its sheafification is 0, as it vanishes on the affine base.) The same argument will show more generally that for any map $f : X \to Y$, there exist natural maps $H^i(X, \mathcal{F}) \to H^i(X, f^*\mathcal{F})$; I should have asked this instead.

2.4. *Exercise.* Show that if $\mathcal{F} \to \mathcal{G}$ is a morphism of quasicoherent sheaves on separated and quasicompact X then we have natural maps $H^i(X, \mathcal{F}) \to H^i(X, \mathcal{G})$. Hence $H^i(X, \cdot)$ is a covariant functor from quasicoherent sheaves on X to abelian groups (or even R-modules).

In particular, we get the following facts.

1. If $X \hookrightarrow Y$ is a closed subscheme then $H^i(X, \mathcal{F}) = H^i(Y, f_*\mathcal{F})$, as promised at start of our discussion on cohomology.

2. Also, if X can be covered by n affine open sets, then $H^i(X, \mathcal{F}) = 0$ for all quasicoherent $\mathcal{F}$, and $i \geq n$. In particular, $H^i(\operatorname{Spec} R, \mathcal{F}) = 0$ for $i > 0$.

3. Cohomology behaves well for arbitrary direct sums of quasicoherent sheaves.

2.5. Dimensional vanishing for projective k-schemes.

2.6. *Theorem. — Suppose X is a projective k-scheme, and $\mathcal{F}$ is a quasicoherent sheaf on X. Then $H^i(X, \mathcal{F}) = 0$ for $i > \dim X$.*

In other words, cohomology vanishes above the dimension of X. We will later show that this is true when X is a *quasiprojective* k-scheme.

Proof. Suppose $X \hookrightarrow \mathbb{P}^N$, and let $n = \dim X$. We show that X may be covered by n affine open sets. Long ago, we had an exercise saying that we could find n Cartier divisors on $\mathbb{P}^N$ such that their complements $U_0, \dots, U_n$ covered X. (We did this as follows. Lemma: Suppose $Y \hookrightarrow \mathbb{P}^N$ is a projective scheme. Then Y is Noetherian, and hence has a finite number of components. We can find a hypersurface H containing none of their associated points. Then H contains no component of Y, the dimension of $H \cap Y$ is strictly smaller than Y, and if $\dim Y = 0$, then $H \cap Y = \emptyset$.) Then U_i is affine, so $U_i \cap X$ is affine, and thus we have covered X with n affine open sets. □

Remark. We actually *need* n affine open sets to cover X, but I don't see an easy way to prove it. One way of proving it is by showing that the complement of an affine set is always pure codimension 1.

3. Cohomology of line bundles on projective space

I'll now pay off that last IOU.

3.1. *Proposition. —*

- $H^0(\mathbb{P}^n_R, \mathcal{O}_{\mathbb{P}^n_R}(m))$ *is a free R-module of rank $\binom{n+m}{n}$ if $i = 0$ and $m \geq 0$, and 0 otherwise.*
- $H^n(\mathbb{P}^n_R, \mathcal{O}_{\mathbb{P}^n_R}(m))$ *is a free R-module of rank $\binom{-m-1}{-n-m-1}$ if $m \leq -n-1$, and 0 otherwise.*
- $H^i(\mathbb{P}^n_R, \mathcal{O}_{\mathbb{P}^n_R}(m)) = 0$ *if $0 < i < n$.*

It is more helpful to say the following imprecise statement: $H^0(\mathbb{P}^n_R, \mathcal{O}_{\mathbb{P}^n_R}(m))$ should be interpreted as the homogeneous degree m polynomials in $x_0, \dots, x_n$ (with R-coefficients), and $H^n(\mathbb{P}^n_R, \mathcal{O}_{\mathbb{P}^n_R}(m))$ should be interpreted as the homogeneous degree m Laurent polynomials in $x_0, \dots, x_n$, where in each monomial, each x_i appears with degree at most -1.

Proof. The H^0 statement was an (important) exercise last quarter.

Rather than consider $\mathcal{O}(m)$ for various m, we consider them all at once, by considering $\mathcal{F} = \oplus_m \mathcal{O}(m)$.

Of course we take the standard cover $U_0 = D(x_0), \ldots, U_n = D(x_n)$ of $\mathbb{P}^n_R$. Notice that if $I \subset \{1, \ldots, n\}$, then $\mathcal{F}(U_I)$ corresponds to the Laurent monomials where each x_i for $i \notin I$ appears with non-negative degree.

We consider the H^n statement. $H^n(\mathbb{P}^n_R, \mathcal{F})$ is the cokernel of the following surjection

$$\oplus_{i=0}^n \mathcal{F}(U_{\{1,\ldots,n\}-\{i\}}) \to \mathcal{F}_{U_{\{1,\ldots,n\}}}$$

i.e.

$$\oplus_{i=0}^n R[x_0, \ldots, x_n, x_0^{-1}, \ldots, \widehat{x_i^{-1}}, \ldots x_n^{-1}] \to R[x_0, \ldots, x_n, x_0^{-1}, \ldots, x_n^{-1}].$$

This cokernel is precisely as described.

We last consider the H^i statement ($0 < i < n$). We prove this by induction on n. The cases $n = 0$ and 1 are trivial. Consider the exact sequence of quasicoherent sheaves:

$$0 \longrightarrow \mathcal{F} \xrightarrow{\times x_n} \mathcal{F} \longrightarrow \mathcal{F}' \longrightarrow 0$$

where $\mathcal{F}'$ is analogous sheaf on the hyperplane $x_n = 0$ (isomorphic to $\mathbb{P}^{n-1}_R$). (This exact sequence is just the direct sum over all m of the exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n_R}(m-1) \xrightarrow{\times x_n} \mathcal{O}_{\mathbb{P}^n_R}(m) \longrightarrow \mathcal{O}_{\mathbb{P}^{n-1}_R}(m) \longrightarrow 0\ ,$$

which in turn is obtained by twisting the closed subscheme exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n_R}(m-1) \xrightarrow{\times x_n} \mathcal{O}_{\mathbb{P}^n_R}(m) \longrightarrow \mathcal{O}_{\mathbb{P}^{n-1}_R}(m) \longrightarrow 0$$

by $\mathcal{O}_{\mathbb{P}^n_R}(m)$.)

The long exact sequence in cohomology gives us:

$$0 \longrightarrow H^0(\mathbb{P}^n_R, \mathcal{F}) \xrightarrow{\times x_n} H^0(\mathbb{P}^n_R, \mathcal{F}) \longrightarrow H^0(\mathbb{P}^{n-1}_R, \mathcal{F}') \quad .$$

$$\longrightarrow H^1(\mathbb{P}^n_R, \mathcal{F}) \xrightarrow{\times x_n} H^1(\mathbb{P}^n_R, \mathcal{F}) \longrightarrow H^1(\mathbb{P}^{n-1}_R, \mathcal{F}')$$

$$\cdots \longrightarrow H^{n-1}(\mathbb{P}^n_R, \mathcal{F}) \xrightarrow{\times x_n} H^{n-1}(\mathbb{P}^n_R, \mathcal{F}) \longrightarrow H^{n-1}(\mathbb{P}^{n-1}_R, \mathcal{F}')$$

$$\longrightarrow H^n(\mathbb{P}^n_R, \mathcal{F}) \xrightarrow{\times x_n} H^n(\mathbb{P}^n_R, \mathcal{F}) \longrightarrow 0$$

We will now show that this gives an isomorphism

$$\boxed{\times x_n : H^i(\mathbb{P}^n_R, \mathcal{F}) \to H^i(\mathbb{P}^n_R, \mathcal{F})} \tag{3}$$

for $0 < i < n$. The inductive hypothesis gives us this except for $i = 1$ and $i = n-1$, where we have to pay a bit more attention. For the first, note that $H^0(\mathbb{P}^n_R, \mathcal{F}) \longrightarrow H^0(\mathbb{P}^{n-1}_R, \mathcal{F}')$ is surjective: this map corresponds to taking the set of all polynomials in $x_0, \ldots, x_n$, and setting $x_n = 0$. The last is slightly more subtle: $H^{n-1}(\mathbb{P}^{n-1}_R, \mathcal{F}') \to H^n(\mathbb{P}^n_R, \mathcal{F})$ is injective, and corresponds to taking a Laurent polynomial in $x_0, \dots, x_{n-1}$ (where in each monomial, each x_i appears with degree at most -1) and multiplying by x_n^{-1}, which indeed describes the kernel of $H^n(\mathbb{P}^n_R, \mathcal{F}) \xrightarrow{\times x_n} H^n(\mathbb{P}^n_R, \mathcal{F})$. (This is a worthwhile calculation! See the exercise after the end of this proof.) We have thus established (3) above.

We will now show that the localization $H^i(\mathbb{P}^n_R, \mathcal{F})_{x_n} = 0$. (Here's what we mean by localization. Notice $H^i(\mathbb{P}^n_R, \mathcal{F})$ is naturally a module over $R[x_0, \dots, x_n]$ — we know how to multiply by elements of R, and by (3) we know how to multiply by x_i. Then we localize this at x_n to get an $R[x_0, \dots, x_n]_{x_n}$-module.) This means that each element $\alpha \in H^i(\mathbb{P}^n_R, \mathcal{F})$ is killed by some power of x_i. But by (3), this means that $\alpha = 0$, concluding the proof of the theorem.

Consider the Cech complex computing $H^i(\mathbb{P}^n_R, \mathcal{F})$. Localize it at x_n. Localization and cohomology commute (basically because localization commutes with operations of taking quotients, images, etc.), so the cohomology of the new complex is $H^i(\mathbb{P}^n_R, \mathcal{F})_{x_n}$. But this complex computes the cohomology of $\mathcal{F}_{x_n}$ on the affine scheme U_n, and the higher cohomology of *any* quasicoherent sheaf on an affine scheme vanishes (by Theorem 2.2 which we've just proved — in fact we used the same trick there), so $H^i(\mathbb{P}^n_R, \mathcal{F})_{x_n} = 0$ as desired. □

3.2. *Exercise.* Verify that $H^{n-1}(\mathbb{P}^{n-1}_R, \mathcal{F}') \to H^n(\mathbb{P}^n_R, \mathcal{F})$ is injective (likely by verifying that it is the map on Laurent monomials we claimed above).

4. Application of cohomology: Hilbert polynomials and Hilbert functions; degrees

We've already seen some powerful uses of this machinery, to prove things about spaces of global sections, and to prove Serre vanishing. We'll now see some classical constructions come out very quickly and cheaply.

In this section, we will work over a field k. Define $h^i(X, \mathcal{F}) := \dim_k H^i(X, \mathcal{F})$.

Suppose $\mathcal{F}$ is a coherent sheaf on a projective k-scheme X. Define the *Euler characteristic*

$$\chi(X, \mathcal{F}) = \sum_{i=0}^{\dim X} (-1)^i h^i(X, \mathcal{F}).$$

We will see repeatedly here and later that while Euler characteristics behave better than individual cohomology groups. As one sign, notice that for fixed n, and $m \geq 0$,

$$h^0(\mathbb{P}^n_k, \mathcal{O}(m)) = \binom{n+m}{m} = \frac{(m+1)(m+2)\cdots(m+n)}{n!}.$$

Notice that the expression on the right is a polynomial in m of degree n. (For later reference, I want to point out that the leading term is $m^n/n!$.) But it is not true that

$$h^0(\mathbb{P}^n_k, \mathcal{O}(m)) = \frac{(m+1)(m+2)\cdots(m+n)}{n!}$$

for *all* m — it breaks down for $m \leq -n-1$. Still, you can check that

$$\chi(\mathbb{P}^n_k, \mathcal{O}(m)) = \frac{(m+1)(m+2)\cdots(m+n)}{n!}.$$

So one lesson is this: if one cohomology group (usual the top or bottom) behaves well in a certain range, and then messes up, likely it is because (i) it is actually the Euler characteristic which is behaving well *always*, and (ii) the other cohomology groups vanish in that range.

In fact, we will see that it is often hard to calculate cohomology groups (even h^0), but it is often easier calculating Euler characteristics. So one important way of getting a hold of cohomology groups is by computing the Euler characteristics, and then showing that all the *other* cohomology groups vanish. Hence the ubiquity and importance of *vanishing theorems*. (A vanishing theorem usually states that a certain cohomology group vanishes under certain conditions.)

The following exercise already shows that Euler characteristic behaves well.

4.1. *Exercise.* Show that Euler characteristic is additive in exact sequences. In other words, if $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$ is an exact sequence of coherent sheaves on X, then $\chi(X, \mathcal{G}) = \chi(X, \mathcal{F}) + \chi(X, \mathcal{H})$. (Hint: consider the long exact sequence in cohomology.) More generally, if

$$0 \to \mathcal{F}_1 \to \cdots \to \mathcal{F}_n \to 0$$

is an exact sequence of sheaves, show that

$$\sum_{i=1}^{n} (-1)^i \chi(X, \mathcal{F}_i) = 0.$$

4.2. *Exercise.* Prove the *Riemann-Roch theorem* for line bundles on a nonsingular projective curve C over k: suppose $\mathcal{L}$ is an invertible sheaf on C. Show that $\chi(\mathcal{L}) = \deg \mathcal{L} + \chi(C, \mathcal{O}_C)$. (Possible hint: Write $\mathcal{L}$ as the difference of two effective Cartier divisors, $\mathcal{L} \cong \mathcal{O}(Z - P)$ ("zeros" minus "poles"). Describe two exact sequences $0 \to \mathcal{O}_C(-P) \to \mathcal{O}_C \to \mathcal{O}_P \to 0$ and $0 \to \mathcal{L}(-Z) \to \mathcal{L} \to \mathcal{O}_Z \otimes \mathcal{L} \to 0$, where $\mathcal{L}(-Z) \cong \mathcal{O}_C(P)$.)

If $\mathcal{F}$ is a coherent sheaf on X, define the *Hilbert function of* $\mathcal{F}$:

$$h_{\mathcal{F}}(n) := h^0(X, \mathcal{F}(n)).$$

The *Hilbert function of* X is the Hilbert function of the structure sheaf. The ancients were aware that the Hilbert function is "eventually polynomial", i.e. for large enough n, it agrees with some polynomial, called the *Hilbert polynomial* (and denoted $p_{\mathcal{F}}(n)$ or $p_X(n)$). In modern language, we expect that this is because the Euler characteristic should be a polynomial, and that for $n \gg 0$, the higher cohomology vanishes. This is indeed the case, as we now verify.

I ended by stating the following, which we will prove next day.

4.3. *Claim. — For* $n \gg 0$, $h^0(X, \mathcal{F}(n))$ *is a polynomial of degree equal to the dimension of the support of* $\mathcal{F}$. *In particular,* $h^0(X, \mathcal{O}_X(n))$ *is "eventually polynomial" with degree* = $\dim X$.

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FOUNDATIONS OF ALGEBRAIC GEOMETRY CLASS 31

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CONTENTS

Last day: Cohomology continued. Beginning of Hilbert functions

Today: Hilbert polynomials and Hilbert functions. Higher direct image sheaves.

1. APPLICATION OF COHOMOLOGY: HILBERT POLYNOMIALS AND HILBERT FUNCTIONS; DEGREES

We're in the process of seeing applications of cohomology. In this section, we will work over a field k. We defined $h^i(X, \mathcal{F}) := \dim_k H^i(X, \mathcal{F})$. If $\mathcal{F}$ is a coherent sheaf on a projective k-scheme X, we defined the *Euler characteristic*

$$\chi(X, \mathcal{F}) = \sum_{i=0}^{\dim X} (-1)^i h^i(X, \mathcal{F}).$$

We will see repeatedly here and later that Euler characteristics behave better than individual cohomology groups.

If $\mathcal{F}$ is a coherent sheaf on X, define the *Hilbert function of* $\mathcal{F}$:

$$h_{\mathcal{F}}(m) := h^0(X, \mathcal{F}(m)).$$

The *Hilbert function of* X is the Hilbert function of the structure sheaf $\mathcal{O}_X$. The ancients were aware that the Hilbert function is "eventually polynomial", i.e. for large enough n, it agrees with some polynomial, called the *Hilbert polynomial* (and denoted $p_{\mathcal{F}}(m)$ or $p_X(m)$). In modern language, we expect that this is because the Euler characteristic should be a polynomial, and that for $m \gg 0$, the higher cohomology vanishes. This is indeed the case, as we now verify.

Date: Tuesday, February 14, 2006. Tiny update April 9.

1.1. *Theorem. — If $\mathcal{F}$ is a coherent sheaf on a projective* k*-scheme* $X \hookrightarrow \mathbb{P}^n_k$*, for* $m \gg 0$*,* $h^0(X, \mathcal{F}(m))$ *is a polynomial of degree equal to the dimension of the support of* $\mathcal{F}$*. In particular, for* $m \gg 0$*,* $h^0(X, \mathcal{O}_X(m))$ *is polynomial with degree* = $\dim X$.

(Here $\mathcal{O}_X(m)$ is the restriction or pullback of $\mathcal{O}_{\mathbb{P}^n_k}(1)$.)

I realize now that I will use the notion of associated primes of a *module*, not just of a ring. I think I only discussed associated primes of a ring last quarter, because I had hoped not to need this slightly more general case. Now I really don't need it, and if you want to ignore this issue, you can just prove the second half of the theorem, which is all we will use anyway. But the argument carries through with no change, so please follow along if you can.

Proof. For $m \gg 0$, $h^i(X, \mathcal{F}(m)) = 0$ by Serre vanishing (class 29 Theorem 4.2(ii)), so instead we will prove that for *all* m, $\chi(X, \mathcal{F}(m))$ is a polynomial of degree equal to the dimension of the support of $\mathcal{F}$. Define $p_{\mathcal{F}}(m) = \chi(X, \mathcal{F}(m))$; we'll show that $p_{\mathcal{F}}(m)$ is a polynomial of the desired degree.

Our approach will be a little weird. We'll have two steps, and they will be very similar. If you can streamline, please let me know.

Step 1. We first show that for all n, if $\mathcal{F}$ is scheme-theoretically supported a linear subspace of dimension k (i.e. $\mathcal{F}$ is the pushforward of a coherent sheaf on some linear subspace of dimension k), then $p_{\mathcal{F}}(m)$ is a polynomial of degree at most k. (In particular, for any coherent $\mathcal{F}$, $p_{\mathcal{F}}(m)$ is a polynomial of degree at most n.)

We prove this by induction on the dimension of the support. I'll leave the base case $k = 0$ (or better yet, $k = -1$) to you (*exercise*). Suppose now that X is supported in a linear space Λ of dimension k, and we know the result for all $k' < k$. Then let $x = 0$ be a hyperplane not containing Λ, so $\Lambda' = \dim(x = 0) \cap \Lambda = k - 1$. Then we have an exact sequence

$$0 \longrightarrow \mathcal{K} \longrightarrow \mathcal{F} \xrightarrow{\times x} \mathcal{F}(1) \longrightarrow \mathcal{K}' \longrightarrow 0 \qquad (1)$$

where $\mathcal{K}$ (resp. $\mathcal{K}'$) is the kernel (resp. cokernel) of the map $\times x$. Notice that $\mathcal{K}$ and $\mathcal{K}'$ are both supported on Λ'. (This corresponds to an algebraic fact: over an affine open $\operatorname{Spec} A$, the exact sequence is

$$0 \longrightarrow K \longrightarrow M \xrightarrow{\times x} M \longrightarrow K' \longrightarrow 0$$

and both $K = \ker(\times x) = (0 : x)$ and $K' \cong M/xM$ are (A/x)-modules.) Twist (1) by $\mathcal{O}(m)$ and take Euler-characteristics to obtain $p_{\mathcal{F}}(m+1) - p_{\mathcal{F}}(m) = p_{\mathcal{K}'}(m) - p_{\mathcal{K}}(m)$. By the inductive hypothesis, the right side of this equation is a polynomial of degree at most $k - 1$. Hence (by an easy induction) $p(m)$ is a polynomial of degree at most k.

Step 2. We'll now show that the degree of this polynomial is precisely $\dim \operatorname{Supp} \mathcal{F}$. As $\mathcal{F}$ is a coherent sheaf on a Noetherian scheme, it has a finite number of associated points, so we can find a hypersurface $H = (f = 0)$ not containing any of the associated points. (This is that problem from last quarter that we have been repeatedly using recently: problem 24(c) on set 5, which was exercise 1.19 in the class 11 notes.) In particular, $\dim H \cap \operatorname{Supp} \mathcal{F}$

is strictly less than $\dim \operatorname{Supp} \mathcal{F}$, and in fact one less by Krull's Principal Ideal Theorem. Let $d = \deg f$. Then I claim that $\times f : \mathcal{F}(-d) \to \mathcal{F}$ is an inclusion. Indeed, on any affine open set, the map is of the form $\times \overline{f} : M \to M$ (where $\overline{f}$ is the restriction of f to this open set), and the fact that $f = 0$ contains no associated points *means* that this is an injection of modules. (Remember that those ring elements annihilating elements of M are precisely the associated primes, and $\overline{f}$ is contained in none of them.) Then we have

$$0 \to \mathcal{F}(-d) \to \mathcal{F} \to \mathcal{K}' \to 0.$$

Twisting by $\mathcal{O}(m)$ yields

$$0 \to \mathcal{F}(m-d) \to \mathcal{F}(m) \to \mathcal{K}'(m) \to 0.$$

Taking Euler characteristics gives $p_{\mathcal{F}}(m) - p_{\mathcal{F}}(m-d) = p_{\mathcal{K}'}(m)$. Now by step 1, we know that $p_{\mathcal{F}}(m)$ is a polynomial. Also, by our inductive hypothesis, and Exercise 1.2 below, the right side is a polynomial of degree of precisely $\dim \operatorname{Supp} \mathcal{F} - 1$. Hence $p(m)$ is a polynomial of degree $\dim \operatorname{Supp} \mathcal{F}$. □

1.2. *Exercise.* Consider the short exact sequence of A-modules $0 \longrightarrow M \xrightarrow{\times f} M \longrightarrow K' \longrightarrow 0$. Show that $\operatorname{Supp} K' = \operatorname{Supp}(M) \cap \operatorname{Supp}(A/f)$.

Notice that we needed the first part of the proof to ensure that $p_{\mathcal{F}}(m)$ is in fact a polynomial; otherwise, the second part would just show that $p_{\mathcal{F}}(m)$ is just a polynomial when m is fixed modulo d.

(For experts: here is a different way to avoid having two similar steps. If k is an infinite field, e.g. if it were algebraically closed, then we could find a hypersurface as in step 2 of degree 1, using that problem from last quarter mentioned in the proof. So what to do if k is not infinite? Note that if you have a complex of k-vector spaces, and you take its cohomology, and then tensor with $\overline{k}$, you get the same thing as if you tensor first, and then take the cohomology. By this trick, we can assume that k is algebraically closed. In fancy language: we have taken a *faithfully flat* base extension. I won't define what this means here; it will turn up early in the third quarter.)

Example 1. $p_{\mathbb{P}^n}(m) = \binom{m+n}{n}$, where we interpret this as the polynomial $(m+1)\cdots(m+n)/n!$.

Example 2. Suppose H is a degree d hypersurface in $\mathbb{P}^n$. Then from

$$0 \to \mathcal{O}_{\mathbb{P}^n}(-d) \to \mathcal{O}_{\mathbb{P}^n} \to \mathcal{O}_H \to 0,$$

we have

$$p_H(m) = p_{\mathbb{P}^n}(m) - p_{\mathbb{P}^n}(m-d) = \binom{m+n}{n} - \binom{m+n-d}{n}.$$

1.3. *Exercise.* Show that the twisted cubic (in $\mathbb{P}^3$) has Hilbert polynomial $3m+1$.

1.4. *Exercise.* Find the Hilbert polynomial for the dth Veronese embedding of $\mathbb{P}^n$ (i.e. the closed immersion of $\mathbb{P}^n$ in a bigger projective space by way of the line bundle $\mathcal{O}(d)$).

From the Hilbert polynomial, we can extract many invariants, of which two are particularly important. The first is the *degree*. Classically, the degree of a complex projective variety of dimension n was defined as follows. We slice the variety with n generally chosen hyperplane. Then the intersection will be a finite number of points. The degree is this number of points. Of course, this requires showing all sorts of things. Instead, we will define the *degree of a projective k-scheme of dimension* n to be leading coefficient of the Hilbert polynomial (the coefficient of m^n) times $n!$.

For example, the degree of $\mathbb{P}^n$ in itself is 1. The degree of the twisted cubic is 3.

1.5. *Exercise.* Show that the degree of a degree d hypersurface is d (preventing a notational crisis).

1.6. *Exercise.* Suppose a curve C is embedded in projective space via an invertible sheaf of degree d. (In other words, this line bundle determines a closed immersion.) Show that the degree of C under this embedding is d (preventing another notational crisis). (Hint: Riemann-Roch.)

1.7. *Exercise.* Find the degree of the dth Veronese embedding of $\mathbb{P}^n$.

1.8. *Exercise (Bezout's theorem).* Suppose X is a projective scheme of dimension at least 1, and H is a degree d hypersurface not containing any associated points of X. (For example, if X is a projective variety, then we are just requiring H not to contain any irreducible components of X.) Show that $\deg H \cap X = d \deg X$.

This is a very handy theorem! For example: if two projective plane curves of degree m and degree n share no irreducible components, then they intersect in mn points, counted with appropriate multiplicity. The notion of multiplicity of intersection is just the degree of the intersection as a k-scheme.

We trot out a useful example for a third time: let $k = \mathbb{Q}$, and consider the parabola $x = y^2$. We intersect it with the four usual suspects: $x = 1$, $x = 0$, $x = -1$, and $x = 2$, and see that we get 2 each time (counted with the same convention as with the last time we saw this example).

If we intersect it with $y = 2$, we only get one point — but that's of course because this isn't a projective curve, and we really should be doing this intersection on $\mathbb{P}^2_k$ — and in this case, the conic meets the line in two points, one of which is "at ∞".

1.9. *Exercise.* Determine the degree of the d-fold Veronese embedding of $\mathbb{P}^n$ in a different way as follows. Let $v_d : \mathbb{P}^n \to \mathbb{P}^N$ be the Veronese embedding. To find the degree of the image, we intersect it with n hyperplanes in $\mathbb{P}^N$ (scheme-theoretically), and find the number of intersection points (counted with multiplicity). But the pullback of a hyperplane in $\mathbb{P}^N$ to $\mathbb{P}^n$ is a degree d hypersurface. Perform this intersection in $\mathbb{P}^n$, and use Bezout's theorem. (If already you know the answer by the earlier exercise on the degree of the Veronese embedding, this will be easier.)

There is another nice bit of information residing in the Hilbert polynomial. Notice that $p_X(0) = \chi(X, \mathcal{O}_X)$, which is an *intrinsic* invariant of the scheme X, which does not depend on the projective embedding.

Imagine how amazing this must have seemed to the ancients: they defined the Hilbert function by counting how many "functions of various degrees" there are; then they noticed that when the degree gets large, it agrees with a polynomial; and then when they plugged 0 into the polynomial — extrapolating backwards, to where the Hilbert function and Hilbert polynomials didn't agree — they found a magic invariant!

And now I can give you a nonsingular curve over an algebraically closed field that is not $\mathbb{P}^1$! Note that the Hilbert polynomial of $\mathbb{P}^1$ is $(m+1)/1 = m+1$, so $\chi(\mathcal{O}_{\mathbb{P}^1}) = 1$. Suppose C is a degree d curve in $\mathbb{P}^2$. Then the Hilbert polynomial of C is

$$p_{\mathbb{P}^2}(m) - p_{\mathbb{P}^2}(m-d) = (m+1)(m+2)/2 - (m-d+1)(m-d+2)/2.$$

Plugging in $m = 0$ gives us $-(d^2 - 3d)/2$. Thus when $d > 2$, we have a curve that cannot be isomorphic to $\mathbb{P}^1$! (I think I gave you an earlier exercise that there is a *nonsingular* degree d curve. Note however that the calculation above didn't use nonsingularity.)

Now from $0 \to \mathcal{O}_{\mathbb{P}^2}(-d) \to \mathcal{O}_{\mathbb{P}^2} \to \mathcal{O}_C \to 0$, using $h^1(\mathcal{O}_{\mathbb{P}^2}(d)) = 0$, we have that $h^0(C, \mathcal{O}_C) = 1$. As $h^0 - h^1 = \chi$, we have

$$h^1(C, \mathcal{O}_C) = (d-1)(d-2)/2.$$

Motivated by geometry, we define the *arithmetic genus* of a scheme X as $1 - \chi(X, \mathcal{O}_X)$. This is sometimes denoted $p_a(X)$. In the case of nonsingular complex curves, this corresponds to the topological genus. For irreducible reduced curves (or more generally, curves with $h^0(X, \mathcal{O}_X) \cong k$), $p_a(X) = h^1(X, \mathcal{O}_X)$. (In higher dimension, this is a less natural notion.)

We thus now have examples of curves of genus $0, 1, 3, 6, 10, \ldots$ (corresponding to degree 1 or $2, 3, 4, 5, \ldots$).

This begs some questions, such as: are there curves of other genera? (Yes.) Are there other genus 1 curves? (Not if k is algebraically closed, but yes otherwise.) Do we have all the curves of genus 3? (Almost all, but not quite all.) Do we have all the curves of genus 6? (We're missing most of them.)

Caution: The Euler characteristic of the structure sheaf doesn't distinguish between isomorphism classes of nonsingular projective schemes over algebraically closed fields — for example, $\mathbb{P}^1 \times \mathbb{P}^1$ and $\mathbb{P}^2$ both have Euler characteristic 1, but are not isomorphic (as for example $\operatorname{Pic} \mathbb{P}^2 \cong \mathbb{Z}$ while $\operatorname{Pic} \mathbb{P}^1 \times \mathbb{P}^1 \cong \mathbb{Z} \oplus \mathbb{Z}$).

Important Remark. We can restate the Riemann-Roch formula as:

$$h^0(C, \mathcal{L}) - h^1(C, \mathcal{L}) = \deg \mathcal{L} - p_a + 1.$$

This is the most common formulation of the Riemann-Roch formula.

1.10. *Complete intersections.* We define a *complete intersection* in $\mathbb{P}^n$ as follows. $\mathbb{P}^n$ is a complete intersection in itself. A closed subscheme $X_r \hookrightarrow \mathbb{P}^n$ of dimension r (with $r < n$) is a complete intersection if there is a complete intersection X_{r+1}, and X_r is a Cartier divisor in class $\mathcal{O}_{X_{r+1}}(d)$.

Exercise. Show that if X is a complete intersection of dimension r in $\mathbb{P}^n$, then $H^i(X, \mathcal{O}_X(m)) = 0$ for all $0 < i < r$ and all m. Show that if $r > 0$, then $H^0(\mathbb{P}^n, \mathcal{O}(m)) \to H^0(X, \mathcal{O}(m))$ is surjective.

Now in my definition, X_r is the zero-divisor of a section of $\mathcal{O}_{X_{r+1}}(m)$ for some m. But this section is the restriction of a section of $\mathcal{O}(m)$ on $\mathbb{P}^n$. Hence X_r is the scheme-theoretic intersection of X_{r+1} with a hypersurface. Thus inductively we can show that X_r is the scheme-theoretic intersection of $n - r$ hypersurfaces. (By Bezout's theorem, $\deg X_r$ is the product of the degree of the defining hypersurfaces.)

Exercise. Show that complete intersections of positive dimension are connected. (Hint: show $h^0(X, \mathcal{O}_X) = 1$.)

Exercise. Find the genus of the intersection of 2 quadrics in $\mathbb{P}^3$. (We get curves of more genera by generalizing this!)

Exercise. Show that the rational normal curve of degree d in $\mathbb{P}^d$ is *not* a complete intersection if $d > 2$.

Exercise. Show that the union of 2 distinct planes in $\mathbb{P}^4$ is not a complete intersection. (This is the first appearance of another universal counterexample!) Hint: it is connected, but you can slice with another plane and get something not connected.

This is another important scheme in algebraic geometry that is an example of many sorts of behavior. We will see more of it later!

2. Higher direct image sheaves

I'll now introduce a notion generalizing these Cech cohomology groups. Cohomology groups were defined for $X \to \operatorname{Spec} A$ where the structure morphism is quasicompact and separated; for any quasicoherent $\mathcal{F}$ on X, we defined $H^i(X, \mathcal{F})$.

We'll now do something similar for quasicompact and separated morphisms $\pi : X \to Y$: for any quasicoherent $\mathcal{F}$ on X, we'll define $R^i\pi_*\mathcal{F}$, a quasicoherent sheaf on Y.

We have many motivations for doing this. In no particular order:

(1) It "globalizes" what we were doing anywhere.
(2) If $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$ is a short exact sequence of quasicoherent sheaves on X, then we know that $0 \to \pi_*\mathcal{F} \to \pi_*\mathcal{G} \to \pi_*\mathcal{H}$ is exact, and higher pushforwards will extend this to a long exact sequence.

(3) We'll later see that this will show how cohomology groups vary in families, especially in "nice" situations. Intuitively, if we have a nice family of varieties, and a family of sheaves on them, we could hope that the cohomology varies nicely in families, and in fact in "nice" situations, this is true. (As always, "nice" usually means "flat", whatever that means.)

There will be no extra work involved for us.

Suppose $\pi : X \to Y$, and $\mathcal{F}$ is a quasicoherent sheaf on X. For each $\operatorname{Spec} A \subset Y$, we have A-modules $H^i(\pi^{-1}(\operatorname{Spec} A), \mathcal{F})$. We will show that these patch together to form a quasicoherent sheaf. We need check only one fact: that this behaves well with respect to taking distinguished opens. In other words, we must check that for each $f \in A$, the natural map $H^i(\pi^{-1}(\operatorname{Spec} A), \mathcal{F}) \to H^i(\pi^{-1}(\operatorname{Spec} A), \mathcal{F})_f$ (induced by the map of spaces in the opposite direction — H^i is contravariant in the space) is precisely the localization $\otimes_A A_f$. But this can be verified easily: let $\{U_i\}$ be an affine cover of $\pi^{-1}(\operatorname{Spec} A)$. We can compute $H^i(\pi^{-1}(\operatorname{Spec} A), \mathcal{F})$ using the Cech complex. But this induces a cover $\operatorname{Spec} A_f$ in a natural way: If $U_i = \operatorname{Spec} A_i$ is an affine open for $\operatorname{Spec} A$, we define $U_i' = \operatorname{Spec}(A_i)_f$. The resulting Cech complex for $\operatorname{Spec} A_f$ is the localization of the Cech complex for $\operatorname{Spec} A$. As taking cohomology of a complex commutes with localization, we have defined a quasicoherent sheaf on Y by one of our definitions of quasicoherent sheaves.

2.1. (Something important happened in that last sentence — localization commuting with taking cohomology. If you want practice with this notion, here is an *exercise:* suppose $C^0 \to C^1 \to C^2$ is a complex in an abelian category, and F is an exact functor to another abelian category. Show that F applied to the cohomology of this complex is naturally isomorphic to the cohomology of F of this complex. Translation: taking cohomology commutes with exact functors. In the particular case of this construction, the exact functor in equation is the localization functor $\otimes_A A_f$ from A-modules to A_f-modules. I'll discuss this a bit more at the start of the class 32 notes.)

Define the i**th higher direct image sheaf** or the i**th (higher) pushforward sheaf** to be this quasicoherent sheaf.

2.2. *Theorem. —*

(a) $R^0\pi_*\mathcal{F}$ *is canonically isomorphic to* $\pi_*\mathcal{F}$.
(b) $R^i\pi_*$ *is a covariant functor from the category of quasicoherent sheaves on* X *to the category of quasicoherent sheaves on* Y, *and a contravariant functor in* Y*-schemes* X.
(c) *A short exact sequence* $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$ *of sheaves on* X *induces a long exact sequence*

$$0 \longrightarrow R^0\pi_*\mathcal{F} \longrightarrow R^0\pi_*\mathcal{G} \longrightarrow R^0\pi_*\mathcal{H} \longrightarrow$$

$$R^1\pi_*\mathcal{F} \longrightarrow R^1\pi_*\mathcal{G} \longrightarrow R^1\pi_*\mathcal{H} \longrightarrow \cdots$$

of sheaves on Y. *(This is often called the corresponding* **long exact sequence of higher pushforward sheaves***.)*

(d) *(projective pushforwards of coherent are coherent) If* π *is a projective morphism and* $\mathcal{O}_Y$ *is coherent on* Y *(this hypothesis is automatic for* Y *locally Noetherian), and* $\mathcal{F}$ *is a coherent sheaf on* X*, then for all* i, $R^i\pi_*\mathcal{F}$ *is a coherent sheaf on* Y.

Proof. Because it suffices to check each of these results on affine opens, they all follow from the analogous statements in Cech cohomology. □

The following result is handy (and essentially immediate from our definition).

2.3. *Exercise.* Show that if π is affine, then for $i > 0$, $R^i\pi_*\mathcal{F} = 0$. Moreover, show that if Y is quasicompact and quasiseparated then the natural morphism $H^i(X, \mathcal{F}) \to H^i(Y, f_*\mathcal{F})$ is an isomorphism. (A special case of the first sentence is a special case we showed earlier, when π is a closed immersion. Hint: use any affine cover on Y, which will induce an affine cover of X.)

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FOUNDATIONS OF ALGEBRAIC GEOMETRY CLASS 32

RAVI VAKIL

CONTENTS

Last day: Hilbert polynomials and Hilbert functions. Higher direct image sheaves.

Today: Applications of higher pushforwards; crash course in spectral sequences; towards the Leray spectral sequence.

1. A USEFUL ALGEBRAIC FACT

I'd like to start with an algebra exercise that is very useful.

1.1. *Exercise (Important algebra exercise).* Suppose $M_1 \xrightarrow{\alpha} M_2 \xrightarrow{\beta} M_3$ is a complex of A-modules (i.e. $\beta \circ \alpha = 0$), and N is an A-module. (a) Describe a natural homomorphism of the cohomology of the complex, tensored with N, with the cohomology of the complex you get when you tensor with N, $H(M_*) \otimes_A B \to H(M_* \otimes_A N)$, i.e.

$$\left(\frac{\ker \beta}{\operatorname{im} \alpha}\right) \otimes_A N \to \frac{\ker(\beta \otimes N)}{\operatorname{im}(\alpha \otimes N)}.$$

I always forget which way this map is supposed to go.
(b) If N is *flat*, i.e. $\otimes N$ is an exact functor, show that the morphism defined above is an isomorphism. (Hint: This is actually a categorical question: if M_* is an exact sequence in an abelian category, and F is a right-exact functor, then (a) there is a natural morphism $FH(M_*) \to H(FM_*)$, and (b) if F is an exact functor, this morphism is an isomorphism.)

Example: localization is exact, so $S^{-1}A$ is a *flat* A-algebra for all multiplicative sets S. In particular, A_f is a flat A-algebra. We used (b) implicitly last day, when I said that given a quasicompact, separated morphism $\pi : X \to Y$, and an affine open subset $\operatorname{Spec} A$ of Y, and a distinguished affine open $\operatorname{Spec} A_f$ of that, the cohomology of any Cech complex computing the cohomology $\pi^{-1}(\operatorname{Spec} A)$, tensored with A_f, would be naturally isomorphic to the cohomology of the complex you get when you tensor with A_f.

Date: Thursday, February 16, 2006. Updated June 26.

Here is another example.

1.2. *Exercise (Higher pushforwards and base change).* (a) Suppose $f : Z \to Y$ is any morphism, and $\pi : X \to Y$ as usual is quasicompact and separated. Suppose $\mathcal{F}$ is a quasicoherent sheaf on X. Let

$$\begin{array}{ccc} W & \xrightarrow{f'} & X \\ \downarrow{\scriptstyle \pi'} & & \downarrow{\scriptstyle \pi} \\ Z & \xrightarrow{f} & Y \end{array}$$

is a fiber diagram. Describe a natural morphism $f^*(R^i\pi_*\mathcal{F}) \to R^i\pi'_*(f')^*\mathcal{F}$.
(b) If $f : Z \to Y$ is an affine morphism, and for a cover $\operatorname{Spec} A_i$ of Y, where $f^{-1}(\operatorname{Spec} A_i) = \operatorname{Spec} B_i$, B_i is a *flat* A-algebra, show that the natural morphism of (a) is an isomorphism. (You can likely generalize this immediately, but this will lead us into the concept of flat morphisms, and we'll hold off discussing this notion for a while.)

A useful special case if the following. If f is a closed immersion of a closed point in Y, the right side is the cohomology of the fiber, and the left side is the fiber of the cohomology. In other words, the fiber of the higher pushforward maps naturally to the cohomology of the fiber. We'll later see that in good situations this is an isomorphism, and thus the higher direct image sheaf indeed "patches together" the cohomology on fibers.

Here is one more consequence of our algebraic fact.

1.3. *Exercise (projection formula).* Suppose $\pi : X \to Y$ is quasicompact and separated, and $\mathcal{E}, \mathcal{F}$ are quasicoherent sheaves on X and Y respectively. (a) Describe a natural morphism

$$(R^i\pi_*\mathcal{E}) \otimes \mathcal{F} \to R^i\pi_*(\mathcal{E} \otimes \pi^*\mathcal{F}).$$

(b) If $\mathcal{F}$ is locally free, show that this natural morphism is an isomorphism.

Here is another consequence, that I stated in class 33. (It is still also in the class 33 notes.)

Exercise. Suppose that X is a quasicompact separated k-scheme, where k is a field. Suppose $\mathcal{F}$ is a quasicoherent sheaf on X. Let $X_{\overline{k}} = X \times_{\operatorname{Spec} k} \operatorname{Spec} \overline{k}$, and $f : X_{\overline{k}} \to X$ the projection. Describe a natural isomorphism $H^i(X, \mathcal{F}) \otimes_k \overline{k} \to H^i(X_{\overline{k}}, f^*\mathcal{F})$. Recall that a k-scheme X is *geometrically integral* if $X_{\overline{k}}$ is integral. Show that if X is geometrically integral and projective, then $H^0(X, \mathcal{O}_X) \cong k$. (This is a clue that $\mathbb{P}^1_{\mathbb{C}}$ is not a geometrically integral $\mathbb{R}$-scheme.)

2. Fun applications of the higher pushforward

Last day we proved that if $\pi : X \to Y$ is a projective morphism, and $\mathcal{F}$ is a coherent sheaf on X, then $\pi_*\mathcal{F}$ is coherent (under a technical assumption: if either Y and hence X are Noetherian; or more generally if $\mathcal{O}_Y$ is a coherent sheaf).

As a nice immediate consequence is the following. Finite morphisms are affine (from the definition) and projective (an earlier exercise); the converse also holds.

2.1. *Corollary. — If $\pi : X \to Y$ is projective and affine and $\mathcal{O}_Y$ is coherent, then π is finite.*

In fact, more generally, if π is universally closed and affine, then π is finite. We won't use this, so I won't explain why, but you can read about it in Atiyah-Macdonald, Exercise 5.35.

Proof. By the theorem from last day, $\pi_* \mathcal{O}_X$ is coherent and hence finitely generated. □

Here is another handy theorem.

2.2. *Theorem (relative dimensional vanishing). — If $f : X \to Y$ is a projective morphism and $\mathcal{O}_Y$ is coherent, then the higher pushforwards vanish in degree higher than the maximum dimension of the fibers.*

This is false without the projective hypothesis. Here is an example of why.

Exercise. Consider the open immersion $\pi : \mathbb{A}^n - 0 \to \mathbb{A}^n$. By direct calculation, show that $R^{n-1} f_* \mathcal{O}_{\mathbb{A}^n - 0} \neq 0$.

Proof. Let m be the maximum dimension of all the fibers.

The question is local on Y, so we'll show that the result holds near a point p of Y. We may assume that Y is affine, and hence that $X \hookrightarrow \mathbb{P}^n_Y$.

Let k be the residue field at p. Then $f^{-1}(p)$ is a projective k-scheme of dimension at most m. Thus we can find affine open sets $D(f_1), \dots, D(f_{m+1})$ that cover $f^{-1}(p)$. In other words, the intersection of $V(f_i)$ does not intersect $f^{-1}(p)$.

If $Y = \operatorname{Spec} A$ and $p = [\mathfrak{p}]$ (so $k = A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p}$), then arbitrarily lift each f_i from an element of $k[x_0, \dots, x_n]$ to an element f_i' of $A_\mathfrak{p}[x_0, \dots, x_n]$. Let F be the product of the denominators of the f_i'; note that $F \notin \mathfrak{p}$, i.e. $p = [\mathfrak{p}] \in D(F)$. Then $f_i' \in A_F[x_0, \dots, x_n]$. The intersection of their zero loci $\cap V(f_i') \subset \mathbb{P}^n_{A_F}$ is a closed subscheme of $\mathbb{P}^n_{A_F}$. Intersect it with X to get another closed subscheme of $\mathbb{P}^n_{A_F}$. Take its image under f; as projective morphisms are closed, we get a closed subset of $D(F) = \operatorname{Spec} A_F$. But this closed subset does not include p; hence we can find an affine neighborhood $\operatorname{Spec} B$ of p in Y missing the image. But if f_i'' are the restrictions of f_i' to $B[x_0, \dots, x_n]$, then $D(f_i'')$ cover $f^{-1}(\operatorname{Spec} B)$; in other words, over $f^{-1}(\operatorname{Spec} B)$ is covered by $m+1$ affine open sets, so by the affine-cover vanishing theorem, its cohomology vanishes in degree at least $m+1$. But the higher-direct image sheaf is computed using these cohomology groups, hence the higher direct image sheaf $R^i f_* \mathcal{F}$ vanishes on $\operatorname{Spec} B$ too. □

2.3. *Important Exercise.* Use a similar argument to prove *semicontinuity of fiber dimension of projective morphisms*: suppose $\pi : X \to Y$ is a projective morphism where $\mathcal{O}_Y$ is coherent. Show that $\{y \in Y : \dim f^{-1}(y) > k\}$ is a Zariski-closed subset. In other words, the dimension of the fiber "jumps over Zariski-closed subsets". (You can interpret the case $k = -1$ as the fact that projective morphisms are closed.) This exercise is rather important for having a sense of how projective morphisms behave! Presumably the result is true more generally for proper morphisms.

Here is another handy theorem, that is proved by a similar argument. We know that finite morphisms are projective, and have finite fibers. Here is the converse.

2.4. *Theorem (projective + finite fibers = finite).* — *Suppose $\pi : X \to Y$ is such that $\mathcal{O}_Y$ is coherent. Then π is projective and finite fibers if and only if it is finite. Equivalently, π is projective and quasifinite if and only it is finite.*

(Recall that quasifinite = finite fibers + finite type. But projective includes finite type.)

It is true more generally that proper + quasifinite = finite. (We may see that later.)

Proof. We show it is finite near a point $y \in Y$. Fix an affine open neighborhood $\operatorname{Spec} A$ of y in Y. Pick a hypersurface H in $\mathbb{P}^n_A$ missing the preimage of y, so $H \cap X$ is closed. (You can take this as a hint for Exercise 2.3!) Let $H' = \pi_*(H \cap X)$, which is closed, and doesn't contain y. Let $U = \operatorname{Spec} R - H'$, which is an open set containing y. Then above U, π is projective and affine, so we are done by the previous Corollary 2.1. $\square$

Here is one last potentially useful fact. (To be honest, I'm not sure if we'll use it in this course.)

2.5. *Exercise.* Suppose $f : X \to Y$ is a projective morphism, with $\mathcal{O}(1)$ on X. Suppose Y is quasicompact and $\mathcal{O}_Y$ is coherent. Let $\mathcal{F}$ be coherent on X. Show that

(a) $f^* f_* \mathcal{F}(n) \to \mathcal{F}(n)$ is surjective for $n \gg 0$. (First show that there is a natural map for any n! Hint: by adjointness of f_* with f_*.) Translation: for $n \gg 0$, $\mathcal{F}(n)$ is relatively generated by global sections.
(b) For $i > 0$ and $n \gg 0$, $R^i f_* \mathcal{F}(n) = 0$.

3. Toward the Leray spectral sequence: Crash course in spectral sequences

My goal now is to tell you enough about spectral sequences that you'll have a good handle on how to use them in practice, and why you shouldn't be frightened when they come up in a seminar. There will be some key points that I will not prove; it would be good, once in your life, to see a proof of these facts, or even better, to prove it yourself. Then in good conscience you'll know how the machine works, and you can close the hood once and for all and just happily drive the powerful machine.

My philosophy will be to tell you just about a stripped down version of spectral sequences, which frankly is what is used most of the time. You can always gussy it up later on. But it will be enough to give a quick proof of the Leray spectral sequence.

A good reference as always is Weibel. I learned it from Lang's *Algebra*. I don't necessarily endorse that, but at least his exposition is just a few pages long.

Let's get down to business.

For me, a *double complex* (in an abelian category) will be a bunch of objects $A^{p,q}$ ($p, q \in \mathbb{Z}$), which are zero unless $p, q \geq 0$, and morphisms $d^{p,q} : A^{p,q} \to A^{p+1,q}$ and $\delta^{p,q} : A^{p,q} \to A^{p,q+1}$ (we will always write these as d and δ and ignore the subscripts) satisfying $d^2 = 0$ and $\delta^2 = 0$, and one more condition: either $d\delta = \delta d$ ("all the squares commute") or $d\delta + \delta d = 0$ (they all "anticommute"). Both come up, and you can switch from one to the other by replacing $\delta^{p,q}$ with $(-1)^p \delta^{p,q}$. So I'll hereafter presume that all the squares anticommute, but that you know how to turn the commuting case into this one.

Also, there are variations on this definition, where for example the vertical arrows go downwards, or some different subset of the $A^{p,q}$ are required to be zero, but I'll leave these straightforward variations to you.

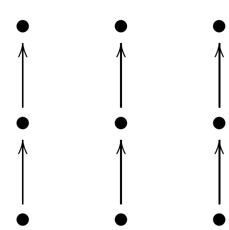
From the double complex (with the anticommuting convention), we construct a corresponding (single) complex A^* with $A^k = \oplus_i A^{i,k-i}$, with $D = d + \delta$. Note that $D^2 = (d+\delta)^2 = d^2 + (d\delta + \delta d) + \delta^2 = 0$, so A^* is indeed a complex. (Be sure you see how to interpret this in $A^{*,*}$!)

The *cohomology* of the single complex is sometimes called the *hypercohomology* of the double complex.

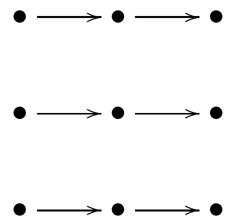
Our motivating goal will be to find the hypercohomology of the double complex. (You'll see later that we'll have other real goals, and that this is a red herring.)

Then here is recipe for computing (information) about the cohomology. We create a countable sequence of tables as follows. Table 0, denoted $E_0^{p,q}$, is defined as follows: $E_0^{p,q} = A^{p,q}$.

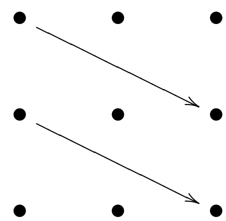
We then look just at the vertical arrows (the δ-arrows).



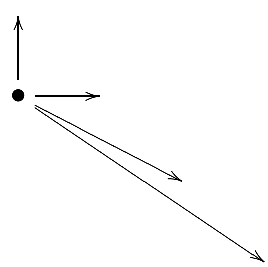
The columns are complexes, so we take cohomology of these vertical complexes, resulting in a new table, $E_1^{p,q}$. Then there are natural morphisms from each entry of the new table to the entry on the right. (This needs to be checked!)



The composition of two of these morphisms is again zero, so again we have complexes. We take cohomology of these as well, resulting in a new table, $E_2^{p,q}$. It turns out that there are natural morphisms from each entry to the entry two to the right and one below, and that the composition of these two is 0.



This can go on until the cows come home. The order of the morphisms is shown pictorially below.



(Notice that the map always is "degree 1" in the grading of the single complex.)

Now if you follow any entry in our original table, eventually the arrow into it will come from outside of the first quadrant, and the arrow out of it will go to outside the first quadrant, so after a certain stage the complex will look like $0 \to E_?^{p,q} \to 0$. Then after that stage, the (p,q)-entry will never change. We define $E_\infty^{p,q}$ to be the table whose (p,q)th entry is this object. We say that $E_k^{p,q}$ *converges to* $E_\infty^{p,q}$.

Then it is a fact (or even a theorem) that there is a filtration of $H^k(A^*)$ by More precisely you can filter $H^k(A^*)$ with $k+1$ objects whose successive quotients are $E_\infty^{i,k-i}$, where the sub-object is $E_\infty^{k,0}$, and the quotient $H^k(A^*)$ by the next biggest object is $E_\infty^{0,k}$. I hope that is clear; please let me know if I can say it better! The following may help:

	$E_\infty^{0,k}$		$E_\infty^{1,k-1}$		$E_\infty^{k-1,1}$		$E_\infty^{k,0}$	
$H^k(A^*)$	$\supset$	?	$\supset$	$? \supset \cdots \supset ?$	$\supset$	?	$\supset$	0

(I always forget which way the quotients are supposed to go. One way of remembering it is by having some idea of how the result is proved. The picture here is that the double complex is filtered by subcomplexes $\oplus_{p \geq k, q \geq 0} A^{p,q}$, and the first term corresponding by taking the cohomology of the subquotients of this filtration. Then the "biggest quotient" corresponds to the left column, which remains true at the level of cohomology. If this doesn't help you, just ignore this parenthetical comment. If you have a better way of remembering this, even a mnemonic trick, please let me know!)

The sequence $E_k^{p,q}$ is called a *spectral sequence*, and we say that it *abuts* to $H^*(A^*)$. We often say that $E_2^{p,q}$ (or any other term) abuts to H.

Unfortunately, you only get partial information about $H^*(A^*)$. But there are some cases where you get more information: if all $E_\infty^{i,k-i}$ are zero, or if all but one of them are zero; or if we are in the category of vector spaces over a field k, and are interested only in the dimension of $H^*(A^*)$.

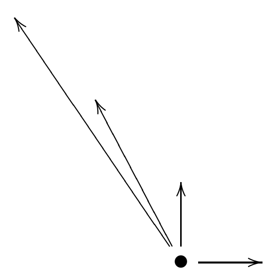
Also, in good circumstances, E_2 (or some other low term) already equals E_∞.

3.1. *Exercise.* Show that $H^0(A^*) = E_\infty^{0,0} = E_2^{0,0}$ and

$$0 \to E_2^{1,0} \to H^1(A^*) \to E_2^{0,1} \to E_2^{2,0} \to H^2(A^*).$$

3.2. *Exercise.* Suppose we are working in the category of vector spaces over a field k, and $\oplus_{p,q} E_2^{p,q}$ is a finite-dimensional vector space. Show that $\chi(H^*(A^*))$ is well-defined, and equals $\sum_{p,q}(-1)^{p+q} E_2^{p,q}$. (It will sometimes happen that $\oplus E_0^{p,q}$ will be an infinite-dimensional vector space, but that $E_2^{p,q}$ will be finite-dimensional!)

Eric pointed out that I was being a moron, and I could just as well have done everything in the opposite direction, i.e. reversing the roles of horizontal and vertical morphisms. Then the sequences of arrows giving the spectral sequence would look like this:



Then we would again get pieces of a filtration of $H^*(A^*)$ (where we have to be a bit careful with the order with which $E_\infty^{p,q}$ corresponds to the subquotients — it in the opposite order to the previous case).

I tried unsuccessfully to convince that Eric that I am *not* a moron, and that this was my secret plan all along. Both algorithms compute the same thing, and usually we don't care about the final answer — we often care about the answer we get in one way, and we get at it by doing the spectral sequence in the *other* way.

Now we're ready to try this out, and see how to use it in practice.

The moral of these examples is what follows: in the past, you've had prove various facts involving various sorts of diagrams, which involved chasing elements all around. Now, you'll just plug them into a spectral sequence, and let the spectral sequence machinery do your chasing for you.

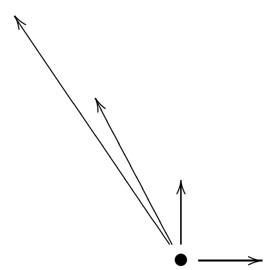
Example: Proving the snake lemma. Consider the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & D & \longrightarrow & E & \longrightarrow & F & \longrightarrow & 0 \\ & & \alpha\uparrow & & \beta\uparrow & & \gamma\uparrow & & \\ 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \end{array}$$

where the rows are exact and the squares commute. (Normally the snake lemma is described with the vertical arrows pointing downwards, but I want to fit this into my spectral sequence conventions.) We wish to show that there is an exact sequence

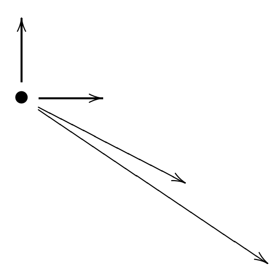
$$0 \to \ker\alpha \to \ker\beta \to \ker\gamma \to \operatorname{im}\alpha \to \operatorname{im}\beta \to \operatorname{im}\gamma \to 0. \tag{1}$$

We plug this into our spectral sequence machinery. We first compute the hypercohomology by taking the rightward morphisms first, i.e. using the order



Then because the rows are exact, $E_1^{p,q} = 0$, so the spectral sequence has already converged: $E_\infty^{p,q} = 0$.

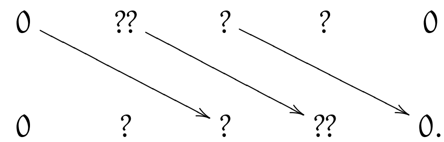
We next compute this "0" in another way, by computing the spectral sequence starting in the other direction.



Then $E_1^{*,*}$ (with its arrows) is:

$$\begin{array}{c} 0 \longrightarrow \operatorname{im}\alpha \longrightarrow \operatorname{im}\beta \longrightarrow \operatorname{im}\gamma \longrightarrow 0 \\ \\ 0 \longrightarrow \ker\alpha \longrightarrow \ker\beta \longrightarrow \ker\gamma \longrightarrow 0. \end{array}$$

Then we compute $E_2^{*,*}$ and find:



Then we see that after E_2, all the terms will stabilize except for the double question marks; and after E_3, even these two will stabilize. But in the end our complex must be the 0 complex. This means that in E_2, all the entries must be zero, except for the two double question marks; and these two must be the same. This means that $0 \to \ker \alpha \to \ker \beta \to \ker \gamma$ and $\operatorname{im} \alpha \to \operatorname{im} \beta \to \operatorname{im} \gamma \to 0$ are both exact (that comes from the vanishing of the single-question-marks), and

$$\operatorname{coker}(\ker \beta \to \ker \gamma) \cong \ker(\operatorname{im} \alpha \to \operatorname{im} \beta)$$

is an isomorphism (that comes from the equality of the double-question-marks). Taken together, we have proved the snake lemma (1)!

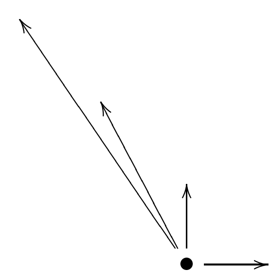
Example: the Five Lemma. Suppose

(2)
$$\begin{array}{ccccccccc} F & \longrightarrow & G & \longrightarrow & H & \longrightarrow & I & \longrightarrow & J \\ \alpha \uparrow & & \beta \uparrow & & \gamma \uparrow & & \delta \uparrow & & \epsilon \uparrow \\ A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & D & \longrightarrow & E \end{array}$$

where the rows are exact and the squares commute.

Suppose α, β, δ, ϵ are isomorphisms. We'll show that γ is an isomorphism.

We first compute the cohomology of the total complex by starting with the rightward arrows:

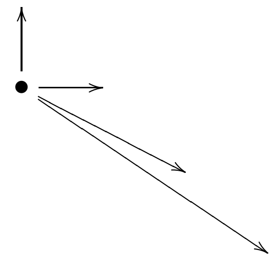


(I chose this because I see that we will get lots of zeros.) Then E_1 looks like this:

$$\begin{array}{ccccc} ? & 0 & 0 & 0 & ? \\ \uparrow & \uparrow & \uparrow & \uparrow & \uparrow \\ ? & 0 & 0 & 0 & ? \end{array}$$

Then E_2 looks similar, and the sequence will converge by E_2 (as we'll never get any arrows between two non-zero entries in a table thereafter). We can't conclude that the cohomology of the total complex vanishes, but we can note that it vanishes in all but four degrees — and most important, in the two degrees corresponding to the entries C and H (the source and target of γ).

We next compute this in the other direction:



Then E_1 looks like this:

$$0 \longrightarrow 0 \longrightarrow ? \longrightarrow 0 \longrightarrow 0$$

$$0 \longrightarrow 0 \longrightarrow ? \longrightarrow 0 \longrightarrow 0$$

and the spectral sequence converges at this step. We wish to show that those two ?'s are zero. But they are precisely the cohomology groups of the total complex that we just showed *were* zero — so we're done!

Exercise. By looking at this proof, prove a subtler version of the five lemma, where one of the isomorphisms can instead just be required to be an injection, and another can instead just be required to be a surjection. (I'm deliberately not telling you which ones, so you can see how the spectral sequence is telling you how to improve the result.) I've heard this called the "subtle five lemma", but I like calling it the $4\frac{1}{2}$-lemma.

Exercise. If β and δ (in (2)) are injective, and α is surjective, show that γ is injective. State the dual statement. (The proof of the dual statement will be essentially the same.)

Exercise. Use spectral sequences to show that a short exact sequence of complexes gives a long exact sequence in cohomology.

3.3. *Exercise.* Suppose $\mu : A^* \to B^*$ is a morphism of complexes. Suppose C^* is the single complex associated to the double complex $A^* \to B^*$. (C^* is called the *mapping cone* of μ.) Show that there is a long exact sequence of complexes:

$$\cdots \to H^{i-1}(C^*) \to H^i(A^*) \to H^i(B^*) \to H^i(C^*) \to H^{i+1}(A^*) \to \cdots .$$

(There is a slight notational ambiguity here; depending on how you index your double complex, your long exact sequence might look slightly different.) In particular, people often use the fact μ induces an isomorphism on cohomology if and only if the mapping cone is exact.

(Does anyone else have some classical important fact that would be useful practice for people learning spectral sequences?)

Next day, I'll state and prove the Leray spectral sequence in algebraic geometry.

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FOUNDATIONS OF ALGEBRAIC GEOMETRY CLASS 33

RAVI VAKIL

CONTENTS

Last day: Applications of higher pushforwards; crash course in spectral sequences.

Today: The Leray spectral sequence. Beginning fun with curves: the Riemann-Hurwitz formula.

Before I start, here is one small comment I should have made earlier. In the notation $R^j f_* \mathcal{F}$ for higher pushforward sheaves, the "R" stands for "right derived functor", and "corresponds" to the fact that we get a long exact sequence in cohomology extending to the right (from the 0th terms). More generally, next quarter we will see that in good circumstances, if we have a left-exact functor, there may be a long exact sequence going off to the right, in terms of right derived functors. Similarly, if we have a right-exact functor (e.g. if M is an A-module, then $\otimes_A M$ is a right-exact functor from the category of A-modules to itself), there may be a long exact sequence going off to the left, in terms of left derived functors.

Here is another exercise that I should have asked earlier. I have also now included it in the class 32 notes (in section 1).

Exercise. Suppose that X is a quasicompact separated k-scheme, where k is a field. Suppose $\mathcal{F}$ is a quasicoherent sheaf on X. Let $X_{\overline{k}} = X \times_{\operatorname{Spec} k} \operatorname{Spec} \overline{k}$, and $f : X_{\overline{k}} \to X$ the projection. Describe a natural isomorphism $H^i(X, \mathcal{F}) \otimes_k \overline{k} \to H^i(X_{\overline{k}}, f^* \mathcal{F})$. Recall that a k-scheme X is *geometrically integral* if $X_{\overline{k}}$ is integral. Show that if X is geometrically integral and projective, then $H^0(X, \mathcal{O}_X) \cong k$. (This is a clue that $\mathbb{P}^1_{\mathbb{C}}$ is not a geometrically integral $\mathbb{R}$-scheme.)

Date: Tuesday, February 21, 2006. Updated June 26.

Suppose

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & {\scriptstyle h}\searrow \quad \swarrow{\scriptstyle g} & \\ & Z, & \end{array}$$

with f and g (and hence h) quasicompact and separated. Suppose $\mathcal{F}$ is a quasicoherent sheaf on X. The Leray spectral sequence lets us find out about the higher pushforwards of h in terms of the higher pushforwards under g of the higher pushforwards under f.

1.1. *Theorem (Leray spectral sequence). — There is a spectral sequence whose* $E_2^{p,q}$*-term is* $R^j g_*(R^i f_* \mathcal{F})$*, abutting to* $R^{i+j} h_* \mathcal{F}$*.*

An important special case is if $Z = \operatorname{Spec} k$, or Z is some other base ring. Then this gives us handle on the cohomology of $\mathcal{F}$ on X in terms of the cohomology of its higher pushforwards to Y.

Proof. We assume Z is an affine ring, say $\operatorname{Spec} A$. Our construction will be "natural" and will hence glue. (At worst, we you can check that it behaves well under localization.)

Fix a finite affine cover of X, U_i. Fix a finite affine cover of Y, V_j. Create a double complex

$$E_0^{a,b} = \oplus_{|I|=a+1,|J|=b+1} \mathcal{F}(U_I \cap \pi^{-1} V_J)$$

for $a, b \geq 0$, with obvious Cech differential maps. By exercise 15 on problem set 11 (class 25, exercise 1.31), $U_I \cap \pi^{-1} V_J$ is affine (for all I, J).

Let's choose the filtration that corresponds to first taking the arrow in the vertical (V) direction. For each I, we'll get a Cech covering of U_I. The Cech cohomology of an affine is trivial except for H^0, so the E_1 term will be 0 except when $j = 0$. There, we'll get $\oplus \mathcal{F}(U_I)$. Then the E_2 term will be $E_2^{p,q} = H^p(X, \mathcal{F}) = \Gamma(Z, R^p h_* \mathcal{F})$ if $q = 0$ and 0 otherwise, and it will converge there.

Let's next choose the filtration that corresponds to first taking the arrow in the horizontal (U) direction. For each V_J, we will get a Cech covering of $\pi^{-1} V_J$. The entries of E_1 will thus be $\oplus_J H^i(f^{-1} V_j, \mathcal{F}) = \oplus_j \Gamma(V_j, R^i \pi_* \mathcal{F})$. Thus E_2 will be as advertised in the statement of Leray. $\square$

Here are some useful examples.

Consider $h^i(\mathbb{P}_k^m \times_k \mathbb{P}_k^n, \mathcal{O}_{\mathbb{P}_k^m \times_k \mathbb{P}_k^n})$. We get 0 unless $i = 0$, in which case we get 1. (The same argument shows that $h^i(\mathbb{P}_A^m \times_A \mathbb{P}_A^n, \mathcal{O}_{\mathbb{P}_A^m \times_A \mathbb{P}_A^n}) \cong A$ if $i = 0$, and 0 otherwise.) You should make this precise:

Exercise. Suppose Y is any scheme, and $\pi : \mathbb{P}^n_Y \to Y$ is the trivial projective bundle over Y. Show that $\pi_* \mathcal{O}_{\mathbb{P}^n_Y} \cong \mathcal{O}_Y$. More generally, show that $R^j \pi_* \mathcal{O}(m)$ is a finite rank free sheaf on Y, and is 0 if $j \neq 0, n$. Find the rank otherwise.

More generally, let's consider $H^i(\mathbb{P}^m_k \times_k \mathbb{P}^n_k, \mathcal{O}(a,b))$. I claim that for each (a,b) at most one cohomology group is non-trivial, and it will be $i = 0$ if $a, b \geq 0$; $i = m+n$ if $a \leq -m-1$, $b \leq -n-1$; $i = m$ if $a \geq 0$, $b \leq -n-1$, and $i = n$ if $a \leq -m-1$, $b = 0$. I attempted to show this to you in a special case, in the hope that you would see how the argument goes. I tried to show that $h^i(\mathbb{P}^2_k \times_k \mathbb{P}^1_k, \mathcal{O}(-4,1))$ is 6 if $i = 2$ and 0 otherwise. The following exercise will help you see if you understood this.

Exercise. Let A be any ring. Suppose a is a negative integer and b is a positive integer. Show that $H^i(\mathbb{P}^m_A \times_A \mathbb{P}^n_A, \mathcal{O}(a,b))$ is 0 unless $i = m$, in which case it is a free A-module. Find the rank of this free A-module. (Hint: Use the previous exercise, and the projection formula, which was Exercise 1.3 of class 32, and exercise 17 of problem set 14.)

We can now find curves of any (non-negative) genus, over any algebraically closed field. An integral projective nonsingular curve over k is *hyperelliptic* if admits a finite degree 2 morphism (or "cover") of $\mathbb{P}^1$.

1.2. *Exercise.* (a) Find the genus of a curve in class $(2,n)$ on $\mathbb{P}^1_k \times_k \mathbb{P}^1_k$. (A curve in class $(2,n)$ is any effective Cartier divisor corresponding to invertible sheaf $\mathcal{O}(2,n)$. Equivalently, it is a curve whose ideal sheaf is isomorphic to $\mathcal{O}(-2,-n)$. Equivalently, it is a curve cut out by a non-zero form of bidegree $(2,n)$.)
(b) Suppose for convenience that k is algebraically closed of characteristic not 2. Show that there exists an integral nonsingular curve in class $(2,n)$ on $\mathbb{P}^1_k \times \mathbb{P}^1_k$ for each $n > 0$.

1.3. *Exercise.* Suppose X and Y are projective k-schemes, and $\mathcal{F}$ and $\mathcal{G}$ are coherent sheaves on X and Y respectively. Recall that if $\pi_1 : X \times Y \to X$ and $\pi_2 : X \times Y \to Y$ are the two projections, then $\mathcal{F} \boxtimes \mathcal{G} := \pi_1^* \mathcal{F} \otimes \pi_2^* \mathcal{G}$. Prove the following, adding additional hypotheses if you find them necessary.
(a) Show that $H^0(X \times Y, \mathcal{F} \boxtimes \mathcal{G}) = H^0(X, \mathcal{F}) \otimes H^0(Y, \mathcal{G})$.
(b) Show that $H^{\dim X + \dim Y}(X \times Y, \mathcal{F} \boxtimes \mathcal{G}) = H^{\dim X}(X, \mathcal{F}) \otimes_k H^{\dim Y}(Y, \mathcal{G})$.
(c) Show that $\chi(X \times Y, \mathcal{F} \boxtimes \mathcal{G}) = \chi(X, \mathcal{F}) \chi(Y, \mathcal{G})$.

I suspect that this Leray spectral sequence converges in this case at E^2, so that $h^n(X \times Y, \mathcal{F} \boxtimes \mathcal{G}) = \sum_{i+j=n} h^i(X, \mathcal{F}) h^j(Y, \mathcal{G})$. Or if this is false, I'd like to see a counterexample. It might even be true that

$$H^n(X \times Y, \mathcal{F} \boxtimes \mathcal{G}) = \oplus_{i+j=n} H^i(X, \mathcal{F}) \otimes H^j(Y, \mathcal{G}).$$

2. Fun with Curves

We already know enough to study curves in a great deal of detail, so this seems like a good way to end this quarter. We get much more mileage if we have a few facts involving differentials, so I'll introduce these facts, and take them as a black box. The actual black boxes we'll need are quite small, but I want to tell you some of the background behind them.

For this topic, we will assume that all curves are projective geometrically integral nonsingular curves over a field k. We will sometimes add the hypothesis that k is algebraically closed.

Most people are happy with working over algebraically closed fields, and all of you should ignore the adverb "geometrically" in the previous paragraph. For those interested in non-algebraically closed fields, an example of a curve that is integral but not geometrically integral is $\mathbb{P}^1_{\mathbb{C}}$ over $\mathbb{R}$. Upon base change to the algebraic closure $\mathbb{C}$ of $\mathbb{R}$, this curve has two components.

2.1. **Differentials on curves.** There is a sheaf of differentials on a curve C, denoted Ω_C, which is an invertible sheaf. (In general, nonsingular k-varieties of dimension d will have a sheaf of differentials over k that will be locally free of rank k. And differentials will be defined in vastly more generality.) We will soon see that this invertible sheaf has degree equal to twice the genus minus 2: $\boxed{\deg \Omega_C = 2g_c - 2}$. For example, if $C = \mathbb{P}^1$, then $\Omega_C \cong \mathcal{O}(-2)$.

Differentials pull back: any surjective morphism of curves $f : C \to C'$ induces a natural map $f^*\Omega_{C'} \to \Omega_C$.

2.2. **The Riemann-Hurwitz formula.** Whenever we invoke this formula (in this section), we will assume that k is algebraically closed and characteristic 0. These conditions aren't necessary, but save us some extra hypotheses. Suppose $f : C \to C'$ is a dominant morphism. Then it turns out $f^*\Omega_{C'} \hookrightarrow \Omega_C$ is an inclusion of invertible sheaves. (This is a case when inclusions of invertible sheaves does not mean what people normally mean by inclusion of line bundles, which are always isomorphisms.) Its cokernel is supported in dimension 0:

$$0 \to f^*\Omega_{C'} \to \Omega_C \to [\text{dimension } 0] \to 0.$$

The divisor R corresponding to those points (with multiplicity), is called the *ramification divisor*.

We can study this in local coordinates. We don't have the technology to describe this precisely yet, but you might still find this believable. If the map at $q \in C'$ looks like $u \mapsto u^n = t$, then $dt \mapsto d(u^n) = nu^{n-1}du$, so dt when pulled back vanishes to order $n-1$. Thus branching of this sort $u \mapsto u^n$ contributes $n-1$ to the ramification divisor. (More correctly, we should look at the map of Spec's of discrete valuation rings, and then u is a uniformizer for the stalk at q, and t is a uniformizer for the stalk at $f(q)$, and t is actually a unit times u^n. But the same argument works.)

Now in a recent exercise on pullbacks of invertible sheaves under maps of curves, we know that a degree of the pullback of an invertible sheaf is the degree of the map times the degree of the original invertible sheaf. Thus if d is the degree of the cover, $\deg \Omega_C =$

$d \deg \Omega_{C'} + \deg R$. Conclusion: if $C \to C'$ is a degree d cover of curves, then

$$\boxed{2g_c - 2 = d(2g_{C'} - 2) + \deg R}$$

Here are some applications.

Example. When I drew a sample branched cover of one complex curve by another, I showed a genus 2 curve covering a genus 3 curve. Show that this is impossible. (Hint: $\deg R \geq 0$.)

Example: Hyperelliptic curves. Hyperelliptic curves are curves that are double covers of $\mathbb{P}^1_k$. If they are genus g, then they are branched over $2g+2$ points, as each ramification can happen to order only 1. (Caution: we are in characteristic 0!) You may already have heard about genus 1 complex curves double covering $\mathbb{P}^1$, branched over 4 points.

Application 1. First of all, the degree of R is even: any cover of a curve must be branched over an even number of points (counted with multiplicity).

Application 2. The only connected unbranched cover of $\mathbb{P}^1_k$ is the isomorphism. Reason: if $\deg R = 0$, then we have $2 - 2g_C = 2d$ with $d \geq 1$ and $g_c \geq 0$, from which $d = 1$ and $g_C = 0$.

Application 3: Luroth's theorem. Suppose $g(C) = 0$. Then from the Riemann-Hurwitz formula, $g(C') = 0$. (Otherwise, if $g_{C'}$ were at least 1, then the right side of the Riemann-Hurwitz formula would be non-negative, and thus couldn't be -2, which is the left side. This has a non-obvious algebraic consequence, by our identification of covers of curves with field extensions (class 28 Theorem 1.5). Hence all subfields of $k(x)$ containing k are of the form $k(y)$ where $y = f(x)$. (Here we have the hypothesis where k is algebraically closed. We'll patch that later.) Kirsten said that an algebraic proof was given in Math 210.

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FOUNDATIONS OF ALGEBRAIC GEOMETRY CLASS 34

RAVI VAKIL

CONTENTS

Last day: The Leray spectral sequence. Beginning fun with curves: Ω_C, and the Riemann-Hurwitz formula.

Today: More fun with curves: Serre duality, criterion for closed immersion, series of useful remarks, curves of genus 0 and 2

1. LAST DAY

Last day we began to talk about curves over a field k. Our standing assumptions will be that a curve C is projective, geometrically integral and nonsingular over a field k.

(People happy to work over algebraically closed fields can continue to ignore the adverb "geometrically".)

I'm in the process of telling you a few facts that we will prove next quarter. We will use these facts to prove lots of things about curves.

Last day I defined Ω_C, sheaf of differentials on C. I really should have called it $\boxed{\Omega_{C/k}}$, to make clear that this sheaf on C depends on the structure morphism $C \to k$. I stated that $\Omega_{C/k}$ is an invertible sheaf, and told you that we will soon see that has degree $\boxed{\deg \Omega_C = 2g_c - 2}$. I stated that differentials pullback under covers $f : C \to C'$ (i.e. that there is a morphism $f^*\Omega_{C'/k} \to \Omega_{C/k}$), and if we are in characteristic 0, then this yields an

Date: Thursday, February 23, 2006. Minor correction June 25, 2007.

inclusion of invertible sheaves, which yields $0 \to f^*\Omega_{C'} \to \Omega_C \to R \to 0$, where R corresponds to the *ramification divisor* on C, which keeps track of the branching of $C \to C'$. From this I claimed that we will deduce the *Riemann-Hurwitz formula*

$$\boxed{2g_C - 2 = d(2g_{C'} - 2) + \deg R}$$

1.1. **Serre duality.** (We are not requiring k to be algebraically closed.) In general, nonsingular varieties will have a special invertible sheaf $\mathcal{K}_X$ which is the determinant of Ω_X. This invertible sheaf is called the *canonical bundle*, and will later be defined in much greater generality. In our case, $X = C$ is a curve, so $\mathcal{K}_C = \Omega_C$, and from here on in, we'll use $\mathcal{K}_C$ instead of Ω_C. The reason it is called the dualizing sheaf is because it arises in Serre duality. Serre duality states that $H^1(C, \mathcal{K}) \cong k$, or more precisely that there is a *trace morphism* $\boxed{H^1(C,\mathcal{K}) \to k}$ that is an isomorphism. (Example: if $C = \mathbb{P}^1$, then we indeed have $h^1(\mathbb{P}^1, \mathcal{O}(-2)) = 1$.)

Further, for any coherent sheaf $\mathcal{F}$, the natural map

$$\boxed{H^0(C,\mathcal{F}) \otimes_k H^1(C, \mathcal{K} \otimes \mathcal{F}^\vee) \to H^1(C,\mathcal{K})}$$

is a perfect pairing. Thus in particular, $h^0(C,\mathcal{F}) = h^1(C, \mathcal{K} \otimes \mathcal{F}^\vee)$. Recall we defined the arithmetic genus of a curve to be $h^1(C, \mathcal{O}_C)$. Then $h^0(C,\mathcal{K}) = g$ as well.

Recall that Riemann-Roch for a invertible sheaf $\mathcal{L}$ states that

$$h^0(C,\mathcal{L}) - h^1(C,\mathcal{L}) = \deg \mathcal{L} - g + 1.$$

Applying this to $\mathcal{L} = \mathcal{K}$, we get

$$\deg \mathcal{K} = h^0(C,\mathcal{K}) - h^1(C,\mathcal{K}) + g - 1 = h^1(C,\mathcal{O}) - h^0(C,\mathcal{O}) + g - 1 = g - 1 + g - 1 = 2g - 2$$

as promised earlier.

1.2. **A criterion for when a morphism is a closed immersion.** We'll also need a criterion for when something is a closed immersion. To help set it up, let's observe some facts about closed immersions. Suppose $f : X \to Y$ is a closed immersion. Then f is projective, and it is injective on points. This is not enough to ensure that it is a closed immersion, as the example of the normalization of the cusp shows (Figure 1). Another example is the Frobenius morphism from $\mathbb{A}^1$ to $\mathbb{A}^1$, given by $k[t] \to k[u]$, $u \to t^p$, where k has characteristic p.

The additional information you need is that the tangent map is an isomorphism at all closed points. (Exercise: show this is false in those two examples.)

1.3. *Theorem. — Suppose k is an algebraically closed field, and $f : X \to Y$ is a projective morphism of finite-type k-schemes that is injective on closed points and injective on tangent vectors of closed points. Then f is a closed immersion.*

The example of $\operatorname{Spec} \mathbb{C} \to \operatorname{Spec} \mathbb{R}$ shows that we need the hypothesis that k is algebraically closed.

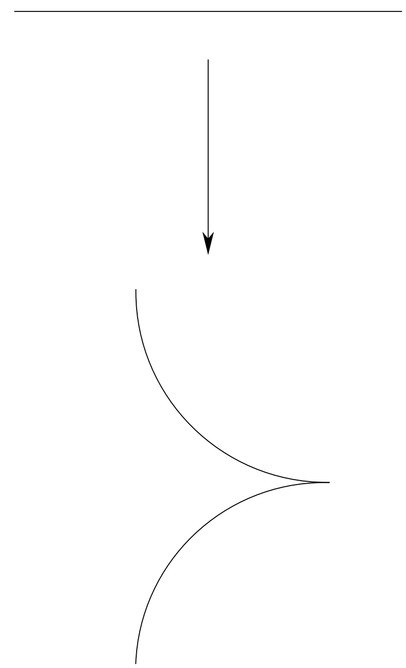

FIGURE 1. Projective morphisms that are injective on points need not be closed immersions

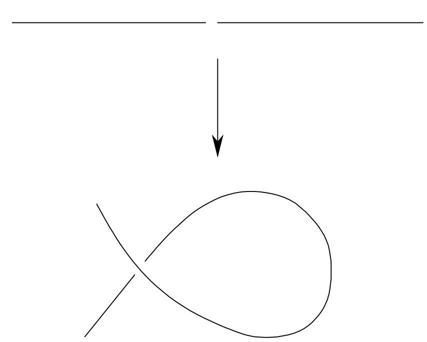

FIGURE 2. We need the projective hypothesis in Theorem 1.3

We need the hypothesis of projective morphism, as shown by the following example (which was described at the blackboard, see Figure 2). We map $\mathbb{A}^1$ to the plane, so that its image is a curve with one node. We then consider the morphism we get by discarding one of the preimages of the node. Then this morphism is an injection on points, and is also injective on tangent vectors, but it is not a closed immersion. (In the world of differential geometry, this fails to be an embedding because the map doesn't give a homeomorphism onto its image.)

Suppose $f(p) = q$, where p and q are closed points. We will use the hypothesis that X and Y are k-schemes where k is algebraically closed at only one point of the argument: that the map induces an isomorphism of residue fields at p and q.

(For those of you who are allergic to algebraically closed fields: still pay attention, as we'll use this to prove things about curves over k where k is *not* necessarily algebraically closed.)

This is the hardest result of today. We will kill the problem in old-school French style: death by a thousand cuts.

Proof. We may assume that Y is affine, say $\operatorname{Spec} B$.

I next claim that f has finite fibers, not just finite fibers above closed points: the fiber dimension for projective morphisms is upper-semicontinuous (Class 32 Exercise 2.3), so the locus where the fiber dimension is at least 1 is a closed subset, so if it is non-empty, it must contain a closed point of Y. Thus the fiber over any point is a dimension 0 finite type scheme over that point, hence a finite set.

Hence f is a projective morphism with finite fibers, thus affine, and even finite (Class 32 Corollary 2.4).

Thus X is affine too, say $\operatorname{Spec} A$, and f corresponds to a ring morphism $B \to A$. We wish to show that this is a surjection of rings, or (equivalently) of B-modules. We will show that for any maximal ideal $\mathfrak{n}$ of B, $B_{\mathfrak{n}} \to A_{\mathfrak{n}}$ is a surjection of $B_{\mathfrak{n}}$-modules. (This will show that $B \to A$ is a surjection. Here is why: if K is the cokernel, so $B \to A \to K \to 0$, then we wish to show that $K = 0$. Now A is a finitely generated B-module, so K is as well, being a homomorphic image of A. Thus $\operatorname{Supp} K$ is a closed set. If $K \neq 0$, then $\operatorname{Supp} K$ is non-empty, and hence contains a closed point $[\mathfrak{n}]$. Then $K_{\mathfrak{n}} \neq 0$, so from the exact sequence $B_{\mathfrak{n}} \to \mathbb{A}_{\mathfrak{n}} \to K_{\mathfrak{n}} \to 0$, $B_{\mathfrak{n}} \to A_{\mathfrak{n}}$ is not a surjection.)

If $A_{\mathfrak{n}} = 0$, then clearly $B_{\mathfrak{n}}$ surjects onto $A_{\mathfrak{n}}$, so assume otherwise. I claim that $A_{\mathfrak{n}} = A \otimes_B B_{\mathfrak{n}}$ is a local ring. Proof: $\operatorname{Spec} A_{\mathfrak{n}} \to \operatorname{Spec} B_{\mathfrak{n}}$ is a finite morphism (as it is obtained by base change from $\operatorname{Spec} A \to \operatorname{Spec} B$), so we can use the going-up theorem. $A_{\mathfrak{n}} \neq 0$, so $A_{\mathfrak{n}}$ *has* a prime ideal. Any point p of $\operatorname{Spec} A_{\mathfrak{n}}$ maps to some point of $\operatorname{Spec} B_{\mathfrak{n}}$, which has $[\mathfrak{n}]$ in its closure. Thus there is a point q in the closure of p that maps to $[\mathfrak{n}]$. But there is only one point of $\operatorname{Spec} A_{\mathfrak{n}}$ mapping to $[\mathfrak{n}]$, which we denote $[\mathfrak{m}]$. Thus we have shown that $\mathfrak{m}$ contains all other prime ideals of $\operatorname{Spec} A_{\mathfrak{n}}$, so $A_{\mathfrak{n}}$ is a local ring.

Injectivity of tangent vectors *means* surjectivity of cotangent vectors, i.e. $\mathfrak{n}/\mathfrak{n}^2 \to \mathfrak{m}/\mathfrak{m}^2$ is a surjection, i.e. $\mathfrak{n} \to \mathfrak{m}/\mathfrak{m}^2$ is a surjection. Claim: $\mathfrak{n}A_{\mathfrak{n}} = \mathfrak{m}A_{\mathfrak{n}}$. Reason: By Nakayama's lemma for the local ring $A_{\mathfrak{n}}$ and the $A_{\mathfrak{n}}$-module $\mathfrak{m}A_{\mathfrak{n}}$, we conclude that $\mathfrak{n}A_{\mathfrak{n}} = \mathfrak{m}A_{\mathfrak{n}}$.

Next apply Nakayama's Lemma to the $B_{\mathfrak{n}}$-module $A_{\mathfrak{n}}$. The element $1 \in A_{\mathfrak{n}}$ gives a generator for $A_{\mathfrak{n}}/\mathfrak{n}A_{\mathfrak{n}} = A_{\mathfrak{n}}/\mathfrak{m}A_{\mathfrak{n}}$, which equals $B_{\mathfrak{n}}/\mathfrak{n}B_{\mathfrak{n}}$ (as both equal k), so we conclude that 1 also generates $A_{\mathfrak{n}}$ as a $B_{\mathfrak{n}}$-module as desired. □

1.4. *Exercise.* Use this to show that the dth Veronese morphism from $\mathbb{P}^n_k$, corresponding to the complete linear series (see Class 22) $|\mathcal{O}_{\mathbb{P}^n_k}(d)|$, is a closed immersion. Do the same for the Segre morphism from $\mathbb{P}^m_k \times_{\operatorname{Spec} k} \mathbb{P}^n_k$. (This is just for practice for using this criterion.

This is a weaker result than we had before; we've earlier checked this over an arbitrary base ring, and we are now checking it only over algebraically closed fields.)

2. A SERIES OF USEFUL REMARKS

Suppose now that $\mathcal{L}$ is an invertible sheaf on a curve C (which as always in this discussion is projective, geometrically integral and nonsingular, over a field k which is not necessarily algebraically closed). I'll give a series of small useful remarks that we will soon use to great effect.

2.1. $h^0(C, \mathcal{L}) = 0$ if $\deg \mathcal{L} < 0$. Reason: if there is a non-zero section, then the degree of $\mathcal{L}$ can be interpreted as the number of zeros minus the number of poles. But there are no poles, so this would have to be non-negative. A slight refinement gives:

2.2. $h^0(C, \mathcal{L}) = 0$ or 1 if $\deg \mathcal{L} = 0$. This is because if there is a section, then the degree of $\mathcal{L}$ is the number of zeros minus the number of poles. Then as there are no poles, there can be no zeros. Thus the section (call it s) vanishes nowhere, and gives a trivialization for the invertible sheaf. (Recall how this works: we have a natural bijection for any open set $\Gamma(U, \mathcal{L}) \leftrightarrow \Gamma(U, \mathcal{O}_U)$, where the map from left to right is $s' \mapsto s'/s$, and the map from right to left is $f \mapsto sf$.) Thus if there is a section, $\mathcal{L} \cong \mathcal{O}$. But we've already checked that for a geometrically integral and nonsingular curve C, $h^0(C, \mathcal{L}) = 1$.

2.3. Suppose p is any closed point of degree 1. (In other words, the residue field of p is k.) Then $h^0(C, \mathcal{L}) - h^0(C, \mathcal{L}(-p)) = 0$ or 1. Reason: consider $0 \to \mathcal{O}_C(-p) \to \mathcal{O}_C \to \mathcal{O}_p \to 0$, tensor with $\mathcal{L}$ (this is exact as $\mathcal{L}$ is locally free) to get

$$0 \to \mathcal{L}(-p) \to \mathcal{L} \to \mathcal{L}|_p \to 0.$$

Then $h^0(C, \mathcal{L}|_p) = 1$, so as the long exact sequence of cohomology starts off

$$0 \to H^0(C, \mathcal{L}(-p)) \to H^0(C, \mathcal{L}) \to H^0(C, \mathcal{L}|_p),$$

we are done.

2.4. Suppose for this remark that k is algebraically closed. (In particular, *all* closed points have degree 1 over k.) Then if $h^0(C, \mathcal{L}) - h^0(C, \mathcal{L}(-p)) = 1$ for *all* closed points p, then $\mathcal{L}$ is base-point-free, and hence induces a morphism from C to projective space. (Note that $\mathcal{L}$ has a finite-dimensional vector space of sections: all cohomology groups of all coherent sheaves on a projective k-scheme are finite-dimensional.) Reason: given any p, our equality shows that there exists a section of $\mathcal{L}$ that does not vanish at p.

2.5. Next, suppose p and q are distinct points of degree 1. Then $h^0(C, \mathcal{L}) - h^0(C, \mathcal{L}(-p - q)) = 0, 1,$ or 2 (by repeating the argument of 2.3 twice). If $h^0(C, \mathcal{L}) - h^0(C, \mathcal{L}(-p-q)) = 2$, then necessarily

$$h^0(C, \mathcal{L}) = h^0(C, \mathcal{L}(-p)) + 1 = h^0(C, \mathcal{L}(-q)) + 1 = h^0(C, \mathcal{L}(-p - q)) + 2. \quad (1)$$

I claim that the linear system $\mathcal{L}$ separates points p and q, by which I mean that the corresponding map f to projective space satisfies $f(p) \neq f(q)$. Reason: there is a hyperplane of projective space passing through p but not passing through q, or equivalently, there is a section of $\mathcal{L}$ vanishing at p but not vanishing at q. This is because of the last equality in (1).

2.6. By the same argument as above, if p is a point of degree 1, then $h^0(C, \mathcal{L}) - h^0(C, \mathcal{L}(-2p)) = 0, 1$, or 2. I claim that if this is 2, then map corresponds to $\mathcal{L}$ (which is already seen to be base-point-free from the above) separates the tangent vectors at p. To show this, I need to show that the cotangent map is *surjective*. To show surjectivity onto a one-dimensional vector space, I just need to show that the map is non-zero. So I need to give a function on the target vanishing at the image of p that pulls back to a function that vanishes at p to order 1 but not 2. In other words, I want a section of $\mathcal{L}$ vanishing at p to order 1 but not 2. But that is the content of the statement $h^0(C, \mathcal{L}(-p)) - h^0(C, \mathcal{L}(-2p)) = 1$.

2.7. Combining some of our previous comments: suppose C is a curve over an *algebraically closed* field k, and $\mathcal{L}$ is an invertible sheaf such that for *all* closed points p and q, *not necessarily distinct*, $h^0(C, \mathcal{L}) - h^0(C, \mathcal{L}(-p-q)) = 2$, then $\mathcal{L}$ gives a closed immersion into projective space, as it separates points and tangent vectors, by Theorem 1.3.

2.8. We now bring in Serre duality. I claim that $\deg \mathcal{L} > 2g - 2$ implies

$$\boxed{h^0(C, \mathcal{L}) = \deg \mathcal{L} - g - 1.}$$

This is important — remember this! Reason: $h^1(C, \mathcal{L}) = h^0(C, \mathcal{K} \otimes \mathcal{L}^\vee)$; but $\mathcal{K} \otimes \mathcal{L}^\vee$ has negative degree (as $\mathcal{K}$ has degree $2g - 2$), and thus this invertible sheaf has no sections. Thus Riemann-Roch gives us the desired result.

Exercise. Suppose $\mathcal{L}$ is a degree $2g-2$ invertible sheaf. Show that it has $g-1$ or g sections, and it has g sections if and only if $\mathcal{L} \cong \mathcal{K}$.

2.9. We now come to our most important conclusion. Thus if k is algebraically closed, then $\deg \mathcal{L} \geq 2g$ implies that $\mathcal{L}$ is basepoint free (and hence determines a morphism to projective space). Also, $\deg \mathcal{L} \geq 2g + 1$ implies that this is in fact a closed immersion. Remember this! [k need not be algebraically closed.]

2.10. I now claim (for the people who like fields that are not algebraically closed) that *the previous remark holds true even if* k *is not algebraically closed.* Here is why: suppose C is our curve, and $C_{\overline{k}} := C \otimes_k \overline{k}$ is the base change to the algebraic closure (which we are assuming is connected and nonsingular), with $\pi : C_{\overline{k}} \to C$ (which is an affine morphism, as it is obtained by base change from the affine morphism $\operatorname{Spec} \overline{k} \to \operatorname{Spec} k$). Then $H^0(C, \mathcal{L}) \otimes_k \overline{k} \cong H^0(C_{\overline{k}}, \pi^* \mathcal{L})$ for reasons I explained last day (see the first exercise on the class 33 notes, and also on problem set 15).

$$\begin{array}{ccc} C_{\overline{k}} & \xrightarrow{\pi} & C \\ \downarrow & & \downarrow \\ \operatorname{Spec} \overline{k} & \longrightarrow & \operatorname{Spec} k \end{array}$$

Let $s_0, \dots, s_n$ be a basis for the k-vector space $H^0(C, \mathcal{L})$; they give a basis for the $\overline{k}$-vector space $H^0(C_{\overline{k}}, \pi^*\mathcal{L})$. If $\mathcal{L}$ has degree at least $2g$, then these sections have no common zeros on $C_{\overline{k}}$; but this means that they have no common zeros on C. If $\mathcal{L}$ has degree at least $2g+1$, then these sections give a closed immersion $C_{\overline{k}} \hookrightarrow \mathbb{P}^n_{\overline{k}}$. Then I claim that $f : C \to \mathbb{P}^n_k$ (given by the same sections) is also a closed immersion. Reason: we can check this on each affine open subset $U = \operatorname{Spec} A \subset \mathbb{P}^n_k$. Now f has finite fibers, and is projective, hence is a finite morphism (and in particular affine). Let $\operatorname{Spec} B = f^{-1}(U)$. We wonder if $A \to B$ is a surjection of rings. But we know that this is true upon base changing by $\overline{k}$: $A \otimes_k \overline{k} \to B \otimes_k \overline{k}$ is surjective. So we are done.

We're now ready to take these facts and go to the races.

3. Genus 0

3.1. *Claim. — Suppose* C *is genus* 0*, and* C *has a* k*-valued point. Then* $C \cong \mathbb{P}^1_k$.

Of course C automatically has a k-point if k is algebraically closed. Thus we see that all genus 0 (integral, nonsingular) curves over an algebraically closed field are isomorphic to $\mathbb{P}^1$.

If k is not algebraically closed, then C needn't have a k-valued point: witness $x^2 + y^2 + z^2 = 0$ in $\mathbb{P}^2_{\mathbb{R}}$. We have already observed that this curve is *not* isomorphic to $\mathbb{P}^1_{\mathbb{R}}$, because it doesn't have an $\mathbb{R}$-valued point.

Proof. Let p be the point, and consider $\mathcal{L} = \mathcal{O}(p)$. Then $\deg \mathcal{L} = 1$, so we can apply what we know above: first of all, $h^0(C, \mathcal{L}) = 2$, and second of all, these two sections give a closed immersion in to $\mathbb{P}^1_k$. But the only closed immersion of a curve into $\mathbb{P}^1_k$ is the isomorphism! □

As a fun bonus, we see that the weird real curve $x^2 + y^2 + z^2 = 0$ in $\mathbb{P}^2_{\mathbb{R}}$ has no *divisors* of degree 1 over $\mathbb{R}$; otherwise, we could just apply the above argument to the corresponding line bundle.

Our weird curve shows us that over a non-algebraically closed field, there can be genus 0 curves that are not isomorphic to $\mathbb{P}^1_k$. The next result lets us get our hands on them as well.

3.2. *Claim. — All genus* 0 *curves can be described as conics in* $\mathbb{P}^2_k$.

Proof. Any genus 0 curve has a degree −2 line bundle — the canonical bundle $\mathcal{K}$. Thus any genus 0 curve has a degree 2 line bundle: $\mathcal{L} = \mathcal{K}^\vee$. We apply our machinery to this bundle: $h^0(C, \mathcal{L}) = 3 \geq 2g + 1$, so this line bundle gives a closed immersion into $\mathbb{P}^2$. [This proof is not complete if $k = \overline{k}$, as the criterion we are using requires this hypothesis. Exercise: Use §2.10 to give a complete proof.] □

3.3. *Exercise.* Suppose C is a genus 0 curve (projective, geometrically integral and nonsingular). Show that C has a point of degree at most 2.

We will use the following result later.

3.4. *Claim. — Suppose* C *is not isomorphic to* $\mathbb{P}^1_k$ *(with no restrictions on the genus of* C*), and* $\mathcal{L}$ *is an invertible sheaf of degree* 1*. Then* $h^0(C, \mathcal{L}) < 2$.

Proof. Otherwise, let s_1 and s_2 be two (independent) sections. As the divisor of zeros of s_i is the degree of $\mathcal{L}$, each vanishes at a single point p_i (to order 1). But $p_1 \neq p_2$ (or else s_1/s_2 has no poles or zeros, i.e. is a constant function, i.e. s_1 and s_2 are dependent). Thus we get a map $C \to \mathbb{P}^1$ which is basepoint free. This is a finite degree 1 map of nonsingular curves, which induces a degree 1 extension of function fields, i.e. an isomorphism of function fields, which means that the curves are isomorphic. But we assumed that C is not isomorphic to $\mathbb{P}^1_k$. □

4. Genus ≥ 2

It might make most sense to jump to genus 1 at this point, but the theory of elliptic curves is especially rich and beautiful, so I'll leave it for the end.

In general, the curves have quite different behaviors (topologically, arithmetically, geometrically) depending on whether $g = 0$, $g = 1$, or $g > 2$. This trichotomy extends to varieties of higher dimension. I gave a very brief discussion of this trichotomy for curves. For example, arithmetically, genus 0 curves can have lots and lots of points, genus 1 curves can have lots of points, and by Faltings' Theorem (Mordell's Conjecture) any curve of genus at least 2 has at most finitely many points. (Thus we knew before Wiles that $x^n + y^n = z^n$ in $\mathbb{P}^2$ has at most finitely many solutions for $n \geq 4$, as such curves have genus $\binom{n-1}{2} > 1$.) Geometrically, Riemann surfaces of genus 0 are positively curved, Riemann surfaces of genus 1 are flat, and Riemann surfaces of genus 1 are negatively curved. We will soon see that curves of genus at least 2 have finite automorphism groups, while curves of genus 1 have some automorphisms (a one-dimensional family), and (we've seen earlier) curves of genus 1 (over an algebraically closed field) have a three-dimensional automorphism group.

4.1. **Genus 2.** Fix a curve C of genus 2. Then $\mathcal{K}$ is degree 2, and has 2 sections. I claim that $\mathcal{K}$ is base-point-free. Otherwise, if p is a base point, then $\mathcal{K}(-p)$ is a degree 1 invertible sheaf with 2 sections, and we just showed (Claim 3.4) that this is impossible. Thus we have a double cover of $\mathbb{P}^1$. Conversely, any double cover $C \to \mathbb{P}^1$ arises from a degree 2 invertible sheaf with at least 2 sections, so by one of our useful facts, if $g(C) = 2$, this invertible sheaf must be the canonical bundle (as the only degree 2 invertible sheaf on a genus 2 curve with at least 2 sections is $\mathcal{K}_C$). Hence we have a natural bijection between genus 2 curves and genus 2 double covers of $\mathbb{P}^1$.

We now specialize to the case where $k = \overline{k}$, and the characteristic of k is 0. (All we will need, once we actually prove the Riemann-Hurwitz formula, is that the characteristic be distinct from 2.) Then the Riemann-Hurwitz formula shows that the cover is branched over 6 points. We will see next day that a double cover is determined by its branch points. Hence genus 2 curves are in bijection with unordered sextuples of points on $\mathbb{P}^1$. There is thus a 3-dimensional family of genus 2 curves — we have found them all!

(This is still a little imprecise; we would like to say that the moduli space of genus 2 curves is of dimension 3, but we haven't defined what we mean by moduli space!)

More generally, we may see next week (admittedly informally) that if $g > 1$, the curves of genus g "form a family" of dimension $3g - 3$. (If we knew the meaning of "moduli space", we would say that the dimension of the moduli space of genus g curves $\mathcal{M}_g$ is $3g - 3$.) What goes wrong in genus 0 and 1? The following table (as yet unproved by us!) might help.

genus	dimension of family of curves	dimension of automorphism group of curve
0	0	3
1	1	1
2	3	0
3	6	0
4	9	0
5	12	0
⋮	⋮	⋮

You can probably see the pattern. This is a little like the behavior of the Hilbert function: the dimension of the moduli space is "eventually polynomial", so there is something that is better-behaved that is an alternating sum, and once the genus is sufficiently high, the "error term" becomes zero. The interesting question then becomes: why is the "right" notion the second column of the table minus the third? (In fact the second column is $h^1(C, T_C)$, where T_C is the tangent bundle — not yet defined — and the third column is $h^0(C, T_C)$. All other cohomology groups of the tangent bundle vanish by dimensional vanishing.)

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FOUNDATIONS OF ALGEBRAIC GEOMETRY CLASS 34 CRIB SHEET

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This is a summary of useful facts we proved or assumed. We will use them in the next two classes.

All curves C are projective, and geometrically integral and nonsingular over a field k.

There is an invertible sheaf (rank bundle) $\mathcal{K}$, called the *dualizing sheaf*; it is also the sheaf of differentials (in this guise it is called $\Omega_{C/k}$), and the cotangent bundle. $\deg \mathcal{K} = 2g-2$.

The *Riemann-Hurwitz formula* is $2g_C - 2 = d(2g_{C'} - 2) + \deg R$, where R is the *ramification divisor*.

Serre duality. There is an isomorphism $H^0(C, \mathcal{K}) \xrightarrow{\sim} k$ For any coherent sheaf $\mathcal{F}$, the natural map

$$\boxed{H^0(C, \mathcal{F}) \otimes_k H^1(C, \mathcal{K} \otimes \mathcal{F}^\vee) \to H^0(C, \mathcal{K})}$$

is a perfect pairing, so in particular, $h^0(C, \mathcal{F}) = h^1(C, \mathcal{K} \otimes \mathcal{F}^\vee)$. (As $g := h^1(C, \mathcal{O}_C)$, we get $h^0(C, \mathcal{K}) = g$ as well.) Hence Riemann-Roch now states:

$$h^0(C, \mathcal{L}) - h^1(C, \mathcal{L}) = \deg \mathcal{L} - g + 1.$$

Applying this to $\mathcal{L} = \mathcal{K}$, we get $\deg \mathcal{K} = 2g - 2$ (promised earlier).

Suppose now that $\mathcal{L}$ is an invertible sheaf on C.

0.1. $h^0(C, \mathcal{L}) = 0$ if $\deg \mathcal{L} < 0$. $h^0(C, \mathcal{L}) = 0$ or 1 if $\deg \mathcal{L} = 0$.

0.2. Suppose p is any closed point of degree 1. (In other words, the residue field of p is k.) Then $h^0(C, \mathcal{L}) - h^0(C, \mathcal{L}(-p)) = 0$ or 1.

0.3. Suppose for this remark that k is algebraically closed. (In particular, *all* closed points have degree 1 over k.) Then if $h^0(C, \mathcal{L}) - h^0(C, \mathcal{L}(-p)) = 1$ for *all* closed points p, then $\mathcal{L}$ is base-point-free, and hence induces a morphism from C to projective space.

0.4. Suppose p and q are distinct points of degree 1. Then $h^0(C, \mathcal{L}) - h^0(C, \mathcal{L}(-p-q)) = 0$, 1, or 2. If $h^0(C, \mathcal{L}) - h^0(C, \mathcal{L}(-p-q)) = 2$, then $\mathcal{L}$ separates points p and q, by which I mean that the corresponding map f to projective space satisfies $f(p) \neq f(q)$.

Date: Thursday, February 23, 2006.

0.5. If p is a point of degree 1, then $h^0(C, \mathcal{L}) - h^0(C, \mathcal{L}(-2p)) = 0$, 1, or 2. If it is 2, then the map corresponding to $\mathcal{L}$ separates the tangent vectors at p.

0.6. Combining some of our previous comments: suppose C is a curve over an *algebraically closed* field k, and $\mathcal{L}$ is an invertible sheaf such that for *all* closed points p and q, *not necessarily distinct*, $h^0(C, \mathcal{L}) - h^0(C, \mathcal{L}(-p-q)) = 2$, then $\mathcal{L}$ gives a closed immersion into projective space.

0.7. We now bring in Serre duality. $\deg \mathcal{L} > 2g - 2$ implies

$$\boxed{h^0(C, \mathcal{L}) = \deg \mathcal{L} - g - 1.}$$

If $\mathcal{L}$ is a degree $2g - 2$ invertible sheaf, then $\mathcal{L}$ has $g - 1$ or g sections, and it has g sections if and only if $\mathcal{L} \cong \mathcal{K}$.

0.8. *Our most important conclusion.* $\deg \mathcal{L} \geq 2g$ implies that $\mathcal{L}$ is basepoint free (and hence determines a morphism to projective space). Also, $\deg \mathcal{L} \geq 2g + 1$ implies that this is in fact a closed immersion. Remember this!

0.9. Suppose C is not isomorphic to $\mathbb{P}^1_k$ (with no restrictions on the genus of C), and $\mathcal{L}$ is an invertible sheaf of degree 1. Then $h^0(C, \mathcal{L}) < 2$.

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FOUNDATIONS OF ALGEBRAIC GEOMETRY CLASS 35

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CONTENTS

Last day: More fun with curves: Serre duality, criterion for closed immersion, a series of useful remarks, curves of genus 0 and 2.

Today: hyperelliptic curves; curves of genus at least 2; elliptic curves take 1.

Last day we started studying curves in detail, using things we'd proved. Today, we'll continue to use these things. (See the "Class 34 crib sheet" for a reminder of what we know.)

1. HYPERELLIPTIC CURVES

As usual, we begin by working over an arbitrary field k, and specializing only when we need to. A curve C of genus at least 2 is *hyperelliptic* if it admits a degree 2 cover of $\mathbb{P}^1$. This map is often called the *hyperelliptic map*.

Equivalently, C is hyperelliptic if it admits a degree 2 invertible sheaf $\mathcal{L}$ with $h^0(C, \mathcal{L}) = 2$.

1.1. *Exercise..* Verify that these notions are the same. Possibly in the course of doing this, verify that if C is a curve, and $\mathcal{L}$ has a degree 2 invertible sheaf with at least 2 (linearly independent) sections, then $\mathcal{L}$ has precisely two sections, and that this $\mathcal{L}$ is base-point free and gives a hyperelliptic map.

The degree 2 map $C \to \mathbb{P}^1$ gives a degree 2 extension of function fields $FF(C)$ over $FF(\mathbb{P}^1) \cong k(t)$. If the characteristic is not 2, this extension is necessarily Galois, and the induced involution on C is called the *hyperelliptic involution*.

Date: Tuesday, February 28, 2006. Superficial update June 25, 2007.

1.2. *Proposition. — If $\mathcal{L}$ corresponds to a hyperelliptic cover $C \to \mathbb{P}^1$, then $\mathcal{L}^{\otimes(g-1)} \cong \mathcal{K}_C$.*

Proof. Compose the hyperelliptic map with the $(g-1)$th Veronese map:

$$C \xrightarrow{\mathcal{L}} \mathbb{P}^1 \xrightarrow{\mathcal{O}_{\mathbb{P}^1}(g-1)} \mathbb{P}^{g-1}.$$

The composition corresponds to $\mathcal{L}^{\otimes(g-1)}$. This invertible sheaf has degree $2g-2$, and the image is nondegenerate in $\mathbb{P}^{g-1}$, and hence has at least g sections. But one of our useful facts (and indeed an exercise) was that the only invertible sheaf of degree $2g-2$ with (at least) g sections is the canonical sheaf. □

1.3. *Proposition. — If a curve (of genus at least 2) is hyperelliptic, then it is hyperelliptic in "only one way". In other words, it admits only one double cover of $\mathbb{P}^1$.*

Proof. If C is hyperelliptic, then we can recover the hyperelliptic map by considering the canonical map: it is a double cover of a degree $g-1$ rational normal curve (by the previous Proposition), and this double cover is the hyperelliptic cover (also by the proof of the previous Proposition). □

Next, we invoke the Riemann-Hurwitz formula. We assume the $\operatorname{char} k = 0$, and $k = \overline{k}$, so we can invoke this black box. However, when we actually discuss differentials, and prove the Riemann-Hurwitz formula, we will see that we can just require $\operatorname{char} k \neq 2$ (and $k = \overline{k}$).

The Riemann-Hurwitz formula implies that hyperelliptic covers have precisely $2g+2$ (distinct) branch points. We will see in a moment that the branch points determine the curve (Claim 1.4).

Assuming this, we see that hyperelliptic curves of genus g correspond to precisely $2g+2$ points on $\mathbb{P}^1$ modulo S_{2g+2}, and modulo automorphisms of $\mathbb{P}^1$. Thus "the space of hyperelliptic curves" has dimension

$$2g+2-\dim \operatorname{Aut}\mathbb{P}^1 = 2g-1.$$

(As usual, this is not a well-defined statement, because as yet we don't know what we mean by "the space of hyperelliptic curves". For now, take it as a plausibility statement.) If we believe that the curves of genus g form a family of dimension $3g-3$, we have shown that "most curves are not hyperelliptic" if $g > 2$ (or on a milder note, there exists a hyperelliptic curve of each genus $g > 2$).

1.4. *Claim. — Assume $\operatorname{char} k \neq 2$ and $k = \overline{k}$. Given n distinct points on $\mathbb{P}^1$, there is precisely one cover branched at precisely these points if n is even, and none if n is odd.*

In particular, the branch points determine the hyperelliptic curve. (We also used this fact when discussing genus 2 curves last day.)

Proof. Suppose we have a double cover of $\mathbb{A}^1$, $C \to \mathbb{A}^1$, where x is the coordinate on $\mathbb{A}^1$. This induces a quadratic field extension K over $k(x)$. As $\operatorname{char} k \neq 2$, this extension is Galois. Let σ be the hyperelliptic involution. Let y be an element of K such that $\sigma(y) = -y$, so 1 and y form a basis for K over the field $k(x)$ (and are eigenvectors of σ). Now $y^2 \in k(x)$, so we can replace y by an appropriate $k(x)$-multiple so that y^2 is a polynomial, with no repeated factors, and monic. (This is where we use the hypothesis that k is algebraically closed, to get leading coefficient 1.) Thus $y^2 = x^n + a_{n-1}x^{n-1} + \cdots + a_0$. The branch points correspond to those values of x for which there is exactly one value of y, i.e. the roots of the polynomial. As we have no double roots, the curve is nonsingular. Let this cover be $C' \to \mathbb{A}^1$. Both C and C' are normalizations of $\mathbb{A}^1$ in this field extension, and are thus isomorphic. Thus every double cover can be written in this way, and in particular, if the branch points are $r_1, \ldots, r_n$, the cover is $y^2 = (x - r_1) \cdots (x - r_n)$.

We now consider the situation over $\mathbb{P}^1$. A double cover can't be branched over an odd number of points by the Riemann-Hurwitz formula. Given an even number of points r_1, $\ldots$, r_n in $\mathbb{P}^1$, choose an open subset $\mathbb{A}^1$ containing all n points. Construct the double cover of $\mathbb{A}^1$ as explained in the previous paragraph: $y^2 = (x - r_1) \cdots (x - r_n)$. Then take the normalization of $\mathbb{P}^1$ in this field extension. Over the open $\mathbb{A}^1$, we recover this cover. We just need to make sure we haven't accidentally acquired a branch point at the missing point $\infty = \mathbb{P}^1 - \mathbb{A}^1$. But the total number of branch points is even, and we already have an even number of points, so there is no branching at ∞. □

Remark. If k is not algebraically closed (but of characteristic not 2), the above argument shows that if we have a double cover of $\mathbb{A}^1$, then it is of the form $y^2 = af(x)$, where f is monic, and $a \in k^*/(k^*)^2$. So (assuming the field doesn't contain all squares) a double cover does *not* determine the same curve. Moreover, see that this failure is classified by $k^*/(k^*)^2$. Thus we have lots of curves that are not isomorphic over k, but become isomorphic over $\overline{k}$. These are often called *twists* of each other.

(In particular, even though haven't talked about elliptic curves yet, we definitely have two elliptic curves over $\mathbb{Q}$ with the same j-invariant, that are not isomorphic.)

2. Curves of genus 3

Suppose C is a curve of genus 3. Then $\mathcal{K}$ has degree $2g - 2 = 4$, and has $g = 3$ sections.

2.1. *Claim.* — *$\mathcal{K}$ is base-point-free, and hence gives a map to $\mathbb{P}^2$.*

Proof. We check base-point-freeness by working over the algebraic closure $\overline{k}$. For any point p, by Riemann-Roch,

$$h^0(C, \mathcal{K}(-p)) - h^0(C, \mathcal{O}(p)) = \deg(\mathcal{K}(-p)) - g + 1 = 3 - 3 + 1 = 1.$$

But $h^0(C, \mathcal{O}(p)) = 0$ by one of our useful facts, so

$$h^0(C, \mathcal{K}(-p)) = 2 = h^0(C, \mathcal{K}) - 1.$$

Thus p is not a base-point of $\mathcal{K}$, so $\mathcal{K}$ is base-point-free. □

The next natural question is: Is this a closed immersion? Again, we can check over algebraic closure. We use our "closed immersion test" (again, see our useful facts). If it *isn't* a closed immersion, then we can find two points p and q (possibly identical) such that

$$h^0(C, \mathcal{K}) - h^0(C, \mathcal{K}(-p-q)) = 2,$$

i.e. $h^0(C, \mathcal{K}(-p-q)) = 2$. But by Serre duality, this means that $h^0(C, \mathcal{O}(p+q)) = 2$. We have found a degree 2 divisor with 2 sections, so C is hyperelliptic. (Indeed, I could have skipped that sentence, and made this observation about $\mathcal{K}(-p-q)$, but I've done it this way in order to generalize to higher genus.) Conversely, if C is hyperelliptic, then we already know that $\mathcal{K}$ gives a double cover of a nonsingular conic in $\mathbb{P}^2$ (also known as a rational normal curve of degree 2).

Thus we conclude that if C is not hyperelliptic, then the canonical map describes C as a degree 4 curve in $\mathbb{P}^2$.

Conversely, any quartic plane curve is canonically embedded. Reason: the curve has genus 3 (we can compute this — see our discussion of Hilbert functions), and is mapped by an invertible sheaf of degree 4 with 3 sections. Once again, we use the useful fact saying that the only invertible sheaf of degree $2g-2$ with g sections is $\mathcal{K}$.

Exercise. Show that the nonhyperelliptic curves of genus 3 form a family of dimension 6. (Hint: Count the dimension of the family of nonsingular quartics, and quotient by $\operatorname{Aut} \mathbb{P}^2 = PGL(3)$.)

The genus 3 curves thus seem to come in two families: the hyperelliptic curves (a family of dimension 5), and the nonhyperelliptic curves (a family of dimension 6). This is misleading — they actually come in a single family of dimension 6.

In fact, hyperelliptic curves are naturally limits of nonhyperelliptic curves. We can write down an explicit family. (This next paragraph will necessarily require some handwaving, as it involves topics we haven't seen yet.) Suppose we have a hyperelliptic curve branched over $2g+2 = 8$ points of $\mathbb{P}^1$. Choose an isomorphism of $\mathbb{P}^1$ with a conic in $\mathbb{P}^2$. There is a nonsingular quartic meeting the conic at precisely those 8 points. (This requires Bertini's theorem, so I'll skip that argument.) Then if f is the equation of the conic, and g is the equation of the quartic, then $f^2 + t^2 g$ is a family of quartics that are nonsingular for most t (nonsingular is an open condition as we will see). The $t = 0$ case is a double conic. Then it is a fact that if you normalize the family, the central fiber (above $t = 0$) turns into our hyperelliptic curve. Thus we have expressed our hyperelliptic curve as a limit of nonhyperelliptic curves.

3. Genus at least 3

We begin with two exercises in general genus, and then go back to genus 4.

Exercise Suppose C is a genus g curve. Show that if C is not hyperelliptic, then the canonical bundle gives a closed immersion $C \hookrightarrow \mathbb{P}^{g-1}$. (In the hyperelliptic case, we have already seen that the canonical bundle gives us a double cover of a rational normal curve.) Hint: follow the genus 3 case. Such a curve is called a *canonical curve*.

Exercise. Suppose C is a curve of genus $g > 1$, over a field k that is not algebraically closed. Show that C has a closed point of degree at most $2g - 2$ over the base field. (For comparison: if $g = 1$, there is no such bound!)

We next consider nonhyperelliptic curves C of genus 4. Note that $\deg \mathcal{K} = 6$ and $h^0(C, \mathcal{K}) = 4$, so the canonical map expresses C as a sextic curve in $\mathbb{P}^3$. We shall see that all such C are complete intersections of quadric surfaces and cubic surfaces, and vice versa.

By Riemann-Roch, $\mathcal{K}^{\otimes 2}$ has $\deg \mathcal{K}^{\otimes 2} - g + 1 = 12 - 4 + 1 = 9$ sections. That's one less than $\dim \operatorname{Sym}^2 \Gamma(C, \mathcal{K}) = \binom{4+1}{2}$. Thus there is at least one quadric in $\mathbb{P}^3$ that vanishes on our curve C. Translation: C lies on at least on quadric Q. Now quadrics are either double planes, or the union of two planes, or cones, or nonsingular quadrics. (They corresponds to quadric forms of rank 1, 2, 3, and 4 respectively.) Note that C can't lie in a plane, so Q must be a cone or nonsingular. In particular, Q is irreducible.

Now C can't lie on *two* (distinct) such quadrics, say Q and Q'. Otherwise, as Q and Q' have no common components (they are irreducible and not the same!), $Q \cap Q'$ is a curve (not necessarily reduced or irreducible). By Bezout's theorem, it is a curve of degree 4. Thus our curve C, being of degree 6, cannot be contained in $Q \cap Q'$.

We next consider cubics. By Riemann-Roch, $\mathcal{K}^{\otimes 3}$ has $\deg \mathcal{K} \otimes 3 - g + 1 = 18 - 4 + 1 = 15$ sections. Now $\dim \operatorname{Sym}^3 \Gamma(C, \mathcal{K})$ has dimension $\binom{4+2}{3} = 20$. Thus C lies on at least a 5-dimensional vector space of cubics. Admittedly 4 of them come from multiplying the quadric Q by a linear form $(?w + ?x + ?y + ?z)$. But hence there is still one cubic K whose underlying form is not divisible by the quadric form Q (i.e. K doesn't contain Q.) Then K and Q share no component, so $K \cap Q$ is a complete intersection. By Bezout's theorem, we obtain a curve of degree 6. Our curve C has degree 6. This suggests that $C = K \cap Q$. In fact, $K \cap Q$ and C have the same Hilbert polynomial, and $C \subset K \cap Q$. Hence $C = K \cap Q$ by the following exercise.

Exercise. Suppose $X \subset Y \subset \mathbb{P}^n$ are a sequence of closed subschemes, where X and Y have the same Hilbert polynomial. Show that $X = Y$. Hint: consider the exact sequence

$$0 \to \mathcal{I}_{X/Y} \to \mathcal{O}_Y \to \mathcal{O}_X \to 0.$$

Show that if the Hilbert polynomial of $\mathcal{I}_{X/Y}$ is 0, then $\mathcal{I}_{X/Y}$ must be the 0 sheaf.

We now consider the converse, and who that any nonsingular complete intersection C of a quadric surface with a cubic surface is a canonically embedded genus 4 curve. It is not hard to check that it has genus 3 (again, using our exercises involving Hilbert functions).
Exercise. Show that $\mathcal{O}_C(1)$ has 4 sections. (Translation: C doesn't lie in a hyperplane.) Hint: long exact sequences! Again, the only degree $2g-2$ invertible sheaf with g sections is the canonical sheaf, so $\mathcal{O}_C(1) \cong \mathcal{K}_C$, and C is indeed canonically embedded.

Exercise. Conclude that nonhyperelliptic curves of genus 4 "form a family of dimension $9 = 3g - 3$". (Again, this isn't a mathematically well-formed question. So just give a plausibility argument.)

On to genus 5!

Exercise. Suppose C is a nonhyperelliptic genus 5 curve. The canonical curve is degree 8 in $\mathbb{P}^4$. Show that it lies on a three-dimensional vector space of quadrics (i.e. it lies on 3 independent quadrics). Show that a nonsingular complete intersection of 3 quadrics is a canonical genus 5 curve.

In fact a canonical genus 5 is always a complete intersection of 3 quadrics.

Exercise. Show that the complete intersections of 3 quadrics in $\mathbb{P}^4$ form a family of dimension $12 = 3g - 3$.

This suggests that the nonhyperelliptic curves of genus 5 form a dimension 12 family.

So we've managed to understand curves of genus up to 5 (starting with 3) by thinking of canonical curves as complete intersections. Sadly our luck has run out.

Exercise. Show that if $C \subset \mathbb{P}^{g-1}$ is a canonical curve of genus $g \geq 6$, then C is *not* a complete intersection. (Hint: Bezout.)

4. Genus 1

Finally, we come to the very rich case of curves of genus 1.

Note that $\mathcal{K}$ is an invertible sheaf of degree $2g - 2 = 0$ with $g = 1$ section. But the only degree 0 invertible sheaf with a section is the trivial sheaf, so we conclude that $\mathcal{K} \cong \mathcal{O}$.

Next, note that if $\deg \mathcal{L} > 0$, then Riemann-Roch and Serre duality gives

$$h^0(C, \mathcal{L}) = h^0(C, \mathcal{L}) - h^0(C, \mathcal{K} \otimes \mathcal{L}^\vee) = h^0(C, \mathcal{L}) - h^0(C, \mathcal{L}^\vee) = \deg \mathcal{L}$$

as an invertible sheaf $\mathcal{L}^\vee$ of negative degree necessarily has no sections.

An *elliptic curve* is a genus 1 curve E with a choice of k-valued point p. (Note: it is *not* the same as a genus 1 curve — some genus 1 curves have no k-valued points. However, if $k = \overline{k}$, then any closed point is k-valued; but still, the choice of a closed point should always be considered part of the definition of an elliptic curve.)

Note that $\mathcal{O}_E(2p)$ has 2 sections, so the argument given in the hyperelliptic section shows that E admits a double cover of $\mathbb{P}^1$. One of the branch points is $2p$: one of the sections of $\mathcal{O}_E(2p)$ vanishes to p of order 2, so there is a point of $\mathbb{P}^1$ consists of p (with multiplicity 2). Assume now that $k = \overline{k}$, so we can use the Riemann-Hurwitz formula. Then the Riemann-Hurwitz formula shows that E has 4 branch points (p and three others). Conversely, given 4 points in $\mathbb{P}^1$, we get a map ($y^2 = \cdots$). This determines C (as shown in the hyperelliptic section). Thus elliptic curves correspond to 4 points in $\mathbb{P}^1$, where one is marked p, up to automorphisms of $\mathbb{P}^1$. (Equivalently, by placing p at ∞, elliptic curves correspond to 3 points in $\mathbb{A}^1$, up to affine maps $x \mapsto ax + b$.)

If the three other points are temporarily labeled q_1, q_2, q_3, there is a unique automorphism of $\mathbb{P}^1$ taking p, q_1, q_2 to $(\infty, 0, 1)$ respectively (as $\operatorname{Aut} \mathbb{P}^1$ is three-transitive). Suppose that q_3 is taken to some number λ under this map. Notice that $\lambda \neq 0, 1, \infty$.

- If we had instead sent p, q_2, q_1 to $(\infty, 0, 1)$, then q_3 would have been sent to $1 - \lambda$.
- If we had instead sent p, q_1, q_3 to $(\infty, 0, 1)$, then q_2 would have been sent to $1/\lambda$.
- If we had instead sent p, q_3, q_1 to $(\infty, 0, 1)$, then q_2 would have been sent to $1 - 1/\lambda = (\lambda - 1)/\lambda$.
- If we had instead sent p, q_2, q_3 to $(\infty, 0, 1)$, then q_2 would have been sent to $1/(1 - \lambda)$.
- If we had instead sent p, q_3, q_2 to $(\infty, 0, 1)$, then q_2 would have been sent to $1 - 1/(1 - \lambda) = \lambda/(\lambda - 1)$.

Thus these six values (in bijection with S_3) yield the same elliptic curve, and this elliptic curve will (upon choosing an ordering of the other 3 branch points) yield one of these six values.

Thus the elliptic curves over k corresponds to k-valued points of $\mathbb{P}^1 - \{0, 1, \lambda\}$, modulo the action of S_3 on λ given above. Consider the subfield of $k(\lambda)$ fixed by S_3. By Luroth's theorem, it must be of the form $k(j)$ for some $j \in k(\lambda)$. Note that λ should satisfy a sextic polynomial over $k(\lambda)$, as for each j-invariant, there are six values of λ in general.

At this point I should just give you j:

$$j = 2^8 \frac{(\lambda^2 - \lambda + 1)^3}{\lambda^2(\lambda - 1)^2}.$$

But this begs the question: where did this formula come from? How did someone think of it?

Far better is to guess what j is. We want to come up with some $j(\lambda)$ such that $j(\lambda) = j(1/\lambda) = \cdots$. Hence we want some expression in λ that is invariant under this S_3-action. A silly choice would be the product of the six numbers $\lambda(1/\lambda) \cdots$ as this is 1.

A better idea is to add them all together. Unfortunately, if you do this, you'll get 3. (Here is one reason to realize this can't work: if you look at the sum, you'll realize that you'll get something of the form "degree at most 3" divided by "degree at most 2" (before cancellation). Then if $j' = p(\lambda)/q(\lambda)$, then λ satisfies (at most) a cubic over j. But we said that λ should satisfy a sextic over j'. The only way we avoid a contradiction is if $j' \in k$.

Our next attempt is to add up the six squares. When you do this by hand (it isn't hard), you get

$$j'' = \frac{2\lambda^6 - 6\lambda^5 + 9\lambda^4 - 8\lambda^3 + 9\lambda^2 - 6\lambda + 2}{\lambda^2(\lambda - 1)^2}.$$

This works just fine: $k(j) \cong k(j'')$. If you really want to make sure that I'm not deceiving you, you can check (again by hand) that

$$2j/2^8 = \frac{2\lambda^6 - 6\lambda^5 + 12\lambda^4 - 14\lambda^3 + 12\lambda^2 - 6\lambda + 2}{\lambda^2(\lambda-1)^2}.$$

The difference is 3.

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FOUNDATIONS OF ALGEBRAIC GEOMETRY CLASS 36

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CONTENTS

Last day: More fun with curves: hyperelliptic curves; curves of genus at least 2; elliptic curves take 1.

Today: elliptic curves; the Picard variety; "the moduli space of curves has dimension $3g-3$."

This is the last class of the quarter! We'll finish off using what we know (and a little of what we'll know soon) to learn a great deal about curves.

There will be one more homework out early next week, due Thursday of the week after, covering this week's notes. We may well have a question-and-answer question on the last morning of class.

Once again, I'm going to use those important facts that we proved a couple of days ago, so I'll refer you to the class 34 crib sheet.

Let me first give you an exercise I should have given you last day.

Exercise. (a) Suppose C is a projective curve. Show that $C-p$ is affine. (Hint: show that $n \gg 0$, $\mathcal{O}(np)$ gives an embedding of C into some projective space $\mathbb{P}^m$, and that there is some hyperplane H meeting C precisely at p. Then $C-p$ is a closed subscheme of $\mathbb{P}^n - H$.) (b) If C is a geometrically integral nonsingular curve over a field k (i.e. all of our standing assumptions, minus projectivity), show that it is projective or affine.

Date: Thursday, March 2, 2006. Minor update January 30, 2007.

We're in the process of studying elliptic curves, i.e. curves E (projective, geometrically integral and nonsingular, over a field k) of genus 1, with a choice of a k-valued point p. (It is typical to use the letter E for the curve rather than C.)

So far we have seen that they admit double covers of $\mathbb{P}^1$, and that if $k = \overline{k}$, then the elliptic curves are classified by the j-invariant. The double cover corresponded to the invertible sheaf $\mathcal{O}_E(2p)$. We'll now consider $\mathcal{O}_E(np)$ for larger n.

1.1. **Degree 3.** Consider the degree 3 invertible sheaf $\mathcal{O}_E(3p)$. We consult our useful facts. By Riemann-Roch, $h^0(E, \mathcal{O}_E(3p)) = \deg(3p) - g + 1 = 3$. As $\deg E > 2g$, this gives a closed immersion. Thus we have a closed immersion $E \hookrightarrow \mathbb{P}^2_k$ as a cubic curve. Moreover, there is a line in $\mathbb{P}^2_k$ meeting E at point p with multiplicity 3. (Remark: a line in the plane meeting a smooth curve with multiplicity at least 2 is said to be a *tangent line*. A line in the plane meeting a smooth curve with multiplicity at least 3 is said to be a *flex line*.)

We can choose projective coordinates on $\mathbb{P}^2_k$ so that p maps to $[0; 1; 0]$, and the flex line is the line at infinity $z = 0$. Then the cubic is of the following form:

$$
\begin{array}{ccccccccc}
?x^3 & + & 0x^2y & + & 0xy^2 & + & 0y^3 & & \\
+ & ?x^2z & + & ?xyz & + & ?y^2z & & & \\
 & + & ?xz^2 & + & ?yz^2 & & & & \\
 & & + & ?z^3 & & & & & = 0
\end{array}
$$

The co-efficient of x is not 0 (or else this cubic is divisible by z). We can scale x so that the coefficient of x^3 is 1. The coefficient of y^2z is not 0 either (or else this cubic is singular at $x = z = 0$). As k is algebraically closed, we can scale y so that the coefficient of y^2z is 1. (More precisely, we are changing variables, say $y' = ay$ for some $a \in k$.) If the characteristic of k is not 2, then we can then replace y by $y + ?x + ?z$ so that the coefficients of xyz and yz^2 are 0, and if the characteristic of k is not 3, we can replace x by $x + ?z$ so that the coefficient of x^2z is also 0. In conclusion, if k is algebraically closed of characteristic not 2 or 3, we can write our elliptic curve in the form

$$y^2z = x^3 + ax^2z + bz^3.$$

This is called *Weierstrass normal form.* (If only some of the "bonus hypotheses" $k = \overline{k}$, $\operatorname{char} k \neq 2, 3$ is true, then we can perform only some of the reductions of course.)

Notice that we see the hyperelliptic description of the curve (by setting $z = 1$, or more precisely, by working in the distinguished open set $z \neq 0$ and using inhomogeneous coordinates). In particular, we can compute the j-invariant.

Here is the geometric explanation of why the double cover description is visible in the cubic description.

I drew a picture of the projective plane, showing the cubic, and where it met the z-axis (the line at infinity) — where the z-axis and x-axis meet — it has a flex there. I drew the lines through that point — vertical lines. Equivalently, you're just taking 2 of the 3 sections: x and z. These are two sections of $\mathcal{O}(3p)$, but they have a common zero — a base point at p. So you really get two sections of $\mathcal{O}(2p)$.

Exercise. Show that $\mathcal{O}(4p)$ embeds E in $\mathbb{P}^3$ as the complete intersection of two quadrics.

1.2. The group law.

1.3. *Theorem. — The closed points of* E *are in natural bijection with* $\text{Pic}^0(E)$, *via* $x \leftrightarrow x - p$. *In particular, as* $\text{Pic}^0(E)$ *is a group, we have endowed the closed points of* E *with a group structure.*

For those of you familiar with the complex analytic picture, this isn't surprising: E is isomorphic to the complex numbers modulo a lattice: $E \cong \mathbb{C}/\Lambda$.

This is currently just a bijection of sets. Given that E has a much richer structure (it has a generic point, and the structure of a variety), this is a sign that there should be a way of defining some *scheme* $\text{Pic}^0(E)$, and that this should be an isomorphism of schemes.

Proof. For injectivity: $\mathcal{O}(x - p) \cong \mathcal{O}(y - p)$ implies $\mathcal{O}(x - y) \cong \mathcal{O}$. But as E is not genus 0, this is possible only if $x = y$.

For surjectivity: any degree 1 invertible sheaf has a section, so if $\mathcal{L}$ is any degree 0 invertible sheaf, then $\mathcal{O}(\mathcal{L}(p)) \cong \mathcal{O}(x)$ for some x. □

Note that more naturally, $\text{Pic}^1(E)$ is in bijection with the points of E (without any choice of point p).

From now on, we will conflate closed points of E with degree 0 invertible sheaves on E.

Remark. The 2-torsion points in the group are the branch points in the double cover! Reason: q is a 2-torsion point if and only if $2q \sim 2p$ if and only if there is a section of $\mathcal{O}(2p)$ vanishing at q to order 2. (This is characteristic-independent.) Now assume that the characteristic is 0. (In fact, we'll only be using the fact that the characteristic is not 2.) By the Riemann-Hurwitz formula, there are 3 non-trivial torsion points. (Again, given the complex picture $E \cong \mathbb{C}/\Lambda$, this isn't surprising.)

Follow-up remark. An elliptic curve with *full level* n*-structure* is an elliptic curve with an isomorphism of its n-torsion points with $(\mathbb{Z}/n)^2$. (This notion will have problems if n is divisible by $\text{char}\, k$.) Thus an elliptic curve with *full level 2 structure* is the same thing as an elliptic curve with an ordering of the three other branch points in its degree 2 cover description. Thus (if $k = \overline{k}$) these objects are parametrized by the λ-line (see the discussion last day).

Follow-up to the follow-up. There is a notion of moduli spaces of elliptic curves with full level n structure. Such moduli spaces are smooth curves (where this is interpreted appropriately), and have smooth compactifications. A *weight* k *level* n *modular form* is a section of $\mathcal{K}^{\otimes k}$ where $\mathcal{K}$ is the canonical sheaf of this "modular curve".

But let's get back down to earth.

1.4. *Proposition. — There is a morphism of varieties* $E \to E$ *sending a (degree* 1*) point to its inverse.*

In other words, the "inverse map" in the group law actually arises from a morphism of schemes — it isn't just a set map. This is another clue that $\mathrm{Pic}^0(E)$ really wants to be a scheme.

Proof. It is the hyperelliptic involution $y \mapsto -y$! Here is why: if q and r are "hyperelliptic conjugates", then $q + r \sim 2p = 0$. □

We can describe addition in the group law using the cubic description. (Here a picture is absolutely essential, and at some later date, I hope to add it.) To find the sum of q and r on the cubic, we draw the line through q and r, and call the third point it meets s. Then we draw the line between p and s, and call the third point it meets t. Then $q + r = t$. Here's why: $q + r + s = p + s + t$ gives $(q - p) + (r - p) = (s - p)$.

(When the group law is often defined on the cubic, this is how it is done. Then you have to show that this is indeed a group law, and in particular that it is associative. We don't need to do this — $\mathrm{Pic}^0 E$ is a group, so it is automatically associative.)

Note that this description works in all characteristics; we haven't required the cubic to be in Weierstrass normal form.

1.5. *Proposition. — There is a morphism of varieties* $E \times E \to E$ *that on degree* 1 *points sends* (q, r) *to* $q + r$.

"Proof". We just have to write down formulas for the construction on the cubic. This is no fun, so I just want to convince you that it can be done, rather than writing down anything explicit. They key idea is to define another map $E \times E \to E$, where if the input is (a, b), the output is the third point where the cubic meets the line, with the natural extension if the line doesn't meet the curve at three distinct points. Then we can use this to construct addition on the cubic. □

Aside: Discussion on group varieties and group schemes.

A *group variety* X over k is something that can be defined as follows: We are given an element $e \in X(k)$ (a k-valued point of X), and maps $i : X \to X$, $m : X \times X \to X$. They satisfy the hypotheses you'd expect from the definition of a group.

(i) associativity:

$$\begin{array}{ccc} X \times X \times X & \xrightarrow{(m,id)} & X \times X \\ \big\downarrow{\scriptstyle (id,m)} & & \big\downarrow{\scriptstyle m} \\ X \times X & \xrightarrow{m} & X \end{array}$$

commutes.

(ii) $X \xrightarrow{e,id} X \times X \xrightarrow{m} X$ and $X \xrightarrow{id,e} X \times X \xrightarrow{m} X$ are both the identity.

(iii) $X \xrightarrow{i,id} X \times X \xrightarrow{m} X$ and $X \xrightarrow{id,i} X \times X \xrightarrow{m} X$ are both e.

More generally, a *group scheme over a base* B is a scheme $X \to B$, with a section $e : B \to X$, and B-morphisms $i : X \to X$, $m : X \times_B X \to X$, satisfying the three axioms above.

More generally still, a *group object in a category* $\mathcal{C}$ is the above data (in a category $\mathcal{C}$), satisfying the same axioms. The e map is from the final object in the category to the group object.

You can check that a group object in the category of sets is in fact the same thing as a group. (This is symptomatic of how you take some notion and make it categorical. You write down its axioms in a categorical way, and if all goes well, if you specialize to the category of sets, you get your original notion. You can apply this to the notion of "rings" in an exercise below.)

1.6. *The functorial description.* It is often cleaner to describe this in a functorial way. Notice that if X is a group object in a category $\mathcal{C}$, then for any other element of the category, the set $\mathrm{Hom}(Y, X)$ is a group. Moreover, given any $Y_1 \to Y_2$, the induced map $\mathrm{Hom}(Y_2, X) \to \mathrm{Hom}(Y_1, X)$ is group homomorphism.

We can instead define a group object in a category to be an object X, along with morphisms $m : X \times X \to X$, $i : X \to X$, and e : final object $\to X$, such that these induce a natural group structure on $\mathrm{Hom}(Y, X)$ for each Y in the category, such that the forgetful maps are group homomorphisms. This is much cleaner!

Exercise. Verify that the axiomatic definition and the functorial definition are the same.

Exercise. Show that (E, p) is a group scheme. (Caution! we've stated that only the closed points form a group — the group Pic^0. So there is something to show here. The main idea is that with varieties, lots of things can be checked on closed points. First assume that $k = \overline{k}$, so the closed points are dimension 1 points. Then the associativity diagram is commutative on closed points; argue that it is hence commutative. Ditto for the other categorical requirements. Finally, deal with the case where k is not algebraically closed, by working over the algebraic closure.)

We've seen examples of group schemes before. For example, $\mathbb{A}^1_k$ is a group scheme under addition. $\mathbb{G}_m = \mathrm{Spec}\, k[t, t^{-1}]$ is a group scheme.

Easy exercise. Show that $\mathbb{A}^1_k$ is a group scheme under addition, and $\mathbb{G}_m$ is a group scheme under multiplication. You'll see that the functorial description trumps the axiomatic description here! (Recall that $\mathrm{Hom}(X, \mathbb{A}^1_k)$ is canonically $\Gamma(X, \mathcal{O}_X)$, and $\mathrm{Hom}(X, \mathbb{G}_m)$ is canonically $\Gamma(X, \mathcal{O}_X)^*$.)

Exercise. Define the group scheme $GL(n)$ over the integers.

Exercise. Define μ_n to be the kernel of the map of group schemes $\mathbb{G}_m \to \mathbb{G}_m$ that is "taking nth powers". In the case where n is a prime p, which is also $\mathrm{char}\, k$, describe μ_p. (I.e. how many points? How "big" = degree over k?)

Exercise. Define a *ring scheme*. Show that $\mathbb{A}^1_k$ is a ring scheme.

1.7. *Hopf algebras.* Here is a notion that we'll certainly not use, but it is easy enough to define now. Suppose $G = \mathrm{Spec}\, A$ is an affine group scheme, i.e. a group scheme that is an affine scheme. The categorical definition of group scheme can be restated in terms of the ring A. Then these axioms define a *Hopf algebra*. For example, we have a "comultiplication map" $A \to A \otimes A$. *Exercise.* As $\mathbb{A}^1_k$ is a group scheme, $k[t]$ has a Hopf algebra structure. Describe the comultiplication map $k[t] \to k[t] \otimes_k k[t]$.

2. Fun counterexamples using elliptic curves

We have a morphism $(\times n) : E \to E$ that is "multiplication by n", which sends p to np. If $n = 0$, this has degree 0. If $n = 1$, it has degree 1. Given the complex picture of a torus, you might not be surprised that the degree of $\times n$ is n^2. If $n = 2$, we have almost shown that it has degree 4, as we have checked that there are precisely 4 points q such that $2p = 2q$. All that really shows is that the degree is at least 4.

2.1. *Proposition. — For each $n > 0$, the "multiplication by n" map has positive degree. In other words, there are only a finite number of n torsion points.*

Proof. We prove the result by induction; it is true for $n = 1$ and $n = 2$.

If n is odd, then assume otherwise that $nq = 0$ for all closed points q. Let r be a non-trivial 2-torsion point, so $2r = 0$. But $nr = 0$ as well, so $r = (n - 2[n/2])r = 0$, contradicting $r \neq 0$.

If n is even, then $[\times n] = [\times 2] \circ [\times(n/2)]$, and by our inductive hypothesis both $[\times 2]$ and $[\times(n/2)]$ have positive degree. □

In particular, the total number of torsion points on E is countable, so if k is an uncountable field, then E has an uncountable number of closed points (consider an open subset of the curve as $y^2 = x^3 + ax + b$; there are uncountably many choices for x, and each of them has 1 or 2 choices for y).

Thus *almost all* points on E are non-torsion. I'll use this to show you some pathologies.

An example of an affine open set that is not distinguished. I can give you an affine scheme X and an affine open subset Y that is not distinguished in X. Let $X = E - p$, which is affine (easy, or see Exercise).

Let q be another point on E so that $q-p$ is non-torsion. Then $E-p-q$ is affine (Exercise). Assume that it is distinguished. Then there is a function f on $E-p$ that vanishes on q (to some positive order d). Thus f is a rational function on E that vanishes at q to order d, and (as the total number of zeros minus poles of f is 0) has a pole at p of order d. But then $d(p-q) = 0$ in $\mathrm{Pic}^0 E$, contradicting our assumption that $p - q$ is non-torsion.

An Example of a scheme that is locally factorial at a point p, but such that no affine open neighborhood of p has ring that is a Unique Factorization Domain.

Consider $p \in E$. Then an open neighborhood of E is of the form $E - q_1 - \cdots - q_n$. I claim that its Picard group is nontrivial. Recall the exact sequence:

$$\mathbb{Z}^n \xrightarrow{(a_1,\dots,a_n)\mapsto a_1 q_1 + \cdots + a_n q_n} \mathrm{Pic}\, E \longrightarrow \mathrm{Pic}(E - q_1 - \cdots - q_n) \longrightarrow 0 .$$

But the group on the left is countable, and the group in the middle is uncountable, so the group on the right is non-zero.

Example of variety with non-finitely-generated space of global sections.

This is related to Hilbert's fourteenth problem, although I won't say how.

Before we begin we have a preliminary exercise.

Exercise. Suppose X is a scheme, and L is the total space of a line bundle corresponding to invertible sheaf $\mathcal{L}$, so $L = \mathrm{Spec} \oplus_{n \geq 0} (\mathcal{L}^\vee)^{\otimes n}$. Show that $H^0(L, \mathcal{O}_L) = \oplus H^0(X, (\mathcal{L}^\vee)^{\otimes n})$.

Let E be an elliptic curve over some ground field k, N a degree 0 non-torsion invertible sheaf on E, and P a positive-degree invertible sheaf on E. Then $H^0(E, N^m \otimes P^n)$ is nonzero if and only if either (i) $n > 0$, or (ii) $m = n = 0$ (in which case the sections are elements of k). Thus the ring $R = \oplus_{m,n \geq 0} H^0(E, N^m \otimes P^n)$ is not finitely generated.

Now let X be the total space of the vector bundle $N \oplus P$ over E. Then the ring of global sections of X is R.

3. More serious stuff

I'll conclude the quarter by showing the following.

- If C has genus g, then "$\mathrm{Pic}^0(C)$ has dimension g".
- "The moduli space of curves of genus g "is dimension $3g - 3$."

We'll work over an algebraically closed field k. We haven't yet made the above notions precise, so what follows are just plausibility arguments. (It is worth trying to think of a way of making these notions precise! There are several ways of doing this usefully.)

3.1. The Picard group has dimension g: "$\dim \mathrm{Pic}^0 C = g$". There are quotes around this equation because so far, $\mathrm{Pic}^0 C$ is simply a set, so this will just be a plausibility argument. Let p be any (closed, necessarily degree 1) point of C. Then twisting by p gives an isomorphism of $\mathrm{Pic}^d C$ and $\mathrm{Pic}^{d+1} C$, via $\mathcal{L} \leftrightarrow \mathcal{L}(p)$. Thus we'll consider $\mathrm{Pic}^d C$, where $d \gg 0$ (in fact $d > \deg \mathcal{K} = 2g - 2$ will suffice). Say $\dim \mathrm{Pic}^d C = h$. We ask: how many degree d *effective divisors* are there (i.e. what is the dimension of this family)? The answer is clearly d, and C^d surjects onto this set (and is usually $d!$-to-1).

But we can count effective divisors in a different way. There is an h-dimensional family of line bundles by hypothesis, and each one of these has a $(d - g + 1)$-dimensional family of non-zero sections, each of which gives a divisor of zeros. But two sections yield the same divisor if one is a multiple of the other. Hence we get: $h + (d - g + 1) - 1 = h + d - g$.

Thus $d = h + d - g$, from which $h = g$ as desired.

Note that we get a bit more: if we believe that Pic^d has an algebraic structure, we have a fibration $(C^d)/S_d \to \mathrm{Pic}^d$, where the fibers are isomorphic to $\mathbb{P}^{d-g}$. In particular, Pic^d is reduced, and irreducible.

3.2. The moduli space of genus g curves has dimension $3g - 3$. Let $\mathcal{M}_g$ be the set of nonsingular genus g curves, and pretend that we can give it a variety structure. Say $\mathcal{M}_g$ has dimension p. By our useful Riemann-Roch facts, if $d \gg 0$, and D is a divisor of degree d, then $h^0(C, \mathcal{O}(D)) = d - g + 1$. If we take two general sections s, t of the line bundle $\mathcal{O}(D)$, we get a map to $\mathbb{P}^1$, and this map is degree d. Conversely, any degree d cover $f : C \to \mathbb{P}^1$ arises from two linearly independent sections of a degree d line bundle. Recall that (s, t) gives the same map to $\mathbb{P}^1$ as (s', t') if and only (s, t) is a scalar multiple of (s', t'). Hence the number of maps to $\mathbb{P}^1$ arising from a fixed curve C and a fixed line bundle $\mathcal{L}$ correspond to the choices of two sections ($2(d - g + 1)$), minus 1 to forget the scalar multiple, for a total of $2d - 2g + 1$. If we let the the line bundle vary, the number of maps from a fixed curve is $2d - 2g + 1 + \dim \mathrm{Pic}^d(C) = 2d - g + 1$. If we let the curve also vary, we see that the number of degree d genus g covers of $\mathbb{P}^1$ is $\boxed{p + 2d - g + 1}$.

But we can also count this number using the Riemann-Hurwitz formula. I'll need one believable fact: there are a finite number of degree d covers with a given set of branch points. (In the complex case, this is believable for the following reason. If $C \to \mathbb{P}^1$ is a branched cover of $\mathbb{P}^1$, branched over p_1, ..., p_r, then by discarding the branch points and their preimages, we have an unbranched cover $C' \to \mathbb{P}^1 - \{p_1, \dots, p_r\}$. Then you can check that (i) the original map $C \to \mathbb{P}^1$ is determined by this map (because C is the normalization of $\mathbb{P}^1$ in this function field extension $\mathsf{FF}(C')/\mathsf{FF}(\mathbb{P}^1)$), and (ii) there are a finite number of such covers (corresponding to the monodromy data around these r points; we have r elements of S_d once we take branch cuts). This last step is where the characteristic 0 hypothesis is necessary.)

By the Riemann-Hurwitz formula, for a fixed g and d, the total amount of branching is $2g + 2d - 2$ (including multiplicity). Thus if the branching happens at no more than $2g + 2d - 2$ points, and if we have the simplest possible branching at $2g + 2d - 2$ points, the covering curve is genus g. Thus

$$p + 2d - g + 1 = 2g + 2d - 2,$$

from which $\boxed{p = 3g - 3}$.

Thus there is a $3g-3$-dimensional family of genus g curves! (By showing that the space of branched covers is reduced and irreducible, we could again "show" that the moduli space is reduced and irreducible.)

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