FOUNDATIONS OF ALGEBRAIC GEOMETRY CLASS 1

CONTENTS

Today: About this course. Why algebraic geometry? Motivation and program. Crash course in category theory: universal properties, Yoneda's lemma.

1. WELCOME

Welcome! This is Math 216A, Foundations of Algebraic Geometry, the first of a three-quarter sequence on the topic. I'd like to tell you a little about what I intend with this course.

Algebraic geometry is a subject that somehow connects and unifies several parts of mathematics, including obviously algebra and geometry, but also number theory, and depending on your point of view many other things, including topology, string theory, etc. As a result, it can be a handy thing to know if you are in a variety of subjects, notably number theory, symplectic geometry, and certain kinds of topology. The power of the field arises from a point of view that was developed in the 1960's in Paris, by the group led by Alexandre Grothendieck. The power comes from rather heavy formal and technical machinery, in which it is easy to lose sight of the intuitive nature of the objects under consideration. This is one reason why it used to strike fear into the hearts of the uninitiated.

The rough edges have been softened over the ensuing decades, but there is an inescapable need to understand the subject on its own terms.

This class is intended to be an experiment. I hope to try several things, which are mutually incompatible. Over the year, I want to cover the foundations of the subject fairly completely: the idea of varieties and schemes, the morphisms between them, their properties, cohomology theories, and more. I would like to do this rigorously, while

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trying hard to keep track of the geometric intuition behind it. This is the second time I will have taught such a class, and the first time I'm going to try to do this without working from a text. So in particular, I may find that I talk myself into a corner, and may tell you about something, and then realize I'll have to go backwards and say a little more about an earlier something.

Some of you have asked what background will be required, and how fast this class will move. In terms of background, I'm going to try to assume as little as possible, ideally just commutative ring theory, and some comfort with things like prime ideals and localization. *(All my rings will be commutative, and have unit!)* The more you know, the better, of course. But if I say things that you don't understand, please slow me down in class, and also talk to me after class. Given the amount of material that there is in the foundations of the subject, I'm afraid I'm going to move faster than I would like, which means that for you it will be like drinking from a firehose, as one of you put it. If it helps, I'm very happy to do my part to make it easier for you, and I'm happy to talk about things outside of class. I also intend to post notes for as many classes as I can. They will usually appear before the next class, but not always.

In particular, this will not be the type of class where you can sit back and hope to pick up things casually. The only way to avoid losing yourself in a sea of definitions is to become comfortable with the ideas by playing with examples.

To this end, I intend to give problem sets, to be handed in. They aren't intended to be onerous, and if they become so, please tell me. But they *are* intended to force you to become familiar with the ideas we'll be using.

Okay, I think I've said enough to scare most of you away from coming back, so I want to emphasize that I'd like to do everything in my power to make it better, short of covering less material. The best way to get comfortable with the material is to talk to me on a regular basis about it.

One other technical detail: you'll undoubtedly have noticed that this class is schedule for Mondays, Wednesdays, and Fridays, 9–10:30, $4\frac{1}{2}$ hours per week, not the usual 3. That's not because I'm psychotic; it was presumably a mistake. So I'm going to take advantage of it, and most weeks just meet two days a week, and I'll propose usually meeting on Mondays and Wednesday. I'll be away for some days, and so I'll make up for it by meeting on Fridays as well some weeks. I'll warn you well in advance.

Office hours: I haven't decided if it will be useful to have formal office hours rather than being available to talk after class, and also on many days by appointment. One possibility would be to have office hours on the 3rd day of the week during the time scheduled for class. Another is to have it some afternoon. I'm open to suggestions.

Okay, let's get down to business. I'd like to say a few words about what algebraic geometry is about, and then to start discussing the machinery.

Texts: Here are some books to have handy. Hartshorne's *Algebraic Geometry* has most of the material that I'll be discussing. It isn't a book that you should sit down and read, but you might find it handy to flip through for certain results. It should be at the bookstore, and is on 2-day reserve at the library. Mumford's *Red Book of Varieties and Schemes* has a good deal of the material I'll be discussing, and with a lot of motivation too. That is also on 2-day reserve in the library. The second edition is strictly worse than the 1st, because someone at Springer retyped it without understanding the math, introducing an irritating number of errors. If you would like something gentler, I would suggest Shafarevich's books on algebraic geometry. Another excellent foundational reference is Eisenbud and Harris' book *The geometry of schemes*, and Harris' earlier book *Algebraic geometry* is a beautiful tour of the subject.

For background, it will be handy to have your favorite commutative algebra book around. Good examples are Eisenbud's *Commutative Algebra with a View to Algebraic Geometry*, or Atiyah and Macdonald's *Commutative Algebra*. If you'd like something with homological algebra, category theory, and abstract nonsense, I'd suggest Weibel's book *Introduction to Homological Algebra*.

2. WHY ALGEBRAIC GEOMETRY?

It is hard to define algebraic geometry in its vast generality in a couple of sentences. So I'll talk around it a bit.

As a motivation, consider the study of manifolds. Real manifolds are things that locally look like bits of real n-space, and they are glued together to make interesting shapes. There is already some subtlety here — when you glue things together, you have to specify what kind of gluing is allowed. For example, if the transition functions are required to be differentiable, then you get the notion of a differentiable manifold.

A great example of a manifold is a submanifold of $\mathbb{R}^n$ (consider a picture of a torus). In fact, any compact manifold can be described in such a way. You could even make this your definition, and not worry about gluing. This is a good way to think about manifolds, but not the best way. There is something arbitrary and inessential about defining manifolds in this way. Much cleaner is the notion of an *abstract manifold*, which is the current definition used by the mathematical community.

There is an even more sophisticated way of thinking about manifolds. A differentiable manifold is obviously a topological space, but it is a little bit more. There is a very clever way of summarizing what additional information is there, basically by declaring what functions on this topological space are differentiable. The right notion is that of a sheaf, which is a simple idea, that I'll soon define for you. It is true, but non-obvious, that this ring of functions that we are declaring to be differentiable determines the differentiable manifold structure.

Very roughly, algebraic geometry, at least in its geometric guise, is the kind of geometry you can describe with polynomials. So you are allowed to talk about things like $y^2 = x^3 + x$, but not $y = \sin x$. So some of the fundamental geometric objects under consideration are things in n-space cut out by polynomials. Depending on how you define them, they are called *affine varieties* or *affine schemes*. They are the analogues of the patches on a manifold. Then you can glue these things together, using things that you can describe with polynomials, to obtain more general varieties and schemes. So then we'll have these algebraic objects, that we call varieties or schemes, and we can talk about maps between them, and things like that.

In comparison with manifold theory, we've really restricted ourselves by only letting ourselves use polynomials. But on the other hand, we have gained a huge amount too. First of all, we can now talk about things that aren't smooth (that are *singular*), and we can work with these things. (One thing we'll have to do is to define what we mean by smooth and singular!) Also, we needn't work over the real or complex numbers, so we can talk about arithmetic questions, such as: what are the rational points on $y^2 = x^3 + x^2$? (Here, we work over the field $\mathbb{Q}$.) More generally, the recipe by which we make geometric objects out of things to do with polynomials can generalize drastically, and we can make a geometric object out of rings. This ends up being surprisingly useful — all sorts of old facts in algebra can be interpreted geometrically, and indeed progress in the field of commutative algebra these days usually requires a strong geometric background.

Let me give you some examples that will show you some surprising links between geometry and number theory. To the ring of integers $\mathbb{Z}$, we will associate a smooth curve $\operatorname{Spec}\mathbb{Z}$. In fact, to the ring of integers in a number field, there is always a smooth curve, and to its orders (subrings), we have singular = non-smooth curves.

An old flavor of Diophantine question is something like this. Given an equation in two variables, $y^2 = x^3 + x^2$, how many rational solutions are there? So we're looking to solve this equation over the field $\mathbb{Q}$. Instead, let's look at the equation over the field $\mathbb{C}$. It turns out that we get a complex surface, perhaps singular, and certainly non-compact. So let me separate all the singular points, and compactify, by adding in points. The resulting thing turns out to be a compact oriented surface, so (assuming it is connected) it has a genus g, which is the number of holes it has. For example, $y^2 = x^3 + x^2$ turns out to have genus 0. Then Mordell conjectured that if the genus is at least 2, then there are at most a finite number of rational solutions. The set of complex solutions somehow tells you about the number of rational solutions! Mordell's conjecture was proved by Faltings, and earned him a Fields Medal in 1986. As an application, consider Fermat's Last Theorem. We're looking for integer solutions to $x^n + y^n = z^n$. If you think about it, we are basically looking for rational solutions to $X^n + Y^n = 1$. Well, it turns out that this has genus $\binom{n-1}{2}$ — we'll verify something close to this at some point in the future. Thus if n is at least 4, there are only a finite number of solutions. Thus Falting's Theorem implies that for each $n \geq 4$, there are only a finite number of counterexamples to Fermat's last theorem. Of course, we now know that Fermat is true — but Falting's theorem applies much more widely — for example, in more variables. The equations $x^3 + y^2 + z^14 + xy + 17 = 0$ and $3x^{14} + x^{34}y + \cdots = 0$, assuming their complex solutions form a surface of genus at least 2, which they probably do, have only a finite number of solutions.

So here is where we are going. Algebraic geometry involves a new kind of "space", which will allow both singularities, and arithmetic interpretations. We are going to define these spaces, and define maps between them, and other geometric constructions such as vector bundles and sheaves, and pretty soon, cohomology groups.

In order to think about these notions clearly and cleanly, it really helps to use the language of categories. There is not much to know about categories to get started; it is just a very useful language.

Here is an informal definition. I won't give you the precise definition unless you really want me to. A category has some *objects*, and some maps, or *morphisms*, between them. (For the pedants, I won't worry about sets and classes. And I'm going to accept the axiom of choice.) *The prototypical example to keep in mind is the category of sets.* The objects are sets, and the morphisms are maps of sets. Another good example is that of vector spaces over your favorite filed k. The objects are k-vector spaces, and the morphisms are linear transformations.

For each object, there is always an *identity morphism* from from the object to itself. There is a way of composing morphisms: if you have a morphism $f : A \to B$ and another $g : B \to C$, then there is a composed morphism $g \circ f : A \to C$. I could be pedantic and say that we have a map of sets $\mathrm{Mor}(A, B) \times \mathrm{Mor}(B, C) \to \mathrm{Mor}(A, C)$. Composition is associative: $(h \circ g) \circ f = h \circ (g \circ f)$. When you compose with the identity, you get the same thing.

Exercise. A category in which each morphism is an isomorphism is called a *groupoid*. (a) A perverse definition of a group is: a groupoid with one element. Make sense of this. (b) Describe a groupoid that is not a group. (This isn't an important notion for this course. The point of this exercise is to give you some practice with categories, by relating them to an object you know well.)

Here are a couple of other important categories. If R is a ring, then R-modules form a category. In the special case where R is a field, we get the category of vector spaces. There is a category of rings, where the objects are rings, and the morphisms are morphisms of rings (which I'll assume send 1 to 1).

If we have a category, then we have a notion of isomorphism between two objects (if we have two morphisms $f : A \to B$ and $g : B \to A$, *both* of whose compositions are the identity on the appropriate object), and a notion of automorphism.

3.1. **Functors.** A *covariant functor* is a map from one category to another, sending objects to objects, and morphisms to morphisms, such that everything behaves the way you want it to; if $F : \mathcal{A} \to \mathcal{B}$, and $a_1, a_2 \in \mathcal{A}$, and $m : a_1 \to a_2$ is a morphism in $\mathcal{A}$, then $F(m)$ is a morphism from $F(a_1) \to F(a_2)$ in B. Everything composes the way it should.

Example: If $\mathcal{A}$ is the category of complex vector spaces, and $\mathcal{B}$ is the category of sets, then there is a forgetful functor where to a complex vector space, we associate the set of its elements. Then linear transformations certainly can be interpreted as set maps.

A *contravariant functor* is just the same, except the arrows switch directions: in the above language, $F(m)$ is now an arrow from $F(a_2)$ to $F(a_1)$.

Example: If $\mathcal{A}$ is the category complex vector spaces, then taking duals gives a contravariant functor $\mathcal{A} \to \mathcal{A}$. Indeed, to each linear transformation $V \to W$, we have a dual transformation $W^* \to V^*$.

3.2. **Universal properties.** Given some category that we come up with, we often will have ways of producing new objects from old. In good circumstances, such a definition can be made using the notion of a *universal property*. Informally, we wish that there is an object with some property. We first show that if it exists, then it is essentially unique, or more precisely, is unique up to unique isomorphism. Then we go about constructing an example of such an object.

A good example of this, that you may well have seen, is the notion of a tensor product of R-modules. The way in which it is often defined is as follows. Suppose you have two R-modules M and N. Then the tensor product $M \otimes_R N$ is often first defined for people as follows: elements are of the form $m \otimes n$ ($m \in M$, $n \in N$), subject to relations $(m_1+m_2)\otimes n = m_1 \otimes n + m_2 \otimes n$, $m \otimes (n_1+n_2) = m \otimes n_1 + m \otimes n_2$, $r(m \otimes n) = (rm) \otimes n = m \otimes n$ (where $r \in R$).

Special case: if R is a field k, we get the tensor product of vector spaces.

Exercise (if you haven't seen tensor products before). Calculate $\mathbb{Z}/10 \otimes_{\mathbb{Z}} \mathbb{Z}/12$. (The point of this exercise is to give you a very little hands-on practice with tensor products.)

This is a weird definition!! And this is a clue that it is a "wrong" definition. A better definition: notice that there is a natural R-bilinear map $M \times N \to M \otimes_R N$. *Any* R-bilinear map $M \times N \to C$ factors through the tensor product uniquely: $M \times N \to M \otimes_R N \to C$. This is kind of clear when you think of it.

I could almost take this as the *definition* of the tensor product. Because if I could create something satisfying this property, $(M \otimes_R N)'$, and you were to create something else $(M \otimes_R N)''$, then by my universal property for $C = (M \otimes_R N)''$, there would be a unique map $(M \otimes_R N)' \to (M \otimes_R N)''$ interpolating $M \times N \to (M \otimes_R N)''$, and similarly by your universal property there would be a unique universal map $(M \otimes_R N)'' \to (M \otimes_R N)'$. The composition of these two maps in one order

$$(M \otimes_R N)' \to (M \otimes_R N)'' \to (M \otimes_R N)'$$

has to be the identity, by the universal property for $C = (M \otimes_R N)'$, and similarly for the other composition. Thus we have shown that these two maps are inverses, and our two spaces are isomorphic. In short: our two definitions may not be the *same*, but there is a canonical isomorphism between them. Then the "usual" construction works, but someone else may have another construction which works just as well.

I want to make three remarks. First, if you have never seen this sort of argument before, then you might think you get it, but you don't. So you should go back over the notes, and think about it some more, because it is rather amazing. Second, the language I would use to describe this is as follows: There is an R-bilinear map $t : M \times N \to M \otimes_R N$, unique up to unique isomorphism, defined by the following universal property: for any R-bilinear map $s : M \times N \to C$ there is a unique $f : M \otimes_R N \to C$ such that $s = f \circ t$. Third, you might notice that I didn't use much about the R-module structure, and indeed I can vary this to get a very general statement. This takes us to a powerful fact, that is very zen: it is very deep, but also very shallow. It's hard, but easy. It is black, but white. I'm going to tell you about it, and it will be mysterious, but then I'll show you some concrete examples.

Here is a motivational example: the notion of **product**. You have likely seen product defined in many cases, for example the notion of a product of manifolds. In each case, the definition agreed with your intuition of what a product should be. We can now make this precise. I'll describe product in the category of sets, in a categorical manner. Given two sets M and N, there is a unique set $M \times N$, along with maps to M and N, such that for *any other set* S *with maps to* M *and* N, this map must factor *uniquely* through $M \times N$:

$$\begin{tikzcd} S \arrow[dr, dotted, "\exists!"] & & \\ & M \times N \arrow[r] \arrow[d] & N \\ & M. & \end{tikzcd}$$

You can immediately check that this agrees with the usual definition. But it has the advantage that we now have a definition in any category! The product may not exist, but if it does, then we know that it is unique up to unique isomorphism! (Explain.) This is handy even in cases that you understand. For example, one way of defining the product of two manifolds M and N is to cut them both up in to charts, then take products of charts, then glue them together. But if I cut up the manifolds in one way, and you cut them up in another, how do we know our resulting manifolds are the "same"? We could wave our hands, or make an annoying argument about refining covers, but instead, we should just show that they are indeed products, and hence the "same" (aka isomorphic).

3.3. **Yoneda's Lemma.** I want to begin with an easy fact that I'll state in a complicated way. Suppose we have a category $\mathcal{C}$. This isn't scary — just pick your favorite friendly low-brow category. Pick an object in your category $A \in \mathcal{C}$. Then for any object $C \in \mathcal{C}$, we have a set of morphisms $\mathrm{Mor}(C, A)$. If we have a morphism $f : B \to C$, we get a map of sets

$$\mathrm{Mor}(C, A) \to \mathrm{Mor}(B, A), \tag{1}$$

just by composition: given a map from C to A, we immediately get a map from B to A by precomposing with f. In fancy language, we have a contravariant functor from the category $\mathcal{C}$ to the category of sets $\underline{\mathrm{Sets}}$. Yoneda's lemma, or at least part of it, says that this functor determines A up to unique isomorphism. Translation: If we have two objects A and A', and isomorphisms

$$i_C : \mathrm{Mor}(C, A) \to \mathrm{Mor}(C, A') \tag{2}$$

that commute with the maps (1), then the i_C must be induced from a unique morphism $A \to A'$.

Important Exercise. Prove this. This sounds hard, but it really is not. This statement is so general that there are really only a couple of things that you could possibly try. For example, if you're hoping to find an isomorphism $A \to A'$, where will you find it? Well, you're looking for an element $\mathrm{Mor}(A, A')$. So just plug in $C = A$ to (2), and see where the identity goes. (Everyone should prove Yoneda's Lemma once in their life. This is your chance.)

Remark. There is an analogous statement with the arrows reversed, where instead of maps into A, you think of maps *from* A.

Example: Fibered products. Suppose we have morphisms $X, Y \to Z$. Then the *fibered product* is an object $X \times_Z Y$ along with morphisms to X and Y, where the two compositions $X \times_Z Y \to Z$ agree, such that given any other object W with maps to X and Y (whose compositions to Z agree), these maps factor through some unique $W \to X \times_Z Y$:

$$\begin{tikzcd}
W \arrow[dr, dotted, "\exists!"] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& X \times_Z Y \arrow[r, "\pi_Y"] \arrow[d, "\pi_X"] & Y \arrow[d, "g"] \\
& X \arrow[r, "f"] & Z
\end{tikzcd}$$

The right way to interpret this is first to think about what it means in the category of sets. I'll tell you it, and let you figure out why I'm right: $X \times_Z Y = \{(x \in X, y \in Y) : f(x) = g(y)\}$.

In any category, we can make this definition, and we know thanks to Yoneda that if it exists, then it is unique up to unique isomorphism, and so we should reasonably be allowed to give it the name $X \times_Z Y$. We know what maps to it are: they are precisely maps to X and maps to Y that agree on maps to Z.

(Remark for experts: if our category has a final object, then the fibered product over the final object is just the product.)

The notion of fibered product will be important for us later.

Exercises on fibered product. (a) Interpret fibered product in the category of sets: If we are given maps from sets X and Y to the set Z, interpret $X \times_Z Y$. (This will help you build intuition about this concept.)
(b) A morphism $f : X \to Y$ is said to be a a **monomorphism** if any two morphisms $g_1, g_2 : Z \to X$ such that $f \circ g_1 = f \circ g_2$ must satisfy $g_1 = g_2$. This is the generalization of an injection of sets. Prove that a morphism is a monomorphism if and only if the natural morphism $X \to X \times_Y X$ is an isomorphism. (We may then take this as the definition of monomorphism.) (Monomorphisms aren't very central to future discussions, although they will come up again. This exercise is just good practice.)
(c) Suppose $X \to Y$ is a monomorphism, and $W, Z \to X$ are two morphisms. Show that $W \times_X Z$ and $W \times_Y Z$ are canonically isomorphic. (We will use this later when talking about fibered products.)
(d) Given $X \to Y \to Z$, show that there is a natural morphism $X \times_Y X \to X \times_Z X$. (This is trivial once you figure out what it is saying. The point of this exercise is to see why it is trivial.)

Important Exercise. Suppose $T \to R, S$ are two ring morphisms. Let I be an ideal of R. We get a morphism $R \to R \otimes_T S$ by definition. Let I^e be the extension of I to $R \otimes_T S$. (These are the elements $\sum_j i_j \otimes s_j$ where $i_j \in I$, $s_j \in S$. But it is more elegant to solve this exercise using the universal property.) Show that there is a natural isomorphism

$$R/I \otimes_T S \cong (R \otimes_T S)/I^e.$$

Hence the natural morphism $S \otimes_T R \to S \otimes_T R/I$ is a surjection. As an application, we can compute tensor products of finitely generated k algebras over k. For example,

$$k[x_1, x_2]/(x_1^2 - x_2) \otimes_k k[y_1, y_2]/(y_1^3 + y_2^3) \cong k[x_1, x_2, y_1, y_2]/(x_1^2 - x_2, y_1^3 + y_2^3).$$

Exercise. Define coproduct in a category by reversing all the arrows in the definition of product. Show that coproduct for sets is disjoint union.

I then discussed adjoint functors briefly. I will describe them again briefly next day.

Next day: more examples of universal properties, including direct and inverse limits. Groupification. Sheaves!

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FOUNDATIONS OF ALGEBRAIC GEOMETRY CLASS 2

RAVI VAKIL

CONTENTS

Last day: What is algebraic geometry? Crash course in category theory, ending with Yoneda's Lemma.

Today: more examples of things defined using universal properties: inverse limits, direct limits, adjoint functors, groupification. Sheaves: the motivating example of differentiable functions. Definition of presheaves and sheaves.

At the start of the class, everyone filled out a sign-up sheet, giving their name, e-mail address, mathematical interests, and odds of attending. You're certainly not committing yourself to anything by signing it; it will just give me a good sense of who is in the class. Also, I'll use this for an e-mail list. If you didn't fill out the sheet, but want to, e-mail me.

I'll give out some homework problems on Monday, that will be based on this week's lectures. Most or all of the problems will already have been asked in class. Most likely I will give something like ten problems, and ask you to do five of them.

If you have any questions, please ask me, both in and out of class. Other people you can ask include the other algebraic geometers in the class, including Rob Easton, Andy Schultz, Jarod Alper, Joe Rabinoff, Nikola Penev, and others.

1. SOME CONSTRUCTIONS USING UNIVERSAL PROPERTIES, OLD AND NEW

Last day, I defined categories and functors. I hope I convinced you that you already have a feeling for categories, because you know so many examples.

A key point was **Yoneda's Lemma**, which says informally that you can essentially recover an object in a category by knowing the maps into it. For example, the data of maps to $X \times Y$ are precisely the data of maps to X and to Y. The data of maps to $X \times_Z Y$ are

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precisely the data of maps to X and Y that commute with maps to Z. (Explain how the universal property says this.)

1.1. Example: Inverse limits. Here is another example of something defined by universal properties. Suppose you have a sequence

$$\cdots \longrightarrow A_3 \longrightarrow A_2 \longrightarrow A_1$$

of morphisms in your category. Then the **inverse limit** is an object $\lim_{\leftarrow} A_i$ along with commuting morphisms to all the A_i

$$\begin{tikzcd}
 & \lim_{\leftarrow} A_i \arrow[d] \arrow[dr] \arrow[drr] & & \\
\cdots \arrow[r] & A_3 \arrow[r] & A_2 \arrow[r] & A_1
\end{tikzcd}$$

so that any other object along with maps to the A_i factors through $\lim_{\leftarrow} A_i$

$$\begin{tikzcd}
W \arrow[r, dotted, "\exists!"] \arrow[dr] \arrow[drr] \arrow[drrr] & \lim_{\leftarrow} A_i \arrow[d] \arrow[dr] \arrow[drr] & & \\
\cdots \arrow[r] & A_3 \arrow[r] & A_2 \arrow[r] & A_1
\end{tikzcd}$$

You've likely seen such a thing before. How many of you have seen the p-adics ($\mathbb{Z}_p = ? + ?p + ?p^2 + ?p^3 + \cdots$)? Here's an example in the categories of rings.

$$\begin{tikzcd}
 & \mathbb{Z}_p \arrow[d] \arrow[dr] \arrow[drr] & & \\
\cdots \arrow[r] & \mathbb{Z}/p^3 \arrow[r] & \mathbb{Z}/p^2 \arrow[r] & \mathbb{Z}/p
\end{tikzcd}$$

You can check using your universal property experience that if it exists, then it is unique, up to unique isomorphism. It will boil down to the following fact: we know precisely what the maps to $\lim_{\leftarrow} A_i$ are: they are the same as maps to all the A_i's.

A few quick comments.

(1) We don't know in general that they have to exist.

(2) Often you can see that it exists. If these objects in your categories are all sets, as they are in this case of $\mathbb{Z}_p$, you can interpret the elements of the inverse limit as an element of $a_i \in A_i$ for each i, satisfying $f(a_i) = a_{i-1}$. From this point of view, $2 + 3p + 2p^2 + \cdots$ should be understood as the sequence $(2, 2 + 3p, 2 + 3p + 2p^2, \ldots)$.

(3) We could generalize the system in any different ways. We could basically replace it with any category, although this is way too general. (Most often this will be a *partially ordered set*, often called *poset* for short.) If you wanted to say it in a fancy way, you could say the system could be indexed by an arbitrary category. Example: the product is an example of an inverse limit. Another example: the fibered product is an example of an inverse limit. (What are the partially ordered sets in each case?) Infinite products, or indeed products in general, are examples of inverse limits.

1.2. Example: Direct limits. More immediately relevant for us will be the dual of this notion. We just flip all the arrows, and get the notion of a *direct limit*. Again, if it exists, it is unique up to unique isomorphism.

Here is an example. $5^{-\infty}\mathbb{Z} = \lim_{\rightarrow} 5^{-i}\mathbb{Z}$ is an example. (These are the rational numbers whose denominators are required to be powers of 5.)

$$\begin{tikzcd}
\mathbb{Z} \arrow[r] \arrow[d] & 5^{-1}\mathbb{Z} \arrow[r] \arrow[dl] & 5^{-2}\mathbb{Z} \arrow[r] \arrow[dll] & \cdots \\
5^{-\infty}\mathbb{Z} & & &
\end{tikzcd}$$

Even though we have just flipped the arrows, somehow it behaves quite differently from the inverse limit.

Some observations:

(1) In this example, each element of the direct limit is an element of something upstairs, but you can't say in advance what it is an element of. For example, 17/125 is an element of the $5^{-3}\mathbb{Z}$ (or $5^{-4}\mathbb{Z}$, or later ones), but not $5^{-2}\mathbb{Z}$.

(2) We can index this by any partially ordered set (or *poset*). (Or even any category, although I don't know if we care about this generality.)

1.3. Remark. (3) That first remark applies in some generality for the category of A-modules, where A is a ring. (See Atiyah-Macdonald p. 32, Exercise 14.) We say a partially ordered set I is a *directed set* if for $i, j \in I$, there is some $k \in I$ with $i, j \leq k$. We can show that the direct limit of any system of R-modules indexed by I exists, by constructing it. Say the system is given by M_i ($i \in I$), and $f_{ij} : M_i \to M_j$ ($i \leq j$ in I). Let $M = \oplus_i M_i$, where each M_i is associated with its image in M, and let R be the submodule generated by all elements of the form $m_i - f_{ij}(m_i)$ where $m_i \in M_i$ and $i \leq j$. **Exercise.** Show that M/R (with the inclusion maps from the M_i) is $\lim_{\rightarrow} M_i$. (This example will come up soon.) You will notice that the same argument works in other interesting categories, such as: sets; groups; and abelian groups. (Less important question for the experts: what hypotheses do we need for this to work more generally?)

(4) (Infinite) sums are examples of direct limits.

2. Adjoint functors

Let me re-define adjoint functors (Weibel Definition 2.3.9). Two *covariant* functors $L : \mathcal{A} \to \mathcal{B}$ and $R : \mathcal{B} \to \mathcal{A}$ are *adjoint* if there is a natural bijection for all $A \in \mathcal{A}$ and $B \in \mathcal{B}$

$$\tau_{AB} : \mathrm{Hom}_{\mathcal{B}}(L(A), B) \to \mathrm{Hom}_{\mathcal{A}}(A, R(B)).$$

In this instance, let me make precise what "natural" means, which will also let us see why the functors here are covariant. For all $f : A \to A'$ in $\mathcal{A}$, we require

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{B}}(L(A'), B) \arrow[r, "Lf^*"] \arrow[d, "\tau"] & \mathrm{Hom}_{\mathcal{B}}(L(A), B) \arrow[d, "\tau"] \\
\mathrm{Hom}_{\mathcal{A}}(A', R(B)) \arrow[r, "f^*"] & \mathrm{Hom}_{\mathcal{A}}(A, R(B))
\end{tikzcd}$$

to commute, and for all $g : B \to B'$ in $\mathcal{B}$ we want a similar commutative diagram to commute. (Here f^* is the map induced by $f : A \to A'$, and Lf^* is the map induced by $Lf : L(A) \to L(A')$.)

Exercise. Write down what this diagram should be.

We could figure out what this should mean if the functors were both contravariant. I haven't tried to see if this could make sense.

You've actually seen this before, in linear algebra, when you have seen adjoint matrices. But I've long forgotten how they work, so let me show you another example. (Question for the audience: is there a very nice example out there?)

2.1. Example: groupification. Motivating example: getting a group from a semigroup. A semigroup is just like a group, except you don't require an inverse. Examples: the non-negative integers $0, 1, 2, \dots$ under addition, or the positive integers under multiplication $1, 2, \dots$. From a semigroup, you can create a group, and this could be called groupification. Here is a formalization of that notion. If S is a semigroup, then its groupification is a map of semigroups $\pi : S \to G$ such that G is a group, and any other map of semigroups from S to a *group* G' factors *uniquely* through G.

$$\begin{tikzcd}
S \arrow[r] \arrow[dr, "\pi"] & G \arrow[d, dashed, "\exists!"] \\
 & G'
\end{tikzcd}$$

(Thanks Jack for explaining how to make dashed arrows in `\xymatrix`.)

(General idea for experts: We have a full subcategory of a category. We want to "project" from the category to the subcategory. We have $\mathrm{Hom}_{\text{category}}(S, H) = \mathrm{Hom}_{\text{subcategory}}(G, H)$ automatically; thus we are describing the left adjoint to the forgetful functor. How the argument worked: we constructed something which was in the small category, which automatically satisfies the universal property.)

Example of a universal property argument: If a semigroup is *already* a group then groupification is the identity morphism, by the universal property.

Exercise to get practice with this. Suppose R is a ring, and S is a multiplicative subset. Then $S^{-1}R$-modules are a full subcategory of the category of R-modules. Show that $M \to$

$S^{-1}M$ satisfies a universal property. Translation: Figure out what the universal property is.

2.2. Additive and abelian categories. There is one last concept that we will use later. It is convenient to give a name to categories with some additional structure. Here are some definitions.

Initial object of a category. It is an object with a unique map to any other object. (By a universal property argument, if it exists, it is unique up to unique isomorphism.) Example: the empty set, in the category of sets.

Final object of a category. It is an object with a unique map from any other object. (By a universal property argument, if it exists, it is unique up to unique isomorphism.) Question: does the category of sets have a final object?

Exercise. If Z is the final object in a category $\mathcal{C}$, and $X, Y \in \mathcal{C}$, then "$X \times_Z Y = X \times Y$" ("the" fibered product over Z is canonically isomorphic to "the" product). (This is an exercise about unwinding the definition.)

Additive categories (Weibel, p. 5). (I think I forgot to say part of this definition in class.) A category $\mathcal{C}$ is said to be *additive* if it has the following properties. For each $A, B \in \mathcal{C}$, $\mathrm{Hom}_{\mathcal{C}}(A, B)$ is an abelian group, such that composition of morphisms distributes over addition (think about what this could mean). It has a 0-object (= simultaneously initial object and final object), and products (a product $A \times B$ for any pair of objects). (Why is the 0-object called the 0-object?)

Yiannis points out that Banach spaces form an additive category. Another example are R-modules for a ring R, but they have even more structure.

Abelian categories (Weibel, p. 6). I deliberately didn't give a precise definition in class, as you should first get used to this concept before reading the technical definition.

But here it is. Let $\mathcal{C}$ be an additive category. First, a *kernel* of a morphism $f : B \to C$ is a map $i : A \to B$ such that $f \circ i = 0$, and that is universal with respect to this property. (Hence it is unique up to unique isomorphism by universal property nonsense. Note that we said "a" kernel, not "the" kernel.) A *cokernel* is defined dually by reversing the arrows — do this yourself. We say a morphism i in $\mathcal{C}$ is *monic* if $i \circ g = 0$, where the source of g is the target of i, implies $g = 0$. Dually, there is the notion of *epi* — reverse the arrows to find out what that is.

An *abelian category* is an additive category satisfying three properties. 1. Every map has a kernel and cokernel. 2. Every monic is the kernel of its cokernel. 3. Every epi is the cokernel of its kernel.

It is a non-obvious and imprecise fact that every property you want to be true about kernels, cokernels, etc. follows from these three.

An abelian category has kernels and cokernels and images, and they behave the way you expect them to. So you can have exact sequences: we say

$$A \xrightarrow{f} B \xrightarrow{g} C$$

is exact if $\ker g = \operatorname{im} f$.

The key example of an abelian category is the category of R-modules (where R is a ring).

3. Sheaves

I now want to discuss an important new concept, the notion of a *sheaf*. A sheaf is the kind of object you will automatically consider if you are interested in something like the continuous functions on a space X, or the differentiable functions on a space X, or things like that. Basically, you want to consider all continuous functions on all open sets all at once, and see what properties this sort of collection of information has. I'm going to motivate it for you, and tell you the definition. I find this part quite intuitive. Then I will do things with this concept (for example talking about cokernels of maps of sheaves), and things become less intuitive.

3.1. Motivating example: sheaf of differentiable functions.

We'll consider differentiable functions on $X = \mathbb{R}^n$, or a more general manifold X. To each open set $U \subset X$, we have a ring of differentiable functions. I will denote this ring $\mathcal{O}(U)$.

If you take a differentiable function on an open set, you can restrict it to a smaller open set, and you'll get a differentiable function there. In other words, if $U \subset V$ is an inclusion of open sets, we have a map $\mathrm{res}_{V,U}; \mathcal{O}(V) \to \mathcal{O}(U)$.

If you take a differentiable function on a big open set, and restrict it to a medium open set, and then restrict that to a small open set, then you get the same thing as if you restrict the differentiable function on the big open set to the small open set all at once. In other words, if $U \hookrightarrow V \hookrightarrow W$, then the following diagram commutes:

$$\begin{tikzcd} \mathcal{O}(W) \arrow[rr, "\mathrm{res}_{W,V}"] \arrow[dr, "\mathrm{res}_{W,U}"'] & & \mathcal{O}(V) \arrow[dl, "\mathrm{res}_{V,U}"] \\ & \mathcal{O}(U) & \end{tikzcd}$$

Now say you have two differentiable functions f_1 and f_2 on a big open set U, and you have an open cover of U by some U_i. Suppose that f_1 and f_2 agree on each of these U_i. Then they must have been the same function to begin with. Right? In other words, if $\{U_i\}_{i \in I}$ is a cover of U, and $f_1, f_2 \in \mathcal{O}(U)$, and $\mathrm{res}_{U,U_i} f_1 = \mathrm{res}_{U,U_i} f_2$, then $f_1 = f_2$. In other words, I can *identify* functions locally.

Finally, suppose I still have my U, and my cover U_i of U. Suppose I've got a differentiable function on each of the U_i — a function on U_1, a function on U_2, and so on — and they agree on the overlaps. Then I can glue all of them together to make one function on all of U. Right? In other words: given $f_i \in \mathcal{O}(U_i)$ for all i, such that $\mathrm{res}_{U_i, U_i \cap U_j} f_i = \mathrm{res}_{U_j, U_i \cap U_j} f_j$ for all i, j, then there is some $f \in \mathcal{O}(U)$ such that $\mathrm{res}_{U, U_i} f = f_i$ for all i.

Great. Now I could have done all this with continuous functions. [Go back over it all, with differentiable replaced by continuous.] Or smooth functions. Or just functions. That's the idea that we'll formalize soon into a sheaf.

3.2. Motivating example continued: the germ of a differentiable function. Before we do, I want to point out another definition, that of the germ of a differentiable function at a point $x \in X$. Intuitively, it is a shred of a differentiable function at x. Germs are objects of the form $\{(f, \text{open } U) : x \in U, f \in \mathcal{O}(U)\}$ modulo the relation that $(f, U) \sim (g, V)$ if there is some open set $W \subset U, V$ where $f|_W = g|_W$ (or in our earlier language, $\mathrm{res}_{U,W} f = \mathrm{res}_{V,W} g$). In other words, two functions that are the same here near x but differ way over there have the same germ. Let me call this set of germs $\mathcal{O}_x$. Notice that this forms a ring: you can add two germs, and get another germ: if you have a function f defined on U, and a function g defined on V, then $f + g$ is defined on $U \cap V$. Notice also that if $x \in U$, you get a map

$$\mathcal{O}(U) \to \{\text{germs at } x\}.$$

Aside for the experts: this is another example of a direct limit, and I'll tell you why in a bit.

Fact: $\mathcal{O}_x$ is a local ring. Reason: Consider those germs vanishing at x. That certainly is an ideal: it is closed under addition, and when you multiply something vanishing at x by any other function, you'll get something else vanishing at x. Anything not in this ideal is invertible: given a germ of a function f not vanishing at x, then f is non-zero near x by continuity, so $1/f$ is defined near x. The residue map should map onto a field, and in this case it does: we have an exact sequence:

$$0 \longrightarrow \mathfrak{m} := \text{ideal of germs vanishing at } x \longrightarrow \mathcal{O}_x \xrightarrow{f \mapsto f(x)} \mathbb{R} \longrightarrow 0$$

If you have never seen exact sequences before, this is a good chance to figure out how they work. This is what is called a short exact sequence. **Exercise.** Check that this is an exact sequence, i.e. that the image of each map is the kernel of the next. Show that this implies that the map on the left is an injection, and the one on the right is a surjection.

(Interesting fact, for people with a little experience with a little geometry: $\mathfrak{m}/\mathfrak{m}^2$ is a module over $\mathcal{O}_x/\mathfrak{m} \cong \mathbb{R}$, i.e. it is a real vector space. It turns out to be "naturally" — whatever that means — the cotangent space to the manifold at x. This will turn out to be handy later on, when we define tangent and cotangent spaces of schemes.)

Conclusion: We can interpret the value of a function at a point, or the value of a germ at a point, as an element of the local ring modulo the maximal ideal. (However, this can be a bit more problematic for more general sheaves.)

3.3. Definition of sheaf and presheaf.

We are now ready to formalize these notions.

Definition: Sheaf on a topological space X. (A note on language: this is called a sheaf because of an earlier, different perspective on the definition, see Serre's *Faisceaux Algébriques Cohérents*. I'm not going to discuss this earlier definition, so you'll have to take this word without any motivation.)

I will define a sheaf of sets , just to be concrete. But you can have sheaves of groups, rings, modules, etc. without changing the definitions at all. Indeed, if you want to be fancy, you can say that you can have a sheaf with values in any category.

A *presheaf* $\mathcal{F}$ is the following data. To each open set $U \subset X$, we have a set $\mathcal{F}(U)$ (e.g. the set of differentiable functions). (Notational warning: Several notations are in use, for various good reasons: $\mathcal{F}(U) = \Gamma(U, \mathcal{F}) = H^0(U, \mathcal{F})$. I will use them all. I forgot to say this in class, but will say it next day.) The elements of $\mathcal{F}(U)$ are called *sections of $\mathcal{F}$ over* U.

For each inclusion $U \hookrightarrow V$, we have a restriction map $\operatorname{res}_{V,U} : \mathcal{F}(V) \to \mathcal{F}(U)$ (just as we did for differentiable functions). The map $\operatorname{res}_{U,U}$ should be the identity. If $U \hookrightarrow V \hookrightarrow W$, then the restriction maps commute, i.e. the following diagram commutes.

$$\begin{tikzcd}
\mathcal{F}(W) \arrow[rr, "\operatorname{res}_{W,V}"] \arrow[dr, "\operatorname{res}_{W,U}"'] & & \mathcal{F}(V) \arrow[dl, "\operatorname{res}_{V,U}"] \\
& \mathcal{F}(U) &
\end{tikzcd}$$

That ends the definition of a presheaf.

3.4. Useful exercise for experts liking category theory: "A presheaf is the same as a contravariant functor". Given any topological space X, we can get a category, which I will call the "category of open sets". The objects are the open sets. The morphisms are the inclusions $U \hookrightarrow V$. (What is the initial object? What is the final object?) Verify that the data of a presheaf is precisely the data of a contravariant functor from the category of open sets of X to the category of sets.

Note for pedants, which can be ignored by everyone else. An annoying question is: what is $\mathcal{F}(\emptyset)$. We will see that it can be convenient to have $\mathcal{F}(\emptyset) = \{1\}$, or more generally, if we are having sheaves with value in some category $\mathcal{C}$ (such as Groups), we would like $\mathcal{F}(\emptyset)$ to be the final object in the category. This should probably be part of the definition of presheaf. (For example, Weibel, p. 26, takes it as such; Weibel seems to define sheaves with values only in an abelian category.) I hope to be fairly scrupulous in this course, so I hope people who care keep me honest on issues like this.

We add two more axioms to make this into a sheaf.

Identity axiom. If $\{U_i\}_{i \in I}$ is a open cover of U, and $f_1, f_2 \in \mathcal{F}(U)$, and $\operatorname{res}_{U,U_i} f_1 = \operatorname{res}_{U,U_i} f_2$, then $f_1 = f_2$.

(A presheaf + identity axiom is sometimes called a *separated sheaf*, but we will not use that notation here.)

Gluability axiom. If $\{U_i\}_{i\in I}$ is a open cover of U, then given $f_i \in \mathcal{F}(U_i)$ for all i, such that $\mathrm{res}_{U_i,U_i\cap U_j} f_i = \mathrm{res}_{U_j,U_i\cap U_j} f_j$ for all i, j, then there is some $f \in \mathcal{F}(U)$ such that $\mathrm{res}_{U,U_i} f = f_i$ for all i.

(Philosophical note: identity means there is at most one way to glue. Gluability means that there is at least one way to glue.)

Remark for people enjoying category theory for the first time — as opposed to learning it for the first time. The gluability axiom may be interpreted as saying that $\mathcal{F}(\cup_{i\in I}U_i)$ is a certain inverse limit.

Example. If U and V are disjoint, then $\mathcal{F}(U \cup V) = \mathcal{F}(U) \times \mathcal{F}(V)$. (Here we use the fact that $F(\emptyset)$ is the final object, from the "note for pedants" above.)

3.5. Exercise. Suppose Y is a topological space. Show that "continuous maps to Y" form a sheaf of sets on X. More precisely, to each open set U of X, we associate the set of continuous maps to Y. Show that this forms a sheaf.
(Fancier versions that you can try:
(b) Suppose we are given a continuous map $f : Y \to X$. Show that "sections of f" form a sheaf. More precisely, to each open set U of X, associate the set of continuous maps s to Y such that $f \circ s = id|_U$. Show that this forms a sheaf.
(c) (If you know what a topological group is.) Suppose that Y is a topological group. Show that maps to Y form a sheaf of *groups*. (If you don't know what a topological group is, you might be able to guess.)

Example: skyscraper sheaf. Suppose X is a topological space, with $x \in X$, and G is a group. Then $\mathcal{F}$ defined by $\mathcal{F}(U) = G$ if $x \in U$ and $\mathcal{F}(U) = \{e\}$ if $x \notin U$ forms a sheaf. (Check this if you don't see how.) This is called a *skyscraper sheaf*, because the informal picture of it looks like a skyscraper at x.

Important example/exercise: the pushforward . Suppose $f : X \to Y$ is a continuous map, and $\mathcal{F}$ is a sheaf on X. Then define $f_*\mathcal{F}$ by $f_*\mathcal{F}(V) = \mathcal{F}(f^{-1}(V))$, where V is an open subset of Y. Show that $f_*\mathcal{F}$ is a sheaf. This is called a *pushforward sheaf*. More precisely, $f_*\mathcal{F}$ is called the *pushforward of $\mathcal{F}$ by f*.

Example / exercise. (a) Let X be a topological space, and S a set with more than one element, and define $\mathcal{F}(U) = S$ for all open sets U. Show that this forms a presheaf (with the obvious restriction maps), and even satisfies the identity axiom. Show that this needn't form a sheaf. (Actually, for this to work, here we need $\mathcal{F}(\emptyset)$ to be the final object, not S. Without this patch, the constant presheaf *is* a sheaf. You can already see how the empty set is giving me a headache.) This is called the *constant presheaf with values in* S. We will denote this presheaf by $\underline{S}^{\mathrm{pre}}$.
(b) Now let $\mathcal{F}(U)$ be the maps to S that are *locally constant*, i.e. for any point x in U, there is a neighborhood of x where the function is constant. A better description is this: endow S with the discrete topology, and let $\mathcal{F}(U)$ be the continuous maps $U \to S$. Show that this is a *sheaf*. (Here we need $\mathcal{F}(\emptyset)$ to be the final object again, *not* S.) Using the "better description", this follows immediately from Exercise 3.5. We will try to call this the *locally*

constant sheaf. (Unfortunately, in the real world, this is stupidly called the *constant sheaf*.) We will denote this sheaf by $\underline{S}$.

3.6. Stalks. We define **stalk** = set of germs of a (pre)sheaf $\mathcal{F}$ in just the same way as before: Elements are $\{(f, \text{open } U) : x \in U, f \in \mathcal{O}(U)\}$ modulo the relation that $(f, U) \sim (g, V)$ if there is some open set $W \subset U, V$ where $\operatorname{res}_{U,W} f = \operatorname{res}_{V,W} g$. In other words, two section that are the same near x but differ far away have the same germ. This set of germs is denoted $\mathcal{F}_x$.

A useful equivalent definition is as a direct limit, of all $\mathcal{F}(U)$ where $x \in U$:

$$\mathcal{F}_x := \{\text{germs at } x\} = \lim_{\rightarrow} \mathcal{F}(U).$$

(All such U into a partially ordered set using inclusion. People having thought about the category of open sets, §3.4, will have a warm feeling in their stomachs.) This poset is a directed set (§1.3: given any two such sets, there is a third such set contained in both), so these two definitions are the same by Remark/Exercise 1.3. It would be good for you to think this through. Hence by that Remark/Exercise, we can have stalks for sheaves of sets, groups, rings, and other things for which direct limits exist for directed sets.

Let me repeat: it is useful to think of stalks in both ways, as direct limits, and also as something extremely explicit: an element of a stalk at p has as a representative a section over an open set near p.

Caution: Value at a point doesn't yet make sense.

3.7. Exercise. Show that pushforward induces maps of stalks.

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FOUNDATIONS OF ALGEBRAIC GEOMETRY CLASS 3

RAVI VAKIL

CONTENTS

Last day: end of category theory background. Motivation for and definitions of presheaf, sheaf, stalk.

Today: Presheaves and sheaves. Morphisms thereof. Sheafification.

I will be away Wednesday Oct. 5 to Thursday Oct. 13. The *next* class will be Friday, October 14. That means there will be *no class* this Wednesday, or next Monday or Wednesday. If you want to be on the e-mail list (low traffic), and didn't sign up last day, please let me know.

Problem set 1 out today, due Monday Oct. 17.

1. WHERE WE WERE

At this point, you're likely wondering when we're going to get to some algebraic geometry. We'll start that next class. We're currently learning how to think about things correctly. When we define interesting new objects, we'll learn how we want them to behave because we know a little category theory.

1.1. **Category theory.** I think in the heat of the last lecture, I skipped something I shouldn't have. An abelian category has several properties. One of these is that the morphisms form abelian groups: $\mathrm{Hom}(A, B)$ is an abelian group. This behaves well with respect to composition. For example if $f, g : A \to B$, and $h : B \to C$, then $h \circ (f + g) = h \circ f + h \circ g$. There is an obvious dual statement, that I'll leave to you. This implies other things, such as for example $0 \circ f = 0$. I think I forgot to say the above. An abelian category also has

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a 0-object (an object that is both a final object and initial object). An abelian category has finite products. If you stopped there, you'd have the definition of an additive category.

In an additive category, you can define things like kernels, cokernels, images, epimorphisms, monomorphisms, etc. In an abelian category, these things behave just way you expect them to, from your experience with R-modules. I've put the definition in the last day's notes.

2. Presheaves and Sheaves

We then described presheaves and sheaves on a topological space X. I'm going to remind you of two examples, and introduce a third. The **first example** was of a sheaf of nice functions, say differentiable functions, which I will temporarily call $\mathcal{O}_X$. This is an example of a sheaf of rings.

The axioms are as follows. We can have sheaves of rings, groups, abelian groups, and sets.

To each open set, we associate a ring $\mathcal{F}(U)$. Elements of this ring are called *sections of the sheaf over* U. (Notational warning: Several notations are in use, for various good reasons: $\mathcal{F}(U) = \Gamma(\mathcal{F}, U) = H^0(\mathcal{F}, U)$. I will use them all.)

If $U \subset V$ is an inclusion of open sets, we have restriction maps $res_{V,U}; \mathcal{F}(V) \to \mathcal{F}(U)$.

The map $res_{U,U}$ must be the identity for all U.

If you take a section over a big open set, and restrict it to a medium open set, and then restrict that to a small open set, then you get the same thing as if you restrict the section on the big open set to the small open set all at once. In other words, if $U \hookrightarrow V \hookrightarrow W$, then the following diagram commutes:

$$\begin{tikzcd}
\mathcal{F}(W) \arrow[rr, "res_{W,V}"] \arrow[dr, "res_{W,U}"'] & & \mathcal{F}(V) \arrow[dl, "res_{V,U}"] \\
& \mathcal{F}(U) &
\end{tikzcd}$$

A subtle point that you shouldn't worry about at the start are the sections over the empty set. $\mathcal{F}(\emptyset)$ should be the final object in the category under consideration (sets: a set with one element; abelian groups: 0; rings: the 0-ring). (I'm tentatively going to say that there is a 1-element ring. In other words, I will not assume that rings satisfy $1 \neq 0$. Every ring maps to the 0-ring. But it doesn't map to any other ring, because in a ring morphisms, 0 goes to 0, and 1 goes to 1, but in every ring beside this one, $0 \neq 1$. I think this convention will solve some problems, but it will undoubtedly cause others, and I may eat my words, so only worry about it if you really want to.)

Something satisfying the properties I've described is a *presheaf*. (For experts: a presheaf of rings is the same thing as a contravariant functor from the category of open sets to the category of rings, plus that final object annoyance, see problem set 1.)

Sections of presheaves $\mathcal{F}$ have *germs* at each point $x \in X$ where they are defined, and the set of germs is denoted $\mathcal{F}_x$, and is called the *stalk of $\mathcal{F}$ at* x. Elements of the stalk correspond to sections over some open set containing x. Two of these sections are considered the same if they agree on some smaller open set. If $\mathcal{F}$ is a sheaf of rings, then $\mathcal{F}_x$ is a ring, and ditto for rings replaced by other categories we like.

We add two more axioms to make this into a sheaf.

Identity axiom. If $\{U_i\}_{i \in I}$ is a cover of U, and $f_1, f_2 \in \mathcal{F}(U)$, and $\operatorname{res}_{U,U_i} f_1 = \operatorname{res}_{U,U_i} f_2$, then $f_1 = f_2$.

Gluability axiom. given $f_i \in \mathcal{F}(U_i)$ for all i, such that $\operatorname{res}_{U_i, U_i \cap U_j} f_i = \operatorname{res}_{U_j, U_i \cap U_j} f_j$ for all i, j, then there is some $f \in \mathcal{F}(U)$ such that $\operatorname{res}_{U,U_i} f = f_i$ for all i.

Example 2 (on problem set 1). Suppose we are given a continuous map $f : Y \to X$. The "sections of f" form a sheaf. More precisely, to each open set U of X, associate the set of continuous maps s to Y such that $f \circ s = id|_U$. This forms a sheaf. (Example for those who know this language: a vector bundle.)

Example 3: **Sheaf of $\mathcal{O}_X$-modules.** Suppose $\mathcal{O}_X$ is a sheaf of rings on X. Then we define the notion of a sheaf of $\mathcal{O}_X$-modules. We have a metaphor: rings is to modules, as sheaves of rings is to sheaves of modules.

There is only one possible definition that could go with this name, so let's figure out what it is. For each U, $\mathcal{F}(U)$ should be a $\mathcal{O}_X(U)$-module. Furthermore, this structure should behave well with respect to restriction maps. This means the following. If $U \hookrightarrow V$, then

$$\begin{tikzcd}
\mathcal{O}_X(V) \times \mathcal{F}(V) \arrow[r, "\text{action}"] \arrow[d, "\operatorname{res}_{V,U}"] & \mathcal{F}(V) \arrow[d, "\operatorname{res}_{V,U}"] \\
\mathcal{O}_X(U) \times \mathcal{F}(U) \arrow[r, "\text{action}"] & \mathcal{F}(U)
\end{tikzcd}$$

commutes. You should think about this later, and convince yourself that I haven't forgotten anything.

For category theorists: the notion of R-module generalizes the notion of abelian group, because an abelian group is the same thing as a $\mathbb{Z}$-module. It is similarly immediate that the notion of $\mathcal{O}_X$-module generalizes the notion of sheaf of abelian groups, because the latter is the same thing as a $\underline{\mathbb{Z}}$-module, where $\underline{\mathbb{Z}}$ is the locally constant sheaf with values in $\mathbb{Z}$. Hence when we are proving things about $\mathcal{O}_X$-modules, we are also proving things about sheaves of abelian groups. For experts: Someone pointed out that we can make the same notion of *presheaf* of $\mathcal{O}_X$-modules, where $\mathcal{O}_X$ is a *presheaf* of rings. In this setting, *presheaves* of abelian groups are the same as modules over the *constant presheaf* $\underline{\mathbb{Z}}^{\text{pre}}$. I doubt we will use this, so feel free to ignore it.

3. MORPHISMS OF PRESHEAVES AND SHEAVES

I'll now tell you how to map presheaves to each other; and similarly for sheaves. In other words, I am describing the *category of presheaves* and the *category of sheaves*.

A morphism of presheaves of sets $f : \mathcal{F} \to \mathcal{G}$ is a collection of maps $f_U : \mathcal{F}(U) \to \mathcal{G}(U)$ that commute with the restrictions, in the sense that: if $U \hookrightarrow V$ then

$$\begin{tikzcd} \mathcal{F}(V) \arrow[r, "f_V"] \arrow[d, "\mathrm{res}_{V,U}"] & \mathcal{G}(V) \arrow[d, "\mathrm{res}_{V,U}"] \\ \mathcal{F}(U) \arrow[r, "f_U"] & \mathcal{G}(U) \end{tikzcd}$$

commutes. (Notice: the underlying space remains X!) A morphism of sheaves is defined in the same way. (For category-lovers: a morphism of presheaves on X is a natural transformation of functors. This definition describes the category of sheaves on X as a full subcategory of the category of presheaves on X.)

A morphism of presheaves (or sheaves) of rings (or groups, or abelian groups, or $\mathcal{O}_X$-modules) is defined in the same way.

Exercise. Show morphisms of (pre)sheaves induces morphisms of stalks.

Interesting examples of morphisms of presheaves of abelian groups. Let $X = \mathbb{C}$ with the usual (analytic) topology, and define $\mathcal{O}_X$ to be the sheaf of holomorphic functions, and $\mathcal{O}_X^*$ to be the sheaf of *invertible (= nowhere 0) holomorphic functions*. This is a sheaf of abelian groups under multiplication. We have maps of presheaves

$$1 \longrightarrow \underline{\mathbb{Z}}^{\text{pre}} \xrightarrow{\times 2\pi i} \mathcal{O}_X \xrightarrow{\exp} \mathcal{O}_X^* \longrightarrow 1$$

where $\underline{\mathbb{Z}}^{\text{pre}}$ is the constant presheaf. This is *not* an exact sequence of presheaves, and it is worth figuring out why. (Hint: it is not exact at $\mathcal{O}_X$ or $\mathcal{O}_X^*$. Replacing $\underline{\mathbb{Z}}^{\text{pre}}$ with the locally constant sheaf $\underline{\mathbb{Z}}$ remedies the first, but not the second.)

Now abelian groups, and R-modules, form an abelian category — by which I just mean that you are used to taking kernels, images, etc. — and you might hope for the same for sheaves of abelian groups, and sheaves of $\mathcal{O}_X$-modules. That is indeed the case. Presheaves are easier to understand in this way.

The presheaves of abelian groups on X*, or* $\mathcal{O}_X$*-modules on* X*, form an abelian category.* If $f : \mathcal{F} \to \mathcal{G}$ is a morphism of presheaves, then $\ker f$ is a presheaf, with $(\ker f)(U) = \ker f_U$, and $(\operatorname{im} f)(U) = \operatorname{im} f_U$. The resulting things are indeed presheaves. For example, if $U \hookrightarrow V$, there is a natural map $\mathcal{G}(V)/f_V(\mathcal{F}(V)) \to \mathcal{G}(U)/f_U(\mathcal{F}(U))$, as we observe by chasing the following diagram:

$$\begin{tikzcd}
\mathcal{F}(V) \arrow[r] \arrow[d, "\mathrm{res}_{V,U}"] & \mathcal{G}(V) \arrow[r] \arrow[d, "\mathrm{res}_{V,U}"] & \mathcal{G}(V)/\mathcal{F}(V) \arrow[r] \arrow[d, dotted] & 0 \\
\mathcal{F}(U) \arrow[r] & \mathcal{G}(U) \arrow[r] & \mathcal{G}(U)/\mathcal{F}(U) \arrow[r] & 0.
\end{tikzcd}$$

Thus I have defined $\mathcal{G}/\mathcal{F}$, by showing what its sections are, and what its restriction maps are. I have to check that it restriction maps compose — **exercise**. Hence I've defined a presheaf. I still have to convince you that it deserves to be called a cokernel. **Exercise.** Do this. It is less hard than you might think. Here is the definition of cokernel of $g : \mathcal{F} \to \mathcal{G}$. It is a morphism $h : \mathcal{G} \to \mathcal{H}$ such that $h \circ g = 0$, and for any $i : \mathcal{G} \to \mathcal{I}$ such that $i \circ g = 0$, there is a unique morphism $j : \mathcal{H} \to \mathcal{I}$ such that $j \circ h = i$:

$$\begin{tikzcd}
\mathcal{F} \arrow[r] & \mathcal{G} \arrow[r] \arrow[dr] & \mathcal{H} \arrow[d, dotted, "\exists!"] \\
& & \mathcal{I}
\end{tikzcd}$$

(Translation: cokernels in an additive category are defined by a universal property. Hence if they exist, they are unique. We are checking that our construction satisfies the universal property.)

Punchline: The presheaves of $\mathcal{O}_X$-modules is an abelian category, and as nice as can be. We can define terms such as *subpresheaf*, *image presheaf*, *quotient presheaf*, *cokernel presheaf*. You construct kernels, quotients, cokernels, and images open set by open set. (Quotients are special cases of cokernels.)

Exercise. In particular: if $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \cdots \to \mathcal{F}_n \to 0$ is an exact sequence of presheaves, then $0 \to \mathcal{F}_1(U) \to \mathcal{F}_2(U) \to \cdots \to \mathcal{F}_n(U) \to 0$ is also an exact sequence for all U, and vice versa.

However, we are interested in more geometric objects, sheaves, where things are can be understood in terms of their local behavior, thanks to the identity and gluing axioms.

3.1. The category of sheaves of $\mathcal{O}_X$-modules is trickier. It turns out that the kernel of a morphism of sheaves is also sheaf. **Exercise.** Show that this is true. (Confusing translation: this subpresheaf of a sheaf is in fact also a sheaf.) Thus we have the notion of a **subsheaf**.

But other notions behave weirdly.

Example: image sheaf. We don't need an abelian category to talk about images — the notion of image makes sense for a map of sets. And the notion of image is a bit problematic even for sheaves of sets. Let's go back to our example of $\mathcal{O}_X \xrightarrow{\exp} \mathcal{O}_X^*$. What is the image *presheaf*? Well, if U is a *simply connected* open set, then this is surjective: every non-zero holomorphic function on a simply connected set has a logarithm (in fact many). However, this is not true if U is not simply connected — the function $f(z) = z$ on $\mathbb{C} - 0$ does *not* have a logarithm.

However, it locally does.

So what do we do? Answer 1: throw up our hands. Answer 2: Develop a new definition of image. We can't just define anything — we need to figure out what we want the image to be. Answer: category theoretic definition.

The patch: sheafification. Define **sheafification** of a presheaf by universal property: $\mathcal{F} \to \mathcal{F}^{\mathrm{sh}}$. Hence if it is exists, it is unique up to unique isomorphism. (This is analogous to the method of getting a group from a semigroup, see last day's notes.)

(Category-lovers: this says that sheafification is left-adjoint to the forgetful functor. This is just like groupification.)

Theorem (later today): Sheafification exists. (The specific construction will later be useful, but we won't need anything but the universal property right now.)

In class, I attempted to show that the sheafification of the image presheaf satisfies the universal property of the image sheaf, but I realized that I misstated the property. Instead, I will let you show that the sheaf of the cokernel presheaf satisfies the universal property of the cokernel sheaf. See the notes about one page previous for the definition of the cokernel.

Exercise. Do this.

Possible exercise. I'll tell you the definition of the image sheaf, and you can check.

Remark for experts: someone pointed out in class that likely the same arguments apply without change whenever you have an adjoint to a forgetful functor.

In short: $\mathcal{O}_X$-modules form an abelian category. To define image and cokernel (and quotient), you need to sheafify.

3.2. *Exercise.* Suppose $f : \mathcal{F} \to \mathcal{G}$ is a morphism of sheaves. Show that there are natural isomorphisms $\operatorname{im} f \cong \mathcal{F}/\ker f$ and $\operatorname{coker} f \cong \mathcal{G}/\operatorname{im} f$.

Tensor products of $\mathcal{O}_X$-modules: also requires sheafification.

3.3. *Exercise.* Define what we should mean by tensor product of two $\mathcal{O}_X$-modules. Verify that this construction satisfies your definition. (Hint: sheafification is required.)

3.4. *Left-exactness of the global section functor.* Left-exactness of global sections; hints of cohomology. More precisely:

$$0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$$

implies

$$0 \to \mathcal{F}(U) \to \mathcal{G}(U) \to \mathcal{H}(U)$$

is exact. Give example where not right exact, (Hint: $0 \to \mathbb{Z} \to \mathcal{O} \to \mathcal{O}^* \to 0$.)

Caution: The cokernel in the category of sheaves *is* a presheaf, but it is *isn't* the cokernel in the category of presheaves.

3.5. Important Exercise. Show the same thing (3.4) is true for pushforward sheaves. (The previous case is the case of a map from U to a point.)

4. Stalks, and sheafification

4.1. *Important exercise.* Prove that a section of a sheaf is determined by its germs, i.e.

$$\Gamma(U, \mathcal{F}) \to \prod_{x \in U} \mathcal{F}_x$$

is injective. (Hint: you won't use the gluability axiom. So this is true for separated presheaves.) [Answer: Suppose $f, g \in \Gamma(U, \mathcal{F})$, with $f_x = g_x$ in $\mathcal{F}_x$ for all $x \in U$. In terms of the concrete interpretation of stalks, $f_x = (U, f)$ and $g_x = (U, g)$, and $(U, f) = (U, g)$ means that there is an open subset U_x of U, containing x, such that $f|_{U_x} = g|_{U_x}$. The U_x cover U, so by the identity axiom for this cover of U, $f = g$.]

Corollary. In particular, if a sheaf has all stalks 0, then it is the 0-sheaf.

4.2. Morphisms and stalks.

4.3. *Exercise.* Show that morphisms of presheaves induce morphisms of stalks.

4.4. *Exercise.* Show that morphisms of sheaves are determined by morphisms of stalks. Hint # 1: you won't use the gluability axiom. So this is true of morphisms of separated presheaves.) Hint # 2: study the following diagram.

(1)
$$\begin{tikzcd} \mathcal{F}(U) \arrow[r] \arrow[d, hook] & \mathcal{G}(U) \arrow[d, hook] \\ \prod_{x \in U} \mathcal{F}_x \arrow[r] & \prod_{x \in U} \mathcal{G}_x \end{tikzcd}$$

4.5. *Exercise.* Show that a morphism of sheaves is an isomorphism if and only if it induces an isomorphism of all stalks. (Hint: Use (1). Injectivity uses from the previous exercise. Surjectivity will use gluability.)

4.6. *Exercise.* (a) Show that Exercise 4.1 is false for general presheaves. (Hint: take a 2-point space with the discrete topology, i.e. every subset is open.)
(b) Show that Exercise 4.4 is false for general presheaves. (Hint: a 2-point space suffices.)
(c) Show that Exercise 4.5 is false for general presheaves.

4.7. Description of sheafification. Suppose $\mathcal{F}$ is a presheaf on a topological space X. We define $\mathcal{F}^{sh}$ as follows. Sections over $U \subset X$ are stalks at each point, with compatibility conditions (to each element of the stalk, there is a representative (g, U) with g restricting correctly to all stalks in U). More explicitly:

$$\mathcal{F}^{sh}(U) := \{(f_x \in \mathcal{F}_x)_{x \in U} : \forall x \in U, \exists x \subset U_x \subset U, F^x \in \mathcal{F}(U_x) : F^x_y = f_y \forall y \in U_x\}.$$

(Those who want to worry about the empty set are welcome to.)

This is clearly a sheaf: we have restriction maps; they commute; we have identity and gluability.

4.8. For any morphism of presheaves $\phi : \mathcal{F} \to \mathcal{G}$, we get a natural induced morphism of sheaves $\phi^{sh} : \mathcal{F}^{sh} \to \mathcal{G}^{sh}$.

We have a natural presheaf morphism $\mathcal{F} \to \mathcal{F}^{sh}$. This induces a natural morphism of stalks $\mathcal{F}_x \to \mathcal{F}^{sh}_x$ (Exercise 4.3). Hence if $\mathcal{F}$ is a sheaf already, then $\mathcal{F} \to \mathcal{F}^{sh}$ is an isomorphism, by Exercise 4.5. If we knew that $\mathcal{F}^{sh}$ satisfied the universal property of sheafification, this would have been immediate by abstract nonsense, but we don't know that. In fact, we'll show that now. Suppose we have the solid arrows in

$$\begin{tikzcd} \mathcal{F} \arrow[r] \arrow[dr] & \mathcal{F}^{sh} \arrow[d, dotted] \\ & \mathcal{G}. \end{tikzcd}$$

We want to show that there exists a dashed arrow as in the diagram, making the diagram commute, and we want to show that it is unique. By 4.8, $\mathcal{F} \to \mathcal{G}$ induces a morphism $\mathcal{F}^{sh} \to \mathcal{G}^{sh} = \mathcal{G}$, so we have existence.

For uniqueness: as morphisms of sheaves are determined by morphisms of stalks (Exercise 4.4), and for any $x \in X$, we have a commutative diagram

$$\begin{tikzcd} \mathcal{F}_x \arrow[r, "="] \arrow[dr] & \mathcal{F}^{sh}_x \arrow[d] \\ & \mathcal{G}_x, \end{tikzcd}$$

we are done. Thus $\mathcal{F} \to \mathcal{F}^{sh}$ is indeed the sheafification.

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FOUNDATIONS OF ALGEBRAIC GEOMETRY CLASS 4

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CONTENTS

Last day: Presheaves and sheaves. Morphisms thereof. Sheafification.

Today: Understanding sheaves via stalks. Understanding sheaves via "sheaves on a nice base of a topology". Affine schemes $\operatorname{Spec} R$: the set.

Here's where we are. I introduced you to some of the notions of category theory. Our motivation is as follows. We will be creating some new mathematical objects, and we expect them to act like object we have seen before. We could try to nail down precisely what we mean, and what minimal set of things we have to check in order to verify that they act the way we expect. Fortunately, we don't have to — other people have done this before us, by defining key notions, like abelian categories, which behave like modules over a ring.

We then defined presheaves and sheaves. We have seen sheaves of sets and rings. We have also seen sheaves of abelian groups and of $\mathcal{O}_X$-modules, which form an abelian category. Let me contrast again presheaves and sheaves. Presheaves are simpler to define, and notions such as kernel and cokernel are straightforward, and are defined open set by open set. Sheaves are more complicated to define, and some notions such as cokernel require the notion of sheafification. But we like sheaves because they are in some sense geometric; you can get information about a sheaf locally. Today, I'd like to go over some of the things we talked about last day in more detail. I'm going to talk again about stalks, and how information about sheaves are contained in stalks.

First, a small comment I should have said earlier. Suppose we have an exact sequence of sheaves of abelian groups (or $\mathcal{O}_X$-modules) on X

$$0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H}.$$

Date: Friday, October 14, 2005. Small updates January 31, 2007.

If $U \subset X$ is any open set, then

$$0 \to \mathcal{F}(U) \to \mathcal{G}(U) \to \mathcal{H}(U)$$

is exact. Translation: taking sections over U is a *left-exact functor*. Reason: the kernel sheaf of $\mathcal{G} \to \mathcal{H}$ is in fact the kernel presheaf (see the previous lectures). Note that $\mathcal{G}(U) \to \mathcal{H}(U)$ is not necessarily surjective (the functor is not exact); a counterexample is given by our old friend

$$0 \to \underline{\mathbb{Z}} \to \mathcal{O}_X \to \mathcal{O}_X^* \to 0.$$

(By now you should be able to guess what U to use.)

1. Stalks, and sheafification

1.1. *Important exercise.* Prove that a section of a sheaf is determined by its germs, i.e.

$$\mathcal{F}(U) \to \prod_{x \in U} \mathcal{F}_x$$

is injective. (Hint: you won't use the gluability axiom. So this is true for separated presheaves.)

Corollary. In particular, if a sheaf has all stalks 0, then it is the 0-sheaf.

1.2. Morphisms and stalks.

1.3. *Exercise.* Show that morphisms of presheaves (and sheaves) induce morphisms of stalks.

1.4. *Exercise.* Show that morphisms of sheaves are determined by morphisms of stalks. Hint # 1: you won't use the gluability axiom. So this is true of morphisms of separated presheaves. Hint # 2: study the following diagram.

(1)
$$\begin{tikzcd} \mathcal{F}(U) \arrow[r] \arrow[d, hook] & \mathcal{G}(U) \arrow[d, hook] \\ \prod_{x \in U} \mathcal{F}_x \arrow[r] & \prod_{x \in U} \mathcal{G}_x \end{tikzcd}$$

1.5. *Exercise.* Show that a morphism of sheaves is an isomorphism if and only if it induces an isomorphism of all stalks. (Hint: Use (1). Injectivity uses the previous exercise. Surjectivity will use gluability.)

1.6. *Exercise.* (a) Show that Exercise 1.1 is false for general presheaves. (Hint: take a 2-point space with the discrete topology, i.e. every subset is open.)
(b) Show that Exercise 1.4 is false for general presheaves. (Hint: a 2-point space suffices.)
(c) Show that Exercise 1.5 is false for general presheaves.

1.7. Description of sheafification. I described sheafification a bit quickly last time. I will do it again now.

Suppose $\mathcal{F}$ is a presheaf on a topological space X. We define $\mathcal{F}^{sh}$ as follows. Sections over $U \subset X$ are stalks at each point, with compatibility conditions (to each element of the stalk, there is a representative (g, U) with g restricting correctly to all stalks in U). More explicitly:

$$\mathcal{F}^{sh}(U) := \{(f_x \in \mathcal{F}_x)_{x \in U} : \forall x \in U, \exists U_X \text{ with } x \subset U_x \subset U, F^x \in \mathcal{F}(U_x) : F^x_y = f_y \forall y \in U_x\}.$$

(Those who want to worry about the empty set are welcome to.)

This is less confusing than it seems. $\mathcal{F}^{sh}(U)$ is clearly a sheaf: we have restriction maps; they commute; we have identity and gluability. It would be good to know that it satisfies the universal property of sheafification.

1.8. *Exercise.* The stalks of $\mathcal{F}^{sh}$ are the same as the stalks of $\mathcal{F}$. Reason: Use the concrete description of the stalks.

1.9. *Exercise.* For any morphism of presheaves $\phi : \mathcal{F} \to \mathcal{G}$, we get a natural induced morphism of sheaves $\phi^{sh} : \mathcal{F}^{sh} \to \mathcal{G}^{sh}$.

We have a natural presheaf morphism $\mathcal{F} \to \mathcal{F}^{sh}$. This induces a natural morphism of stalks $\mathcal{F}_x \to \mathcal{F}^{sh}_x$ (Exercise 1.3). This is an isomorphism by remark a couple of paragraphs previous. Hence if $\mathcal{F}$ is a sheaf already, then $\mathcal{F} \to \mathcal{F}^{sh}$ is an isomorphism, by Exercise 1.5. If we knew that $\mathcal{F}^{sh}$ satisfied the universal property of sheafification, this would have been immediate by abstract nonsense, but we don't know that yet. In fact, we'll show that now. Suppose we have the solid arrows in

$$\begin{tikzcd}
\mathcal{F} \arrow[r] \arrow[dr] & \mathcal{F}^{sh} \arrow[d, dotted] \\
 & \mathcal{G}.
\end{tikzcd}$$

We want to show that there exists a dashed arrow as in the diagram, making the diagram commute, and we want to show that it is unique. By 1.9, $\mathcal{F} \to \mathcal{G}$ induces a morphism $\mathcal{F}^{sh} \to \mathcal{G}^{sh} = \mathcal{G}$, so we have existence.

For uniqueness: as morphisms of sheaves are determined by morphisms of stalks (Exercise 1.4), and for any $x \in X$, we have a commutative diagram

$$\begin{tikzcd}
\mathcal{F}_x \arrow[r, "="] \arrow[dr] & \mathcal{F}^{sh}_x \arrow[d] \\
 & \mathcal{G}_x,
\end{tikzcd}$$

we are done. Thus $\mathcal{F} \to \mathcal{F}^{sh}$ is indeed the sheafification.

Four properties of morphisms of sheaves that you can check on stalks.

You can verify the following.

- A morphism of sheaves of sets is injective (monomorphism) if and only if it is injective on all stalks.
- Same with surjective (epimorphism).
- Same with isomorphic — we've already seen this.
- Suppose $\mathcal{F} \to \mathcal{G} \to \mathcal{H}$ is a complex of sheaves of abelian groups (or $\mathcal{O}_X$-modules). Then it is exact if and only if it is on stalks.

I'll prove one of these, to show you how it works: surjectivity.

Suppose first that we have surjectivity on all stalks for a morphism $\phi : \mathcal{F} \to \mathcal{G}$. We want to check the definition of epimorphism. Suppose we have $\alpha : \mathcal{F} \to \mathcal{H}$, and $\beta, \gamma : \mathcal{G} \to \mathcal{H}$ such that $\alpha = \beta \circ \phi = \gamma \circ \phi$.

$$\begin{tikzcd} \mathcal{F} \arrow[r, "\phi"] \arrow[dr, "\alpha"'] & \mathcal{G} \arrow[d, dotted, "\leq 1?"] \\ & \mathcal{H} \end{tikzcd}$$

Then by taking stalks at x, we have

$$\begin{tikzcd} \mathcal{F}_x \arrow[r, "\phi_x"] \arrow[dr, "\alpha_x"'] & \mathcal{G}_x \arrow[d, dotted, "\beta_x, \gamma_x"] \\ & \mathcal{H}_x \end{tikzcd}$$

By surjectivity (epimorphism-ness) of the morphisms of stalks, $\beta_x = \gamma_x$. But as morphisms are determined by morphisms at stalks (Exercise 1.4), we must have $\beta = \gamma$.

Next assume that $\phi : \mathcal{F} \to \mathcal{G}$ is an epimorphism of sheaves, and $x \in X$. We will show that $\phi_x : \mathcal{F}_x \to \mathcal{G}_x$ is a epimorphism for any given $x \in X$. Choose for $\mathcal{H}$ any skyscraper sheaf supported at x. (the stalk of a skyscraper sheaf at the skyscraper point is just the skyscraper set/group/ring). Then the maps α, β, γ factor through the stalk maps:

$$\begin{tikzcd} \mathcal{F} \arrow[r] \arrow[d] & \mathcal{G} \arrow[d] & \\ i_*\mathcal{F}_x \arrow[r] \arrow[dr] & i_*\mathcal{G}_x \arrow[d] & \text{skyscrapers} \\ & \mathcal{H} & \end{tikzcd}$$

and then we are basically done.

2. Recovering sheaves from a "sheaf on a base"

Sheaves are natural things to want to think about, but hard to get one's hands on. We like the identity and gluability axioms, but they make proving things trickier than for presheaves. We've just talked about how we can understand sheaves using stalks. I now want to introduce a second way of getting a hold of sheaves, by introducing the notion of a *sheaf on a nice base*.

First, let me define the notion of a *base of a topology*. Suppose we have a topological space X, i.e. we know which subsets of X are open $\{U_i\}$. Then a base of a topology is a subcollection of the open sets $\{B_j\} \subset \{U_i\}$, such that each U_i is a union of the B_j. There is one example that you have seen early in your mathematical life. Suppose $X = \mathbb{R}^n$. Then the way the usual topology is often first defined is by defining *open balls* $B_r(x) = \{y \in \mathbb{R}^n : |y - x| < r\}$ *are open sets*, and declaring that any union of balls is open. So the balls form a base of the usual topology. Equivalently, we often say that they *generate* the usual topology. As an application of how we use them, if you want to check continuity of some map $f : X \to \mathbb{R}^n$ for example, you need only think about the pullback of balls on $\mathbb{R}^n$.

There is a slightly nicer notion I want to use. A base is particularly pleasant if the intersection of any two elements is also an element of the base. (Does this have a name?) I will call this a *nice base*. For example if $X = \mathbb{R}^n$, then a base would be *convex open sets*. Certainly the intersection of two convex open sets is another convex open set. Also, this certainly forms a base, because it includes the balls.

Now suppose we have a sheaf $\mathcal{F}$ on X, and a nice base $\{B_i\}$ on X. Then consider the information $(\{\mathcal{F}(B_i)\}, \{\phi_{ij} : \mathcal{F}(B_i) \to \mathcal{F}(B_j)\}$, which is a subset of the information contained in the sheaf — we are only paying attention to the information involving elements of the base, not all open sets.

Observation. We can recover the entire sheaf from this information. Proof:

$$\mathcal{F}(U) = \{(f_i \in \mathcal{F}(B_i))_{B_i \subset U} : \phi_{ij}(f_i) = f_j\}.$$

The map from the left side to the right side is clear. We get a map from the right side to the left side as follows. By gluability, each element gives at least one element of the left side. By identity, it gives a unique element.

Conclusion: we can recover a sheaf from less information. This even suggests a notion, that of a *sheaf on a nice base*.

A sheaf of sets (rings etc.) on a nice base $\{B_i\}$ is the following. For each B_i in the base, we have a set $\mathcal{F}(B_i)$. If $B_i \subset B_j$, we have maps $\mathrm{res}_{ji} : \mathcal{F}(B_j) \to \mathcal{F}(B_i)$. (Everywhere things called B are assumed to be in the base.) If $B_i \subset B_j \subset B_k$, then $\mathrm{res}_{B_k,B_i} = \mathrm{res}_{B_j,B_i} \circ \mathrm{res}_{B_k,B_j}$. For the pedants, $\mathcal{F}(\emptyset)$ is a one-element set (a final object). So far we have defined a *presheaf on a nice base*.

We also have base identity: If $B = \cup B_i$, then if $f, g \in \mathcal{F}(B)$ such that $\mathrm{res}_{B,B_i} f = \mathrm{res}_{B,B_i} g$ for all i, then $f = g$.

And base gluability: If $B = \cup B_i$, and we have $f_i \in \mathcal{F}(B_i)$ such that f_i agrees with f_j on basic open set $B_i \cap B_j$ (i.e. $\mathrm{res}_{B_i,B_i \cap B_j} f_i = \mathrm{res}_{B_j,B_i \cap B_j} f_j$) then there exist $f \in \mathcal{F}(B)$ such that $\mathrm{res}_{B,B_i} = f_i$ for all i.

2.1. *Theorem.* — *Suppose we have data* $F(U_i)$, ϕ_{ij}, *satisfying "base presheaf", "base identity" and "base gluability". Then (if the base is nice) this uniquely determines a sheaf of sets (or rings, etc.)* $\mathcal{F}$, *extending this.*

This argument will later get trumped by one given in Class 13.

Proof. Step 1: define the sections over an arbitrary U. For $U \neq \emptyset$, define

$$\mathcal{F}(U) = \{f_i \in F(B_i) \text{ for all } B_i \subset U : \operatorname{res}_{B_i, B_i \cap B_j} f_i = \operatorname{res}_{B_j, B_i \cap B_j} f_j \text{ in } F(B_i \cap B_j)\}$$

where if the set is empty, then we use the final object in our category; this is the only place where we needed to determine our category in advance. We get $\operatorname{res}_{U,V}$ in the obvious way. We get a presheaf.

$\mathcal{F}(B_i) = F(B_i)$ and $\operatorname{res}_{B_i, B_j}$ is as expected; both are clear.

Step 2: check the identity axiom. Take $f, g \in \mathcal{F}(U)$ restricting to $f_i \in \mathcal{F}(U_i)$. Then f, g agree on any base element contained in some U_i. We'll show that for each $B_j \subset U$, they agree. Take a cover of B_j by base elements each contained entirely in some U_i. The intersection of any two is also contained some U_i; they agree there too. Hence by "base identity" we get identity.

Step 3: check the gluability axiom. Suppose we have some $f_i \in \mathcal{F}(U_i)$ that agree on overlaps. Take any $B_j \subset \cup U_i$. Take a cover by basic opens that each lie in some U_i. Then they agree on overlaps. By "base gluability", we get a section over B_j. (Unique by "base identity".) Any two of the f_j's agree on the overlap. □

2.2. *Remark.* In practice, to find a section of such a sheaf over some open set U we may choose a smaller (finite if possible) cover of U.

Eventually, we will define a sheaf on a base in general, not just on a nice base. Experts may want to ponder the definition, and how to prove the above theorem in that case.

2.3. *Important Exercise.* (a) Verify that a morphism of sheaves is determined by a morphism on the base. (b) Show that a "morphism of sheaves on the base" (i.e. such that the diagram

$$\begin{tikzcd} \Gamma(B_i, \mathcal{F}) \arrow[r] \arrow[d] & \Gamma(B_i, \mathcal{G}) \arrow[d] \\ \Gamma(B_j, \mathcal{F}) \arrow[r] & \Gamma(B_j, \mathcal{G}) \end{tikzcd}$$

commutes) gives a morphism of sheaves.

2.4. *Remark.* Suppose you have a presheaf you want to sheafify, and when restricted to a base it is already a sheaf. Then the sheafification is obtained by taking this process.

Example: Let $X = \mathbb{C}$, and consider the sequence

$$1 \longrightarrow \underline{\mathbb{Z}} \xrightarrow{\times 2\pi i} \mathcal{O}_X \xrightarrow{\exp} \mathcal{O}_X^* \longrightarrow 1.$$

Let's check that it is exact, using our new knowledge. We instead work on the nice base of convex open sets. then on these open sets, this is indeed exact. The key fact here is that on any convex open set B, every element of $\mathcal{O}_X^*(B)$ has a logarithm, so we have surjectivity here.

3. Toward schemes

We're now ready to define schemes! Here is where we are going. After some more motivation for what kind of objects affine schemes are, I'll define affine schemes, which are like balls in the analytic topology. We'll generalize in three transverse directions. I'll define schemes in general, including projective schemes. I'll define morphisms between schemes. And I'll define sheaves on schemes. These notions will take up the rest of the quarter.

We will define schemes as a *topological space* along with a *sheaf of "algebraic functions"* (that we'll call the *structure sheaf*). Thus our construction will have three steps: we'll describe the *set*, then the *topology*, and then the *sheaf*.

We will try to draw pictures throughout; geometric intuition can guide algebra (and vice versa). Pictures develop geometric intuition. We learn to draw them; the algebra tells how to think about them geometrically. So these comments are saying: "this is a good way to think". Eventually the picture tells you some algebra.

4. Motivating examples

As motivation for why this is a good foundation for a kind of "space", we'll reinterpret differentiable manifolds in this way. We will feel free to be informal in this section.

Usual definition of differentiable manifold: atlas, and gluing functions. (There is also a Hausdorff axiom, which I'm going to neglect for now.)

A fancier definition is as follows: as a topological space, with a sheaf of differentiable functions. (Some observations: Functions are determined by values at points. This is an obvious statement, but won't be true for schemes in general. Note: Stalks are local rings $(\mathcal{O}_x, \mathfrak{m}_x)$; the residue map is "value at a point" $0 \to \mathfrak{m}_x \to \mathcal{O}_x \to \mathbb{R} \to 0$, as I described in an earlier class, probably class 1 or class 2.)

There is an interesting fact that I'd like to mention now, but that you're not quite ready for. So don't write this down, but hopefully let some of it subconsciously sink into your head. The tangent space at a point x can be naturally identified with $(\mathfrak{m}_x/\mathfrak{m}_x^2)^*$. Let's make this a bit explicit. Every function vanishing at p canonically gives a functional on the tangent space to X at p. If $X = \mathbb{R}^2$, the function $\sin x - y + y^2$ gives the functional $x - y$.

Morphisms $X \to Y$: these are certain continuous maps — but which ones? We can pull back functions along continuous maps. Differentiable functions pull back to differentiable functions. We haven't defined the inverse image of sheaves yet — if you're curious, that will be in the second problem set — but if we had, we would have a map $f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$. (I don't want to call it "pullback" because that word is used for a slightly different concept.) Inverse image is left-adjoint to pushforward, which we *have* seen, so we get map $f^\# : \mathcal{O}_Y \to f_*\mathcal{O}_X$.

Interesting question: which continuous maps are differentiable? Answer: Precisely those for which the induced map of functions sends differentiable functions to differentiable functions. (Check on local patches.)

4.1. *Unimportant Exercise.* Show that a morphism of differentiable manifolds $f : X \to Y$ with $f(p) = q$ induces a morphism of stalks $f^\# : \mathcal{O}_{Y,q} \to \mathcal{O}_{X,p}$. Show that $f^\#(\mathfrak{m}_{Y,q}) \subset \mathfrak{m}_{X,p}$. (In other words, if you pull back a function that vanishes at q, you get a function that vanishes at p.)

More for experts: Notice that this induces a map on tangent spaces $(\mathfrak{m}_{X,p}/\mathfrak{m}_{X,p}^2)^* \to (\mathfrak{m}_{Y,q}/\mathfrak{m}_{Y,q}^2)^*$. This is the tangent map you would geometrically expect. Interesting fact: the cotangent space, and cotangent map, is somehow more algebraically natural, despite the fact that tangent spaces, and tangent maps, are more geometrically natural. Rhetorical questions: How to check for submersion ("smooth morphism")? How to check for inclusion, but not just set-theoretically? Answer: differential information.

[Then we have a normal exact sequence.

Vector bundle can be rewritten in terms of sheaves; explain how.]

Side Remark. Manifolds are covered by disks that are all isomorphic. Schemes will not have isomorphic open sets, even varieties won't. An example will be given later.

5. Affine schemes I: the underlying set

For any ring R, we are going to define something called $\operatorname{Spec} R$. First I'll define it as a set, then I'll tell you its topology, and finally I'll give you a sheaf of rings on it, which I'll call the sheaf of functions. Such an object is called an affine scheme. In the future, $\operatorname{Spec} R$ will denote the set, the topology, and the structure sheaf, and I might use the notation $\operatorname{sp}(\operatorname{Spec} R)$ to mean the underlying topological space. But for now, as there is no possibility of confusion, $\operatorname{Spec} R$ will just be the set.

The set $\operatorname{sp}(\operatorname{Spec} R)$ is the set of prime ideals of R.

Let's do some examples. Along with the examples, I'll say a few things that aren't yet rigorously defined. But I hope they will motivate the topological space we'll eventually define, and also the structure sheaf.

Example 1. $\mathbb{A}^1_{\mathbb{C}} = \operatorname{Spec} \mathbb{C}[x]$. "The affine line", "the affine line over $\mathbb{C}$". What are the prime ideals? 0. $(x-a)$ where $a \in \mathbb{C}$. There are no others. Proof: $\mathbb{C}[x]$ is a Unique Factorization Domain. Suppose $\mathfrak{p}$ were prime. If $\mathfrak{p} \neq 0$, then suppose $f(x) \in \mathfrak{p}$ is an element of smallest degree. If $f(x)$ is not linear, then factor $f(x) = g(x)h(x)$, where $g(x)$ and $h(x)$ have positive degree. Then $g(x)$ or $h(x) \in \mathfrak{p}$, contradicting the minimality of the degree of f. Hence there is a linear element $(x-a)$ of $\mathfrak{p}$. Then I claim that $\mathfrak{p} = (x-a)$. Suppose $f(x) \in \mathfrak{p}$. Then the division algorithm would give $f(x) = g(x)(x-a) + m$ where $m \in \mathbb{C}$. Then $m = f(x) - g(x)(x-a) \in \mathfrak{p}$. If $m \neq 0$, then $1 \in \mathfrak{p}$, giving a contradiction: prime ideals can't contain 1.

Thus we have a picture of $\operatorname{Spec} \mathbb{C}[x]$. There is one point for each complex number, plus one extra point. Where is this point? How do we think of it? We'll soon see; but it is a special kind of point. Because (0) is contained in all of these primes, I'm going to somehow identify it with this line passing through all the other points. Here is one way to think of it. You can ask me: is it on the line? I'd answer yes. You'd say: is it here? I'd answer no. This is kind of zen.

To give you an idea of this space, let me make some statements that are currently undefined. The functions on $\mathbb{A}^1_{\mathbb{C}}$ are the polynomials. So $f(x) = x^2 - 3x + 1$ is a function. What is its value at $(x-1)$ = "1"? Plug in 1! Or evaluate mod $x-1$ — same thing by division algorithm! (What is its value at (0)? We'll see later. In general, values at maximal ideals are immediate, and we'll have to think a bit more when primes aren't maximal.)

Here is a "rational function": $(x-3)^3/(x-2)$. This function is defined everywhere but $x = 2$; it is an element of the structure sheaf on the open set $\mathbb{A}^1_{\mathbb{C}} - \{2\}$. It has a triple zero at 3, and a single pole at 2.

Example 2. Let k be an algebraically closed field. $\mathbb{A}^1_k = \operatorname{Spec} k[x]$. The same thing works, without change.

Example 3. $\operatorname{Spec} \mathbb{Z}$. One amazing fact is that from our perspective, this will look a lot like the affine line. Another unique factorization domain. Prime ideals: (0), (p) where p is prime. (Do this if you don't know it!!) Hence we have a picture of this Spec (omitted from notes).

Fun facts: 100 is a function on this space. It's value at (3) is "$1 \pmod 3$". It's value at (2) is "$0 \pmod 2$", and in fact it has a double 0. We will have to think a little bit to make sense of its value at (0).

27/4 is a rational function on $\operatorname{Spec} \mathbb{Z}$. It has a double pole at (2), a triple zero at (3). What is its value at (5)? Answer

$$27 \times 4^{-1} \equiv 2 \times (-1) \equiv 3 \pmod 5.$$

Example 4: stupid examples. $\operatorname{Spec} k$ where k is any field is boring: only one point. $\operatorname{Spec} 0$, where 0 is the zero-ring: the empty set, as 0 has no prime ideals.

Exercise. Describe the set $\operatorname{Spec} k[x]/x^2$. The ring $k[x]/x^2$ is called the *ring of dual numbers* (over k).

Example 5: $\mathbb{R}[x]$. The primes are (0), $(x-a)$ where $a \in \mathbb{R}$, and (x^2+ax+b) where x^2+ax+b is an irreducible quadratic **(exercise)**. The latter two are maximal ideals, i.e. their quotients are fields. Example: $\mathbb{R}[x]/(x-3) \cong \mathbb{R}$, $\mathbb{R}[x]/(x^2+1) \cong \mathbb{C}$. So things are a bit more complicated: we have points corresponding to real numbers, and points corresponding to *conjugate pairs* of complex numbers. So consider the "function" x^3-1 at the point $(x-2)$. We get 7. How about at (x^2+1)? We get

$$x^3 - 1 \equiv x - 1 \pmod{x^2+1}.$$

This is profitably thought of as $i-1$.

One moral of this example is that we can work over a non-algebraically closed field if we wish. (i) It is more complicated, (ii) but we can recover much of the information we wanted.

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FOUNDATIONS OF ALGEBRAIC GEOMETRY CLASS 5

CONTENTS

Last day: Understanding sheaves via stalks, and via "nice base" of topology. $\operatorname{Spec} R$**: the set.**

Today: $\operatorname{Spec} R$**: the set, and the Zariski topology.**

Here is a reminder of where we are going. Affine schemes $\operatorname{Spec} R$ will be defined as a topological space with a sheaf of rings, that we will refer to as the sheaf of functions, called the structure sheaf. A scheme in general will be such a thing (a topological space with a sheaf of rings) that locally looks like $\operatorname{Spec} R$'s. Last day we defined the set: it is the set of primes of R.

We're in the process of doing lots of examples. In the course of doing these examples, we are saying things that we aren't allowed to say yet, because we're using words that we haven't defined. We're doing this because it will motivate where we're going.

We discussed the example $R = k[x]$ where k is algebraically closed (notation: $\mathbb{A}^1_k = \operatorname{Spec} k[x]$). This has old-fashioned points $(x - a)$ corresponding to $a \in k$. (Such a point is often just called a, rather than $(x - a)$.) But we have a new point, (0). (Notational caution: $0 \neq (0)$.) This is a "smooth irreducible curve". (We don't know what any of these words mean!)

We then discussed $R = \mathbb{Z}$. This has points (p) where p is an old-fashioned prime, and the point (0). This is also a smooth irreducible curve (whatever that means).

We discussed the case $R = k[x]$ where k is not necessarily algebraically closed, in particular $k = \mathbb{R}$. The maximal ideals here correspond to unions of Galois conjugates of points in $\mathbb{A}^1_{\mathbb{C}}$.

Example 6 for more arithmetic people: $\mathbb{F}_p[x]$. As in the previous examples, this is a Unique Factorization Domain, so we can figure out its primes in a hands-on way. The points are (0), and irreducible polynomials, which come in any degree. Irreducible polynomials correspond to sets of Galois conjugates in $\overline{\mathbb{F}}_p$. You should think about this, even if you are

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a geometric person — there is some arithmetic intuition that later turns into geometric intuition.

Example 7. $\mathbb{A}^2_{\mathbb{C}} = \operatorname{Spec} \mathbb{C}[x, y]$. (This discussion will apply with $\mathbb{C}$ replaced by *any* algebraically closed field.) Sadly, $\mathbb{C}[x, y]$ is not a Principal Ideal Domain: (x, y) is not a principal ideal. We can quickly name *some* prime ideals. One is (0), which has the same flavor as the (0) ideals in the previous examples. $(x - 2, y - 3)$ is prime, because $\mathbb{C}[x, y]/(x-2, y-3) \cong \mathbb{C}$, where this isomorphism is via $f(x, y) \mapsto f(2, 3)$. More generally, $(x-a, y-b)$ is prime for any $(a, b) \in \mathbb{C}^2$. Also, if $f(x, y)$ is an irreducible polynomial (e.g. $y - x^2$ or $y^2 - x^3$) then $(f(x, y))$ is prime. We will later prove that we have identified all these primes. Here is a picture: the "maximal primes" correspond to the old-fashioned points in $\mathbb{C}^2$ (I drew it). (0) somehow lives behind all of these points (I drew it). $(y - x^2)$ somehow is associated to this parabola (I drew it). Etc. You can see from this picture that we already want to think about "dimension". The primes $(x - a, y - b)$ are somehow of dimension 0, the primes $(f(x, y))$ are of dimension 1, and (0) is somehow of dimension 2. I won't define dimension today, so every time I say it, you should imagine that I am waving my hands wildly.

(This paragraph will not be comprehensible in the notes.) Let's try to picture this. Where is the prime $(y - x^2)$? Well, is it in the plane? Yes. Is it at $(2, 4)$? No. Is it in the set cut out by $y - x^2$? Yes. Is it in the set cut out by $(y^2 - x^3)$? No. Is it in the set cut out by $xy(y - x^2)$? Yes.

Note: maximal ideals correspond to "smallest" points. Smaller ideals correspond to "bigger" points. "One prime ideal contains another" means that the points "have the opposite containment." All of this will be made precise once we have a topology. This order-reversal can be a little confusing (and will remain so even once we have made the notions precise).

Example: $\mathbb{A}^n_{\mathbb{C}} = \operatorname{Spec} \mathbb{C}[x_1, \dots, x_n]$. (More generally, $\mathbb{A}^n_k = \operatorname{Spec} k[x_1, \dots, x_n]$, and even $A^n_R = \operatorname{Spec} R[x_1, \dots, x_n]$ where R is an arbitrary ring.)

For concreteness, let's consider $n = 3$. We now have an interesting question in algebra: What are the prime ideals of $\mathbb{C}[x, y, z]$? We have $(x-a, y-b, z-c)$. This is a maximal ideal, with residue field $\mathbb{C}$; we think of these as "0-dimensional points". Have we discovered all the maximal ideals? The answer is yes, by Hilbert's Nullstellensatz, which is covered in Math 210.

Hilbert's Nullstellensatz, Version 1. (This is sometimes called the "weak version" of the Nullstellensatz.) Suppose $R = k[x_1, \dots, x_n]$, where k is an algebraically closed field. Then the maximal ideals are precisely those of the form $(x_1 - a_1, \dots, x_n - a_n)$, where $a_i \in k$.

There are other prime ideals too. We have (0), which is a "3-dimensional point". We have $(f(x, y, z))$, where f is irreducible. To this we associate the hypersurface $f = 0$, so this is "2-dimensional" in nature. Do we have them all? No! One clue: we're missing dimension 1 things. Here is a "one-dimensional" prime ideal: (x, y). (Picture: it is the z-axis, which is cut out by $x = y = 0$.) How do we check that this is prime? The easiest way is to check that the quotient is an integral domain, and indeed $\mathbb{C}[x, y, z]/(x, y) \cong \mathbb{C}[z]$ is an integral domain (and visibly the functions on the z-axis). There are lots of one-dimensional primes, and it is not possible to classify them in a reasonable way. It will turn out that they correspond to things that we think of as "irreducible" curves: the natural answer to this algebraic question is geometric.

0.1. We now come to two more general flavors of affine schemes that will be useful in the future. There are two nice ways of producing new rings from a ring R. One is by taking the quotient by an ideal I. The other is by localizing at a multiplicative set. We'll see how Spec behaves with respect to these operations. In both cases, the new ring has a Spec that is a subset of the Spec of the old ring.

Important example: $\operatorname{Spec} R/I$ *in terms of* $\operatorname{Spec} R$. As a motivating example, consider $\operatorname{Spec} R/I$ where $R = \mathbb{C}[x, y]$, $I = (xy)$. We have a picture of $\operatorname{Spec} R$, which is the complex plane, with some mysterious extra "higher-dimensional points". Important algebra fact: The primes of R/I are in bijection with the primes of R containing I. (Here I'm using a prerequisite from Math 210. You should review this fact! This is not a result that you should memorize — you should know why it is true. If you don't remember why it is true, or didn't know this fact, then treat this as an **exercise** and do it yourself.) Thus we can picture $\operatorname{Spec} R/I$ as a subset of $\operatorname{Spec} R$. We have the "0-dimensional points" $(a, 0)$ and $(0, b)$. We also have two "1-dimensional points" (x) and (y).

We get a bit more: the inclusion structure on the primes of R/I corresponds to the inclusion structure on the primes containing I. More precisely, if $J_1 \subset J_2$ in R/I, and K_i is the ideal of R corresopnding to J_i, then $K_1 \subset K_2$.

So the minimal primes of $\mathbb{C}/(xy)$ are the "biggest" points we see, and there are two of them: (x) and (y). Thus we have the intuition that will later be precise: the minimal primes correspond to the "components" of $\operatorname{Spec} R$.

Important example: $\operatorname{Spec} S^{-1}R$ *in terms of* $\operatorname{Spec} R$, where S is a multiplicative subset of R. There are two particularly important flavors of multiplicative subsets. The first is $R - \mathfrak{p}$, where $\mathfrak{p}$ is a prime ideal. (Check that this is a multiplicative set!) The localization $S^{-1}R$ is denoted $R_\mathfrak{p}$. Here is a motivating example: $R = \mathbb{C}[x, y]$, $S = R - (x, y)$. The second is $\{1, f, f^2, \ldots\}$, where $f \in R$. The localization is denoted R_f. (Notational warning: If (f) is a prime ideal, then $R_f \neq R_{(f)}$.) Here is an example: $R = \mathbb{C}[x, y]$, $f = x$.

Important algebra fact (to review and know): The primes of $S^{-1}R$ are in bijection with the primes of R that *don't meet* the multiplicative set S. So if $S = R - \mathfrak{p}$ where $\mathfrak{p}$ is a prime ideal, the primes of $S^{-1}R$ are just the primes of R *contained in* $\mathfrak{p}$. If $S = \{1, f, f^2, \ldots\}$, the primes of $S^{-1}R$ are just those primes not containing f (the points where "f doesn't vanish"). A bit more is true: the inclusion structure on the primes of $S^{-1}R$ corresponds to the inclusion structure on the primes not meeting S. (If you didn't know it, take this as an **exercise** and prove it yourself!)

In each of these two cases, a picture is worth a thousand words. In these notes, I'm not making pictures unfortunately. But I'll try to describe them in less than a thousand words.

The case of $S = \{1, f, f^2, \dots\}$ is easier: we just throw out those points where f vanishes. (We will soon call this a *distinguished open set*, once we know what open sets are.) In our example of $R = k[x,y]$, $f = x$, we throw out the y-axis.

Warning: sometimes people are first introduced to localizations in the special case that R is an integral domain. In this case, $R \hookrightarrow R_f$, but this isn't true in general. Here's the definition of localization (which you should be familiar with). The elements of $S^{-1}R$ are of the form r/s where $r \in R$ and $s \in S$, and $(r_1/s_1) \times (r_2/s_2) = (r_1 r_2/s_1 s_2)$, and $(r_1/s_1) + (r_2/s_2) = (r_1 s_2 + s_1 r_2)/(s_1 s_2)$. We say that $r_1/s_1 = r_2/s_2$ if **for some** $s \in S$ $s(r_1 s_2 - r_2 s_1) = 0$.

Example/warning: $R[1/0] = 0$. Everything in $R[1/0]$ is 0. (Geometrically, this is good: the locus of points where 0 doesn't vanish is the empty set, so certainly $D(0) = \operatorname{Spec} R_0$.)

In general, inverting zero-divisors can make things behave weirdly. Example: $R = k[x,y]/(xy)$. $f = x$. What do you get? It's actually a straightforward ring, and we'll use some geometric intuition to figure out what it is. $\operatorname{Spec} k[x,y]/(xy)$ "is" the union of the two axes in the plane. Localizing means throwing out the locus where x vanishes. So we're left with the x-axis, minus the origin, so we expect $\operatorname{Spec} k[x]_x$. So there should be some natural isomorphism $(k[x,y]/(xy))_x \cong k[x]_x$. **Exercise.** Figure out why these two rings are isomorphic. (You'll see that y on the left goes to 0 on the right.)

In the case of $S = R - \mathfrak{p}$, we keep only those primes contained in $\mathfrak{p}$. In our example $R = k[x,y]$, $\mathfrak{p} = (x,y)$, we keep all those points corresponding to "things through the origin", i.e. the 0-dimensional point (x,y), the 2-dimensional point (0), and those 1-dimensional points $(f(x,y))$ where $f(x,y)$ is irreducible and $f(0,0) = 0$, i.e. those "irreducible curves through the origin". (There is a picture of this in Mumford's Red Book: Example F, Ch. 2, §1, p. 140.)

Caution with notation: If $\mathfrak{p}$ is a prime ideal, then $R_{\mathfrak{p}}$ means you're allowed to divide by elements not in $\mathfrak{p}$. However, if $f \in R$, R_f means you're allowed to divide by f. I find this a bit confusing. Especially when (f) is a prime ideal, and then $R_{(f)} \neq R_f$.

1. Affine schemes II: the underlying topological space

We now define the **Zariski topology** on $\operatorname{Spec} R$. Topologies are often described using open subsets, but it will more convenient for us to define this topology in terms of their complements, closed subsets. If S is a subset of R, define

$$V(S) := \{\mathfrak{p} \in \operatorname{Spec} R : S \subset \mathfrak{p}\}.$$

We interpret this as the **vanishing set** of S; it is the set of points on which all elements of S are zero. We declare that these (and no others) are the closed subsets.

1.1. Exercise. Show that if (S) is the ideal generated by S, then $V(S) = V((S))$. This lets us restrict attention to vanishing sets of ideals.

Let's check that this is a topology. Remember the requirements: the empty set and the total space should be open; the union of an arbitrary collection of open sets should be open; and the intersection of two open sets should be open.

1.2. Exercise. (a) Show that $\emptyset$ and $\operatorname{Spec} R$ are both open.
(b) (The intersection of two open sets is open.) Check that $V(I_1 I_2) = V(I_1) \cup V(I_2)$.
(c) (The union of any collection of open sets is open.) If I_i is a collection of ideals (as i runs over some index set), check that $V(\sum_i I_i) = \cap_i V(I_i)$.

1.3. Properties of "vanishing set" function $V(\cdot)$. The function $V(\cdot)$ is obviously inclusion-reversing: If $S_1 \subset S_2$, then $V(S_2) \subset V(S_1)$. (Warning: We could have equality in the second inclusion without equality in the first, as the next exercise shows.)

1.4. Exercise. If $I \subset R$ is an ideal, then define its *radical* by

$$\sqrt{I} := \{r \in R : r^n \in I \text{ for some } n \in \mathbb{Z}^{\geq 0}\}.$$

Show that $V(\sqrt{I}) = V(I)$. (We say an *ideal is radical* if it equals its own radical.)

Hence: $V(IJ) = V(I \cap J)$. (Reason: $(I \cap J)^2 \subset IJ \subset I \cap J$.) Combining this with Exercise 1.1, we see

$$V(S) = V((S)) = V(\sqrt{(S)}).$$

1.5. Examples. Let's see how this meshes with our examples from earlier.

Recall that $\mathbb{A}^1_{\mathbb{C}}$, as a set, was just the "old-fashioned" points (corresponding to maximal ideals, in bijection with $a \in \mathbb{C}$), and one "weird" point (0). The Zariski topology on $\mathbb{A}^1_{\mathbb{C}}$ is not that exciting. The open sets are the empty set, and $\mathbb{A}^1_{\mathbb{C}}$ minus a finite number of maximal ideals. (It "almost" has the cofinite topology. Notice that the open sets are determined by their intersections with the "old-fashioned points". The "weird" point (0) comes along for the ride, which is a good sign that it is harmless. Ignoring the "weird" point, observe that the topology on $\mathbb{A}^1_{\mathbb{C}}$ is a coarser topology than the analytic topology.)

The case $\operatorname{Spec} \mathbb{Z}$ is similar. The topology is "almost" the cofinite topology in the same way. The open sets are the empty set, and $\operatorname{Spec} \mathbb{Z}$ minus a finite number of "ordinary" ((p) where p is prime) primes.

The case $\mathbb{A}^2_{\mathbb{C}}$ is more interesting. I discussed it in a bit of detail in class, using pictures.

1.6. Topological definitions. We'll now define some words to do with the topology.

A topological space is said to be *irreducible* if it is not the union of two proper closed subsets. In other words, X if irreducible if whenever $X = Y \cup Z$ with Y and Z closed, we have $Y = X$ or $Z = X$.

1.7. Exercise. Show that if R is an integral domain, then $\operatorname{Spec} R$ is an irreducible topological space. (Hint: look at the point $[(0)]$.)

A point of a topological space $x \in X$ is said to be *closed* if $\overline{\{x\}} = \{x\}$.

1.8. Exercise. Show that the closed points of $\operatorname{Spec} R$ correspond to the maximal ideals.

Given two points x, y of a topological space X, we say that x is a *specialization* of y, and y is a *generization* of x, if $x \in \overline{\{y\}}$. This now makes precise our hand-waving about "one point contained another". It is of course nonsense for a point to contain another. But it is no longer nonsense to say that the closure of a point contains another.

1.9. Exercise. If $X = \operatorname{Spec} R$, show that $[\mathfrak{p}]$ is a specialization of $[\mathfrak{q}]$ if and only if $\mathfrak{q} \subset \mathfrak{p}$. Verify to your satisfaction that this is precisely the intuition of "containment of points" that we were talking about before.

We say that a point $x \in X$ is a *generic point* for a closed subset K if $\overline{\{x\}} = K$.

1.10. Exercise. Verify that $[(y - x^2)] \in \mathbb{A}^2$ is a generic point for $V(y - x^2)$.

A topological space X is *quasicompact* if given any cover $X = \cup_{i \in I} U_i$ by open sets, there is a finite subset S of the index set I such that $X = \cup_{i \in S} U_i$. Informally: every cover has a finite subcover. This is "half of the definition of quasicompactness". We will like this condition, because we are afraid of infinity.

1.11. Exercise. Show that $\operatorname{Spec} R$ is quasicompact. (Warning: it can have nonquasicompact open sets.)

1.12. Exercise. If X is a finite union of quasicompact spaces, show that X is quasicompact.

Earlier today, we explained that $\operatorname{Spec} R/I$ and $\operatorname{Spec} S^{-1}R$ are naturally subsets of $\operatorname{Spec} R$. All of these have Zariski topologies, and it is natural to ask if the topology behaves well with respect to these inclusions, and indeed it does.

1.13. Exercise. Suppose that $I, S \subset R$ are an ideal and multiplicative subset respectively. Show that $\operatorname{Spec} R/I$ is naturally a closed subset of $\operatorname{Spec} R$. Show that the Zariski topology on $\operatorname{Spec} R/I$ (resp. $\operatorname{Spec} S^{-1}R$) is the subspace topology induced by inclusion in $\operatorname{Spec} R$. (Hint: compare closed subsets.)

1.14. The function $I(\cdot)$, taking subsets of $\operatorname{Spec} R$ to ideals of R. Here is another notion, that is in some sense "opposite" to the vanishing set function $V(\cdot)$. Given a subset $S \subset \operatorname{Spec} R$, $I(S)$ is the ideal of functions vanishing on S. Three quick points: it is clearly an ideal. $I(\overline{S}) = I(S)$. And $I(\cdot)$ is inclusion-reversing: if $S_1 \subset S_2$, then $I(S_2) \subset I(S_1)$.

1.15. Exercise/Example. Let $R = k[x, y]$. If $S = \{(x), (x-1, y)\}$ (draw this!), then $I(S)$ consists of those polynomials vanishing on the y axis, and at the point $(1, 0)$. Give generators for this ideal.

More generally:

1.16. Exercise. Show that $V(I(S)) = \overline{S}$. Hence $V(I(S)) = S$ for a closed set S.

1.17. Exercise. Suppose $X \subset \mathbb{A}^3$ is the union of the three axes. Give generators for the ideal $I(X)$.

Note that $I(S)$ is always a radical ideal — if $f \in \sqrt{I(S)}$, then f^n vanishes on S for some $n > 0$, so then f vanishes on S, so $f \in I(S)$.

Here is a handy algebraic fact to know. The *nilradical* $\mathfrak{N} = \mathfrak{N}(R)$ of a ring R is defined as $\sqrt{0}$ — it consists of all functions that have a power that is zero. (Checked that this set is indeed an ideal, for example that it is closed under addition!)

1.18. Theorem. *The nilradical $\mathfrak{N}(R)$ is the intersection of all the primes of R.*

If you don't know it, then look it up, or even better, prove it yourself. (Hint: one direction is easy. The other will require knowing that any proper ideal of R is contained in a maximal ideal, which requires the axiom of choice.) As a corollary, $\sqrt{I}$ is the intersection of all the prime ideals containing I. (Hint of proof: consider the ring R/I, and use the previous theorem.)

1.19. Exercise. Prove that if $I \subset R$ is an ideal, then $I(V(I)) = \sqrt{I}$.

Hence in combination with Exercise 1.16, we get the following:

1.20. *Theorem. — $V(\cdot)$ and $I(\cdot)$ give a bijection between closed subsets of $\operatorname{Spec} R$ and radical ideals of R (where a closed subset gives a radical ideal by $I(\cdot)$, and a radical ideal gives a closed subset by $V(\cdot)$).*

1.21. Important Exercise. Show that $V(\cdot)$ and $I(\cdot)$ give a bijection between *irreducible closed subsets* of $\operatorname{Spec} R$ and *prime* ideals of R. From this conclude that in $\operatorname{Spec} R$ there is a bijection between points of $\operatorname{Spec} R$ and irreducible closed subsets of $\operatorname{Spec} R$ (where a point determines an irreducible closed subset by taking the closure). Hence each irreducible closed subset has precisely one generic point.

To drive this point home: Suppose Z is an irreducible closed subset of $\operatorname{Spec} R$. Then there is one and only one $z \in Z$ such that $Z = \overline{\{z\}}$.

If $f \in R$, define the **distinguished open set** $D(f) = \{\mathfrak{p} \in \operatorname{Spec} R : f \notin \mathfrak{p}\}$. It is the locus where f doesn't vanish. (I often privately write this as $D(f \neq 0)$ to remind myself of this. I also private call this a D*oesn't vanish set* in analogy with $V(f)$ being the Vanishing set.) We have already seen this set when discussing $\operatorname{Spec} R_f$ as a subset of $\operatorname{Spec} R$.

2.1. Important exercise. Show that the distinguished opens form a base for the Zariski topology.

2.2. Easy important exercise. Suppose $f_i \in R$ for $i \in I$. Show that $\cup_{i \in I} D(f_i) = \operatorname{Spec} R$ if and only if $(f_i) = R$.

2.3. Easy important exercise. Show that $D(f) \cap D(g) = D(fg)$. Hence the distinguished base is a *nice* base.

2.4. Easy important exercise. Show that if $D(f) \subset D(g)$, then $f^n \in (g)$ for some n.

2.5. Easy important exercise. Show that $f \in \mathfrak{N}$ if and only if $D(f) = \emptyset$.

We have already observed that the Zariski topology on the distinguished open $D(f) \subset \operatorname{Spec} R$ coincides with the Zariski topology on $\operatorname{Spec} R_f$.

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FOUNDATIONS OF ALGEBRAIC GEOMETRY CLASS 6

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CONTENTS

Last day: $\operatorname{Spec} R$**: the set, and the topology**

Today: The structure sheaf, and schemes in general.

Announcements: on problem set 2, there was a serious typo in # 10. $\operatorname{Hom}(\mathcal{F}(U), \mathcal{G}(U))$ should read $\operatorname{Hom}(\mathcal{F}|_U, \mathcal{G}|_U)$. The notation is new, but will likely be clear to you after you think about it a little. If $\mathcal{F}$ is a sheaf on X, and U is an open subset, then we can define the sheaf $\mathcal{F}|_U$ on U in the obvious way. This is sometimes called *the restriction of the sheaf* $\mathcal{F}$ *to the open set* U (not to be confused with restriction maps!). This homomorphism $\operatorname{Hom}(\mathcal{F}|_U, \mathcal{G}|_U)$ is the set of all *sheaf homomorphisms* from $\mathcal{F}|_U$ to $\mathcal{G}|_U$. The revised version is posted on the website.

Also, the final problem set this quarter will be due no later than Monday, December 12, the Monday after the last class.

1. Recap of last day, and further discussion

Last day, we saw lots of examples of the underlying sets of affine schemes, which correspond to primes in a ring. In this dictionary, "an element r of the ring lying in a prime ideal $\mathfrak{p}$" translates to "an element r of the ring vanishing at the point $[\mathfrak{p}]$, and I will use these phrases interchangeably.

There was some language I was using informally, and I've decided to make it more formal: elements $r \in R$ will officially be called "global functions", and their value at the point $[\mathfrak{p}]$ will be $r \pmod{\mathfrak{p}}$. This language will be "justified" by the end of today.

I then defined the Zariski topology. The closed subsets were just those points where some set of ring elements all vanish.

Date: Wednesday, October 19, 2005. Minor update January 30, 2007.

As a reminder, here are the key words that we learned about topological spaces: irreducible; generic point; closed points (points p such that $\overline{\{p\}} = \{p\}$; did I forget to say this last time?); specialization/generization; quasicompact. All of these words can be used on any topological spaces, but they tend to be boring (or highly improbable) on spaces that you knew and loved before.

On $\operatorname{Spec} R$, closed points correspond to maximal ideals of R. Also, I described a bijection between closed subsets and radical ideals. The two maps of this bijection use the "vanishing set" function V and the "ideal of functions vanishing" function I. I also described a bijection between points and closed subsets; one direction involved taking closures, and the other involved taking generic points. Some of this was left to you in the form of **exercises**.

As an example, consider the prime (or point) $(y - x^2)$ in $k[x, y]$ (or $\operatorname{Spec} k[x, y]$). What is its closure? We look at all functions vanishing at this point, and see at what other points they all vanish. In other words, we look for all prime ideals containing all elements of this one. In other words, we look at all prime ideals containing this one. Picture: we get all the closed points on the parabola. We get the closed set corresponding to this point. (Caveat: I haven't proved that I've described all the primes in $k[x, y]$.)

In class, I spontaneously showed you that the Zariski closure of the countable set (n, n^2) in $\mathbb{A}^2_{\mathbb{C}}$ was the parabola. The Zariski closure of a finite set of points will just be itself: a finite union of closed sets is closed.

Last day I showed that the Zariski topology behaves well with respect to taking quotients, and localizing. I said a little more today.

About taking quotients: suppose you have a ring R, and an ideal I. Then there is a bijection of the points of $\operatorname{Spec} R/I$ with the points of $V(I)$ in $\operatorname{Spec} R$. (Just unwind the algebraic definition! Both correspond to primes $\mathfrak{p}$ of R containing I.) My comments of last day showed that this is in fact a homeomorphism: $\operatorname{Spec} R/I$ may be identified with the closed subset of $\operatorname{Spec} R$ as a topological space: the subspace topology induced from $\operatorname{Spec} R$ is indeed the topology of $\operatorname{Spec} R$. The reason was not sophisticated: there is a natural correspondence of closed subsets.

About localizing: this is quite a general procedure, so in general you can't say much besides the fact that $\operatorname{Spec} S^{-1}R$ is naturally a subset of $\operatorname{Spec} R$, with the induced topology. Last day I discussed the important case where $S = R - \mathfrak{p}$, the complement of a prime ideal, so then $S^{-1}R = R_{\mathfrak{p}}$.

But there is a second important example of localization, when $S = \{1, f, f^2, \ldots\}$ for some $f \in R$. In this case we get the ring denoted R_f. In this case, $\operatorname{Spec} R_f$ is $D(f)$, again just by unwinding the definitions: both consist of the primes not containing f (= the points where f doesn't vanish). The Zariski topology on $D(f)$ agrees with the Zariski topology on $\operatorname{Spec} R_f$.

Here is an **exercise** from last day. Show that $(f_1, \ldots, f_n) = (1)$ if and only if $\cup D(f_i) = \operatorname{Spec} R$. I want to do it for you, to show you how it can be interpreted simultaneously in both algebra and geometry. Here is one direction. Suppose $[\mathfrak{p}] \notin \cup D(f_i)$. You can unwind this to get an algebraic statement. I think of it as follows. All of the f_i vanish at $[\mathfrak{p}]$, i.e. all $f_i \in \mathfrak{p}$, so then $(f_1, \dots) \subset \mathfrak{p}$ and hence this ideal can't be all of R. Conversely, consider the ideal $(f_1, \dots)$. If it isn't R, then it is contained in a maximal ideal. (For logic-lovers: we're using the axiom of choice, which I said I'd assume at the very start of this class.) But then there is some prime ideal containing all the f_i. Translation: $[\mathfrak{m}] \notin D(f_i)$ for any i. (As an added bonus: this argument shows that if $\operatorname{Spec} R$ is an infinite union $\cup_{i \in I} D(f_i)$, then in fact it is a union of a finite number of these. This is one way of proving quasicompactness.)

Important comment: This machinery will let us bring our geometric intuition to algebra. There is one point where your intuition will be false, and I want to tell you now, so you can adjust your intuition appropriately. Suppose we have a function (ring element) vanishing at all points. Is it zero? Not necessarily! Translation: is intersection of all prime ideals necessarily just 0? No: $k[\epsilon]/\epsilon^2$ is a good example, as $\epsilon \neq 0$, but $\epsilon^2 = 0$. This is called the *ring of dual numbers* (over the field k). Any function whose power is zero certainly lies in the intersection of all prime ideals. And the converse is true (algebraic fact): the intersection of all the prime ideals consists of functions for which some power is zero, otherwise known as the nilradical $\mathfrak{N}$. (Ring elements that have a power that is 0 are called *nilpotents*.) Summary: "functions on affine schemes" will not be determined by their values at points. (For example: $\operatorname{Spec} k[\epsilon]/\epsilon^2$ has one point. $3 + 4\epsilon$ has value 3 at that point, but the function isn't 3.) In particular, any function vanishing at all points might not be zero, but some power of it is zero. This takes some getting used to.

1.1. *Easy fun unimportant exercise.* Suppose we have a polynomial $f(x) \in k[x]$. Instead, we work in $k[x, \epsilon]/\epsilon^2$. What then is $f(x + \epsilon)$? (Do a couple of examples, and you will see the pattern. For example, if $f(x) = 3x^3 + 2x$, we get $f(x + \epsilon) = (3x^3 + 2x) + \epsilon(9x^2 + 2)$. Prove the pattern!) Useful tip: the dual numbers are a good source of (counter)examples, being the "smallest ring with nilpotents". They will also end up being important in defining differential information.

Here is one more (important!) algebraic fact: suppose $D(f) \subset D(g)$. Then $f^n \in (g)$ for some n. I'm going to let you prove this (**Exercise** from last day), but I want to tell you how I think of it geometrically. Draw a picture of $\operatorname{Spec} R$. Draw the closed subset $V(g) = \operatorname{Spec} R/(g)$. That's where g vanishes, and the complement is $D(g)$, where g doesn't vanish. Then f is a function on this closed subset, and it vanishes at all points of the closed subset. (Translation: Consider f as an element of the ring $R/(g)$.) Now any function vanishing at every point of Spec a ring must have some power which is 0. Translation: there is some n such that $f^n = 0$ in $R/(g)$, i.e. $f^n \equiv 0 \pmod g$ in R, i.e. $f^n \in (g)$.

2. THE FINAL INGREDIENT IN THE DEFINITION OF AFFINE SCHEMES: THE STRUCTURE SHEAF

The final ingredient in the definition of an affine scheme is the **structure sheaf** $\mathcal{O}_{\operatorname{Spec} R}$, which we think of as the "sheaf of algebraic functions". These functions will have values at points, but won't be determined by their values at points. Like all sheaves, they will indeed be determined by their germs.

It suffices to describe it as a sheaf on the nice base of distinguished open sets. We define the sections on the base by

$$\mathcal{O}_{\operatorname{Spec} R}(D(f)) = R_f \tag{1}$$

We define the restriction maps $\operatorname{res}_{D(g),D(f)}$ as follows. If $D(f) \subset D(g)$, then we have shown that $f^n \in (g)$, i.e. we can write $f^n = ag$. There is a natural map $R_g \to R_f$ given by $r/g^m \mapsto (ra^m)/(f^{mn})$, and we define

$$\operatorname{res}_{D(g),D(f)} : \mathcal{O}_{\operatorname{Spec} R}(D(g)) \to \mathcal{O}_{\operatorname{Spec} R}(D(f))$$

to be this map.

2.1. *Exercise.* (a) Verify that (1) is well-defined, i.e. if $D(f) = D(f')$ then R_f is canonically isomorphic to $R_{f'}$. (b) Show that $\operatorname{res}_{D(g),D(f)}$ is well-defined, i.e. that it is independent of the choice of a and n, and if $D(f) = D(f')$ and $D(g) = D(g')$, then

$$\begin{tikzcd}
R_g \arrow[r, "\operatorname{res}_{D(g),D(f)}"] \arrow[d, "\sim"] & R_f \arrow[d, "\sim"] \\
R_{g'} \arrow[r, "\operatorname{res}_{D(g'),D(f')}"] & R_{f'}
\end{tikzcd}$$

commutes.

We now come to the big theorem of today.

2.2. *Theorem. — The data just described gives a sheaf on the (nice) distinguished base, and hence determines a sheaf on the topological space* $\operatorname{Spec} R$*.*

This sheaf is called the **structure sheaf**, and will be denoted $\mathcal{O}_{\operatorname{Spec} R}$, or sometimes $\mathcal{O}$ if the scheme in question is clear from the context. Such a topological space, with sheaf, will be called an **affine scheme**. The notation $\operatorname{Spec} R$ will hereafter be a topological space, with a structure sheaf.

Proof. Clearly this is a presheaf on the base: if $D(f) \subset D(g) \subset D(h)$ then the following diagram commutes:

$$\begin{tikzcd}
R_h \arrow[rr, "\operatorname{res}_{D(h),D(g)}"] \arrow[dr, "\operatorname{res}_{D(h),D(f)}"'] & & R_g \arrow[dl, "\operatorname{res}_{D(g),D(f)}"] \\
& R_f. &
\end{tikzcd} \tag{2}$$

You can check this directly. Here is a trick which helps (and may help you with Exercise 2.1 above). As $D(g) \subset D(h)$, $D(gh) = D(g)$. (Translation: The locus where g doesn't vanish is a subset of where h doesn't vanish, so the locus where gh doesn't vanish is the same as the locus where g doesn't vanish.) So we can replace R_g by R_{gh}, and R_f by R_{fgh}, in (2). The restriction maps are $\operatorname{res}_{D(h),D(gh)} : a/h \mapsto ag/gh$, $\operatorname{res}_{D(gh),D(fgh)} : b/gh \mapsto bf/fgh$, and $\operatorname{res}_{D(h),D(fgh)} : a/h \to afg/fgh$, so they clearly commute as desired.

We next check identity on the base. We deal with the case of a cover of the entire space R, and let the reader verify that essentially the same argument holds for a cover of some R_f. Suppose that $\operatorname{Spec} R = \cup_{i \in A} D(f_i)$ where i runs over some index set I. By quasicompactness, there is some finite subset of I, which we name $\{1, \ldots, n\}$, such that $\operatorname{Spec} R = \cup_{i=1}^n D(f_i)$, i.e. $(f_1, \ldots, f_n) = R$. (Now you see why we like quasicompactness!) Suppose we are given $s \in R$ such that $\operatorname{res}_{\operatorname{Spec} R, D(f_i)} s = 0$ in R_{f_i} for all i. Hence there is some m such that for each $i \in \{1, \ldots, n\}$, $f_i^m s = 0$. (Reminder: $R \to R_f$. What goes to 0? Precisely things killed by some power of f.) Now $(f_1^m, \ldots, f_n^m) = R$ (do you know why?), so there are $r_i \in R$ with $\sum_{i=1}^n r_i f_i^m = 1$ in R, from which

$$s = \left(\sum r_i f_i^m \right) s = 0.$$

Thus we have checked the "base identity" axiom for $\operatorname{Spec} R$.

Remark. Serre has described this as a "partition of unity" argument, and if you look at it in the right way, his insight is very enlightening.

2.3. *Exercise.* Make the tiny changes to the above argument to show base identity for any distinguished open $D(f)$.

We next show base gluability. As with base identity, we deal with the case where we wish to glue sections to produce a section over $\operatorname{Spec} R$. As before, we leave the general case where we wish to glue sections to produce a section over $D(f)$ to the reader (Exercise 2.4).

Suppose $\cup_{i \in I} D(f_i) = \operatorname{Spec} R$, where I is a index set (possibly horribly uncountably infinite). Suppose we are given

$$\frac{a_i}{f_i^{l_i}} \in R_{f_i} \quad (i \in I)$$

such that for all $i, j \in I$, there is some $m_{ij} \geq l_i, l_j$ with

$$(f_i f_j)^{m_{ij}} \left(f_j^{l_j} a_i - f_i^{l_i} a_j \right) = 0 \tag{3}$$

in R. We wish to show that there is some $r \in R$ such that $r = a_i / f_i^{l_i}$ in R_{f_i} for all $i \in I$.

Choose a finite subset $\{1, \ldots, n\} \subset I$ with $(f_1, \ldots, f_n) = R$.

To save ourself some notation, we may take the l_i to all be 1, by replacing f_i with $f_i^{l_i}$ (as $D(f_i) = D(f_i^{l_i})$). We may take m_{ij} ($1 \leq i, j \leq n$) to be the same, say m — take $m = \max m_{ij}$.) The only reason to do this is to have fewer variables.

Let $a_i' = a_i f_i^m$. Then $a_i / f_i = a_i' / f_i^{m+1}$, and (3) becomes

$$f_j^{m+1} a_i' - f_i^{m+1} a_j' = 0 \tag{4}$$

As $(f_1, \ldots, f_n) = R$, we have $(f_1^{m+1}, \ldots, f_n^{m+1}) = R$, from which $1 = \sum b_i f_i^{m+1}$ for some $b_i \in R$. Define

$$r = b_1 a_1' + \cdots + b_n a_n'.$$

This will be the r that we seek. For each $i \in \{1, \ldots, n\}$, we will show that $r - a'_i/f_i^{m+1} = 0$ in D_{f_i}. Indeed,

$$\begin{aligned} rf_i^{m+1} - a'_i &= \sum_{j=1}^{n} a'_j b_j f_i^{m+1} - \sum_{j=1}^{n} a'_i b_j f_j^{m+1} \\ &= \sum_{j \neq i} b_j (a'_j f_i^{m+1} - a'_i f_j^{m+1}) \\ &= 0 \quad \text{(by (4))} \end{aligned}$$

So are we done? No! We are supposed to have something that restricts to $a_i/f_i^{l_i}$ for *all* $i \in I$, not just $i = 1, \ldots, n$. But a short trick takes care of this. We now show that for any $\alpha \in I - \{1, \ldots, n\}$, $r = a_\alpha/f_\alpha^{l_\alpha}$ in R_{f_α}. Repeat the entire process above with $\{1, \ldots, n, \alpha\}$ in place of $\{1, \ldots, n\}$, to obtain $r' \in R$ which restricts to $a_i/f_i^{l_i}$ for $i \in \{1, \ldots, n, \alpha\}$. Then by base identity, $r' = r$. Hence r restricts to $a_\alpha/f_\alpha^{l_\alpha}$ as desired.

2.4. *Exercise.* Alter this argument appropriately to show base gluability for any distinguished open $D(f)$.

$\square$

So now you know what an affine scheme is!

We can even define a scheme in general: it is a topological space X, along with a structure sheaf $\mathcal{O}_X$, that locally looks like an affine scheme: for any $x \in X$, there is an open neighborhood U of x such that $(U, \mathcal{O}_X|_U)$ is an affine scheme.

On Friday, I'll discuss some of the ramifications of this definition. In particular, you'll see that stalks of this sheaf are something familiar, and I'll show you that constructing the sheaf by looking at this nice distinguished base isn't just a kluge, it's something very natural — we'll do this by finding sections of $\mathcal{O}_{\mathbb{A}^2}$ over the open set $\mathbb{A}^2 - (0, 0)$.

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FOUNDATIONS OF ALGEBRAIC GEOMETRY CLASS 7

CONTENTS

Last day: The structure sheaf.

Today: $\tilde{M}$, sheaf associated to R-module M; Chinese remainder theorem; germs and value at a point of the structure sheaf; non-affine schemes $\mathbb{A}^2 - (0,0)$, line with doubled origin, $\mathbb{P}^n$.

Another problem set 2 issue, about the pullback sheaf. First, I think I'd like to call it the inverse image sheaf, because I don't want to confuse it with something that I'll also call the pullback. Second, and more importantly, I didn't give the correct definition.

Here is what I should have said (and what is now in the problem set). Define $f^{-1}\mathcal{G}^{\text{pre}}(U) = \lim_{\to V \supset f(U)} \mathcal{G}(V)$. Then show that this is a *presheaf*. Then the sheafification of this is said to be the *inverse image sheaf* (sometimes called the pullback sheaf) $f^{-1}\mathcal{G} := (f^{-1}\mathcal{G}^{\text{pre}})^{\text{sh}}$. Thanks to Kate for pointing out this important patch!

1. PLAYING WITH THE STRUCTURE SHEAF

Here's where we were last day. We defined the *structure sheaf* $\mathcal{O}_{\operatorname{Spec} R}$ on an affine scheme $\operatorname{Spec} R$. We did this by describing it as a *sheaf on the distinguished base*.

An immediate consequence is that we can recover our ring R from the scheme $\operatorname{Spec} R$ by taking global sections, as the entire scheme is $D(1)$:

$$\begin{aligned}
\Gamma(\operatorname{Spec} R, \mathcal{O}_{\operatorname{Spec} R}) &= \Gamma(D(1), \mathcal{O}_{\operatorname{Spec} R}) \quad \text{as } D(1) = \operatorname{Spec} R \\
&= R_1 \quad \text{(i.e. allow 1's in the denominator) by definition} \\
&= R
\end{aligned}$$

Another easy consequence is that the restriction of the sheaf $\mathcal{O}_{\operatorname{Spec} R}$ to the distinguished open set $D(f)$ gives us the affine scheme $\operatorname{Spec} R_f$. Thus not only does the set restrict, but also the topology (as we've seen), and the structure sheaf (as this exercise shows).

Date: Friday, October 21, 2005. Minor updates January 28, 2007.

1.1. *Important but easy exercise.* Suppose $f \in R$. Show that under the identification of $D(f)$ in $\operatorname{Spec} R$ with $\operatorname{Spec} R_f$, there is a natural isomorphism of sheaves $(D(f), \mathcal{O}_{\operatorname{Spec} R}|_{D(f)}) \cong (\operatorname{Spec} R_f, \mathcal{O}_{\operatorname{Spec} R_f})$.

The proof of Big Theorem of last time (that the object $\mathcal{O}_{\operatorname{Spec} R}$ defined by $\Gamma(D(f), \mathcal{O}_{\operatorname{Spec} R}) = R_f$ forms a sheaf on the distinguished base, and hence a sheaf) immediately generalizes, as the following exercise shows. This exercise will be essential for the definition of a quasicoherent sheaf later on.

1.2. *Important but easy exercise.* Suppose M is an R-module. Show that the following construction describes a sheaf $\tilde{M}$ on the distinguished base. To $D(f)$ we associate $M_f = M \otimes_R R_f$; the restriction map is the "obvious" one. This is a sheaf of $\mathcal{O}_{\operatorname{Spec} R}$-modules! We get a natural bijection: rings, modules $\leftrightarrow$ Affine schemes, $\tilde{M}$.

Useful fact for later: As a consequence, note that if $(f_1, \dots, f_r) = R$, we have identified M with a specific submodule of $M_{f_1} \times \cdots \times M_{f_r}$. Even though $M \to M_{f_i}$ may not be an inclusion for any f_i, $M \to M_{f_1} \times \cdots \times M_{f_r}$ *is* an inclusion. We don't care yet, but we'll care about this later, and I'll invoke this fact. (Reason: we'll want to show that if M has some nice property, then M_f does too, which will be easy. We'll also want to show that if $(f_1, \dots, f_n) = R$, then if M_{f_i} have this property, then M does too.)

1.3. Brief fun: The Chinese Remainder Theorem is a *geometric* fact. I want to show you that the Chinese Remainder theorem is embedded in what we've done, which shouldn't be obvious to you. I'll show this by example. The Chinese Remainder Theorem says that knowing an integer modulo 60 is the same as knowing an integer modulo 3, 4, and 5. Here's how to see this in the language of schemes. What is $\operatorname{Spec} \mathbb{Z}/(60)$? What are the primes of this ring? Answer: those prime ideals containing (60), i.e. those primes dividing 60, i.e. (2), (3), and (5). So here is my picture of the scheme [3 dots]. They are all closed points, as these are all maximal ideals, so the topology is the discrete topology. What are the stalks? You can check that they are $\mathbb{Z}/4$, $\mathbb{Z}/3$, and $\mathbb{Z}/5$. My picture is actually like this (draw a small arrow on (2)): the scheme has nilpotents here ($2^2 \equiv 0 \pmod 4$). So what are global sections on this scheme? They are sections on this open set (2), this other open set (3), and this third open set (5). In other words, we have a natural isomorphism of rings

$$\mathbb{Z}/60 \to \mathbb{Z}/4 \times \mathbb{Z}/3 \times \mathbb{Z}/5.$$

On a related note:

1.4. *Exercise.* Show that the disjoint union of a *finite* number of affine schemes is also an affine scheme. (Hint: say what the ring is.)

This is *always* false for an infinite number of affine schemes:

1.5. *Unimportant exercise.* An infinite disjoint union of (non-empty) affine schemes is not an affine scheme. (One-word hint: quasicompactness.)

1.6. Stalks of this sheaf: germs, and values at a point. Like every sheaf, the structure sheaf has stalks, and we shouldn't be surprised if they are interesting from an algebraic point of view. In fact, we have seen them before.

1.7. *Exercise.* Show that the stalk of $\mathcal{O}_{\operatorname{Spec} R}$ at the point $[\mathfrak{p}]$ is the ring $R_{\mathfrak{p}}$. (Hint: use distinguished open sets in the direct limit you use to define the stalk. In the course of doing this, you'll discover a useful principle. In the concrete definition of stalk, the elements were sections of the sheaf over *some* open set containing our point, and two sections over different open sets were considered the same if they agreed on some smaller open set. In fact, you can just consider elements of your base when doing this. This is called a *cofinal system* in the *directed set*.) This is yet another reason to like the notion of a sheaf on a base.

The *residue field of a scheme at a point* is the local ring modulo its maximal ideal.

Essentially the same argument will show that the stalk of the sheaf $\tilde{M}$ at $[\mathfrak{p}]$ is $M_{\mathfrak{p}}$.

So now we can make precise some of our intuition. Suppose $[\mathfrak{p}]$ is a point in some open set U of $\operatorname{Spec} R$. For example, say $R = k[x, y]$, $\mathfrak{p} = (x)$ (draw picture), and $U = \mathbb{A}^2 - (0, 0)$. (First, make sure you see that this *is* an open set! $(0, 0) = V((x, y))$ is indeed closed. Make sure you see that $[\mathfrak{p}]$ indeed is in U.)

- Then a function on U, i.e. a section of $\mathcal{O}_{\operatorname{Spec} R}$ over U, has a *germ near* $[\mathfrak{p}]$, which is an element of $R_{\mathfrak{p}}$. Note that this is a local ring, with maximal ideal $\mathfrak{p}R_{\mathfrak{p}}$. In our example, consider the function $(3x^4 + x^2 + xy + y^2)/(3x^2 + xy + y^2 + 1)$, which is defined on the open set $D(3x^2 + xy + y^2 + 1)$. Because there are no factors of x in the denominator, it is indeed in $R_{\mathfrak{p}}$.
- A germ has a *value* at $[\mathfrak{p}]$, which is an element of $R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}$. (This is isomorphic to $\operatorname{Frac}(R/\mathfrak{p})$, the fraction field of the quotient domain.) So the value of a function at a point always takes values in a field. In our example, to see the value of our germ at $x = 0$, we simply set $x = 0$! So we get the value $y^2/(y^2 + 1)$, which is certainly in $\operatorname{Frac} k[y]$.
- We say that the germ *vanishes* at $\mathfrak{p}$ if the value is zero. In our example, the germ doesn't vanish at $\mathfrak{p}$.

If anything makes you nervous, you should make up an example to assuage your nervousness. (Example: $27/4$ is a regular function on $\operatorname{Spec} \mathbb{Z} - \{(2), (7)\}$. What is its value at (5)? Answer: $2/(-1) \equiv -2 \pmod 5$. What is its value at (0)? Answer: $27/4$. Where does it vanish? At (3).)

1.8. An extended example. I now want to work through an example with you, to show that this distinguished base is indeed something that you can work with. Let $R = k[x, y]$, so $\operatorname{Spec} R = \mathbb{A}^2_k$. If you want, you can let k be $\mathbb{C}$, but that won't be relevant. Let's work out the space of functions on the open set $U = \mathbb{A}^2 - (0, 0)$.

It is a non-obvious fact that you can't cut out this set with a single equation, so this isn't a distinguished open set. We'll see why fairly soon. But in any case, even if we're not sure if this is a distinguished open set, we can describe it as the union of two things which *are* distinguished open sets. If I throw out the x axis, i.e. the set $y = 0$, I get the distinguished open set $D(y)$. If I throw out the y axis, i.e. $x = 0$, I get the distinguished open set $D(x)$. So $U = D(x) \cup D(y)$. (Remark: $U = \mathbb{A}^2 - V(x, y)$ and $U = D(x) \cup D(y)$. Coincidence? I think not!) We will find the functions on U by gluing together functions on $D(x)$ and $D(y)$.

What are the functions on $D(x)$? They are, by definition, $R_x = k[x, y, 1/x]$. In other words, they are things like this: $3x^2 + xy + 3y/x + 14/x^4$. What are the functions on $D(y)$? They are, by definition, $R_y = k[x, y, 1/y]$. Note that $R \hookrightarrow R_x, R_y$. This is because x and y are not zero-divisors. (In fact, R is an integral domain — it has no zero-divisors, besides 0 — so localization is always an inclusion.) So we are looking for functions on $D(x)$ and $D(y)$ that agree on $D(x) \cap D(y) = D(xy)$, i.e. they are just the same function. Well, which things of this first form are also of the second form? Just old-fashioned polynomials —

$$\Gamma(U, \mathcal{O}_{\mathbb{A}^2}) \equiv k[x, y]. \tag{1}$$

In other words, we get no extra functions by throwing out this point. Notice how easy that was to calculate!

This is interesting: any function on $\mathbb{A}^2 - (0, 0)$ extends over all of $\mathbb{A}^2$. (Aside: This is an analog of Hartogs' theorem in complex geometry: you can extend a holomorphic function defined on the complement of a set of codimension at least two on a complex manifold over the missing set. This will work more generally in the algebraic setting: you can extend over points in codimension at least 2 not only if they are smooth, but also if they are mildly singular — what we will call *normal*.)

We can now see that this is not an affine scheme. Here's why: otherwise, if $(U, \mathcal{O}_{\mathbb{A}^2}|_U) = (\operatorname{Spec} S, \mathcal{O}_{\operatorname{Spec} S})$, then we can recover S by taking global sections:

$$S = \Gamma(U, \mathcal{O}_{\mathbb{A}^2}|_U),$$

which we have already identified in (1) as $k[x, y]$. So if U is affine, then $U = \mathbb{A}^2_k$. But we get more: we can recover the points of $\operatorname{Spec} S$ by taking the primes of S. In particular, the prime ideal (x, y) of S should cut out a point of $\operatorname{Spec} S$. But on U, $V(x) \cap V(y) = \emptyset$. Conclusion: U is *not* an affine scheme. (If you are ever looking for a counterexample to something, and you are expecting one involving a non-affine scheme, keep this example in mind!)

It is however a scheme.

Again, let me repeat the *definition of a* **scheme**. It is a topological space X, along with a sheaf of rings $\mathcal{O}_X$, such that any point $x \in X$ has a neighborhood U such that $(U, \mathcal{O}_X|_U)$ is an affine scheme (i.e. we have a homeomorphism of U with some $\operatorname{Spec} R$, say $f : U \to \operatorname{Spec} R$, and an isomorphism $\mathcal{O}_X|_U \cong \mathcal{O}_R$, where the two spaces are identified). The scheme can be denoted $(X, \mathcal{O}_X)$, although it is often denoted X, with the structure sheaf implicit.

I stated earlier in the notes, Exercise 1.1 (and at roughly at this point in the class): If we take the underlying subset of $D(f)$ with the restriction of the sheaf $\mathcal{O}_{\operatorname{Spec} R}$, we obtain the scheme $\operatorname{Spec} R_f$.

If X is a scheme, and U is *any* open subset, then $(U, \mathcal{O}_X|_U)$ is also a scheme. **Exercise.** Prove this. $(U, \mathcal{O}_X|_U)$ is called an *open subscheme of* U. If U is also an affine scheme, we often say U is an *affine open subset*, or an *affine open subscheme*, or sometimes informally just an *affine open*. For an example, $D(f)$ is an affine open subscheme of $\operatorname{Spec} R$.

1.9. *Exercise.* Show that if X is a scheme, then the affine open sets form a base for the Zariski topology. (Warning: they don't form a nice base, as we'll see in Exercise 1.11 below. However, in "most nice situations" this will be true, as we will later see, when we define the analogue of "Hausdorffness", called separatedness.)

You've already seen two examples of non-affine schemes: an infinite disjoint union of non-empty schemes, and $\mathbb{A}^2 - (0,0)$. I want to give you two more important examples. They are important because they are the first examples of fundamental behavior, the first pathological, and the second central.

First, I need to tell you how to glue two schemes together. Suppose you have two schemes $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$, and open subsets $U \subset X$ and $V \subset Y$, along with a homeomorphism $U \cong V$, and an isomorphism of structure sheaves $(U, \mathcal{O}_X|_U) \cong (V, \mathcal{O}_Y|_V)$. Then we can glue these together to get a single scheme. Reason: let W be X and Y glued together along the isomorphism $U \cong V$. Then problem 9 on the second problem set shows that the structure sheaves can be glued together to get a sheaf of rings. Note that this is indeed a scheme: any point has a neighborhood that is an affine scheme. (Do you see why?)

So I'll give you two non-affine schemes. In both cases, I will glue together two copies of the affine line $\mathbb{A}^1_k$. Again, if it makes you feel better, let $k = \mathbb{C}$, but it really doesn't matter; this is the last time I'll say this.

Let $X = \operatorname{Spec} k[t]$, and $Y = \operatorname{Spec} k[u]$. Let $U = D(t) = \operatorname{Spec} k[t, 1/t] \subset X$ and $V = D(u) = \operatorname{Spec} k[u, 1/u] \subset Y$.

We will get example 1 by gluing X and Y together along U and V. We will get example 2 by gluing X and Y together along U and V.

Example 1: the affine line with the doubled origin. Consider the isomorphism $U \cong V$ via the isomorphism $k[t, 1/t] \cong k[u, 1/u]$ given by $t \leftrightarrow u$. Let the resulting scheme be X. The picture looks like this [line with doubled origin]. This is called the *affine line with doubled origin*.

As the picture suggests, intuitively this is an analogue of a failure of Hausdorffness. $\mathbb{A}^1$ itself is not Hausdorff, so we can't say that it is a failure of Hausdorffness. We see this as weird and bad, so we'll want to make up some definition that will prevent this from happening. This will be the notion of *separatedness*. This will answer other of our prayers as well. For example, on a separated scheme, the "affine base of the Zariski topology" is nice — the intersection of two affine open sets will be affine.

1.10. *Exercise.* Show that X is not affine. Hint: calculate the ring of global sections, and look back at the argument for $\mathbb{A}^2 - (0,0)$.

1.11. *Exercise.* Do the same construction with $\mathbb{A}^1$ replaced by $\mathbb{A}^2$. You'll have defined the *affine plane with doubled origin*. Use this example to show that the affine base of the topology isn't a nice base, by describing two affine open sets whose intersection is not affine.

Example 2: the projective line. Consider the isomorphism $U \cong V$ via the isomorphism $k[t, 1/t] \cong k[u, 1/u]$ given by $t \leftrightarrow 1/u$. The picture looks like this [draw it]. Call the resulting scheme the *projective line over the field* k, $\mathbb{P}^1_k$.

Notice how the points glue. Let me assume that k is algebraically closed for convenience. (You can think about how this changes otherwise.) On the first affine line, we have the closed (= "old-fashioned") points $(t - a)$, which we think of as "a on the t-line", and we have the generic point. On the second affine line, we have closed points that are "b on the u-line", and the generic point. Then a on the t-line is glued to $1/a$ on the u-line (if $a \neq 0$ of course), and the generic point is glued to the generic point (the ideal (0) of $k[t]$ becomes the ideal (0) of $k[t, 1/t]$ upon localization, and the ideal (0) of $k[u]$ becomes the ideal (0) of $k[u, 1/u]$. And (0) in $k[t, 1/t]$ is (0) in $k[u, 1/u]$ under the isomorphism $t \leftrightarrow 1/u$).

We can interpret the closed ("old-fashioned") points of $\mathbb{P}^1$ in the following way, which may make this sound closer to the way you have seen projective space defined earlier. The points are of the form $[a; b]$, where a and b are not both zero, and $[a; b]$ is identified with $[ac; bc]$ where $c \in k^*$. Then if $b \neq 0$, this is identified with a/b on the t-line, and if $a \neq 0$, this is identified with b/a on the u-line.

1.12. *Exercise.* Show that $\mathbb{P}^1_k$ is not affine. Hint: calculate the ring of global sections.

This one I will do for you.

The global sections correspond to sections over X and sections over Y that agree on the overlap. A section on X is a polynomial $f(t)$. A section on Y is a polynomial $g(u)$. If I restrict $f(t)$ to the overlap, I get something I can still call $f(t)$; and ditto for $g(u)$. Now we want them to be equal: $f(t) = g(1/t)$. How many polynomials in t are at the same time polynomials in $1/t$? Not very many! Answer: only the constants k. Thus $\Gamma(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}) = k$. If $\mathbb{P}^1$ were affine, then it would be $\operatorname{Spec} \Gamma(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}) = \operatorname{Spec} k$, i.e. one point. But it isn't — it has lots of points.

Note that we have proved an analog of a theorem: the only holomorphic functions on $\mathbb{CP}^1$ are the constants!

1.13. *Important exercise.* Figure out how to define projective n-space $\mathbb{P}^n_k$. Glue together $n+1$ opens each isomorphic to $\mathbb{A}^n_k$. Show that the only global sections of the structure sheaf are the constants, and hence that $\mathbb{P}^n_k$ is not affine if $n > 0$. (Hint: you might fear that you will need some delicate interplay among all of your affine opens, but you will only need two of your opens to see this. There is even some geometric intuition behind this: the complement of the union of two opens has codimension 2. But "Hartogs' Theorem" says that any function defined on this union extends to be a function on all of projective space. Because we're expecting to see only constants as functions on all of projective space, we should already see this for this union of our two affine open sets.)

Exercise. The closed points of $\mathbb{P}^n_k$ (if k is algebraically closed) may be interpreted in the same way as we interpreted the points of $\mathbb{P}^1_k$. The points are of the form $[a_0; \ldots; a_n]$, where the a_i are not all zero, and $[a_0; \ldots; a_n]$ is identified with $[ca_0; \ldots; ca_n]$ where $c \in k^*$.

We will later give another definition of projective space. Your definition will be handy for computing things. But there is something unnatural about it — projective space is highly symmetric, and that isn't clear from your point of view.

Note that your definition will give a definition of $\mathbb{P}^n_R$ for any *ring* R. This will be useful later.

1.14. *Example.* Here is an example of a function on an open subset of a scheme that is a bit surprising. On $X = \operatorname{Spec} k[w, x, y, z]/(wx - yz)$, consider the open subset $D(y) \cup D(w)$. Show that the function x/y on $D(y)$ agrees with z/w on $D(w)$ on their overlap $D(y) \cap D(w)$. Hence they glue together to give a section. Justin points out that you may have seen this before when thinking about analytic continuation in complex geometry — we have a "holomorphic" function the description x/y on an open set, and this description breaks down elsewhere, but you can still "analytically continue" it by giving the function a different definition.

Follow-up for curious experts: This function has no "single description" as a well-defined expression in terms of w, x, y, z! there is lots of interesting geometry here. Here is a glimpse, in terms of words we have not yet defined. $\operatorname{Spec} k[w, x, y, z]$ is $\mathbb{A}^4$, and is, not surprisingly, 4-dimensional. We are looking at the set X, which is a hypersurface, and is 3-dimensional. It is a cone over a smooth quadric surface in $\mathbb{P}^3$ [show them hyperboloid of one sheet, and point out the two rulings]. $D(y)$ is X minus some hypersurface, so we are throwing away a codimension 1 locus. $D(z)$ involves throwing another codimension 1 locus. You might think that their intersection is then codimension 2, and that maybe failure of extending this weird function to a global polynomial comes because of a failure of our Hartogs'-type theorem, which will be a failure of normality. But that's not true — $V(y) \cap V(z)$ is in fact codimension 1 — so no Hartogs-type theorem holds. Here is what is actually going on. $V(y)$ involves throwing away the (cone over the) union of two lines l and m_1, one in each "ruling" of the surface, and $V(z)$ also involves throwing away the (cone over the) union of two lines l and m_2. The intersection is the (cone over the) line l, which is a codimension 1 set. Neat fact: despite being "pure codimension 1", it is not cut out even set-theoretically by a single equation. (It is hard to get an example of this behavior. This example is the simplest example I know.) This means that any expression $f(w, x, y, z)/g(w, x, y, z)$ for our function cannot correctly describe our function on $D(y) \cup D(z)$ — at some point of $D(y) \cup D(z)$ it must be $0/0$. Here's why. Our function can't be defined on $V(y) \cap V(z)$, so g must vanish here. But then g can't vanish just on the cone over l — it must vanish elsewhere too. (For the experts among the experts: here is why the cone over l is not cut out set-theoretically by a single equation. If $l = V(f)$, then $D(f)$ is affine. Let l' be another line in the same ruling as l, and let $C(l)$ (resp. l' be the cone over l (resp. l'). Then $C(l')$ can be given the structure of a closed subscheme of $\operatorname{Spec} k[w, x, y, z]$, and can be given the structure of $\mathbb{A}^2$. Then $C(l') \cap V(f)$ is a closed subscheme of $D(f)$. Any closed subscheme of an affine scheme is affine. But $l \cap l' = \emptyset$, so the cone over l intersects the cone over l' is a point, so $C(l') \cap V(f)$ is $\mathbb{A}^2$ minus a point, which we've seen is not affine, contradiction.)

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FOUNDATIONS OF ALGEBRAIC GEOMETRY CLASS 8

CONTENTS

Last day: $\tilde{M}$, sheaf associated to R-module M; Chinese remainder theorem; germs and value at a point of the structure sheaf; non-affine schemes $\mathbb{A}^2 - (0,0)$, line with doubled origin, $\mathbb{P}^n$.

Today: irreducible, connected, quasicompact, reduced, dimension.

1. PROPERTIES OF SCHEMES

We're now going to define properties of schemes. We'll start with some topological properties.

I've already defined what it means for a topological space to be *irreducible*: if X is the union of two closed subsets $U \cup V$, then either $X = U$ or $X = V$.

Problem A4 on problem set 3 implies that $\mathbb{A}^n_k$ is irreducible. There is a one-line answer. This argument "behaves well under gluing", yielding:

1.1. Exercise. Show that $\mathbb{P}^n_k$ is irreducible.

1.2. Exercise. You showed earlier that for affine schemes, there is a bijection between irreducible closed subsets and points. Show that this is true of schemes as well.

In the examples we have considered, the spaces have naturally broken up into some obvious pieces. Let's make that a bit more precise.

A topological space X is called *Noetherian* if it satisfies the descending chain condition for closed subsets: any sequence $Z_1 \supseteq Z_2 \supset \cdots$ of closed subsets eventually stabilizes: there is an r such that $Z_r = Z_{r+1} = \cdots$.

I showed some examples on $\mathbb{A}^2$, to show that it can take arbitrarily long to stabilize.

Date: Monday, October 24, 2005. Small correction January 31, 2007.

All of the cases we have considered have this property, but that isn't true of all rings. The key characteristic all of our examples have had in common is that the rings were *Noetherian*. Recall that a ring is *Noetherian* if *ascending* sequence $I_1 \subset I_2 \subset \cdots$ of closed ideals eventually stabilizes: there is an r such that $I_r = I_{r+1} = \cdots$.

Here are some quick facts about Noetherian rings. You should be able to prove them all.

- Fields are Noetherian. $\mathbb{Z}$ is Noetherian.
- If R is Noetherian, and I is any ideal, then R/I is Noetherian.
- If R is Noetherian, and S is any multiplicative set, then $S^{-1}R$ is Noetherian.
- In a Noetherian ring, any ideal is finitely generated. Any submodule of a finitely generated module over a Noetherian ring is finitely generated.

The next fact is non-trivial.

1.3. *The Hilbert basis theorem. — If R is Noetherian, then so is $R[x]$.*

Proof omitted. (This was done in Math 210.)

(I then discussed the game of Chomp. The fact that the game of infinite chomp is guaranteed to end is an analog of the Hilbert basis theorem. In fact, this is a consequence of the Hilbert basis theorem — the fact that infinite chomp is guaranteed to end corresponds to the fact that any ascending chain of monomial ideals in $k[x, y]$ must eventually stabilize. I learned of this cute fact from Rahul Pandharipande. If you prove the Chomp problem, you'll understand how to prove the Hilbert basis theorem.)

Using these results, then any polynomial ring over any field, or over the integers, is Noetherian — and also any quotient or localization thereof. Hence for example any finitely-generated algebra over k or $\mathbb{Z}$, or any localization thereof is Noetherian.

1.4. Exercise. Prove the following. If R is Noetherian, then $\operatorname{Spec} R$ is a Noetherian topological space. If X is a scheme that has a finite cover $X = \cup_{i=1}^{n} \operatorname{Spec} R_i$ where R_i is Noetherian, then X is a Noetherian topological space.

Thus $\mathbb{P}^n_k$ and $\mathbb{P}^n_{\mathbb{Z}}$ are Noetherian topological spaces: we built them by gluing together a finite number of Spec's of Noetherian rings.

If X is a topological space, and Z is an irreducible closed subset not contained in any larger irreducible closed subset, Z is said to be an *irreducible component* of X. (I drew a picture.)

1.5. Exercise. If R is any ring, show that the irreducible components of $\operatorname{Spec} R$ are in bijection with the minimal primes of R. (Here minimality is with respect to inclusion.)

For example, the only minimal prime of $k[x, y]$ is (0). What are the minimal primes of $k[x, y]/(xy)$?

1.6. *Proposition. — Suppose* X *is a Noetherian topological space. Then every non-empty closed subset* Z *can be expressed uniquely as a finite union* $Z = Z_1 \cup \cdots \cup Z_n$ *of irreducible closed subsets, none contained in any other.*

As a corollary, this implies that a Noetherian ring R has only finitely many minimal primes.

Proof. The following technique is often called *Noetherian induction*, for reasons that will become clear.

Consider the collection of closed subsets of X that *cannot* be expressed as a finite union of irreducible closed subsets. We will show that it is empty. Otherwise, let Y_1 be one such. If it properly contains another such, then choose one, and call it Y_2. If this one contains another such, then choose one, and call it Y_3, and so on. By the descending chain condition, this must eventually stop, and we must have some Y_r that cannot be written as a finite union of irreducible closed subsets, but every closed subset contained in it can be so written. But then Y_r is not itself irreducible, so we can write $Y_r = Y' \cup Y''$ where Y' and Y'' are both proper closed subsets. Both of these by hypothesis can be written as the union of a finite number of irreducible subsets, and hence so can Y_r, yielding a contradiction. Thus each closed subset can be written as a finite union of irreducible closed subsets. We can assume that none of these irreducible closed subsets contain any others, by discarding some of them.

We now show uniqueness. Suppose

$$Z = Z_1 \cup Z_2 \cup \cdots \cup Z_r = Z_1' \cup Z_2' \cup \cdots \cup Z_s'$$

are two such representations. Then $Z_1' \subset Z_1 \cup Z_2 \cup \cdots \cup Z_r$, so $Z_1' = (Z_1 \cap Z_1') \cup \cdots \cup (Z_r \cap Z_1')$. Now Z_1' is irreducible, so one of these is Z_1' itself, say (without loss of generality) $Z_1 \cap Z_1'$. Thus $Z_1' \subset Z_1$. Similarly, $Z_1 \subset Z_a'$ for some a; but because $Z_1' \subset Z_1 \subset Z_a'$, and Z_1' is contained in no other Z_i', we must have $a = 1$, and $Z_1' = Z_1$. Thus each element of the list of Z's is in the list of Z''s, and vice versa, so they must be the same list. □

1.7. Connectedness and quasicompactness.

Definition. A topological space X is connected if it cannot be written as the disjoint union of two non-empty open sets.

We say that a subset Y of X is a *connected component* if it is connected, and both open and closed. **Remark added later: Thanks to Anssi for pointing out that this is not the usual definition of connected component. The usual definition, which deals with more pathological situations, implies this one. At some point I might update these notes and say more.**

1.8. Exercise. Show that an irreducible topological space is connected.

1.9. Exercise. Give (with proof!) an example of a scheme that is connected but reducible.

We have already defined **quasicompact**.

1.10. Exercise. Show that a finite union of affine schemes is quasicompact. (Hence $\mathbb{P}^n_k$ is quasicompact.) Show that every closed subset of an affine scheme is quasicompact. Show that every closed subset of a quasicompact scheme is quasicompact.

The last topological property I should discuss is *dimension*. But that will take me some time, and it will involve some non-topological issues, so I'll first talk about an important non-topological property. Remember that one of the alarming things about schemes is that functions are not determined by their values at points, and that was because of the presence of *nilpotents*.

1.11. Definition. We will say that a ring is *reduced* if it has no nilpotents. A scheme is *reduced* if $\mathcal{O}_X(U)$ has no nonzero nilpotents for any open set U of X.

An example of a nonreduced affine scheme is $k[x,y]/(xy,x^2)$. Picture: y-axis with some fuzz at the origin (I drew this). The fuzz indicates that there is some nonreducedness going on at the origin. Here are two different functions: y and $x+y$. Their values agree at all points. They are actually the same function on the open set $D(y)$, which is not surprising, as $D(y)$ is reduced, as the next exercise shows.

1.12. Exercise. Show that $\left(k[x,y]/(xy,x^2)\right)_y$ has no nilpotents. (Hint: show that it is isomorphic to another ring, by considering the geometric picture.)

1.13. Exercise. Show that a scheme is reduced if and only if none of the stalks have nilpotents. Hence show that if f and g are two functions on a reduced scheme that agree at all points, then $f = g$.

Definition. A scheme is *integral* if $\mathcal{O}_X(U)$ is an integral domain for each open set U of X.

1.14. Exercise. Show that an affine scheme $\operatorname{Spec} R$ is integral if and only if R is an integral domain.

1.15. Exercise. Show that a scheme X is integral if and only if it is irreducible and reduced.

1.16. Exercise. Suppose X is an integral scheme. Then X (being irreducible) has a generic point η. Suppose $\operatorname{Spec} R$ is any non-empty affine open subset of X. Show that the stalk at η, $\mathcal{O}_{X,\eta}$, is naturally $\operatorname{Frac} R$. This is called the *function field* of X. It can be computed on any non-empty open set of X (as any such open set contains the generic point).

1.17. Exercise. Suppose X is an integral scheme. Show that the restriction maps $\operatorname{res}_{U,V} : \mathcal{O}_X(U) \to \mathcal{O}_X(V)$ are inclusions so long as $V \neq \emptyset$. Suppose $\operatorname{Spec} R$ is any non-empty affine open subset of X (so R is an integral domain). Show that the natural map $\mathcal{O}_X(U) \to \mathcal{O}_{X,\eta} = \operatorname{Frac} R$ (where U is any non-empty open set) is an inclusion.

2. Dimension

Our goal is to define the dimension of schemes. This should agree with, and generalize, our geometric intuition. (Careful: if you are thinking over the complex numbers, our dimensions will be complex dimensions, and hence half that of the "real" picture.) We will also use it to prove things; as a preliminary example, we will classify the prime ideals of $k[x, y]$.

It turns out that the right definition is purely topological — it just depends on the topological space, and not at all on the structure sheaf. Define **dimension** by Krull dimension: the supremum of lengths of chains of closed irreducible sets, starting indexing with 0. This dimension is allowed to be ∞. Define the Krull dimension of a ring to be the Krull dimension of its topological space. It is one less than the length of the longest chain of nested prime ideals you can find. (You might think a Noetherian ring has finite dimension, but this isn't necessarily true. For a counterexample by Nagata, who is the master of all counterexamples, see Eisenbud's *Commutative Algebra with a View to Algebraic Geometry*, p. 231.)

(Scholars of the empty set can take the dimension of the empty set to be $-\infty$.)

Obviously the Krull dimension of a ring R is the same as the Krull dimension of $R/\mathfrak{N}$: dimension doesn't care about nilpotents.

For example: We have identified the prime ideals of $k[t]$, so we can check that $\dim \mathbb{A}^1 = 1$. Similarly, $\dim \operatorname{Spec} \mathbb{Z} = 1$. Also, $\dim \operatorname{Spec} k = 0$, and $\dim \operatorname{Spec} k[x]/x^2 = 0$.

Caution: if Z is the union of two closed subsets X and Y, then $\dim_Z = \max(\dim X, \dim Y)$. In particular, if Z is the disjoint union of something of dimension 2 and something of dimension 1, then it has dimension 2. Thus dimension is not a "local" characteristic of a space. This sometimes bothers us, so we will often talk about dimensions of irreducible topological spaces. If a topological space can be expressed as a finite union of irreducible subsets, then say that it is *equidimensional* or *pure dimensional* (resp. equidimensional of dimension n or pure dimension n) if each of its components has the same dimension (resp. they are all of dimension n).

The notion of *codimension* of something equidimensional in something equidimensional makes good sense (as the difference of the two dimensions). Caution (added Nov. 6): there is another possible definition of codimension, in terms of *height*, defined later. Hartshorne uses this second definition. These two definitions can disagree — see e.g. the example of "height behaving badly" in the Class 9 notes. So we will be very cautious in using then word "codimension".

An equidimensional dimension 1 (resp. 2, n) topological space is said to be a *curve* (resp. surface, n-fold).

2.1. Reality check. Show that $\dim R/\mathfrak{p} \leq \dim R$, where $\mathfrak{p}$ is prime. Hope: equality holds if and only if $\mathfrak{p} = 0$ or $\dim R/\mathfrak{p} = \infty$. It is immediate that if R is a finite-dimensional domain, and $\mathfrak{p} \neq 0$, then we have inequality.

Warning: in all of the examples we have looked at, they behave well, but dimension can behave quite pathologically. But in good situations, including ones that come up more naturally in nature, it doesn't. For example, in cases involving a finite number variables over a field, dimension follows our intuition. More precisely:

2.2. *Big Theorem of today.* — Suppose R is a finitely-generated domain over a field k. Then $\dim \operatorname{Spec} R$ is the transcendence degree of the fraction field $\operatorname{Frac}(R)$ over k.

(By "finitely generated domain over k", I mean "a finitely generated k-algebra that is an integral domain". I'm just trying to save ink.)

Note that these finitely generated domains over k can each be described as the ring of functions on an irreducible subset of some $\mathbb{A}^n$: given such a domain, choose generators $x_1, \dots, x_n$. Conversely, if $\mathfrak{p} \subset k[x_1, \dots, x_n]$ is any prime ideal, then $\dim \operatorname{Spec} k[x_1, \dots, x_n]/\mathfrak{p}$ is the transcendence degree of $k[x_1, \dots, x_n]/\mathfrak{p}$ over k.

Before getting to the proof, I want to discuss some consequences.

2.3. *Corollary.* — $\dim \mathbb{A}^n_k = n$.

We can now confirm that we have named all the primes of $k[x, y]$ where k is algebraically closed. Recall that we have discovered the primes (0), $f(x, y)$ where f is irreducible, and $(x - a, y - b)$ where $a, b \in k$. By the Nullstellensatz, we have found all the closed points, so we have found all the irreducible subsets of dimension 0. As $\mathbb{A}^2_k$ is irreducible, there is only one irreducible subset of dimension 2. So it remains to show that all the irreducible subsets of dimension 1 are of the form $V(f(x, y))$, where f is an irreducible polynomial. Suppose $\mathfrak{p}$ is a prime ideal corresponding to an irreducible subset of dimension 1. Suppose $g \in \mathfrak{p}$ is non-zero. Factor g into irreducibles: $g_1 \cdots g_n \in \mathfrak{p}$. Then as $\mathfrak{p}$ is prime, one of the g_i's, say g_1, lies in $\mathfrak{p}$. Thus $(g_1) \subset \mathfrak{p}$. Now (g_1) is a prime ideal, and hence cuts out an irreducible subset, which contains $V(\mathfrak{p})$. It can't strictly contain $V(\mathfrak{p})$, as its dimension is no bigger than 1, and the dimension of $V(\mathfrak{p})$ is also 1. Thus $V((g_1)) = V(\mathfrak{p})$. But they are both prime ideals, and by the bijection between irreducible closed subsets and prime ideals, we have $\mathfrak{p} = (g_1)$.

Here are two more exercises added to the notes on November 5.

2.4. Exercise: Nullstellensatz from dimension theory. (a) Prove a microscopically stronger version of the weak Nullstellensatz: Suppose $R = k[x_1, \dots, x_n]/I$, where k is an algebraically closed field and I is some ideal. Then the maximal ideals are precisely those of the form $(x_1 - a_1, \dots, x_n - a_n)$, where $a_i \in k$.
(b) Suppose $R = k[x_1, \dots, x_n]/I$ where k is not necessarily algebraically closed. Show that every maximal ideal of R has a residue field that is a finite extension of k. [Hint for both: the maximal ideals correspond to dimension 0 points, which correspond to transcendence

degree 0 extensions of k, i.e. finite extensions of k. If k is algebraically closed, the maximal ideals correspond to surjections $f : k[x_1, \dots, x_n] \to k$. Fix one such surjection. Let $a_i = f(x_i)$, and show that the corresponding maximal ideal is $(x_1 - a_1, \dots, x_n - a_n)$.]

2.5. Important Exercise. Suppose X is an integral scheme, that can be covered by open subsets of the form $\operatorname{Spec} R$ where R is a finitely generated domain over k. Then $\dim X$ is the transcendence degree of the function field (the stalk at the generic point) $\mathcal{O}_{X,\eta}$ over k. Thus (as the generic point lies in all non-empty open sets) the dimension can be computed in any open set of X.

Here is an application that you might reasonably have wondered about before thinking about algebraic geometry. I don't think there is a simple proof, but maybe I'm wrong.

2.6. *Exercise.* Suppose $f(x, y)$ and $g(x, y)$ are two complex polynomials ($f, g \in \mathbb{C}[x, y]$). Suppose f and g have no common factors. Show that the system of equations $f(x, y) = g(x, y) = 0$ has a finite number of solutions.

Let's start to prove the big theorem! If R is a finitely generated domain over k, temporarily define $\dim_{tr} R = \dim_{tr} \operatorname{Spec} R$ to be the transcendence degree of $\operatorname{Frac}(R)$ over k. We wish to show that $\dim_{tr} R = \dim R$. After proving the big theorem, we will discard the temporary notation $\dim_{tr}$.

2.7. *Lemma. — Suppose R is an integral domain over k (not necessarily finitely generated, although that is the case we will care most about), and $\mathfrak{p} \subset R$ a prime. Then $\dim_{tr} R \geq \dim_{tr} R/\mathfrak{p}$, with equality if and only $\mathfrak{p} = (0)$, or $\dim_{tr} R/\mathfrak{p} = \infty$.*

You should have a picture in your mind when you hear this: if you have an irreducible space of finite dimension, then any proper subspace has strictly smaller dimension — certainly believable!

This implies that $\dim R \leq \dim_{tr} R$.

Proof. You can quickly check that if $\mathfrak{p} = (0)$ or $\dim_{tr} R/\mathfrak{p} = \infty$ then we have equality, so we'll assume that $\mathfrak{p} \neq (0)$, and $\dim_{tr} R/\mathfrak{p} = n < \infty$. Choose $x_1, \dots, x_n$ in R such that their residues $\overline{x}_1, \dots, \overline{x}_n$ are algebraically independent. Choose any $y \neq 0$ in $\mathfrak{p}$. Assume for the sake of contradiction that $\dim_{tr} R = n$. Then $y, x_1, \dots, x_n$ cannot be algebraically independent over k, so there is some irreducible polynomial $f(Y, X_1, \dots, X_n) \in k[Y, X_1, \dots, X_n]$ such that $f(y, x_1, \dots, x_n) = 0$ (in R). This irreducible f is not (a multiple of) Y, as otherwise $f(y, x_1, \dots, x_n) = y \neq 0$ in R. Hence f contains monomials that are not multiples of Y, so $F(X_1, \dots, X_n) := f(0, X_1, \dots, X_n) \in k[X_1, \dots, X_n]$ is non-zero. Reducing $f(y, x_1, \dots, x_n) = 0$ modulo $\mathfrak{p}$ gives us

$$F(\overline{x}_1, \dots, \overline{x}_n) = f(0, \overline{x}_1, \dots, \overline{x}_n) = 0 \qquad \text{in } R/\mathfrak{p}$$

contradicting the algebraic independence of $\overline{x}_1, \dots, \overline{x}_n$. □

At the end of the class, I stated the following, which will play off of our lemma to prove the big theorem.

2.8. *Krull's principal ideal theorem (transcendence degree version). — Suppose* R *is a finitely generated domain over* k*,* $f \in R$*,* $\mathfrak{p}$ *a minimal prime of* R/f*. Then if* $f \neq 0$*,* $\dim_{tr} R/\mathfrak{p} = \dim_{tr} R - 1$*.*

This is best understood geometrically:

2.9. *Theorem (geometric interpretation of Krull). — Suppose* $X = \operatorname{Spec} R$ *where* R *is a finitely generated domain over* k*,* $g \in R$*,* Z *an irreducible component of* $V(g)$*. Then if* $g \neq 0$*,* $\dim_{tr} Z = \dim_{tr} X - 1$*.*

In other words, if you have some irreducible space of finite dimension, then any non-zero function on it cuts out a set of *pure* codimension 1.

We'll see how these two geometric statements will quickly combine to prove our big theorem.

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FOUNDATIONS OF ALGEBRAIC GEOMETRY CLASS 9

CONTENTS

Last day: irreducible, connected, quasicompact, reduced, dimension.

Today: Krull's Principal Ideal Theorem, height, affine communication lemma, properties of schemes: locally Noetherian, Noetherian, finite type S-scheme, locally of finite type S-scheme, normal.

I realize now that you may not have seen the notion of transcendence degree. I'll tell you the main thing you need to know about it, which I hope you will find believable. Suppose K/k is a finitely generated field extension. Then any two maximal sets of algebraically independent elements of K over k (i.e. any set with no algebraic relation) have the same size (a non-negative integer or ∞). If this size is finite, say n, and $x_1, \ldots, x_n$ is such a set, then $K/k(x_1, \ldots, x_n)$ is necessarily a finitely generated algebraic extension, i.e. a finite extension. (Such a set $x_1, \ldots, x_n$ is called a transcendence basis, and n is called the *transcendence degree*.) A short and well-written proof of this fact is in *Mumford's Red Book of Varieties and Schemes*.

1. DIMENSION, CONTINUED

Last day, I defined the dimension of a scheme. I defined the dimension (or Krull dimension) as the supremum of lengths of chains of closed irreducible sets, starting indexing with 0. This dimension is allowed to be ∞. For example: a Noetherian topological space has a finite dimension. The Krull dimension of a ring is the Krull dimension of its topological space. It is one less than the length of the longest chain of nested prime ideals you can find.

We are in the midst of proving the following result, which lets us understand dimension when working in good situations.

1.1. *Big Theorem of last day. — Suppose* R *is a finitely-generated domain over a field* k*. Then* $\dim \operatorname{Spec} R$ *is the transcendence degree of the fraction field* $\operatorname{Frac}(R)$ *over* k*.*

Date: Wednesday, October 26, 2005. Small update January 31, 2007.

1.2. Exercise. Suppose X is an integral scheme, that can be covered by open subsets of the form $\operatorname{Spec} R$ where R is a finitely generated domain over k. Then $\dim X$ is the transcendence degree of the function field (the stalk at the generic point) $\mathcal{O}_{X,\eta}$ over k. Thus (as the generic point lies in all non-empty open sets) the dimension can be computed in any open set of X.

The proof of the big theorem will rely on two different facts pulling in opposite directions. The first is the following lemma, which we proved.

1.3. *Lemma. — Suppose R is an integral domain over k (not necessarily finitely generated, although that is the case we will care most about), and $\mathfrak{p} \subset R$ a prime. Then $\dim_{tr} R \geq \dim_{tr} R/P$, with equality if and only $\mathfrak{p} = (0)$, or $\dim_{tr} R/P = \infty$.*

You should have a picture in your mind when you hear this: if you have an irreducible space of finite dimension, then any proper subspace has strictly smaller dimension — certainly believable!

This implies that $\boxed{\dim R \leq \dim_{tr} R}$. (Think this through!)

The other fact we'll use is Krull's Principal Ideal Theorem. This result is one of the few hard facts I'll not prove. We may prove it later in the class (possibly in a problem set), and you can also read a proof in Mumford's *Red Book*, in §I.7, where you'll find much of this exposition.

1.4. *Krull's Principal Ideal Theorem (transcendence degree version). — Suppose R is a finitely generated domain over k, $f \in R$, $\mathfrak{p}$ a minimal prime of R/f. Then if $f \neq 0$, $\dim_{tr} R/\mathfrak{p} = \dim_{tr} R - 1$.*

This is best understood geometrically: if you have some irreducible space of finite dimension, then any non-zero function on it cuts out a set of *pure* codimension 1. Somewhat more precisely:

1.5. *Theorem (geometric interpretation of Krull). — Suppose $X = \operatorname{Spec} R$ where R is a finitely generated domain over k, $g \in R$, Z an irreducible component of $V(g)$. Then if $g \neq 0$, $\dim_{tr} Z = \dim_{tr} X - 1$.*

Before I get to the proof of the theorem, I want to point out that this is useful on its own. Consider the scheme $\operatorname{Spec} k[w, x, y, z]/(wx - yz)$. What is its dimension? It is cut out by one non-zero equation $wx - yz$ in $\mathbb{A}^4$, so it is a threefold.

1.6. Exercise. What is the dimension of $\operatorname{Spec} k[w, x, y, z]/(wx - yz, x^{17} + y^{17})$? (Be careful to check they hypotheses before invoking Krull!)

1.7. Exercise. Show that $\operatorname{Spec} k[w, x, y, z]/(wz - xy, wy - x^2, xz - y^2)$ is an integral *surface*. You might expect it to be a curve, because it is cut out by three equations in 4-space. (Remark for experts: this is not a random ideal. In language we will later make precise: it is the affine cone over a curve in $\mathbb{P}^3$. This curve is called the *twisted cubic*. It is in some sense the simplest curve in $\mathbb{P}^3$ not contained in a hyperplane. You can think of it as the points of the form (t, t^2, t^3) in $\mathbb{A}^3$. Indeed, you'll notice that $(w, x, y, z) = (a, at, at^2, at^3)$ satisfies the equations above. It turns out that you actually need three equations to cut out this surface. The first equation cuts out a threefold in four-space (by Krull's theorem, see later). The second equation cuts out a surface: our surface, and another surface. The third equation cuts out our surface. One last aside: notice once again that the cone over the quadric surface $k[w, x, y, z]/(wz - xy)$ makes an appearance.)

We'll now put together our lemma, and this geometric interpretation of Krull. Notice the interplay between the two: the first says that the dimension definitely drops when you take a proper irreducible closed subset. The second says that you can arrange for it to drop by precisely 1.

I proved the following result, which I didn't end up using.

1.8. *Proposition. — Suppose* X *is the* Spec *of a finitely generated domain over* k*, and* Z *is an irreducible closed subset, maximal among all proper irreducible closed subsets of* X*. (I gave a picture here.) Then* $\dim_{tr} Z = \dim_{tr} X - 1$.

(We certainly have $\dim_{tr} Z \leq \dim_{tr} X - 1$ by our lemma.)

Proof. Suppose $Z = V(\mathfrak{p})$ where $\mathfrak{p}$ is prime. Choose any non-zero $g \in \mathfrak{p}$. By Krull's theorem, the components of $V(g)$ are have $\dim_{tr} = \dim_{tr} X - 1$. Z is contained in one of the components. By the maximality of Z, Z *is* one of the components. $\square$

1.9. *Proof of big theorem.* We prove it by induction on $\dim_{tr} X$. The base case $\dim_{tr} X = 0$ is easy: by our lemma, $\dim X \leq \dim_{tr} X$, so $\dim X = 0$.

Now assume that $\dim_{tr} X = n$. As $\dim X \leq \dim_{tr} X$, our goal will be to produce a chain of $n + 1$ irreducible closed subsets. Say $X = \operatorname{Spec} R$. Choose any $g \neq 0$ in R. Choose any component Z of $V(g)$. Then $\dim_{tr} Z = n - 1$ by Krull's theorem, and the inductive hypothesis, so we can find a chain of n irreducible closed subsets descending from Z. We're done. $\square$

I gave a geometric picture of both. Note that equality needn't hold in the first case.

The big theorem is about the dimension of finitely generated domains over k. For such rings, dimension is well-behaved. This set of rings behaves well under quotients; I want to show you that it behaves well under localization as well.

1.10. *Proposition. — Suppose* R *is a finitely generated domain over* k*, and* $\mathfrak{p}$ *is a prime ideal. Then* $\dim R_{\mathfrak{p}} = \dim R - \dim R/\mathfrak{p}$.

The scheme-theoretic version of this statement about rings is: $\dim_{\mathcal{O}_{Z,X}} = \dim X - \dim Z$.

1.11. Exercise. Prove this. (I gave a geometric explanation of why this is true, which you can take as a "hint" for this exercise.) In the course of this exercise, you will show the important fact that if $n = \dim R$, then any chain of prime ideals can be extended to a chain of prime ideals of length n. Further, given a prime ideal, you can tell where it is in any chain by looking at the transcendence degree of its quotient field. This is a particularly nice feature of polynomial rings, that will not hold even for Noetherian rings in general (see the next section).

2. Height, and Krull's Principal Ideal Theorem

This is a good excuse to tell you a definition in algebra. Definition: the *height* of the prime ideal $\mathfrak{p}$ in R is $\dim R_{\mathfrak{p}}$. Algebraic translation: it is the supremum of lengths of chains of primes contained in $\mathfrak{p}$.

This is a good but imperfect version of codimension. For finitely generated domains over k, the two notions agree, by Proposition 1.10. An example of a pathology is given below.

With this definition of height, I can state a more general version of Krull's Principal Ideal Theorem.

2.1. *Krull's Principal Ideal Theorem. — Suppose R is a Noetherian ring, and $f \in A$ an element which is not a zero divisor. Then every minimal prime $\mathfrak{p}$ containing f has height 1.* (Atiyah-Macdonald p. 122)

(We could have $V(f) = \emptyset$, if f is a unit — but that doesn't violate the statement.)

The geometric picture is the same as before: "If f is not a zero-divisor, the codimension is 1."

It is possible that I will give a proof later in the course. Either I'll give an algebraic proof in the notes, or I will give a geometric proof in class, using concepts we have not yet developed. (I'll be careful to make sure the argument is not circular!)

2.2. Important Exercise. (This will be useful soon.) (a) Suppose $X = \operatorname{Spec} R$ where R is a Noetherian domain, and Z is an irreducible component of $V(r_1, \dots, r_n)$, where $r_1, \dots, r_n \in R$. Show that the height of (the prime associated to) Z is at most n. Conversely, suppose $X = \operatorname{Spec} R$ where R is a Noetherian domain, and Z is an irreducible subset of height n. Show that there are $f_1, \dots, f_n \in R$ such that Z is an irreducible component of $V(f_1, \dots, f_n)$.
(b) (application to finitely generated k-algebras) Suppose $X = \operatorname{Spec} R$ where R is a finitely generated domain over k, and Z is an irreducible component of $V(r_1, \dots, r_n)$, where $r_1, \dots, r_n \in R$. Show that $\dim Z \geq \dim X - n$. Conversely, suppose $X = \operatorname{Spec} R$ where R is a Noetherian domain, and Z is an irreducible subset of codimension n. Show that there are $f_1, \dots, f_n \in R$ such that Z is an irreducible component of $V(f_1, \dots, f_r)$.

2.3. Important but straightforward exercise. If R is a finitely generated domain over k, show that $\dim R[x] = \dim R + 1$. If R is a Noetherian ring, show that $\dim R[x] \geq \dim R + 1$. (Fact, proved later: if R is a Noetherian ring, then $\dim R[x] = \dim R + 1$. We'll prove this later. You may use this fact in exercises in later weeks.)

We now show how the height can behave badly. Let $R = k[x]_{(x)}[t]$. In other words, elements of R are polynomials in t, whose coefficients are quotients of polynomials in x, where no factors of x appear in the denominator. R is a domain. $(xt-1)$ is not a zero divisor. You can verify that $R/(xt-1) \cong k[x]_{(x)}[1/x] \cong k(x)$ — "in $k[x]_{(x)}$, we may divide by everything but x, and now we are allowed to divide by x as well" — so $R/(xt-1)$ is a field. Thus $(xt-1)$ is not just prime but also maximal. By Krull's theorem, $(xt-1)$ is height 1. Thus $(0) \subset (xt-1)$ is a maximal chain. However, R has dimension at least 2: $(0) \subset (t) \subset (x,t)$ is a chain of primes of length 3. (In fact, R has dimension precisely 2: $k[x]_{(x)}$ has dimension 1, and the fact mentioned in the previous exercise 2.3 implies $\dim k[x]_{(x)}[t] = \dim k[x]_{(x)} + 1 = 2$.) Thus we have a height 1 prime in a dimension 2 ring that is "codimension 2". A picture of this lattice of ideals is below.

$$\begin{tikzcd}[column sep=small]
(x,t) \arrow[d, dash] & & \\
(t) \arrow[dr, dash] & & (xt-1) \arrow[dl, dash] \\
& (0) &
\end{tikzcd}$$

(This example comes from geometry; it is enlightening to draw a picture. $k[x]_{(x)}$ corresponds to a germ of $\mathbb{A}^1_k$ near the origin, and $k[x]_{(x)}[t]$ corresponds to "this $\times$ the affine line".) For this reason, codimension is a badly behaved notion in Noetherian rings in general.

I find it disturbing that this misbehavior turns up even in a relative benign-looking ring.

3. Properties of schemes that can be checked "affine-locally"

Now I want to describe a host of important properties of schemes. All of these are "affine-local" in that they can be checked on any affine cover, by which I mean a covering by open affine sets.

Before I get going, I want to point out something annoying in the definition of schemes. I've said that a scheme is a topological space with a sheaf of rings, that can be covered by affine schemes. There is something annoying about this description that I find hard to express. We have all these affine opens in the cover, but we don't know how to communicate between any two of them. Put a different way, if I have an affine cover, and you have an affine cover, and we want to compare them, and I calculate something on my cover, there should be some way of us getting together, and figuring out how to translate my calculation over onto your cover. (I'm not sure if you buy what I'm trying to sell here.) The affine communication lemma I'll soon describe will do this for us.

3.1. *Remark.* In our limited examples so far, any time we've had an affine open subset of an affine scheme $\operatorname{Spec} S \subset \operatorname{Spec} R$, in fact $\operatorname{Spec} S = D(f)$ for some f. But this is not always true, and we will eventually have an example. (We'll first need to define elliptic curves!)

3.2. *Proposition. — Suppose* $\operatorname{Spec} A$ *and* $\operatorname{Spec} B$ *are affine open subschemes of a scheme* X*. Then* $\operatorname{Spec} A \cap \operatorname{Spec} B$ *is the union of open sets that are simultaneously distinguished open subschemes of* $\operatorname{Spec} A$ *and* $\operatorname{Spec} B$*.*

Proof. (This is best seen with a picture, which unfortunately won't be in the notes.) Given any $\mathfrak{p} \in \operatorname{Spec} A \cap \operatorname{Spec} B$, we produce an open neighborhood of $\mathfrak{p}$ in $\operatorname{Spec} A \cap \operatorname{Spec} B$ that is simultaneously distinguished in both $\operatorname{Spec} A$ and $\operatorname{Spec} B$. Let $\operatorname{Spec} A_f$ be a distinguished open subset of $\operatorname{Spec} A$ contained in $\operatorname{Spec} A \cap \operatorname{Spec} B$. Let $\operatorname{Spec} B_g$ be a distinguished open subset of $\operatorname{Spec} B$ contained in $\operatorname{Spec} A_f$. Then $g \in \Gamma(\operatorname{Spec} B, \mathcal{O}_X)$ restricts to an element $g' \in \Gamma(\operatorname{Spec} A_f, \mathcal{O}_X) = A_f$. The points of $\operatorname{Spec} A_f$ where g vanishes are precisely the points of $\operatorname{Spec} A_f$ where g' vanishes (cf. earlier exercise), so

$$\begin{aligned} \operatorname{Spec} B_g &= \operatorname{Spec} A_f \setminus \{\mathfrak{p} : g' \in \mathfrak{p}\} \\ &= \operatorname{Spec} (A_f)_{g'}. \end{aligned}$$

If $g' = g''/f^n$ ($g'' \in A$) then $\operatorname{Spec}(A_f)_{g'} = \operatorname{Spec} A_{fg''}$, and we are done. □

3.3. *Affine communication lemma. — Let* P *be some property enjoyed by some affine open sets of a scheme* X*, such that*

(i) *if* $\operatorname{Spec} R \hookrightarrow X$ *has* P *then for any* $f \in R$, $\operatorname{Spec} R_f \hookrightarrow X$ *does too.*
(ii) *if* $(f_1, \ldots, f_n) = R$, *and* $\operatorname{Spec} R_{f_i} \hookrightarrow X$ *has* P *for all* i, *then so does* $\operatorname{Spec} R \hookrightarrow X$.

Suppose that $X = \cup_{i \in I} \operatorname{Spec} R_i$ *where* $\operatorname{Spec} R_i$ *is an affine, and* R_i *has property* P*. Then every other open affine subscheme of* X *has property* P *too.*

Proof. (This is best done with a picture.) Cover $\operatorname{Spec} R$ with a finite number of distinguished opens $\operatorname{Spec} R_{g_j}$, each of which is distinguished in some R_{f_i}. This is possible by Proposition 3.2 and the quasicompactness of $\operatorname{Spec} R$. By (i), each $\operatorname{Spec} R_{g_j}$ has P. By (ii), $\operatorname{Spec} R$ has P. □

By choosing P appropriately, we define some important properties of schemes.

3.4. *Proposition. — Suppose* R *is a ring, and* $(f_1, \ldots, f_n) = R$.

(a) *If* R *is a Noetherian ring, then so is* R_{f_i}*. If each* R_{f_i} *is Noetherian, then so is* R*.*
(b) *If* R *has no nonzero nilpotents (i.e. 0 is a radical ideal), then* R_{f_i} *also has no nonzero nilpotents. If no* R_{f_i} *has a nonzero nilpotent, then neither does* R*.* **Do we say "a ring is reduced? radical?"**

(c) *Suppose A is a ring, and R is an A-algebra. If R is a finitely generated A-algebra, then so is R_{f_i}. If each R_{f_i} is a finitely-generated A-algebra, then so is R. (I didn't say this in class, so I'll say it on Monday.)*

(d) *Suppose R is an integral domain. If R is integrally closed, then so is R_{f_i}. If each R_{f_i} is integrally closed, then so is R.*

We'll prove these shortly. But given this, I want to make some definitions.

3.5. Important Definitions. Suppose X is a scheme.

- If X can be covered by affine opens $\operatorname{Spec} R$ where R is Noetherian, we say that X is a *locally Noetherian scheme*. If in addition X is quasicompact, or equivalently can be covered by finitely many such affine opens, we say that X is a *Noetherian scheme* **Exercise.** Show that the underlying topological space of a Noetherian scheme is Noetherian. **Exercise.** Show that all open subsets of a Noetherian scheme are quasicompact.
- If X can be covered by affine opens $\operatorname{Spec} R$ where R is reduced (nilpotent-free), we say that X is *reduced*. **Exercise:** Check that this agrees with our earlier definition. This definition is advantageous: our earlier definition required us to check that the ring of functions over *any* open set is nilpotent free. This lets us check in an affine cover. Hence for example $\mathbb{A}_k^n$ and $\mathbb{P}_k^n$ are reduced.
- Suppose A is a ring (e.g. A is a field k), and $\Gamma(X, \mathcal{O}_X)$ is an A-algebra. Then we say that X is an A-*scheme*, or a *scheme over* A. Suppose X is an A-scheme. (Then for any non-empty U, $\Gamma(U, \mathcal{O}_X)$ is naturally an A-algebra.) If X can be covered by affine opens $\operatorname{Spec} R$ where R is a *finitely generated* A-algebra, we say that X is *locally of finite type over* A, or that it is a *locally of finite type* A-*scheme*. (My apologies for this cumbersome terminology; it will make more sense later.) If furthermore X is quasicompact, X is *finite type over* A, or a *finite type* A-*scheme*.
- If X is integral, and can be covered by affine opens $\operatorname{Spec} R$ where R is a integrally closed, we say that X is *normal*. (Thus in my definition, normality can only apply to integral schemes. I may want to patch this later.) **Exercise.** If R is a unique factorization domain, show that $\operatorname{Spec} R$ is integrally closed. Hence $\mathbb{A}_k^n$ and $\mathbb{P}_k^n$ are both normal.

Proof. (a) (i) If $I_1 \subsetneq I_2 \subsetneq I_3 \subsetneq \cdots$ is a strictly increasing chain of ideals of R_f, then we can verify that $J_1 \subsetneq J_2 \subsetneq J_3 \subsetneq \cdots$ is a strictly increasing chain of ideals of R, where

$$J_j = \{r \in R : r \in I_j\}$$

where $r \in I_j$ means "the image in R_f lies in I_j". (We think of this as $I_j \cap R$, except in general R needn't inject into R_{f_i}.) Clearly J_j is an ideal of R. If $x/f^n \in I_{j+1} \setminus I_j$ where $x \in R$, then $x \in J_{j+1}$, and $x \notin J_j$ (or else $x(1/f)^n \in J_j$ as well). (ii) Suppose $I_1 \subsetneq I_2 \subsetneq I_3 \subset \cdots$ is a strictly increasing chain of ideals of R. Then for each $1 \leq i \leq n$,

$$I_{i,1} \subset I_{i,2} \subset I_{i,3} \subset \cdots$$

is an increasing chain of ideals in R_{f_i}, where $I_{i,j} = I_j \otimes_R R_{f_i}$. We will show that for each j, $I_{i,j} \subsetneq I_{i,j+1}$ for some i; the result will then follow.

(b) **Exercise.**

(c) (I'll present this on Monday.) (i) is clear: if R is generated over S by $r_1, \dots, r_n$, then R_f is generated over S by $r_1, \dots, r_n, 1/f$.

(ii) Here is the idea; I'll leave this as an **exercise** for you to make this work. We have generators of R_i: r_{ij}/f_i^j, where $r_{ij} \in R$. I claim that $\{r_{ij}\}_{ij} \cup \{f_i\}_j$ generate R as a S-algebra. Here's why. Suppose you have any $r \in R$. Then in R_{f_i}, we can write r as some polynomial in the r_{ij}'s and f_i, divided by some huge power of f_i. So "in each R_{f_i}, we have described r in the desired way", except for this annoying denominator. Now use a partition of unity type argument to combine all of these into a single expression, killing the denominator. Show that the resulting expression you build still agrees with r in each of the R_{f_i}. Thus it is indeed r.

(d) (i) is easy. If $x^n + a_{n-1}x^{n-1} + \cdots + a_0 = 0$ where $a_i \in R_f$ has a root in the fraction field. Then we can easily show that the root lies in R_f, by multiplying by enough f's to kill the denominator, then replacing $f^a x$ by y. That is likely incomprehensible, so I'll leave this as an **exercise**.

(ii) (This one involves a neat construction.) Suppose R is not integrally closed. We show that there is some f_i such that R_{f_i} is also not integrally closed. Suppose

$$x^n + a_{n-1}x^{n-1} + \cdots + a_0 = 0 \tag{1}$$

(with $a_i \in R$) has a solution s in $\mathrm{Frac}(R)$. Let I be the "ideal of denominators" of s:

$$I := \{r \in R : rs \in R\}.$$

(Note that I is clearly an ideal of R.) Now $I \neq R$, as $1 \notin I$. As $(f_1, \dots, f_n) = R$, there must be some $f_i \notin I$. Then $s \notin R_{f_i}$, so equation (1) in $R_{f_i}[x]$ shows that R_{f_i} is not integrally closed as well, as desired. $\square$

3.6. *Unimportant Exercise relating to the proof of (d).* One might naively hope from experience with unique factorization domains that the ideal of denominators is principal. This is not true. As a counterexample, consider our new friend $R = k[a,b,c,d]/(ad-bc)$, and $a/c = b/d \in \mathrm{Frac}(R)$. Then it turns out that $I = (c,d)$, which is not principal. We'll likely show that it is not principal at the start of the second quarter. (I could give a one-line explanation right now, but this topic makes the most sense when we talk about Zariski tangent spaces.)

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FOUNDATIONS OF ALGEBRAIC GEOMETRY CLASS 10

CONTENTS

Last day: Krull's Principal Ideal Theorem, height, affine communication lemma, properties of schemes: locally Noetherian, Noetherian, finite type S-scheme, locally of finite type S-scheme, normal

Today: finite type A-scheme, locally of finite type A-scheme, projective schemes over A or k.

Problem set 4 is out today (on the web), and problem set 3 is due today. As always, feedback is most welcome. How are the problem sets pitched? I don't want to make them too grueling, but I'd like to give you enough so that you can get a grip on the concepts. I've noticed that some of you are going after the hardest questions, and others are trying easier questions, and that's fine with me.

There is a notion that I have been using implicitly, and I should have made it explicit by now. It's the notion of what I mean by when two schemes are the *isomorphic*. An *isomorphism* of two schemes $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ is the following data: (i) it is a homeomorphism between X and Y (the identification of the *sets* and *topologies*). Then we can think of $\mathcal{O}_X$ and $\mathcal{O}_Y$ are sheaves (of rings) on the same space, via this homeomorphism. (ii) It is the data of an isomorphism of sheaves $\mathcal{O}_X \leftrightarrow \mathcal{O}_Y$.

Last day, I introduced the affine communication lemma. this lemma will come up repeatedly in the future.

0.1. *Affine communication theorem. — Let P be some property enjoyed by some affine open sets of a scheme X, such that*

(i) *if* $\operatorname{Spec} R \hookrightarrow X$ *has* P *then for any* $f \in R$, $\operatorname{Spec} R_f \hookrightarrow X$ *does too.*
(ii) *if* $(f_1, \dots, f_n) = R$, *and* $\operatorname{Spec} R_{f_i} \hookrightarrow X$ *has* P *for all* i, *then so does* $\operatorname{Spec} R \hookrightarrow X$.

Suppose that $X = \cup_{i \in I} \operatorname{Spec} R_i$ *where* $\operatorname{Spec} R_i$ *is an affine, and* R_i *has property* P. *Then every other open affine subscheme of* X *has property* P *too.*

Date: Monday, October 31 (Hallowe'en), 2005. Small update January 31, 2007.

By choosing P appropriately, we define some important properties of schemes. I gave several examples. Here is one last example.

0.2. *Proposition. — Suppose R is a ring, and $(f_1, \ldots, f_n) = R$. Suppose A is a ring, and R is an A-algebra. (i) If R is a finitely generated A-algebra, then so is R_{f_i}. (ii) If each R_{f_i} is a finitely-generated A-algebra, then so is R.*

This of course leads to a corresponding definition.

0.3. Important Definition. Suppose X is a scheme, and A is a ring (e.g. A is a field k), and $\Gamma(X, \mathcal{O}_X)$ is an A-algebra. Note that $\Gamma(U, \mathcal{O}_X)$ is an A-algebra for all non-empty U. Then we say that X is an *A-scheme*, or a *scheme over A*. Suppose X is an A-scheme. If X can be covered by affine opens $\operatorname{Spec} R$ where R is a *finitely generated* A-algebra, we say that X is *locally of finite type over A*, or that it is a *locally of finite type A-scheme*. (My apologies for this cumbersome terminology; it will make more sense later.) If furthermore X is quasicompact, X is *finite type over A*, or a *finite type A-scheme*.

Proof of Proposition 0.2. (i) is clear: if R is generated over A by $r_1, \ldots, r_n$, then R_f is generated over A by $r_1, \ldots, r_n, 1/f$.

(ii) Here is the idea; I'll leave this as an **exercise** for you to make this work. We have generators of R_{f_i}: r_{ij}/f_i^j, where $r_{ij} \in R$. I claim that $\{r_{ij}\}_{ij} \cup \{f_i\}_i$ generate R as a A-algebra. Here's why. Suppose you have any $r \in R$. Then in R_{f_i}, we can write r as some polynomial in the r_{ij}'s and f_i, divided by some huge power of f_i. So "in each R_{f_i}, we have described r in the desired way", except for this annoying denominator. Now use a partition of unity type argument to combine all of these into a single expression, killing the denominator. Show that the resulting expression you build still agrees with r in each of the R_{f_i}. Thus it is indeed r. □

1. Projective k-schemes and A-schemes: a concrete example

I now want to tell you about an important class of schemes.

Our building blocks of schemes are affine schemes. For example, affine finite type k-schemes correspond to finitely generated k-algebras. Once you pick generators of the algebra, say $x_1, \ldots, x_n$, then you can think of the scheme as sitting in n-space. More precisely, suppose R is a finitely-generated k-algebra, say

$$R = k[x_1, \ldots, x_n]/I.$$

Then at least as a topological space, it is a closed subset of $\mathbb{A}^n$, with set $V(I)$. (We will later be able to say that it is a *closed subscheme*, but we haven't yet defined this phrase.)

Different choices of generators give us different ways of seeing $\operatorname{Spec} R$ as sitting in some affine space. These affine schemes already are very interesting. But when you glue them together, you can get even more interesting things. I'll now tell you about projective schemes.

As a warm-up, let me discuss $\mathbb{P}^n_k$ again.

Intuitive idea: We think of closed points of $\mathbb{P}^n$ as $[x_0; x_1; \cdots; x_n]$, not all zero, with an equivalence relation $[x_0; \cdots; x_n] = [\lambda x_0; \cdots; \lambda x_n]$. $x_0^2 + x_2^2$ isn't a function on $\mathbb{P}^n$. But $x_0^2 + x_2^2 = 0$ makes sense. And $(x_0^2 + x_2^2)/(x_1^2 + x_2 x_3)$ *is* a function on $\mathbb{P}^2 - \{x_1^2 + x_2 x_3 = 0\}$. We have $n+1$ patches, corresponding to $x_i = 0$ $(0 \leq i \leq n)$. Where $x_0 \neq 0$, we have a patch $[x_0; x_1; x_2] = [1; u_1; u_2]$, and similarly for $x_1 \neq 0$ and $x_2 \neq 0$.

More precisely: We defined $\mathbb{P}^n$ by gluing together $n+1$ copies of $\mathbb{A}^n$. Let me show you this in the case of $\mathbb{P}^2_k =$ "$\{[x_0; x_1; x_2]\}$". Let's pick co-ordinates wisely. The first patch is $U_0 = \{x_0 \neq 0\}$. We imagine $[x_0; x_1; x_2] = [1; x_{1/0}; x_{2/0}]$. The patch will have coordinates $x_{1/0}$ and $x_{2/0}$, i.e. it is $\operatorname{Spec} k[x_{1/0}, x_{2/0}]$.

Similarly, the second patch is $U_1 = \{x_1 \neq 0\} = \operatorname{Spec} k[x_{0/1}, x_{2/1}]$. We imagine $[x_0; x_1; x_2] = [x_{0/1}; 1; x_{2/1}]$.

Finally, the third patch is $U_2 = \{x_2 \neq 0\} = \operatorname{Spec} k[x_{0/2}, x_{1/2}]$, with "$[x_0; x_1; x_2] = [x_{0/2}; x_{1/2}; 1]$".

We glue U_0 along $x_{1/0} \neq 0$ to U_1 along $x_{0/1} \neq 0$. Our identification (from $[1; x_{1/0}; x_{2/0}] = [x_{0/1}; 1; x_{2/1}]$) is given by $x_{1/0} = 1/x_{0/1}$ and $x_{2/0} = x_{2/1} x_{1/0}$. $U_{01} := U_0 \cap U_1 = \operatorname{Spec} k[x_{1/0}, x_{2/0}, 1/x_{1/0}] \cong \operatorname{Spec} k[x_{0/1}, x_{2/1}, 1/x_{0/1}]$, where the isomorphism was as just described.

We similarly glue together U_0 and U_2, and U_1 and U_2. You could show that all this is compatible, and you could imagine that this is annoying to show. I'm not going to show you the details, because I'll give you a slick way around this naive approach fairly soon.

Suppose you had a homogeneous polynomial, such as $x_0^2 + x_1^2 = x_2^2$. (Intuition: I want a homogeneous polynomial, because in my intuitive notion of projective space as $[x_0; \cdots; x_n]$, I can make sense of where a homogeneous polynomial vanishes, but I can't make as good sense of where an inhomogeneous polynomial vanishes.)

Then I claim that this defines a scheme "in" projective space (in the same way that $\operatorname{Spec} k[x_1, \ldots, x_n]/I$ was a scheme "in" $\mathbb{A}^n$). Here's how. In the patch U_0, I interpret this as $1 + x_{1/0}^2 = x_{2/0}^2$. In patch U_1, I interpret it as $x_{0/1}^2 + 1 = x_{2/1}^2$. On the overlap U_{01}, these two equations are the same: the first equation in $\operatorname{Spec} k[x_{1/0}, x_{2/0}, 1/x_{1/0}]$ is the second equation in $\operatorname{Spec} k[x_{0/1}, x_{2/1}, 1/x_{0/1}]$ [do algebra], unsurprisingly. So piggybacking on that annoying calculation that $\mathbb{P}^2$ consists of 3 pieces glued together nicely is the fact that this scheme consists of three schemes glued together nicely. Similarly, any homogeneous polynomials $x_0, \ldots, x_n$ describes some nice scheme "in" $\mathbb{P}^n$.

1.1. Exercise. Show that an irreducible homogeneous polynomial in $n+1$ variables (over a field k) describes an integral scheme of dimension $n-1$. We think of this as a "hypersurface in $\mathbb{P}^n_k$". Definition: The degree of the hypersurface is the degree of the polynomial. (Other definitions: degree 1 = hyperplane, degree 2, 3, … = quadric hypersurface, cubic, quartic, quintic, sextic, septic, octic, …; a quadric curve is usually called a conic curve, or a conic for short.) Remark: $x_0^2 = 0$ is degree 2.

I could similarly do this with a bunch of homogeneous polynomials. For example:

1.2. Exercise. Show that $wx = yz, x^2 = wy, y^2 = xz$ describes an irreducible curve in $\mathbb{P}^3_k$ (the twisted cubic!).

1.3. Tentative definitions. Any scheme described in this way ("in $\mathbb{P}^n_k$") is called a *projective* k*-scheme*. We're not using anything about k being a ring, so similarly if A is a ring, we can define a projective A-scheme. (I did the case $A = k$ first because that's the more classical case.) If I is the ideal in $A[x_0, \dots, x_n]$ generated by these homogeneous polynomials, we say that the scheme we have constructed is $\operatorname{Proj} A[x_0, \dots, x_n]/I$.

1.4. Examples of projective k-schemes "in" $\mathbb{P}^2_k$: $x = 0$ ("line"), $x^2 + y^2 = z^2$ ("conic"). $wx = yz$ ("smooth quadric surface"). $y^2z = x^3 - xz$ ("smooth cubic curve"). ($\mathbb{P}^2_k$)

You imagine that we will have a map $\operatorname{Proj} A[x_0, \dots, x_n]/I$ to $\operatorname{Spec} A$. And indeed we will once we have a definition of morphisms of schemes.

The *affine cone* of $\operatorname{Proj} R$ is $\operatorname{Spec} R$. The picture to have in mind is an actual cone. (I described it in the cases above, §1.4.) Intuitively, you could imagine that if you discarded the origin, you would get something that would project onto $\operatorname{Proj} R$. That will be right, but right now we don't know what maps of schemes are.

The *projective cone* of $\operatorname{Proj} R$ is $\operatorname{Proj} R[T]$, where T is one more variable. For example, the cone corresponding to the conic $\operatorname{Proj} k[x, y, z]/(x^2 + y^2 = z^2)$ is $\operatorname{Proj} k[x, y, z, T]/(x^2 + y^2 + z^2)$. I then discussed this in the cases above, in §1.4.

1.5. Exercise. Show that the projective cone of $\operatorname{Proj} R$ has an open subscheme that is the affine cone, and that its complement $V(T)$ can be associated with $\operatorname{Proj} R$ (as a topological space). (More precisely, setting $T = 0$ cuts out a scheme isomorphic to $\operatorname{Proj} R$.)

2. A more general notion of Proj

Let's abstract these notions. In the examples we've been doing, we have a graded ring $S = k[x_0, \dots, x_n]/I$ where I is a *homogeneous ideal* (i.e. I is generated by homogeneous elements of $k[x_0, \dots, x_n]$). Here we are taking the usual grading on $k[x_0, \dots, x_n]$, where each x_i has weight 1. Then S is also a graded ring, and we'll call its graded pieces S_0, S_1, etc. (In a graded ring: $S_m \times S_n \to S_{m+n}$. Note that S_0 is a subring, and S is a S_0-algebra.)

Notice in our example that $S_0 = k$, and S is generated over S_0 by S_1.

2.1. Definition. Assume for the rest of the day that S_* is a graded ring (with grading $\mathbb{Z}^{\geq 0}$). It is automatically a module over S_0. Suppose S_0 is a module over some ring A. (Imagine that $A = S_0 = k$.) Now $S_+ := \oplus_{i>0} S_i$ is an ideal, which we will call the *irrelevant ideal*; suppose that it is a finitely generated ideal.

2.2. Exercise. Show that S_* is a finitely-generated S_0-algebra.

Here is an example to keep in mind: $S_* = k[x_0, x_1, x_2]$ (with the usual grading). In this case we will build $\mathbb{P}^2_k$.

I will now define the scheme, that I will denote $\operatorname{Proj} S_*$. I will define it as a *set*, with a *topology*, and a *structure sheaf*. It will be enlightening to picture this in terms of the *affine cone* $\operatorname{Spec} S_*$. We will think of $\operatorname{Proj} S_*$ as the affine cone, minus the origin, modded out by multiplication by scalars.

The points of $\operatorname{Proj} S_*$ are defined to be the homogeneous prime ideals, except for any ideal containing the irrelevant ideal. (I waved my hands in the air linking this to $\operatorname{Spec} S_*$.)

We'll define the topology by defining the closed subsets. The closed subsets are of the form $V(I)$, where I is a homogeneous ideal. Particularly important open sets will the *distinguished open sets* $D(f) = \operatorname{Proj} S_* - V(f)$, where $f \in S_+$ is homogeneous. They form a base for the same reason as the analogous distinguished open sets did in the affine case.

Note: If $D(f) \subset D(g)$, then $f^n \in (g)$ for some n, and vice versa. We've done this before in the affine case .

Clearly $D(f) \cap D(g) = D(fg)$, by the same immediate argument as in the affine case.

We define $\mathcal{O}_{\operatorname{Proj} S_*}(D(f)) = (S_f)_0$, where $(S_f)_0$ means the 0-graded piece of the graded ring (S_f). As before, we check that this is well-defined (i.e. if $D(f) = D(f')$, then we are defining the same ring). In our example of $S_* = k[x_0, x_1, x_2]$, if we take $f = x_0$, we get $(k[x_0, x_1, x_2]_{x_0})_0 := k[x_{1/0}, x_{2/0}]$.

We now check that this is a sheaf. I could show that this is a sheaf on the base, and the argument would be the same. But instead, here is a trickier argument: I claim that

$$(D(f), \mathcal{O}_{\operatorname{Proj} S_*}) \cong \operatorname{Spec}(S_f)_0.$$

You can do this by showing that the distinguished base elements of $\operatorname{Proj} R$ contained in $D(f)$ are precisely the distinguished base elements of $\operatorname{Spec}(S_f)_0$, and the two sheaves have identifiable sections, and the restriction maps are the same.

2.3. Important Exercise. Do this. (Caution: don't assume $\deg f = 1$.)

2.4. *Example:* $\mathbb{P}^n_A$. $\mathbb{P}^n_A = \operatorname{Proj} A[x_0, \dots, x_n]$. This is great, because we didn't have to do any messy gluing.

2.5. Exercise. Check that this agrees with our earlier version of projective space.

2.6. *Exercise.* Show that $Y = \mathbb{P}^2 - (x^2 + y^2 + z^2 = 0)$ is affine, and find its corresponding ring (= find its ring of global sections).

We like this definition for a more abstract reason. Let V be an $n+1$-dimensional vector space over k. (Here k can be replaced by A as well.) Let $\operatorname{Sym}^{\cdot} V^* = k \oplus V^* \oplus \operatorname{Sym}^2 V^* \oplus \cdots$. The dual here may be confusing; it's here for reasons that will become apparent far later.)

If for example V is the dual of the vector space with basis associated to $x_0, \ldots, x_n$, we would have $\mathrm{Sym}^{\cdot} V^* = k[x_0, \ldots, x_n]$. Then we can define $\mathrm{Proj}\,\mathrm{Sym}^{\cdot} V^*$. (This is often called $\mathbb{P}V$.) I like this definition because it doesn't involved choosing a basis of V. [Picture of vector bundle, and its projectivization.]

If S_* is generated by S_1 (as a S_0-algebra), then $\mathrm{Proj}\, S_*$ "sits in $\mathbb{P}^n_A$". (Terminology: *generated in degree* 1.) $k[\mathrm{Sym}^* S_1] = k[x, y, z] \twoheadrightarrow S_*$ implies $S = k[x, y, z]/I$, where I is a homogeneous ideal. Example: $S_* = k[x, y, z]/(x^2 + y^2 - z^2)$. It sits naturally in $\mathbb{P}^2$.

Next day: I'll describe some nice properties of projective S_0-schemes.

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FOUNDATIONS OF ALGEBRAIC GEOMETRY CLASS 11

CONTENTS

Last day: finite type A-scheme, locally of finite type A-scheme, projective schemes over A or k.

Today: Smoothness=regularity=nonsingularity, Zariski tangent space and related notions, Nakayama's Lemma.

Warning: I've changed problem B6 to make it more general (reposted on web). The proof is the same as the original problem, but I'll use it in this generality.

1. PROJECTIVE k-SCHEMES AND PROJECTIVE A-SCHEMES

Last day, I defined $\operatorname{Proj} S_*$ where: S_* is a graded ring (with grading $\mathbb{Z}^{\geq 0}$). Last day I said: Suppose S_0 is an A-algebra. I've changed my mind: I'd like to take $S_0 = A$. $S_+ := \oplus_{i>0} S_i$ is the *irrelevant ideal*; suppose that it is finitely generated over S.

Set: The points of $\operatorname{Proj} S_*$ are defined to be the homogeneous prime ideals, except for any ideal containing the irrelevant ideal.

Topology: The closed subsets are of the form $V(I)$, where I is a homogeneous ideal. Particularly important open sets will the *distinguished open sets* $D(f) = \operatorname{Proj} S_* - V(f)$, where $f \in S_+$ is homogeneous. They form a base.

Structure sheaf: $\mathcal{O}_{\operatorname{Proj} S_*}(D(f)) := (S_f)_0$, where $(S_f)_0$ means the 0-graded piece of the graded ring (S_f). This is a sheaf. One method:

$$(D(f), \mathcal{O}_{\operatorname{Proj} S_*}|_{D(f)}) \cong \operatorname{Spec}(S_f)_0.$$

1.1. If S_* is generated by S_1 (as an S_0-algebra — we say S_* is *generated in degree* 1), say by $n+1$ elements $x_0, \dots, x_n$, then $\operatorname{Proj} S_*$ "sits in $\mathbb{P}^n_A$" as follows. (X "in" Y currently means that the topological space of X is a subspace of the topological space of Y.) Consider A^{n+1} as a free module with generators $t_0, \dots, t_n$ associated to $x_0, \dots, x_n$.

Date: November 2, 2005. Minor updates January 31, 2007.

$k[\mathrm{Sym}^* A^{n+1}] = k[t_0, t_1, \dots, t_n] \twoheadrightarrow S_*$ implies $S = k[t_0, t_1, \dots t_n]/I$, where I is a homogeneous ideal. Example: $S_* = k[x, y, z]/(x^2 + y^2 - z^2)$ sits naturally in $\mathbb{P}^2$.

1.2. Easy exercise (silly example). $\mathbb{P}^0_A = \mathrm{Proj}\, A[T] \cong \mathrm{Spec}\, A$. Thus "$\mathrm{Spec}\, A$ is a projective A-scheme".

Here are some useful facts.

A **quasiprojective A-scheme** is an open subscheme of a projective A-scheme. The "A" is omitted if it is clear from the context; often A is some field.)

1.3. Exercise. Show that all projective A-schemes are quasicompact. (Translation: show that any projective A-scheme is covered by a finite number of affine open sets.) Show that $\mathrm{Proj}\, S_*$ is finite type over $A = S_0$. If S_0 is a Noetherian ring, show that $\mathrm{Proj}\, S_*$ is a Noetherian scheme, and hence that $\mathrm{Proj}\, S_*$ has a finite number of irreducible components. Show that any quasiprojective scheme is locally of finite type over A. If A is Noetherian, show that any quasiprojective A-scheme is quasicompact, and hence of finite type over A. Show this need not be true if A is not Noetherian.

I'm now going to ask a somewhat rhetorical question. It's going to sound complicated because of all the complicated words in it. But all the complicated words just mean simple concepts.

Question (open for now): are there any quasicompact finite type k-schemes that are not quasiprojective? (Translation: if we're gluing together a finite number of schemes each sitting in some $\mathbb{A}^n$, can we ever get something not quasiprojective?) The difficulty of answering this question shows that this is a good notion! We will see before long that the affine line with the doubled origin is not projective, but we'll call that kind of bad behavior "non-separated", and then the question will still stand: is every separated quasicompact finite type k-scheme quasiprojective?

1.4. Exercise. Show that $\mathbb{P}^n_k$ is normal. More generally, show that $\mathbb{P}^n_R$ is normal if R is a Unique Factorization Domain.

I said earlier that the *affine cone* is $\mathrm{Spec}\, S_*$. (We'll soon see that we'll have a map from cone minus origin to Proj.) The *projective cone* of $\mathrm{Proj}\, S_*$ is $\mathrm{Proj}\, S_*[T]$. We have an intuitive picture of both.

1.5. Exercise (better version of exercise from last day). Show that the projective cone of $\mathrm{Proj}\, S_*$ has an open subscheme $D(T)$ that is the affine cone, and that its complement $V(T)$ can be identified with $\mathrm{Proj}\, S_*$ (as a topological space). More precisely, setting $T = 0$ "cuts out" a scheme isomorphic to $\mathrm{Proj}\, S_*$ — see if you can make that precise.

A lot of what we did for affine schemes generalizes quite easily, as you'll see in these exercises.

1.6. Exercise. Show that the irreducible subsets of dimension $n-1$ of $\mathbb{P}^n_k$ correspond to homogeneous irreducible polynomials modulo multiplication by non-zero scalars.

1.7. Exercise.

(a) Suppose I is any homogeneous ideal, and f is a homogeneous element. Suppose f vanishes on $V(I)$. Show that $f^n \in I$ for some n. (Hint: mimic the proof in the affine case.)
(b) If $Z \subset \operatorname{Proj} S_*$, define $I(\cdot)$. Show that it is a homogeneous ideal. For any two subsets, show that $I(Z_1 \cup Z_2) = I(Z_1) \cap I(Z_2)$.
(c) For any homogeneous ideal I with $V(I) \neq \emptyset$, show that $I(V(I)) = \sqrt{I}$. [They may need the next exercise for this.]
(d) For any subset $Z \subset \operatorname{Proj} S_*$, show that $V(I(Z)) = \overline{Z}$.

1.8. Exercise. Show that the following are equivalent. (a) $V(I) = \emptyset$ (b) for any f_i (i in some index set) generating I, $\cup D(f_i) = \operatorname{Proj} S_*$ (c) $\sqrt{I} \supset S_+$.

Now let's go back to some interesting geometry. Here is a useful construction. Define $S_{n*} := \oplus_i S_{ni}$. (We could rescale our degree, so "old degree" n is "new degree" 1.)

1.9. Exercise. Show that $\operatorname{Proj} S_{n*}$ is isomorphic to $\operatorname{Proj} S_*$.

1.10. Exercise. Suppose S_* is generated over S_0 by $f_1, \ldots, f_n$. Suppose $d = \operatorname{lcm}(\deg f_1, \ldots, \deg f_n)$. Show that S_{d*} is generated in "new" degree 1 (= "old" degree d). (Hint: I like to show this by induction on the size of the set $\{\deg f_1, \ldots, \deg f_n\}$.) This is handy, because we can stick every Proj in some projective space via the construction of 1.1.

1.11. Exercise. If S_* is a Noetherian domain over k, and $\operatorname{Proj} S_*$ is non-empty show that $\dim \operatorname{Spec} S_* = \dim \operatorname{Proj} S_* + 1$. (Hint: throw out the origin. Look at a distinguished $D(f)$ where $\deg f = 1$. Use the fact mentioned in Exercise 2.3 of Class 9. By the previous exercise, you can assume that S_* is generated in degree 1 over $S_0 = A$.)

Example: Suppose $S_* = k[x, y]$, so $\operatorname{Proj} S_* = \mathbb{P}^1_k$. Then $S_{2*} = k[x^2, xy, y^2] \subset k[x, y]$. What is this subring? Answer: let $u = x^2, v = xy, w = y^2$. I claim that $S_{2*} = k[u, v, w]/(uw - v^2)$.

1.12. Exercise. Prove this.

We have a graded ring with three generators; thus we think of it as sitting "in" $\mathbb{P}^2$. This is $\mathbb{P}^1$ as a conic in $\mathbb{P}^2$.

1.13. *Side remark: diagonalizing quadrics.* Suppose k is an algebraically closed field of characteristic not 2. Then any quadratic form in n variables can be "diagonalized" by changing coordinates to be a sum of squares (e.g. $uw - v^2 = ((u+v)/2)^2 + (i(u-v)/2)^2 + (iv)^2$), and the number of such squares is invariant of the change of coordinates. (Reason:

write the quadratic form on $x_1, \ldots, x_n$ as

$$\begin{pmatrix} x_1 & \cdots & x_n \end{pmatrix} M \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

where M is a symmetric matrix — here you are using characteristic $\neq 2$. Then diagonalize M — here you are using algebraic closure.) Thus the conics in $\mathbb{P}^2$, up to change of coordinates, come in only a few flavors: sums of 3 squares (e.g. our conic of the previous exercise), sums of 2 squares (e.g. $y^2 - x^2 = 0$, the union of 2 lines), a single square (e.g. $x^2 = 0$, which looks set-theoretically like a line), and 0 (not really a conic at all). Thus we have proved: any plane conic (over an algebraically closed field of characteristic not 2) that can be written as the sum of three squares is isomorphic to $\mathbb{P}^1$.

We now soup up this example.

1.14. Exercise. Show that $\operatorname{Proj} S_{3*}$ is the *twisted cubic* "in" $\mathbb{P}^3$.

1.15. Exercise. Show that $\operatorname{Proj} S_{d*}$ is given by the equations that

$$\begin{pmatrix} y_0 & y_1 & \cdots & y_{d-1} \\ y_1 & y_2 & \cdots & y_d \end{pmatrix}$$

is rank 1 (i.e. that all the 2×2 minors vanish).

This is called the *degree* d *rational normal curve* "in" $\mathbb{P}^d$.

More generally, if $S_* = k[x_0, \ldots, x_n]$, then $\operatorname{Proj} S_{d*} \subset \mathbb{P}^{N-1}$ (where N is the number of degree d polynomials in x_0, …, x_n) is called the d-*uple embedding* or d-*uple Veronese embedding*. **Exercise.** Show that $N = \binom{n+d}{d}$.

1.16. Exercise. Find the equations cutting out the *Veronese surface* $\operatorname{Proj} S_{2*}$ where $S_* = k[x_0, x_1, x_2]$, which sits naturally in $\mathbb{P}^5$.

1.17. *Example.* If we put a non-standard weighting on the variables of $k[x_1, \ldots, x_n]$ — say we give x_i degree d_i — then $\operatorname{Proj} k[x_1, \ldots, x_n]$ is called *weighted projective space* $\mathbb{P}(d_1, d_2, \ldots, d_n)$.

1.18. Exercise. Show that $\mathbb{P}(m, n)$ is isomorphic to $\mathbb{P}^1$. Show that $\mathbb{P}(1, 1, 2) \cong \operatorname{Proj} k[u, v, w, z]/(uw - v^2)$. Hint: do this by looking at the even-graded parts of $k[x_0, x_1, x_2]$, cf. Exercise 1.9. (Picture: this is a projective cone over a conic curve.)

1.19. *Important exercise for later.* (a) *(Hypersurfaces meet everything of dimension at least* 1 *in projective space — unlike in affine space.)* Suppose X is a closed subset of $\mathbb{P}^n_k$ of dimension at least 1, and H a nonempty hypersurface in $\mathbb{P}^n_k$. Show that H meets X. (Hint: consider the affine cone, and note that the cone over H contains the origin. Use Krull's Principal Ideal Theorem.)

(b) (Definition: Subsets in $\mathbb{P}^n$ cut out by linear equations are called *linear subspaces*. Dimension $1, 2$ linear subspaces are called *lines* and *planes* respectively.) Suppose $X \hookrightarrow \mathbb{P}^n_k$ is a closed subset of dimension r. Show that any codimension r linear space meets X. (Hint: Refine your argument in (a).)
(c) Show that there is a codimension $r+1$ complete intersection (codimension $r+1$ set that is the intersection of $r+1$ hypersurfaces) missing X. (The key step: show that there is a hypersurface of sufficiently high degree that doesn't contain every generic point of X.) If k is infinite, show that there is a codimension $r+1$ linear subspace missing X. (The key step: show that there is a hyperplane not containing any generic point of a component of X.)

1.20. Exercise. Describe all the lines on the quadric surface $wx - yz = 0$ in $\mathbb{P}^3_k$. (Hint: they come in two "families", called the *rulings* of the quadric surface.)

Hence by Remark 1.13, if we are working over an algebraically closed field of characteristic not 2, we have shown that all rank 4 quadric surfaces have two rulings of lines.

2. "SMOOTHNESS" = REGULARITY = NONSINGULARITY

The last property of schemes I want to discuss is something very important: when they are "smooth". For unfortunate historic reasons, *smooth* is a name given to certain morphisms of schemes, but I'll feel free to use this to use it also for schemes themselves. The more correct terms are *regular* and *nonsingular*. A point of a scheme that is not smooth=regular=nonsingular is, not surprisingly, *singular*.

The best way to describe this is by first defining the tangent space to a scheme at a point, what we'll call the *Zariski tangent space*. This will behave like the tangent space you know and love at smooth points, but will also make sense at other points. In other words, geometric intuition at the smooth points guides the definition, and then the definition guides the algebra at all points, which in turn lets us refine our geometric intuition.

This definition is short but surprising. I'll have to convince you that it deserves to be called the tangent space. I've always found this tricky to explain, and that is because we want to show that it agrees with our intuition; but unfortunately, our intuition is crappier than we realize. So I'm just going to define it for you, and later try to convince you that it is reasonable.

Suppose A is a ring, and $\mathfrak{m}$ is a point. Translation: we have a point $[\mathfrak{m}]$ on a scheme $\operatorname{Spec} A$. Let $k = A/\mathfrak{m}$ be the residue field. Then $\mathfrak{m}/\mathfrak{m}^2$ is a vector space over the residue field $A/\mathfrak{m}$: it is an A-module, and $\mathfrak{m}$ acts like 0. This is defined to be the **Zariski cotangent space**. The dual is the **Zariski tangent space**. Elements of the Zariski cotangent space are called **cotangent vectors** or **differentials**; elements of the tangent space are called **tangent vectors**.

Note: This is intrinsic; it doesn't depend on any specific description of the ring itself (e.g. the choice of generators over a field k = choice of embedding in affine space). An interesting feature: in some sense, the cotangent space is more algebraically natural than the tangent space. There is a moral reason for this: the cotangent space is more naturally determined in terms of functions on a space, and we are very much thinking about schemes in terms of "functions on them". This will come up later.

I'm now going to give you a bunch of plausibility arguments that this is a reasonable definition.

First, I'll make a moral argument that this definition is plausible for the cotangent space of the origin of $\mathbb{A}^n$. Functions on $\mathbb{A}^n$ should restrict to a linear function on the tangent space. What function does $x^2 + xy + x + y$ restrict to "near the origin"? You will naturally answer: $x + y$. Thus we "pick off the linear terms". Hence $\mathfrak{m}/\mathfrak{m}^2$ are the linear functionals on the tangent space, so $\mathfrak{m}/\mathfrak{m}^2$ is the cotangent space.

Here is a second argument, for those of you who think of the tangent space as the space of derivations. (I didn't say this in class, because I didn't realize that many of you thought in this way until later.) More precisely, in differential geometry, the tangent space at a point is sometimes defined as the vector space of derivations at that point. A derivation is a function that takes in functions near the point that vanish at the point, and gives elements of the field k, and satisfies the Leibniz rule $(fg)' = f'g + g'f$. Translation: a derivation is a map $\mathfrak{m} \to k$. But $\mathfrak{m}^2 \to 0$, as if $f(p) = g(p) = 0$, then $(fg)'(p) = f'(p)g(p) + g'(p)f(p) = 0$. Thus we have a map $\mathfrak{m}/\mathfrak{m}^2 \to k$, i.e. an element of $(\mathfrak{m}/\mathfrak{m}^2)^*$. **Exercise (for those who think in this way).** Check that this is reversible, i.e. that any map $\mathfrak{m}/\mathfrak{m}^2 \to k$ gives a derivation — i.e., check the Leibniz rule.

2.1. Here is an old-fashioned example to help tie this down to earth. This is not currently intended to be precise. In $\mathbb{A}^3$, we have a curve cut out by $x + y + z^2 + xyz = 0$ and $x - 2y + z + x^2y^2z^3 = 0$. What is the tangent line near the origin? (Is it even smooth there?) Answer: the first surface looks like $x + y = 0$ and the second surface looks like $x - 2y + z = 0$. The curve has tangent line cut out by $x + y = 0$ and $x - 2y + z = 0$. It is smooth (in the analytic sense). I give questions like this in multivariable calculus. The students do a page of calculus to get the answer, because I can't tell them to just pick out the linear terms.

Another example: $x + y + z^2 = 0$ and $x + y + x^2 + y^4 + z^5 = 0$ cuts out a curve, which obviously passes through the origin. If I asked my multivariable calculus students to calculate the tangent line to the curve at the origin, they would do a page of calculus which would boil down to picking off the linear terms. They would end up with the equations $x + y = 0$ and $x + y = 0$, which cuts out a plane, not a line. They would be disturbed, and I would explain that this is because the curve isn't smooth at a point, and their techniques don't work. We on the other hand bravely declare that the cotangent space is cut out by $x + y = 0$, and *define* this as a singular point. (Intuitively, the curve near the origin is very close to lying in the plane $x + y = 0$.) Notice: the cotangent space jumped up in dimension from what it was "supposed to be", not down.

2.2. *Proposition. — Suppose* $(A, \mathfrak{m})$ *is a Noetherian local ring. Then* $\dim A \leq \dim_k \mathfrak{m}/\mathfrak{m}^2$.

We'll prove this on Friday.

If equality holds, we say that A is **regular** at $\mathfrak{p}$. If A is a local ring, then we say that A is a **regular local ring**. If A is regular at all of its primes, we say that A is a **regular ring**.

A scheme X is **regular** or **nonsingular** or **smooth** at a point p if the local ring $\mathcal{O}_{X,p}$ is regular. It is **singular** at the point otherwise. A scheme is **regular** or **nonsingular** or **smooth** if it is regular at all points. It is **singular** otherwise (i.e. if it is singular at *at least one* point.

In order to prove Proposition 2.2, we're going to pull out another algebraic weapon: Nakayama's lemma. This was done in Math 210, so I didn't discuss it in class. You should read this short exposition. If you have never seen Nakayama before, you'll see a complete proof. If you want a refresher, here it is. And even if you are a Nakayama expert, please take a look, because there are several related facts that go by the name of Nakayama's Lemma, and we should make sure we're talking about the same one(s). Also, this will remind you that the proof wasn't frightening and didn't require months of previous results.

2.3. *Nakayama's Lemma version 1.* — *Suppose* R *is a ring,* I *an ideal of* R*, and* M *is a finitely-generated* R*-module. Suppose* $M = IM$*. Then there exists an* $a \in R$ *with* $a \equiv 1 \pmod{I}$ *with* $aM = 0$*.*

Proof. Say M is generated by $m_1, \ldots, m_n$. Then as $M = IM$, we have $m_i = \sum_j a_{ij} m_j$ for some $a_{ij} \in I$. Thus

$$(1) \qquad (\mathrm{Id}_n - A)\begin{pmatrix} m_1 \\ \vdots \\ m_n \end{pmatrix} = 0$$

where Id_n is the $n \times n$ identity matrix in R, and $A = (a_{ij})$. We can't quite invert this matrix, but we almost can. Recall that any $n \times n$ matrix M has an adjoint $\mathrm{adj}(M)$ such that $\mathrm{adj}(M)M = \det(M)\mathrm{Id}_n$. The coefficients of $\mathrm{adj}(M)$ are polynomials in the coefficients of M. (You've likely seen this in the form a formula for M^{-1} when there is an inverse.) Multiplying both sides of (1) on the left by $\mathrm{adj}(\mathrm{Id}_n - A)$, we obtain

$$\det(\mathrm{Id}_n - A)\begin{pmatrix} m_1 \\ \vdots \\ m_n \end{pmatrix} = 0.$$

But when you expand out $\det(\mathrm{Id}_n - A)$, you get something that is $1 \pmod{I}$. $\square$

Here is why you care: Suppose I is contained in all maximal ideals of R. (The intersection of all the maximal ideals is called the *Jacobson radical*, but I won't use this phrase. Recall that the nilradical was the intersection of the *prime ideals* of R.) Then I claim that any $a \equiv 1 \pmod{I}$ is invertible. For otherwise $(a) \neq R$, so the ideal (a) is contained in some maximal ideal $\mathfrak{m}$ — but $a \equiv 1 \pmod{\mathfrak{m}}$, contradiction. Then as a is invertible, we have the following.

2.4. *Nakayama's Lemma version 2. — Suppose* R *is a ring,* I *an ideal of* R *contained in all maximal ideals, and* M *is a finitely-generated* R*-module. (Most interesting case:* R *is a local ring, and* I *is the maximal ideal.) Suppose* $M = IM$*. Then* $M = 0$*.*

2.5. Important exercise (Nakayama's lemma version 3). Suppose R is a ring, and I is an ideal of R contained in all maximal ideals. Suppose M is a *finitely generated* R-module, and $N \subset M$ is a submodule. If $N/IN \hookrightarrow M/IM$ an isomorphism, then $M = N$.

2.6. Important exercise (Nakayama's lemma version 4). Suppose $(R, \mathfrak{m})$ is a local ring. Suppose M is a finitely-generated R-module, and $f_1, \dots, f_n \in M$, with (the images of) $f_1, \dots, f_n$ generating $M/\mathfrak{m}M$. Then $f_1, \dots, f_n$ generate M. (In particular, taking $M = \mathfrak{m}$, if we have generators of $\mathfrak{m}/\mathfrak{m}^2$, they also generate $\mathfrak{m}$.)

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FOUNDATIONS OF ALGEBRAIC GEOMETRY CLASS 12

CONTENTS

Last day: smoothness=regularity=nonsingularity, Zariski tangent space and related notions, Nakayama's Lemma.

Today: Jacobian criterion, Euler test, characterizations of discrete valuation rings = dimension 1 Noetherian regular local rings

1. "SMOOTHNESS" = REGULARITY = NONSINGULARITY, CONTINUED

Last day, I defined the Zariski tangent space. Suppose A is a ring, and $\mathfrak{m}$ is a maximal ideal, with residue field $k = A/\mathfrak{m}$. Then $\mathfrak{m}/\mathfrak{m}^2$, a vector space over k, is the **Zariski cotangent space**. The dual is the **Zariski tangent space**. Elements of the Zariski cotangent space are called **cotangent vectors** or **differentials**; elements of the tangent space are called **tangent vectors**.

I tried to convince you that this was a reasonable definition. I also asked you what your private definition of tangent space or cotangent space was, so I could convince you that this is the right algebraic notion. A couple of you think of tangent vectors as *derivations*, and in this case, the connection is very fast. I've put it in the Class 11 notes, so please check it out if you know what derivations are.

Last day, I stated the following proposition.

1.1. *Proposition. — Suppose* $(A, \mathfrak{m})$ *is a Noetherian local ring. Then* $\dim A \leq \dim_k \mathfrak{m}/\mathfrak{m}^2$.

We'll prove this in a moment.

If equality holds, we say that A is **regular** at $\mathfrak{m}$. If A is a local ring, then we say that A is a **regular local ring**. If A is regular at all of its primes, we say that A is a **regular ring**.

A scheme X is **regular** or **nonsingular** or **smooth** at a point p if the local ring $\mathcal{O}_{X,p}$ is regular. It is **singular** at the point otherwise. A scheme is **regular** or **nonsingular** or

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smooth if it is regular at all points. It is **singular** otherwise (i.e. if it is singular at *at least one* point).

1.2. Exercise. Show that if A is a Noetherian local ring, then A has finite dimension. (*Warning:* Noetherian rings in general could have infinite dimension.)

In order to prove Proposition 1.1, we're going to use Nakayama's Lemma, which hopefully you've looked at.

The version we'll use is:

1.3. Important exercise (Nakayama's lemma version 4). Suppose $(R, \mathfrak{m})$ is a local ring. Suppose M is a finitely-generated R-module, and $f_1, \dots, f_n \in M$, with (the images of) $f_1, \dots, f_n$ generating $M/\mathfrak{m}M$. Then $f_1, \dots, f_n$ generate M. (In particular, taking $M = \mathfrak{m}$, if we have generators of $\mathfrak{m}/\mathfrak{m}^2$, they also generate $\mathfrak{m}$.) Translation: if we have a set of generators of a *finitely generated* module modulo a finite ideal, then they generate the entire module.

Proof of Proposition 1.1: Note that $\mathfrak{m}$ is finitely generated (as R is Noetherian), so $\mathfrak{m}/\mathfrak{m}^2$ is a finitely generated $R/\mathfrak{m} = k$-module, hence finite-dimensional. Say $\dim_k \mathfrak{m}/\mathfrak{m}^2 = n$. Choose n elements of $\mathfrak{m}/\mathfrak{m}^2$, and lift them to elements $f_1, \dots, f_n$ of $\mathfrak{m}$. Then by Nakayama's lemma, $(f_1, \dots, f_n) = \mathfrak{m}$.

Problem B6 on problem set 4 (newest version!) includes the following: Suppose $X = \operatorname{Spec} R$ where R is a Noetherian domain, and Z is an irreducible component of $V(f_1, \dots, f_n)$, where $f_1, \dots, f_n \in R$. Show that the height of Z (as a prime ideal) is at most n.

In this case, $V((f_1, \dots, f_n)) = V(\mathfrak{m})$ is just the point $[\mathfrak{m}]$, so the height of $\mathfrak{m}$ is at most n. Thus the longest chain of prime ideals containing $\mathfrak{m}$ is at most $n + 1$. But this is also the longest chain of prime ideals in X (as $\mathfrak{m}$ is the unique maximal ideal), so $n \geq \dim X$. $\square$

Computing the Zariski-tangent space is actually quite hands-on, because you can compute it in a multivariable calculus way.

For example, last day I gave some motivation, by saying that $x + y + 3z + y^3 = 0$ and $2x + z^3 + y^2 = 0$ cut out a curve in $\mathbb{A}^3$, which is nonsingular at the origin, and that the tangent space at the origin is cut out by $x + y + 3z = 2x = 0$. This can be made precise through the following exercise.

1.4. *Important exercise.* Suppose A is a ring, and $\mathfrak{m}$ a maximal ideal. If $f \in \mathfrak{m}$, show that the dimension of the Zariski tangent space of $\operatorname{Spec} A$ at $[\mathfrak{m}]$ is the dimension of the Zariski tangent space of $\operatorname{Spec} A/(f)$ at $[\mathfrak{m}]$, or one less. (Hint: show that the Zariski tangent space of $\operatorname{Spec} A/(f)$ is "cut out" in the Zariski tangent space of $\operatorname{Spec} A$ by the linear equation $f \pmod{\mathfrak{m}^2}$.)

1.5. *Exercise.* Find the dimension of the Zariski tangent space at the point $[(2, x)]$ of $\mathbb{Z}[2i] \cong \mathbb{Z}[x]/(x^2+4)$. Find the dimension of the Zariski tangent space at the point $[(2, x)]$ of $\mathbb{Z}[\sqrt{2}i] \cong \mathbb{Z}[x]/(x^2+2)$.

1.6. *Exercise (the Jacobian criterion for checking nonsingularity).* Suppose k is an algebraically closed field, and X is a finite type k-scheme. Then locally it is of the form $\operatorname{Spec} k[x_1, \dots, x_n]/(f_1, \dots, f_r)$. Show that the Zariski tangent space at the closed point p (with residue field k, by the Nullstellensatz) is given by the cokernel of the Jacobian map $k^r \to k^n$ given by the Jacobian matrix

$$J = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(p) & \cdots & \frac{\partial f_r}{\partial x_1}(p) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_1}{\partial x_n}(p) & \cdots & \frac{\partial f_r}{\partial x_n}(p) \end{pmatrix}. \tag{1}$$

(This is just making precise our example of a curve in $\mathbb{A}^3$ cut out by a couple of equations, where we picked off the linear terms.) Possible hint: "translate p to the origin," and consider linear terms. See also the exercise two previous to this one.

You might be alarmed: what does $\frac{\partial f}{\partial x_1}$ mean?! Do you need deltas and epsilons? No! Just define derivatives formally, e.g.

$$\frac{\partial}{\partial x_1}(x_1^2 + x_1 x_2 + x_2^2) = 2x_1 + x_2.$$

1.7. Exercise: Dimension theory implies the Nullstellensatz. In the previous exercise, l is necessarily only a finite extension of k, as this exercise shows. (a) Prove a microscopically stronger version of the weak Nullstellensatz: Suppose $R = k[x_1, \dots, x_n]/I$, where k is an algebraically closed field and I is some ideal. Then the maximal ideals are precisely those of the form $(x_1 - a_1, \dots, x_n - a_n)$, where $a_i \in k$.
(b) Suppose $R = k[x_1, \dots, x_n]/I$ where k is not necessarily algebraically closed. Show every maximal ideal of R has residue field that are finite extensions of k. (Hint for both: the maximal ideals correspond to dimension 0 points, which correspond to transcendence degree 0 extensions of k, i.e. finite extensions of k. If $k = \overline{k}$, the maximal ideals correspond to surjections $f : k[x_1, \dots, x_n] \to k$. Fix one such surjection. Let $a_i = f(x_i)$, and show that the corresponding maximal ideal is $(x_1 - a_1, \dots, x_n - a_n)$.) This exercise is a bit of an aside — it belongs in class 8, and I've also put it in those notes.

1.8. Exercise. Show that the singular *closed* points of the hypersurface $f(x_1, \dots, x_n) = 0$ in $\mathbb{A}^n_k$ are given by the equations $f = \frac{\partial f}{\partial x_1} = \cdots = \frac{\partial f}{\partial x_n} = 0$.

1.9. Exercise. Show that $\mathbb{A}^1$ and $\mathbb{A}^2$ are nonsingular. (Make sure to check nonsingularity at the non-closed points! Fortunately you know what all the points of $\mathbb{A}^2$ are; this is trickier for $\mathbb{A}^3$.)

In the previous exercise, you'll use the fact that the local ring at the generic point of $\mathbb{A}^2$ is dimension 0, and the local ring at generic point at a curve in $\mathbb{A}^2$ is 1.

Let's apply this technology to an arithmetic situation.

1.10. Exercise. Show that $\operatorname{Spec}\mathbb{Z}$ is a nonsingular curve.

Here are some fun comments: What is the derivative of 35 at the prime 5? Answer: $35 \pmod{25}$, so 35 has the same "slope" as 10. What is the derivative of 9, which doesn't vanish at 5? Answer: the notion of derivative doesn't apply there. You'd think that you'd want to subtract its value at 5, but you can't subtract "$4 \pmod 5$" from the integer 9. Also, $35 \pmod 25$ you might *think* you want to restate as $7 \pmod 5$, by dividing by 5, but that's morally wrong — you're dividing by a particular choice of generator 5 of the maximal ideal of the 5-adics $\mathbb{Z}_5$; in this case, one appears to be staring you in the face, but in general that won't be true. Follow-up fun: you can talk about the derivative of a function only for functions vanishing at a point. And you can talk about the second derivative of a function only for functions that vanish, and whose first derivative vanishes. For example, 75 has second derivative $75 \pmod 125$ at 5. It's pretty flat.

1.11. Exercise. Note that $\mathbb{Z}[i]$ is dimension 1, as $\mathbb{Z}[x]$ has dimension 2 (problem set exercise), and is a domain, and x^2+1 is not 0, so $\mathbb{Z}[x]/(x^2+1)$ has dimension 1 by Krull. Show that $\operatorname{Spec}\mathbb{Z}[i]$ is a nonsingular curve. (This is intended for people who know about the primes of the Gaussian integers $\mathbb{Z}[i]$.)

1.12. Exercise. Show that there is one singular point of $\operatorname{Spec}\mathbb{Z}[2i]$, and describe it.

1.13. Handy Exercise (the Euler test for projective hypersurfaces). There is an analogous Jacobian criterion for hypersurfaces $f = 0$ in $\mathbb{P}^n_k$. Show that the singular *closed* points correspond to the locus $f = \frac{\partial f}{\partial x_1} = \cdots = \frac{\partial f}{\partial x_n} = 0$. If the degree of the hypersurface is not divisible by the characteristic of any of the residue fields (e.g. if we are working over a field of characteristic 0), show that it suffices to check $\frac{\partial f}{\partial x_1} = \cdots = \frac{\partial f}{\partial x_n} = 0$. (Hint: show that f lies in the ideal $(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n})$). (Fact: this will give the singular points in general. I don't want to prove this, and I won't use it.)

1.14. Exercise. Suppose k is algebraically closed. Show that $y^2 z = x^3 - xz^2$ in $\mathbb{P}^2_k$ is an irreducible nonsingular curve. (This is for practice.) Warning: I didn't say $\operatorname{char} k = 0$.

1.15. Exercises. Find all the singular closed points of the following plane curves. Here we work over a field of characteristic 0 for convenience.

(a) $y^2 = x^2 + x^3$. This is called a *node*.
(b) $y^2 = x^3$. This is called a *cusp*.
(c) $y^2 = x^4$. This is called a *tacnode*.

(I haven't given precise definitions for node, cusp, or tacnode. You may want to think about what they might be.)

1.16. Exercise. Show that the twisted cubic $\operatorname{Proj} k[w,x,y,z]/(wz-xy, wy-x^2, xz-y^2)$ is nonsingular. (You can do this by using the fact that it is isomorphic to $\mathbb{P}^1$. I'd prefer you to do this with the explicit equations, for the sake of practice.)

1.17. Exercise. Show that the only dimension 0 Noetherian regular local rings are fields. (Hint: Nakayama.)

2. Dimension 1 Noetherian regular local rings = discrete valuation rings

The case of dimension 1 is also very important, because if you understand how primes behave that are separated by dimension 1, then you can use induction to prove facts in arbitrary dimension. This is one reason why Krull is so useful.

A dimension 1 Noetherian regular local ring can be thought of as a "germ of a curve". Two examples to keep in mind are $k[x]_{(x)} = \{f(x)/g(x) : x \not| g(x)\}$ and $\mathbb{Z}_{(5)} = \{a/b : 5 \not| b\}$.

The purpose of this section is to give a long series of equivalent definitions of these rings.

Theorem. *Suppose $(R, \mathfrak{m})$ is a Noetherian dimension 1 local ring. The following are equivalent.*
(a) R is regular.

Informal translation: R is a germ of a smooth curve.

(b) $\mathfrak{m}$ is principal. If R is regular, then $\mathfrak{m}/\mathfrak{m}^2$ is one-dimensional. Choose any element $t \in \mathfrak{m} - \mathfrak{m}^2$. Then t generates $\mathfrak{m}/\mathfrak{m}^2$, so generates $\mathfrak{m}$ by Nakayama's lemma. Such an element is called a *uniformizer*. (Warning: we needed to know that $\mathfrak{m}$ was finitely generated to invoke Nakayama — but fortunately we do, thanks to the Noetherian hypothesis!)

Conversely, if $\mathfrak{m}$ is generated by one element t over R, then $\mathfrak{m}/\mathfrak{m}^2$ is generated by one element t over $R/\mathfrak{m} = k$.

(c) All ideals are of the form $\mathfrak{m}^n$ or 0. Suppose $(R, \mathfrak{m}, k)$ is a Noetherian regular local ring of dimension 1. Then I claim that $\mathfrak{m}^n \neq \mathfrak{m}^{n+1}$ for any n. Proof: Otherwise, $\mathfrak{m}^n = \mathfrak{m}^{n+1} = \mathfrak{m}^{n+2} = \cdots$. Then $\cap_i \mathfrak{m}^i = \mathfrak{m}^n$. But $\cap_i \mathfrak{m}^i = (0)$. **(I'd given a faulty reason for this. I owe you this algebraic fact.)** Then as $t^n \in \mathfrak{m}^n$, we must have $t^n = 0$. But R is a domain, so $t = 0$ — but $t \in \mathfrak{m} - \mathfrak{m}^2$.

I next claim that $\mathfrak{m}^n/\mathfrak{m}^{n+1}$ is dimension 1. Reason: $\mathfrak{m}^n = (t^n)$. So $\mathfrak{m}^n$ is generated as as a R-module by one element, and $\mathfrak{m}^n/(\mathfrak{m}\mathfrak{m}^n)$ is generated as a $(R/\mathfrak{m} = k)$-module by 1 element, so it is a one-dimensional vector space.

So we have a chain of ideals $R \supset \mathfrak{m} \supset \mathfrak{m}^2 \supset \mathfrak{m}^3 \supset \cdots$ with $\cap \mathfrak{m}^i = (0)$. We want to say that there is no room for any ideal besides these, because "each pair is "separated by dimension 1", and there is "no room at the end". Proof: suppose $I \subset R$ is an ideal. If $I \neq (0)$, then there is some n such that $I \subset \mathfrak{m}^n$ but $I \not\subset \mathfrak{m}^{n+1}$. Choose some $u \in I - \mathfrak{m}^{n+1}$. Then $(u) \subset I$. But u generates $\mathfrak{m}^n/\mathfrak{m}^{n+1}$, hence by Nakayama it generates $\mathfrak{m}^n$, so we have

$\mathfrak{m}^n \subset I \subset \mathfrak{m}^n$, so we are done. Conclusion: in a Noetherian local ring of dimension 1, regularity implies all ideals are of the form $\mathfrak{m}^n$ or (0).

Conversely, suppose we have a dimension 1 Noetherian local domain that is not regular, so $\mathfrak{m}/\mathfrak{m}^2$ has dimension at least 2. Choose any $u \in \mathfrak{m} - \mathfrak{m}^2$. Then $(u, \mathfrak{m}^2)$ is an ideal, but $\mathfrak{m} \subsetneq (u, \mathfrak{m}^2) \subsetneq \mathfrak{m}^2$. We've thus shown that (c) is equivalent to the previous cases.

(d) R is a principal ideal domain. I didn't do this in class. **Exercise.** Show that (d) is equivalent to (a)–(c).

(e) R is a discrete valuation ring. I will now define something for you that will be a very nice way of describing such rings, that will make precise some of our earlier vague ramblings. We'll have to show that this definition accords with (a)–(d) of course.

Suppose K is a field. A *discrete valuation* on K is a *surjective homomorphism* $\nu : K^* \to \mathbb{Z}$ (homomorphism: $\nu(xy) = \nu(x) + \nu(y)$) satisfying

$$\nu(x+y) \geq \min(\nu(x), \nu(y)).$$

Suggestive examples: (i) *(the 5-adic valuation)* $K = \mathbb{Q}$, $\nu(r)$ is the "power of 5 appearing in r", e.g. $\nu(35/2) = 1$, $\nu(27/125) = -3$.

(ii) $K = k(x)$, $\nu(f)$ is the "power of x appearing in f".

Then $0 \cup \{x \in K^* : \nu(x) \geq 0\}$ is a ring. It is called the *valuation ring* of ν.

2.1. Exercise. Describe the valuation rings in those two examples. Hmm — they are familiar-looking dimension 1 Noetherian local rings. What a coincidence!

2.2. Exercise. Show that $0 \cup \{x \in K^* : \nu(x) \geq 1\}$ is the unique maximal ideal of the valuation ring. (Hint: show that everything in the complement is invertible.) Thus the valuation ring is a local ring.

An integral domain A is called a *discrete valuation ring* if there exists a discrete valuation ν on its fraction field $K = \operatorname{Frac}(A)$.

Now if R is a Noetherian regular local ring of dimension 1, and t is a uniformizer (generator of $\mathfrak{m}$ as an ideal = dimension of $\mathfrak{m}/\mathfrak{m}^2$ as a k-vector space) then any non-zero element r of R lies in some $\mathfrak{m}^n - \mathfrak{m}^{n+1}$, so $r = t^n u$ where u is a unit (as t^n generates $\mathfrak{m}^n$ by Nakayama, and so does r), so $\operatorname{Frac} R = R_t = R[1/t]$. So any element of $\operatorname{Frac} R$ can be written uniquely as ut^n where u is a unit and $n \in \mathbb{Z}$. Thus we can define a valuation $\nu(ut^n) = n$, and we'll quickly see that it is a discrete valuation (**exercise**). Thus (a)-(c) implies (d).

Conversely, suppose $(R, \mathfrak{m})$ is a discrete valuation ring. Then I claim it is a Noetherian regular local ring of dimension 1. **Exercise.** Check this. (Hint: Show that the ideals are all of the form (0) or $I_n = \{r \in R : \nu(r) \geq n\}$, and I_1 is the only prime of the second sort. Then we get Noetherianness, and dimension 1. Show that I_1/I_2 is generated by any element of $I_1 - I_2$.)

Exercise/Corollary. There is only one discrete valuation on a discrete valuation ring.

Thus whenever you see a regular local ring of dimension 1, we have a valuation on the fraction field. If the valuation of an element is $n > 0$, we say that the element has a *zero of order* n. If the valuation is $-n < 0$, we say that the element has a *pole of order* n.

So we can finally make precise the fact that 75/34 has a double zero at 5, and a single pole at 2! Also, you can easily figure out the zeros and poles of $x^3(x+y)/(x^2+xy)^3$ on $\mathbb{A}^2$. Note that we can only make sense of zeros and poles at *nonsingular points of codimension* 1.

Definition. More generally: suppose X is a locally Noetherian scheme. Then for any regular height(=codimension) 1 points (i.e. any point p where $\mathcal{O}_{X,p}$ is a regular local ring of dimension 1), we have a valuation ν. If f is any non-zero element of the fraction field of $\mathcal{O}_{X,p}$ (e.g. if X is integral, and f is a non-zero element of the function field of X), then if $\nu(f) > 0$, we say that the element has a *zero of order* $\nu(f)$, and if $\nu(f) < 0$, we say that the element has a *pole of order* $-\nu(f)$.

Exercise. Suppose X is a regular integral Noetherian scheme, and $f \in \mathrm{Frac}(\Gamma(X, \mathcal{O}_X))^*$ is a non-zero element of its function field. Show that f has a finite number of zeros and poles.

Finally:

(f) $(R, \mathfrak{m})$ *is a unique factorization domain,*

(g) R *is integrally closed in its fraction field* $K = \mathrm{Frac}(R)$.

(a)-(e) clearly imply (f), because we have the following stupid unique factorization: each non-zero element of r can be written uniquely as ut^n where $n \in \mathbb{Z}^{\geq 0}$ and u is a unit.

(f) implies (g), because checked earlier that unique factorization domains are always integrally closed in its fraction field.

So it remains to check that (g) implies (a)-(e). This is straightforward, but for the sake of time, I'm not going to give the proof in class. But in the interests of scrupulousness, I'm going to give you a full proof in the notes. It will take us less than half a page. This is the only tricky part of this entire theorem.

2.3. Fact. Suppose $(S, \mathfrak{n})$ is a Noetherian local domain of dimension 0. Then $\mathfrak{n}^n = 0$ for some n. (I had earlier given this as an exercise, with an erroneous hint. I may later add a proof to the notes.)

2.4. Exercise. Suppose A is a subring of a ring B, and $x \in B$. Suppose there is a faithful $A[x]$-module M that is finitely generated as an A-module. Show that x is integral over A.

(Hint: look carefully at the proof of Nakayama's Lemma version 1 in the Class 11 notes, and change a few words.)

Proof that (f) implies (b). Suppose $(R, \mathfrak{m})$ is a Noetherian local domain of dimension 1, that is integrally closed in its fraction field $K = \mathrm{Frac}(R)$. Choose any $r \in R \neq 0$. Then $S = R/(r)$ is dimension 0, and is Noetherian and local, so if $\mathfrak{n}$ is its maximal ideal, then there is some n such that $\mathfrak{n}^n = 0$ but $\mathfrak{n}^{n-1} \neq 0$ by Exercise 2.3. Thus $\mathfrak{m}^n \subseteq (r)$ but $\mathfrak{m}^{n-1} \not\subset (r)$. Choose $s \in \mathfrak{m}^{n-1} - (r)$. Consider $x = r/s$. Then $x^{-1} \notin R$, so as R is integrally closed, x^{-1} is not integral over R.

Now $x^{-1}\mathfrak{m} \not\subset \mathfrak{m}$ (or else $x^{-1}\mathfrak{m} \subset \mathfrak{m}$ would imply that $\mathfrak{m}$ is a faithful $R[x^{-1}]$-module, contradicting Exercise 2.4). But $x^{-1}\mathfrak{m} \subset R$. Thus $x^{-1}\mathfrak{m} = R$, from which $\mathfrak{m} = xR$, so $\mathfrak{m}$ is principal. $\square$

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FOUNDATIONS OF ALGEBRAIC GEOMETRY CLASS 13

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CONTENTS

Last day: Jacobian criterion, Euler test, characterizations of discrete valuation rings = Noetherian regular local rings

Today: discrete valuation rings (conclusion), cultural facts to know about regular local rings, the distinguished affine base of the topology, 2 definitions of quasicoherent sheaf.

Problem set 5 is out today.

I'd like to start with some words on height versus codimension. Suppose R is an integral domain, and $\mathfrak{p}$ is a prime ideal. Thus geometrically we are thinking of an irreducible topological space $\operatorname{Spec} R$, and an irreducible closed subset $\overline{[\mathfrak{p}]}$. Then we have:

$$\dim R/\mathfrak{p} + \operatorname{height} \mathfrak{p} := \dim R/\mathfrak{p} + \dim R_{\mathfrak{p}} \leq \dim R.$$

The reason is as follows: $\dim R$ is one less than the length of the longest chain of prime ideals of R. $\dim R/\mathfrak{p}$ is one less than the length of the longest chain of prime ideals containing $\mathfrak{p}$. $\dim R_{\mathfrak{p}}$ is the length of the longest chain of prime ideals contained in $\mathfrak{p}$. In the homework, you've shown that if R is a finitely generated domain over k, then we have equality, because we can compute dimension using transcendence degree. Hence through any $\mathfrak{p} \subset R$, we can string a "longest chain". Thus we even know that we have equality if R is a *localization* of a finitely generated domain over k.

However, this is false in general. In the class 9 notes, I've added an elementary example to show that you can have the following strange situation: $R = k[x]_{(x)}[t]$ has dimension 2, it is easy to find a chain of prime ideals of length 3:

$$(0) \subset (t) \subset (x, t).$$

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However, the ideal $(xt-1)$ is prime, and height 1 (there is no prime between it and (0)), and maximal.

$$\begin{tikzcd}
(x,t) \arrow[d, dash] & & \\
(t) \arrow[dr, dash] & & (xt-1) \arrow[dl, dash] \\
& (0) &
\end{tikzcd}$$

The details are easy. Thus we have a dimension 0 subset of a dimension 2 set, but it is height 1. Thus it is dangerous to define codimension as height, because you might say something incorrect accidentally.

This example comes from a geometric picture, and if you're curious as to what it is, ask me after class.

There is one more idea I wanted to mention to you, to advertise a nice consequence of the idea of Zariski tangent space.

Problem. Consider the ring $R = k[x, y, z]/(xy - z^2)$. Show that (x, z) is not a principal ideal.

As $\dim R = 2$, and $R/(x, z) \cong k[y]$ has dimension 1, we see that this ideal is height 1 (as height behaves as codimension should for finitely generated k-domains!). Our geometric picture is that $\operatorname{Spec} R$ is a cone (we can diagonalize the quadric as $xy - z^2 = ((x+y)/2)^2 - ((x-y)/2)^2 - z^2$, at least if $\operatorname{char} k \neq 2$), and that (x, z) is a ruling of the cone. This suggests that we look at the cone point.

Solution. Let $\mathfrak{m} = (x, y, z)$ be the maximal ideal corresponding to the origin. Then $\operatorname{Spec} R$ has Zariski tangent space of dimension 3 at the origin, and $\operatorname{Spec} R/(x, z)$ has Zariski tangent space of dimension 1 at the origin. But $\operatorname{Spec} R/(f)$ must have Zariski tangent space of dimension at least 2 at the origin.

Exercise. Show that $(x, z) \subset k[w, x, y, z]/(wz - xy)$ is a height 1 ideal that is not principal. (What is the picture?)

1. Dimension 1 Noetherian regular local rings = discrete valuation rings

Last day we mostly proved the following.

Theorem. *Suppose $(R, \mathfrak{m})$ is a Noetherian dimension 1 local ring. The following are equivalent.*

(a) R *is regular.*
(b) $\mathfrak{m}$ *is principal.*
(c) *All ideals are of the form* $\mathfrak{m}^n$ *or* 0.
(d) R *is a principal ideal domain.*
(e) R *is a discrete valuation ring.*

(f) $(R, \mathfrak{m})$ *is a unique factorization domain,*
(g) R *is integrally closed in its fraction field* $K = \mathrm{Frac}(R)$.

I didn't state (d) in class, but I included it as an exercise, as it is easy to connect to the others. Other than that, I connected (a) to (e), and showed that they implies (f), which in turn implies (g). All of the arguments were quite short. I didn't show that (g) implies (a)-(e), but I included it in the notes for Friday's class. I find this the trickiest part of the argument, but it is still quite short, less than half a page.

I'd like to repeat what I said on Friday about the consequences of this characterization of discrete valuation rings (DVR's).

Whenever you see a Noetherian regular local ring of dimension 1, we have a valuation on the fraction field. If the valuation of an element is $n > 0$, we say that the element has a *zero of order* n. If the valuation is $-n < 0$, we say that the element has a *pole of order* n.

Definition. More generally: suppose X is a locally Noetherian scheme. Then for any regular height(=codimension) 1 points (i.e. any point p where $\mathcal{O}_{X,p}$ is a regular local ring), we have a valuation ν. If f is any non-zero element of the fraction field of $\mathcal{O}_{X,p}$ (e.g. if X is integral, and f is a non-zero element of the function field of X), then if $\nu(f) > 0$, we say that the element has a *zero of order* $\nu(f)$, and if $\nu(f) < 0$, we say that the element has a *pole of order* $-\nu(f)$.

We aren't yet allowed to discuss order of vanishing at a point that is not regular codimension 1. One can make a definition, but it doesn't behave as well as it does when have you have a discrete valuation.

Exercise. Suppose X is an integral Noetherian scheme, and $f \in \mathrm{Frac}(\Gamma(X, \mathcal{O}_X))^*$ is a non-zero element of its function field. Show that f has a finite number of zeros and poles. (Hint: reduce to $X = \mathrm{Spec}\, R$. If $f = f_1/f_2$, where $f_i \in R$, prove the result for f_i.)

Now I'd like to discuss the geometry of normal Noetherian schemes. Suppose R is an Noetherian integrally closed domain. Then it is *regular in codimension* 1 (translation: all its codimension at most 1 points are regular). If R is dimension 1, then obviously R is nonsingular=regular=smooth.

Example: $\mathrm{Spec}\, \mathbb{Z}[i]$ is smooth. Reason: it is dimension 1, and $\mathbb{Z}[i]$ is a unique factorization domain, hence its Spec is normal.

Remark: A (Noetherian) scheme can be singular in codimension 2 and still normal. Example: you have shown that the cone $x^2+y^2 = z^2$ in $\mathbb{A}^3$ is normal (PS4, problem B4), but it is clearly singular at the origin (the Zariski tangent space is visibly three-dimensional).

So integral (locally Noetherian) schemes can be singular in codimension 2. But their singularities turn out to be not so bad. I mentioned earlier, before we even knew what normal schemes were, that they satisfied "Hartogs Theorem", that you could extend functions over codimension 2 sets.

Remark: We know that for Noetherian rings we have inclusions:

$$\{\text{regular in codimension } 1\} \supset \{\text{integrally closed}\} \supset \{\text{unique factorization domain}\}.$$

Here are two examples to show you that these inclusions are strict.

Exercise. Let R be the subring $k[x^3, x^2, xy, y] \subset k[x, y]$. (The idea behind this example: I'm allowing all monomials in $k[x, y]$ except for x.) Show that it is not integrally closed (easy — consider the "missing x"). Show that it is regular in codimension 1 (hint: show it is dimension 2, and when you throw out the origin you get something nonsingular, by inverting x^2 and y respectively, and considering R_{x^2} and R_y).

Exercise. You have checked that if $k = \mathbb{C}$, then $k[w, x, y, z]/(wx-yz)$ is integrally closed (PS4, problem B5). Show that it is not a unique factorization domain. (The most obvious possibility is to do this "directly", but this might be hard. Another possibility, faster but less intuitive, is to prove the intermediate result that *in a unique factorization domain, any height* 1 *prime is principal*, and considering an exercise from earlier today that $w = z = 0$ is not principal.)

2. Good facts to know about regular local rings

There are some other important facts to know about regular local rings. In this class, I'm trying to avoid pulling any algebraic facts out of nowhere. As a rule of thumb, anything that you wouldn't see in Math 210, I consider "pulled out of nowhere". Even the harder facts from 210, I'm happy to give you a proof of, if you ask — none of those facts require more than a page of proof. To my knowledge, the only facts I've pulled out of nowhere to date are Krull's Principal Ideal Theorem, and its transcendence degree form. I might even type up a short proof of Krull's Theorem, and put it in the notes, so even that won't come out of nowhere.

Now, smoothness is an easy intuitive concept, but it is algebraically hard — harder than dimension. A sign of this is that I'm going to have to pull three facts out of nowhere. I think it's good for you to see these facts, but I'm going to try to avoid using these facts in the future. So consider them as mainly for culture.

Suppose $(R, \mathfrak{m})$ is a Noetherian regular local ring.

Fact 1. Any localization of R at a prime is also a regular local ring (Eisenbud's *Commutative Algebra*, Cor. 19.14, p. 479).

Hence to check if $\operatorname{Spec} R$ is nonsingular, then it suffices to check at closed points (at maximal ideals). For example, to check if $\mathbb{A}^3$ is nonsingular, you can check at all closed points, because all other points are obtained by localizing further. (You should think about this — it is confusing because of the order reversal between primes and closed subsets.)

Exercise. Show that on a Noetherian scheme, you can check nonsingularity by checking at closed points. (Caution: a scheme in general needn't have any closed points!) You are allowed to use the unproved fact from the notes, that any localization of a regular local ring is regular.

2.1. Less important exercise. Show that there is a nonsingular hypersurface of degree d. Show that there is a Zariski-open subset of the space of hypersurfaces of degree d. The two previous sentences combine to show that the nonsingular hypersurfaces form a Zariski-open set. Translation: almost all hypersurfaces are smooth.

Fact 2. ("leading terms", proved in an important case) The natural map $\mathrm{Sym}^n(\mathfrak{m}/\mathfrak{m}^2) \to \mathfrak{m}^n/\mathfrak{m}^{n+1}$ is an isomorphism. Even better, the following diagram commutes:

$$\begin{tikzcd}
(\mathfrak{m}^i/\mathfrak{m}^{i+1}) \times (\mathfrak{m}^j/\mathfrak{m}^{j+1}) \arrow[r, "\times"] \arrow[d, "\sim"] & \mathfrak{m}^{i+j}/\mathfrak{m}^{i+j+1} \arrow[d, "\sim"] \\
\mathrm{Sym}^i(\mathfrak{m}/\mathfrak{m}^2) \times \mathrm{Sym}^j(\mathfrak{m}/\mathfrak{m}^2) \arrow[r, "\times"] & \mathrm{Sym}^{i+j}(\mathfrak{m}/\mathfrak{m}^2)
\end{tikzcd}$$

Easy Exercise. Suppose $(R, \mathfrak{m}, k)$ is a regular Noetherian local ring of dimension n. Show that $\dim_k(\mathfrak{m}^i/\mathfrak{m}^{i+1}) = \binom{n+i-1}{i}$.

Exercise. Show that Fact 2 also implies that $(R, \mathfrak{m})$ is a domain. (Hint: show that if $f, g \neq 0$, then $fg \neq 0$, by considering the leading terms.)

I don't like facts pulled out of nowhere, so I want to prove it in an important case. Suppose $(R, \mathfrak{m})$ is a Noetherian local ring containing its residue field k: $k \hookrightarrow R \twoheadrightarrow R/\mathfrak{m} = k$. (For example, if k is algebraically closed, this is true for all local rings of finite type k-schemes at maximal ideals, by the Nullstellensatz. But it is not true if $(R, \mathfrak{m}) = (\mathbb{Z}_p, p\mathbb{Z}_p)$, as the residue field $\mathbb{F}_p$ is not a subring of $\mathbb{Z}_p$.)

Suppose R is a regular of dimension n, with $x_1, \dots, x_n \in R$ generating $\mathfrak{m}/\mathfrak{m}^2$ as a vector space (and hence $\mathfrak{m}$ as an ideal, by Nakayama's lemma). Then we get a natural map $k[t_1, \dots, t_n] \to R$, taking t_i to x_i.

2.2. *Theorem. — Suppose* $(R, \mathfrak{m})$ *is a Noetherian regular local ring containing its residue field* k*:* $k \hookrightarrow R \twoheadrightarrow R/\mathfrak{m} = k$. *Then* $k[t_1, \dots, t_n]/(t_1, \dots, t_n)^m \to R/\mathfrak{m}^m$ *is an isomorphism for all* m.

Proof: See Section 3.

To interpret this better, and to use it: define the inverse limit $\hat{R} := \lim_{\leftarrow} R/\mathfrak{m}^n$. This is the *completion* of R at $\mathfrak{m}$. (We can complete any ring at any ideal of course.) For example, if $S = k[x_1, \dots, x_n]$, and $\mathfrak{n} = (x_1, \dots, x_n)$, then $\hat{S} = k[[x_1, \dots, x_n]]$, power series in n variables. We have a good intuition for for power series, so we will be very happy with the next result.

2.3. *Theorem. — Suppose* R *contains its residue field* k*:* $k \hookrightarrow R \twoheadrightarrow R/\mathfrak{m} = k$ *. Then the natural map* $k[[t_1, \ldots, t_n]] \to \hat{R}$ *taking* t_i *to* x_i *is an isomorphism.*

This follows immediately from the previous theorem, as both sides are inverse limits of the same things. I'll now give some consequences.

Note that $R \hookrightarrow \hat{R}$. Here's why. (Recall the interpretation of inverse limit: you can interpret $\hat{R}$ as a subring of $R/\mathfrak{m} \times R/\mathfrak{m}^2 \times R/\mathfrak{m}^3 \times \cdots$ such that if $j > i$, the jth element maps to the ith factor under the natural quotient map.) What can go to 0 in $\hat{R}$? It is something that lies in $\mathfrak{m}^n$ for all n. But $\cap_i \mathfrak{m}^i = 0$ (a fact I stated in class when discussing Nakayama — *I owe you a proof of this*), so the map is injective. (Important note: We aren't assuming regularity of R in this argument!!)

Thus we can think of the map $R \to \hat{R}$ as a power series expansion.

This implies the "leading term" fact in this case (where the local ring contains the residue field). (**Exercise:** Prove this. This isn't hard; it's a matter of making sure you see what the definitions are.) Hence in this case we have proved that R is a domain.

We go back to stating important **facts** that we will try not to use.

Fact 3. Not only is $(R, \mathfrak{m})$ a domain, it is a unique factorization domain, which we have shown implies integrally closed in its fraction field. Reference: Eisenbud Theorem 19.19, p. 483. This implies that regular schemes are normal. Reason: integrally closed iff all local rings are integrally closed domains. I'll explain why later.

3. Promised proof of Theorem 2.2

Let's now set up the proof of Theorem 2.2, with a series of exercises.

3.1. Exercise. If S is a Noetherian ring, show that $S[[t]]$ is Noetherian. (Hint: Suppose $I \subset S[[t]]$ is an ideal. Let $I_n \subset S$ be the coefficients of t^n that appear in the elements of I form an ideal. Show that $I_n \subset I_{n+1}$, and that I is determined by $(I_0, I_1, I_2, \ldots)$.)

3.2. Corollary. $k[[t_1, \ldots, t_n]]$ is a Noetherian local ring.

3.3. Exercise. Show that $\dim k[[t_1, \ldots, t_n]]$ is dimension n. (Hint: find a chain of $n+1$ prime ideals to show that the dimension is at least n. For the other inequality, use Krull.)

3.4. Exercise. If R is a Noetherian local ring, show that $\hat{R} := \lim_{\leftarrow} R/\mathfrak{m}^n$ is a Noetherian local ring. (Hint: Suppose $\mathfrak{m}/\mathfrak{m}^2$ is finite-dimensional over k, say generated by $x_1, \ldots, x_n$. Describe a surjective map $k[[t_1, \ldots, t_n]] \to \hat{R}$.)

We now outline the proof of the Theorem, as an extended exercise. (This is hastily and informally written.)

Suppose $\mathfrak{p} \subset R$ is a prime ideal. Define $\hat{\mathfrak{p}} \subset \hat{R}$ by $\mathfrak{p}/\mathfrak{m}^m \subset R/\mathfrak{m}^m$. Show that $\hat{\mathfrak{p}}$ is a prime ideal of $\hat{R}$. (Hint: if $f, g \notin \mathfrak{p}$, then let m_f, m_g be the first "level" where they are not in $\mathfrak{p}$ (i.e. the smallest m such that $f \notin \mathfrak{p}/\mathfrak{m}^{m+1}$). Show that $fg \notin \mathfrak{p}$ by showing that $fg \notin \mathfrak{p}/\mathfrak{m}^{m_f+m_g+1}$.)

Show that if $\mathfrak{p} \subset \mathfrak{q}$, then $\hat{\mathfrak{p}} \subset \hat{\mathfrak{q}}$. Hence show that $\dim \hat{R} \geq \dim R$. But also $\dim \hat{R} \leq \dim \mathfrak{m}/\mathfrak{m}^2 = \dim R$. Thus $\dim \hat{R} = \dim R$.

We're now ready to prove the Theorem. We wish to show that $k[[t_1, \ldots, t_n]] \to \hat{R}$ is injective; we already know it is surjective. Suppose $f \in k[[t_1, \ldots, t_n]] \mapsto 0$, so we get a map $k[[t_1, \ldots, t_n]]/f$ surjects onto $\hat{R}$. Now f is not a zero-divisor, so by Krull, the left side has dimension $n - 1$. But then any quotient of it has dimension at most $n - 1$, contradiction. $\square$

4. Toward quasicoherent sheaves: the distinguished affine base

Schemes generalize and geometrize the notion of "ring". It is now time to define the corresponding analogue of "module", which is a quasicoherent sheaf.

One version of this notion is that of a sheaf of $\mathcal{O}_X$-modules. They form an abelian category, with tensor products. (That might be called a tensor category — I should check.)

We want a better one — a subcategory of $\mathcal{O}_X$-modules. Because these are the analogues of modules, we're going to define them in terms of affine open sets of the scheme. So let's think a bit harder about the structure of affine open sets on a general scheme X. I'm going to define what I'll call the *distinguished affine base* of the Zariski topology. This won't be a base in the sense that you're used to. (For experts: it is a first example of a *Grothendieck topology*.) It is more akin to a base.

The open sets are the affine open subsets of X. We've already observed that this forms a base. But forget about that.

We like distinguished opens $\operatorname{Spec} R_f \hookrightarrow \operatorname{Spec} R$, and we don't really understand open immersions of one random affine in another. So we just remember the "nice" inclusions.

Definition. The *distinguished affine base* of a scheme X is the data of the affine open sets and the distinguished inclusions.

(Remark we won't need, but is rather fundamental: what we are using here is that we have a collection of open subsets, and *some* subsets, such that if we have any $x \in U, V$ where U and V are in our collection of open sets, there is some W containing x, and contained in U and V such that that $W \hookrightarrow U$ and $W \hookrightarrow V$ are both in our collection of inclusions. In the case we are considering here, this is the key Proposition in Class 9 that given any two affine opens $\operatorname{Spec} A$, $\operatorname{Spec} B$ in X, $\operatorname{Spec} A \cap \operatorname{Spec} B$ could be covered by affine opens that were simultaneously distinguished in $\operatorname{Spec} A$ and $\operatorname{Spec} B$. This is a *cofinal* condition.)

We can define a sheaf on the distinguished affine base in the obvious way: we have a set (or abelian group, or ring) for each affine open set, and we know how to restrict to distinguished open sets.

Given a sheaf $\mathcal{F}$ on X, we get a sheaf on the distinguished affine base. You can guess where we're going: we'll show that all the information of the sheaf is contained in the information of the sheaf on the distinguished affine base.

As a warm-up: We can recover stalks. Here's why. $\mathcal{F}_x$ is the direct limit $\lim_{\rightarrow}(f \in \mathcal{F}(U))$ where the limit is over all open sets contained in U. We compare this to $\lim_{\rightarrow}(f \in \mathcal{F}(U))$ where the limit is over all affine open sets, and all distinguished inclusions. You can check that the elements of one correspond to elements of the other. (Think carefully about this! It corresponds to the fact that the basic elements are cofinal in this directed system.)

4.1. Exercise. Show that a section of a sheaf on the distinguished affine base is determined by the section's germs.

4.2. *Theorem. —*

(a) *A sheaf on the distinguished affine base $\mathcal{F}^b$ determines a unique sheaf $\mathcal{F}$, which when restricted to the affine base is $\mathcal{F}^b$. (Hence if you start with a sheaf, and take the sheaf on the distinguished affine base, and then take the induced sheaf, you get the sheaf you started with.)*
(b) *A morphism of sheaves on an affine base determines a morphism of sheaves.*
(c) *A sheaf of $\mathcal{O}_X$-modules "on the distinguished affine base" yields an $\mathcal{O}_X$-module.*

Proof of (a). (Two comments: this is very reminiscent of our sheafification argument. It also trumps our earlier theorem on sheaves on a nice base.)

Suppose $\mathcal{F}^b$ is a sheaf on the distinguished affine base. Then we can define stalks.

For *any* open set U of X, define

$$\mathcal{F}(U) := \{(f_x \in \mathcal{F}^b_x)_{x \in U} : \forall x \in U, \exists U_X \text{ with } x \subset U_x \subset U, F^x \in \mathcal{F}^b(U_x) : F^x_y = f_y \forall y \in U_x\}$$

where each U_x is in our base, and F^x_y means "the germ of F^x at y". (As usual, those who want to worry about the empty set are welcome to.)

This is a sheaf: convince yourself that we have restriction maps, identity, and gluability, really quite easily.

I next claim that if U is in our base, that $\mathcal{F}(U) = \mathcal{F}^b(U)$. We clearly have a map $\mathcal{F}^b(U) \to \mathcal{F}(U)$. For the map $\mathcal{F}(U) \to \mathcal{F}^b(U)$: **gluability exercise** (a bit subtle).

These are isomorphisms, because elements of $\mathcal{F}(U)$ are determined by stalks, as are elements of $\mathcal{F}^b(U)$.

(b) Follows as before.

(c) **Exercise.** $\square$

5. Quasicoherent sheaves

We now define a **quasicoherent sheaf**. In the same way that a scheme is defined by "gluing together rings", a quasicoherent sheaf over that scheme is obtained by "gluing together modules over those rings". We will give two equivalent definitions; each definition is useful in different circumstances. The first just involves the distinguished topology.

Definition 1. An $\mathcal{O}_X$-module $\mathcal{F}$ is a **quasicoherent sheaf** if for every affine open $\operatorname{Spec} R$ and distinguished affine open $\operatorname{Spec} R_f$ thereof, the restriction map $\phi : \Gamma(\operatorname{Spec} R, \mathcal{F}) \to \Gamma(\operatorname{Spec} R_f, \mathcal{F})$ factors as:

$$\phi : \Gamma(\operatorname{Spec} R, \mathcal{F}) \to \Gamma(\operatorname{Spec} R, \mathcal{F})_f \cong \Gamma(\operatorname{Spec} R_f, \mathcal{F}).$$

The second definition is more directly "sheafy". Given a ring R and a module M, we defined a sheaf $\tilde{M}$ on $\operatorname{Spec} R$ long ago — the sections over $D(f)$ were M_f.

Definition 2. An $\mathcal{O}_X$-module $\mathcal{F}$ is a *quasicoherent sheaf* if for every affine open $\operatorname{Spec} R$,

$$\mathcal{F}|_{\operatorname{Spec} R} \cong \Gamma(\widetilde{\operatorname{Spec} R}, \mathcal{F}).$$

(The "wide tilde" is suposed to cover the entire right side $\Gamma(\operatorname{Spec} R, \mathcal{F})$.) This isomorphism is as sheaves of $\mathcal{O}_X$-modules.

Hence by this definition, the sheaves on $\operatorname{Spec} R$ correspond to R-modules. Given an R-module M, we get a sheaf $\tilde{M}$. Given a sheaf $\mathcal{F}$ on $\operatorname{Spec} R$, we get an R-module $\Gamma(X, \mathcal{F})$. These operations are inverse to each other. So in the same way as schemes are obtained by gluing together rings, quasicoherent sheaves are obtained by gluing together modules over those rings.

By Theorem 4.2, we have:

Definition 2′. An $\mathcal{O}_X$-module on the distinguished affine base yields an $\mathcal{O}_X$-module.

5.1. *Proposition. — Definitions 1 and 2 are the same.*

Proof. Clearly Definition 2 implies Definition 1. (Recall that the definition of $\tilde{M}$ was in terms of the distinguished topology on $\operatorname{Spec} R$.) We now show that Definition 1 implies Definition 2. We use Theorem 4.2. By Definition 1, the sections over any distinguished open $\operatorname{Spec} R_f$ of $\mathcal{M}$ on $\operatorname{Spec} R$ is precisely $\Gamma(\operatorname{Spec} R, \mathcal{M})_f$, i.e. the sections of $\Gamma(\widetilde{\operatorname{Spec} R}, \mathcal{M})$ over $\operatorname{Spec} R_f$, and the restriction maps agree. Thus the two sheaves agree. $\square$

We like Definition 1 because it says that to define a quasicoherent sheaf of $\mathcal{O}_X$-modules is that we just need to know what it is on all affine open sets, and that it behaves well under inverting single elements.

One reason we like Definition 2 is that it glues well.

5.2. *Proposition (quasicoherence is affine-local). — Let* X *be a scheme, and* $\mathcal{M}$ *a sheaf of* $\mathcal{O}_X$*-modules. Then let* P *be the property of affine open sets that* $\mathcal{M}|_{\operatorname{Spec} R} \cong \Gamma(\widetilde{\operatorname{Spec} R, \mathcal{M}})$. *Then* P *is an affine-local property.*

We will prove this next day.

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FOUNDATIONS OF ALGEBRAIC GEOMETRY CLASS 14

CONTENTS

Last day: discrete valuation rings (conclusion), cultural facts to know about regular local rings, the distinguished affine base of the topology, 2 definitions of quasicoherent sheaf.

Today: quasicoherence is affine-local, (locally) free sheaves and vector bundles, invertible sheaves and line bundles, torsion-free sheaves, quasicoherent sheaves of ideals and closed subschemes.

Last day, we defined the distinguished affine base of the Zariski topology of a scheme.

We showed that the information contained in a sheaf was precisely the information contained in a sheaf on the distinguished affine base.

0.1. *Theorem. —*

(a) *A sheaf on the distinguished affine base $\mathcal{F}^b$ determines a unique sheaf $\mathcal{F}$, which when restricted to the affine base is $\mathcal{F}^b$. (Hence if you start with a sheaf, and take the sheaf on the distinguished affine base, and then take the induced sheaf, you get the sheaf you started with.)*
(b) *A morphism of sheaves on an affine base determines a morphism of sheaves.*
(c) *A sheaf of $\mathcal{O}_X$-modules "on the distinguished affine base" yields an $\mathcal{O}_X$-module.*

We then gave two definitions of quasicoherent sheaves.

Definition 1. An $\mathcal{O}_X$-module $\mathcal{F}$ is a **quasicoherent sheaf** if for every affine open $\operatorname{Spec} R$ and distinguished affine open $\operatorname{Spec} R_f$ thereof, the restriction map $\phi : \Gamma(\operatorname{Spec} R, \mathcal{F}) \to$

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$\Gamma(\operatorname{Spec} R_f, \mathcal{F})$ factors as:

$$\phi : \Gamma(\operatorname{Spec} R, \mathcal{F}) \to \Gamma(\operatorname{Spec} R, \mathcal{F})_f \cong \Gamma(\operatorname{Spec} R_f, \mathcal{F}).$$

Definition 2. An $\mathcal{O}_X$-module $\mathcal{F}$ is a *quasicoherent sheaf* if for every affine open $\operatorname{Spec} R$,

$$\mathcal{F}|_{\operatorname{Spec} R} \cong \Gamma(\widetilde{\operatorname{Spec} R}, \mathcal{F}).$$

This isomorphism is as sheaves of $\mathcal{O}_X$-modules.

By part (c) of the above Theorem, an $\mathcal{O}_X$-module on the distinguished affine base yields an $\mathcal{O}_X$-module, so these two notions are equivalent. Thus to give a quasicoherent sheaf, I just need to give you a module for each affine open, and have them behave well with respect to restriction. (That's a priori a little weaker than definition 2, where we actually need an $\mathcal{O}_X$-module.)

Last time I proved:

0.2. *Proposition. — Definitions 1 and 2 are the same.*

1. Onwards!

1.1. *Proposition (quasicoherence is affine-local). — Let X be a scheme, and $\mathcal{F}$ a sheaf of $\mathcal{O}_X$-modules. Then let P be the property of affine open sets that $\mathcal{F}|_{\operatorname{Spec} R} \cong \Gamma(\widetilde{\operatorname{Spec} R}, \mathcal{F})$. Then P is an affine-local property.*

Proof. By the Affine Communication Lemma , we must check two things. Clearly if $\operatorname{Spec} R$ has property P, then so does the distinguished open $\operatorname{Spec} R_f$: if M is an R-module, then $\tilde{M}|_{\operatorname{Spec} R_f} \cong \tilde{M}_f$ as sheaves of $\mathcal{O}_{\operatorname{Spec} R_f}$-modules (both sides agree on the level of distinguished opens and their restriction maps).

We next show the second hypothesis of the Affine Communication Lemma. Suppose we have modules $M_1, \ldots, M_n$, where M_i is an R_{f_i}-module, along with isomorphisms $\phi_{ij} : (M_i)_{f_j} \to (M_j)_{f_i}$ of $R_{f_i f_j}$-modules ($i \neq j$; where $\phi_{ij} = \phi_{ji}^{-1}$). We want to construct an M such that $\tilde{M}$ gives us $\tilde{M}_i$ on $D(f_i) = \operatorname{Spec} R_{f_i}$, or equivalently, isomorphisms $\Gamma(D(f_i), \tilde{M}) \cong M_i$, with restriction maps

$$\begin{tikzcd}
\Gamma(D(f_i), \tilde{M}) \arrow[d] & \Gamma(D(f_j), \tilde{M}) \arrow[d] \\
\Gamma(D(f_i), \tilde{M})_{f_j} \arrow[r, leftrightarrow, "\cong"] & \Gamma(D(f_j), \tilde{M})_{f_i}
\end{tikzcd}$$

that agree with ϕ_{ij}.

We already know what M should be. Consider elements of $M_1 \times \cdots \times M_n$ that "agree on overlaps"; let this set be M. Then

$$0 \to M \to M_1 \times \cdots \times M_n \to M_{12} \times M_{13} \times \cdots \times M_{(n-1)n}$$

is an exact sequence (where $M_{ij} = (M_i)_{f_j} \cong (M_j)_{f_i}$, and the latter morphism is the "difference" morphism). So M is a kernel of a morphism of R-modules, hence an R-module. We show that $M_i \cong M_{f_i}$; for convenience we assume $i = 1$. Localization is exact, so

(1) $0 \to M_{f_1} \to M_1 \times (M_2)_{f_1} \times \cdots \times (M_n)_{f_1} \to M_{12} \times \cdots \times (M_{23})_{f_1} \times \cdots \times (M_{(n-1)n})_{f_1}$

Then by interpreting this exact sequence, you can verify that the kernel is M_1. I gave one proof in class, and I'd like to give two proofs here. We know that $\cup_{i=2}^{n} D(f_1 f_i)$ is a distinguished cover of $D(f_1) = \operatorname{Spec} R_1$. So we have an exact sequence

$$0 \to M_1 \to (M_1)_{f_2} \times \cdots \times (M_1)_{f_n} \to (M_1)_{f_2 f_3} \times \cdots \times (M_1)_{f_{n-1} f_n}.$$

Put two copies on top of each other, and add vertical isomorphisms, alternating between identity and the negative of the identity:

$$\begin{tikzcd}
0 \arrow[r] & M_1 \arrow[r] \arrow[d, "\mathrm{id}"] & (M_1)_{f_2} \times \cdots \times (M_1)_{f_n} \arrow[r] \arrow[d, "-\mathrm{id}"] & (M_1)_{f_2 f_3} \times \cdots \times (M_1)_{f_{n-1} f_n} \\
0 \arrow[r] & M_1 \arrow[r] & (M_1)_{f_2} \times \cdots \times (M_1)_{f_n} \arrow[r] & (M_1)_{f_2 f_3} \times \cdots \times (M_1)_{f_{n-1} f_n}
\end{tikzcd}$$

Then the *total complex* of this *double complex* is exact as well (**exercise**). (The total complex is obtained as follows. The terms are obtained by taking the direct sum in each southwest-to-northeast diagonal. This is a baby case of something essential so check it, if you've never seen it before!). But this is the same sequence as (1), except M_{f_1} replaces M_1, so we have our desired isomorphism.

Here is a second proof that the sequence

(2) $0 \to M_1 \to M_1 \times (M_2)_{f_1} \times \cdots \times (M_n)_{f_1} \to M_{12} \times \cdots \times (M_{23})_{f_1} \times \cdots \times (M_{(n-1)n})_{f_1}$

is exact. To check exactness of a complex of R-modules, it suffices to check exactness "at each prime $\mathfrak{p}$". In other words, if a complex is exact once tensored with $R_\mathfrak{p}$ for all $\mathfrak{p}$, then it was exact to begin with. Now note that if N is an R-module, then $(N_{f_i})_\mathfrak{p}$ is 0 if $f_i \in \mathfrak{p}$, and $N_\mathfrak{p}$ otherwise. Hence after tensoring with $R_\mathfrak{p}$, each term in (2) is either 0 or $N_\mathfrak{p}$, and the reader will quickly verify that the resulting complex is exact. (If any reader thinks I should say a few words as to why this is true, they should let me know, and I'll add a bit to these notes. I'm beginning to think that I should re-work some of my earlier arguments, including for example base gluability and base identity of the structure sheaf, in this way.) □

At this point, you probably want an example. I'll give you a boring example, and save a more interesting one for the end of the class.

Example: $\mathcal{O}_X$ is a quasicoherent sheaf. Over each affine open $\operatorname{Spec} R$, it is isomorphic the module $M = R$. This is not yet enough to specify what the sheaf is! We need also to describe the distinguished restriction maps, which are given by $R \to R_f$, where these are the "natural" ones. (This is confusing because this sheaf is too simple!) A variation on this theme is $\mathcal{O}_X^{\oplus n}$ (interpreted in the obvious way). This is called a *rank* n *free sheaf*. It corresponds to a rank n trivial vector bundle.

Joe mentioned an example of an $\mathcal{O}_X$-module that is not a quasicoherent sheaf last day.

1.2. Exercise. (a) Suppose $X = \operatorname{Spec} k[t]$. Let $\mathcal{F}$ be the skyscraper sheaf supported at the origin $[(t)]$, with group $k(t)$. Give this the structure of an $\mathcal{O}_X$-module. Show that this is not a quasicoherent sheaf. (More generally, if X is an integral scheme, and $p \in X$ that is not the generic point, we could take the skyscraper sheaf at p with group the function field of X. Except in a silly circumstances, this sheaf won't be quasicoherent.)
(b) Suppose $X = \operatorname{Spec} k[t]$. Let $\mathcal{F}$ be the skyscraper sheaf supported at the generic point $[(0)]$, with group $k(t)$. Give this the structure of an $\mathcal{O}_X$-module. Show that this *is* a quasicoherent sheaf. Describe the restriction maps in the distinguished topology of X. (Joe remarked that this is a constant sheaf!)

1.3. *Important Exercise for later.* Suppose X is a Noetherian scheme. Suppose $\mathcal{F}$ is a quasicoherent sheaf on X, and let $f \in \Gamma(X, \mathcal{O}_X)$ be a function on X. Let $R = \Gamma(X, \mathcal{O}_X)$ for convenience. Show that the restriction map $\operatorname{res}_{X_f \subset X} : \Gamma(X, \mathcal{F}) \to \Gamma(X_f, \mathcal{F})$ (here X_f is the open subset of X where f doesn't vanish) is precisely localization. In other words show that there is an isomorphism $\Gamma(X, \mathcal{F})_f \to \Gamma(X_f, \mathcal{F})$ making the following diagram commute.

$$\begin{tikzcd}
\Gamma(X, \mathcal{F}) \arrow[rr, "\operatorname{res}_{X_f \subset X}"] \arrow[dr, "\otimes_R R_f"'] & & \Gamma(X_f, \mathcal{F}) \\
& \Gamma(X, \mathcal{F})_f \arrow[ur, "\sim"'] &
\end{tikzcd}$$

All that you should need in your argument is that X admits a cover by a finite number of open sets, and that their pairwise intersections are each quasicompact. We will later rephrase this as saying that X is quasicompact and quasiseparated. (Hint: cover by affine open sets. Use the sheaf property. A nice way to formalize this is the following. Apply the exact functor $\otimes_R R_f$ to the exact sequence

$$0 \to \Gamma(X, \mathcal{F}) \to \oplus_i \Gamma(U_i, \mathcal{F}) \to \oplus \Gamma(U_{ijk}, \mathcal{F})$$

where the U_i form a finite cover of X and U_{ijk} form an affine cover of $U_i \cap U_j$.)

1.4. *Less important exercise.* Give a counterexample to show that the above statement need not hold if X is not quasicompact. (Possible hint: take an infinite disjoint union of affine schemes.)

For the experts: I don't know a counterexample to this when the quasiseparated hypothesis is removed. Using the exact sequence above, I can show that there is a map $\Gamma(X_f, \mathcal{F}) \to \Gamma(X, \mathcal{F})_f$.

2. Locally free sheaves

I want to show you how that quasicoherent sheaves somehow generalize the notion of vector bundles.

(For arithmetic people: don't tune out! Fractional ideals of the ring of integers in a number field will turn out to be an example of a "line bundle on a smooth curve".)

Since this is motivation, I won't make this precise, so you should feel free to think of this in the differentiable category (i.e. the category of differentiable manifolds). A rank n vector bundle on a manifold M is a fibration $\pi : V \to M$ that locally looks like the product with n-space: every point of M has a neighborhood U such that $\pi^{-1}(U) \cong U \times \mathbb{R}^n$, where the projection map is the obvious one, i.e. the following diagram commutes.

$$\begin{tikzcd} \pi^{-1}(U) \arrow[rr, leftrightarrow, "\cong"] \arrow[dr, "\pi|_{\pi^{-1}(U)}"'] & & U \times \mathbb{R}^n \arrow[dl, "\text{projection to first factor}"] \\ & U & \end{tikzcd}$$

This is called a *trivialization over* U. We also want a "consistent vector space structure". Thus given trivializations over U_1 and U_2, over their intersection, the two trivializations should be related by an element of $GL(n)$ with entries consisting of functions on $U_1 \cap U_2$.

Examples of this include for example the tangent bundle on a sphere, and the moebius strip over $\mathbb{R}^1$.

Pick your favorite vector bundle, and consider its sheaf of sections $\mathcal{F}$. Then the sections over any open set form a real vector space. Moreover, given a U and a trivialization, the sections are naturally n-tuples of functions of U. [If I can figure out how to do curly arrows in xymatrix, I'll fix this.]

$$\begin{tikzcd} U \times \mathbb{R}^n \arrow[d, "\pi"', shift right] \\ U \arrow[u, bend right, "f_1, \ldots, f_n"'] \end{tikzcd}$$

The open sets over which V is trivial forms a nice base of the topology.

Motivated by this, we define a *locally free sheaf of rank* n on a scheme X as follows. It is a quasicoherent sheaf that is locally, well, free of rank n. It corresponds to a vector bundle. It is determined by the following data: a cover U_i of X, and for each i, j transition functions T_{ij} lying in $GL(n, \Gamma(U_i \cap U_j, \mathcal{O}_X))$ satisfying

$$T_{ii} = Id_n, T_{ij} T_{jk} = T_{ik}$$

(which implies $T_{ij} = T_{ji}^{-1}$). Given this data, we can find the sections over any open set U as follows. Informally, they are sections of the free sheaves over each $U \cap U_i$ that agree on overlaps. More formally, for each i, they are $\vec{s}^i = \begin{pmatrix} s_1^i \\ \vdots \\ s_n^i \end{pmatrix} \in \Gamma(U \cap U_i, \mathcal{O}_X)^n$, satisfying $T_{ij} \vec{s}^i = \vec{s}^j$ on $U \cap U_i \cap U_j$.

In the differentiable category, locally free sheaves correspond precisely to vector bundles (for example, you can describe them with the same transition functions). So you should really think of these "as" vector bundles, but just keep in mind that they are not the "same", just equivalent notions.

A rank 1 vector bundle is called a *line bundle*. Similarly, a rank 1 locally free sheaf is called an *invertible sheaf*. I'll later explain why it is called invertible; but it is still a somewhat heinous term for something so fundamental.

Caution: Not every quasicoherent sheaf is locally free.

In a few sections, we will define some operations on quasicoherent sheaves that generate natural operations on vector bundles (such as dual, Hom, tensor product, etc.). The constructions will behave particularly well for locally free sheaves. We will see that the invertible sheaves on X will form a group under tensor product, called the *Picard group* of X.

We first make precise our discussion of transition functions. Given a rank n locally free sheaf $\mathcal{F}$ on a scheme X, we get transition functions as follows. Choose an open cover U_i of X so that $\mathcal{F}$ is a free rank n sheaf on each U_i. Choose a basis $e_{i,1}, \ldots, e_{i,n}$ of $\mathcal{F}$ over U_i. Then over $U_i \cap U_j$, for each k, $e_{i,k}$ can be written as a $\Gamma(U_i \cap U_j, \mathcal{O}_X)$-linear combination of the $e_{j,l}$ ($1 \leq l \leq n$), so we get an $n \times n$ "transition matrix" T_{ji} with entries in $\Gamma(U_i \cap U_j, \mathcal{O}_X)$. Similarly, we get T_{ij}, and $T_{ij}T_{ji} = T_{ji}T_{ij} = I_n$, so T_{ij} and T_{ji} are invertible. Also, on $U_i \cap U_j \cap U_k$, we readily have $T_{ik} = T_{ij}T_{jk}$: both give the matrix that expresses the basis vectors of $e_{i,q}$ in terms of $e_{k,q}$. [Make sure this is right!]

2.1. Exercise. Conversely, given transition functions $T_{ij} \in GL(n, \Gamma(U_i \cap U_j, \mathcal{O}_X))$ satisfying the cocycle condition $T_{ij}T_{jk} = T_{ik}$ "on $U_i \cap U_j \cap U_k$", describe the corresponding rank n locally free sheaf.

We end this section with a few stray comments.

Caution: there are new morphisms between locally free sheaves, compared with what people usually say for vector bundles. Give example on $\mathbb{A}^1$:

$$0 \to tk[t] \to k[t] \to k[t]/(t) \to 0.$$

For vector bundle people: the thing on the left isn't a morphism of vector bundles (at least according to some definitions). (If you think it is a morphism of vector bundles, then you should still be disturbed, because its cokernel is not a vector bundle!)

2.2. Remark. Based on your intuition for line bundles on manifolds, you might hope that every point has a "small" open neighborhood on which all invertible sheaves (or locally free sheaves) are trivial. Sadly, this is not the case. We will eventually see that for the curve $y^2 - x^3 - x = 0$ in $\mathbb{A}^2_{\mathbb{C}}$, every nonempty open set has nontrivial invertible sheaves. (This will use the fact that it is an open subset of an *elliptic curve*.)

2.3. Exercise (for arithmetically-minded people only — I won't define my terms). Prove that a fractional ideal on a ring of integers in a number field yields an invertible sheaf. Show that any two that differ by a principal ideal yield the same invertible sheaf.

Thus we have described a map from the class group of the number field to the Picard group of its ring of integers. It turns out that this is an isomorphism. So strangely the number theorists in this class are the first to have an example of a nontrivial line bundle.

2.4. Exercise (for those familiar with Hartogs' Theorem for Noetherian normal schemes). Show that locally free sheaves on Noetherian normal schemes satisfy "Hartogs' theorem": sections defined away from a set of codimension at least 2 extend over that set.

3. Quasicoherent sheaves form an abelian category

The category of R-modules is an abelian category. (Indeed, this is our motivating example of our notion of abelian category.) Similarly, quasicoherent sheaves form an abelian category. I'll explain how.

When you show that something is an abelian category, you have to check many things, because the definition has many parts. However, if the objects you are considering lie in some ambient abelian category, then it is much easier. As a metaphor, there are several things you have to do to check that something is a group. But if you have a subset of group elements, it is much easier to check that it is a subgroup.

You can look at back at the definition of an abelian category, and you'll see that in order to check that a subcategory is an abelian subcategory, you need to check only the following things:

(i) 0 is in your subcategory
(ii) your subcategory is closed under finite sums
(iii) your subcategory is closed under kernels and cokernels

In our case of $\{$quasicoherent sheaves$\} \subset \{\mathcal{O}_X$-modules$\}$, the first two are cheap: 0 is certainly quasicoherent, and the subcategory is closed under finite sums: if $\mathcal{F}$ and $\mathcal{G}$ are sheaves on X, and over $\operatorname{Spec} R$, $\mathcal{F} \cong \tilde{M}$ and $\mathcal{G} \cong \tilde{N}$, then $\mathcal{F} \oplus \mathcal{G} = \widetilde{M \oplus N}$, so $\mathcal{F} \oplus \mathcal{G}$ is a quasicoherent sheaf.

We now check (iii). Suppose $\alpha : \mathcal{F} \to \mathcal{G}$ is a morphism of quasicoherent sheaves. Then on any affine open set U, where the morphism is given by $\beta : M \to N$, define $(\ker \alpha)(U) = \ker \beta$ and $(\operatorname{coker} \alpha)(U) = \operatorname{coker} \beta$. Then these behave well under inversion of a single element: if

$$0 \to K \to M \to N \to P \to 0$$

is exact, then so is

$$0 \to K_f \to M_f \to N_f \to P_f \to 0,$$

from which $(\ker \beta)_f \cong \ker(\beta_f)$ and $(\operatorname{coker} \beta)_f \cong \operatorname{coker}(\beta_f)$. Thus both of these define quasicoherent sheaves. Moreover, by checking stalks, they are indeed the kernel and cokernel of α. Thus the quasicoherent sheaves indeed form an abelian category.

As a side benefit, we see that we may check injectivity, surjectivity, or exactness of a morphism of quasicoherent sheaves by checking on an affine cover.

Warning: If $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$ is an exact sequence of quasicoherent sheaves, then for any open set

$$0 \to \mathcal{F}(U) \to \mathcal{G}(U) \to \mathcal{H}(U)$$

is exact, and we have exactness on the right is guaranteed to hold only if U is affine. (To set you up for cohomology: whenever you see left-exactness, you expect to eventually interpret this as a start of a long exact sequence. So we are expecting H^1's on the right, and now we expect that $H^1(\operatorname{Spec} R, \mathcal{F}) = 0$. This will indeed be the case.)

3.1. *Exercise.* Show that you can check exactness of a sequence of quasicoherent sheaves on an affine cover. (In particular, taking sections over an affine open $\operatorname{Spec} R$ is an exact functor from the category of quasicoherent sheaves on X to the category of R-modules. Recall that taking sections is only left-exact in general.) Similarly, you can check surjectivity on an affine cover (unlike sheaves in general).

4. Module-like constructions on quasicoherent sheaves

In a similar way, basically any nice construction involving modules extends to quasicoherent sheaves.

As an important example, we consider tensor products. **Exercise.** If $\mathcal{F}$ and $\mathcal{G}$ are quasicoherent sheaves, show that $\mathcal{F} \otimes \mathcal{G}$ is given by the following information: If $\operatorname{Spec} R$ is an affine open, and $\Gamma(\operatorname{Spec} R, \mathcal{F}) = M$ and $\Gamma(\operatorname{Spec} R, \mathcal{G}) = N$, then $\Gamma(\operatorname{Spec} R, \mathcal{F} \otimes \mathcal{G}) = M \otimes N$, and the restriction map $\Gamma(\operatorname{Spec} R, \mathcal{F} \otimes \mathcal{G}) \to \Gamma(\operatorname{Spec} R_f, \mathcal{F} \otimes \mathcal{G})$ is precisely the localization map $M \otimes_R N \to (M \otimes_R N)_f \cong M_f \otimes_{R_f} N_f$. (We are using the algebraic fact that that $(M \otimes_R N)_f \cong M_f \otimes_{R_f} N_f$. You can prove this by universal property if you want, or by using the explicit construction.)

Note that thanks to the machinery behind the distinguished affine base, sheafification is taken care of.

For category-lovers: this makes the category of quasicoherent sheaves into a monoid.

4.1. Exercise. If $\mathcal{F}$ and $\mathcal{G}$ are locally free sheaves, show that $\mathcal{F} \otimes \mathcal{G}$ is locally free. (Possible hint for this, and later exercises: check on sufficiently small affine open sets.)

4.2. *Exercise.* (a) Tensoring by a quasicoherent sheaf is right-exact. More precisely, if $\mathcal{F}$ is a quasicoherent sheaf, and $\mathcal{G}' \to \mathcal{G} \to \mathcal{G}'' \to 0$ is an exact sequence of quasicoherent sheaves, then so is $\mathcal{G}' \otimes \mathcal{F} \to \mathcal{G} \otimes \mathcal{F} \to \mathcal{G}'' \otimes \mathcal{F} \to 0$ is exact.
(b) Tensoring by a locally free sheaf is exact. More precisely, if $\mathcal{F}$ is a locally free sheaf, and $\mathcal{G}' \to \mathcal{G} \to \mathcal{G}''$ is an exact sequence of quasicoherent sheaves, then then so is $\mathcal{G}' \otimes \mathcal{F} \to \mathcal{G} \otimes \mathcal{F} \to \mathcal{G}'' \otimes \mathcal{F}$.
(c) The stalk of the tensor product of quasicoherent sheaves at a point is the tensor product of the stalks.

Note: if you have a section s of $\mathcal{F}$ and a section t of $\mathcal{G}$, you get a section $s \otimes t$ of $\mathcal{F} \otimes \mathcal{G}$. If either $\mathcal{F}$ or $\mathcal{G}$ is an invertible sheaf, this section is denoted st.

We now describe other constructions.

4.3. *Exercise.* Sheaf Hom, $\underline{\mathrm{Hom}}$, is quasicoherent, and is what you think it might be. (Describe it on affine opens, and show that it behaves well with respect to localization with respect to f. To show that $\mathrm{Hom}_A(M,N)_f \cong \mathrm{Hom}_{A_f}(M_f, N_f)$, take a "partial resolution" $A^q \to A^p \to M \to 0$, and apply $\mathrm{Hom}(\cdot, N)$ and localize.) ($\underline{\mathrm{Hom}}$ was defined earlier, and was the subject of a homework problem.) Show that $\underline{\mathrm{Hom}}$ is a left-exact functor in both variables.

Definition. $\underline{\mathrm{Hom}}(\mathcal{F}, \mathcal{O}_X)$ is called the *dual* of $\mathcal{F}$, and is denoted $\mathcal{F}^\vee$.

4.4. *Exercise.* The direct sum of quasicoherent sheaves is what you think it is.

5. Some notions especially relevant for locally free sheaves

Exercise. Show that if $\mathcal{F}$ is locally free then $\mathcal{F}^\vee$ is locally free, and that there is a canonical isomorphism $(\mathcal{F}^\vee)^\vee \cong \mathcal{F}$. (Caution: your argument showing that if there is a canonical isomorphism $(\mathcal{F}^\vee)^\vee \cong \mathcal{F}$ better not also show that there is a canonical isomorphism $\mathcal{F}^\vee \cong \mathcal{F}$! We'll see an example soon of a locally free $\mathcal{F}$ that is not isomorphic to its dual. The example will be the line bundle $\mathcal{O}(1)$ on $\mathbb{P}^1$.)

Remark. This is not true for quasicoherent sheaves in general, although your argument will imply that there is always a natural morphism $\mathcal{F} \to (\mathcal{F}^\vee)^\vee$. Quasicoherent sheaves for which this is true are called *reflexive sheaves*. We will not be using this notion. Your argument may also lead to a canonical map $\mathcal{F} \otimes \mathcal{F}^\vee \to \mathcal{O}_X$. This could be called the *trace* map — can you see why?

5.1. *Exercise.* Given transition functions for the locally free sheaf $\mathcal{F}$, describe the transition functions for the locally free sheaf $\mathcal{F}^\vee$. Note that if $\mathcal{F}$ is rank 1 (i.e. locally free), the transition functions of the dual are the inverse of the transition functions of the original; in this case, $\mathcal{F} \otimes \mathcal{F}^\vee \cong \mathcal{O}_X$.

5.2. *Exercise.* If $\mathcal{F}$ and $\mathcal{G}$ are locally free sheaves, show that $\mathcal{F} \otimes \mathcal{G}$ and $\underline{\mathrm{Hom}}(\mathcal{F}, \mathcal{G})$ are both locally free.

5.3. *Exercise.* Show that the invertible sheaves on X, up to isomorphism, form an abelian group under tensor product. This is called the *Picard group* of X, and is denoted $\mathrm{Pic}\, X$. (For arithmetic people: this group, for the Spec of the ring of integers R in a number field, is the class group of R.)

For the next exercises, recall the following. If M is an A-module, then the *tensor algebra* $T^*(M)$ is a non-commutative algebra, graded by $\mathbb{Z}^{\geq 0}$, defined as follows. $T^0(M) = A$, $T^n(M) = M \otimes_A \cdots \otimes_A M$ (where n terms appear in the product), and multiplication is what you expect. The *symmetric algebra* $\operatorname{Sym}^* M$ is a symmetric algebra, graded by $\mathbb{Z}^{\geq 0}$, defined as the quotient of $T^*(M)$ by the (two-sided) ideal generated by all elements of the form $x \otimes y - y \otimes x$ for all $x, y \in M$. Thus $\operatorname{Sym}^n M$ is the quotient of $M \otimes \cdots \otimes M$ by the relations of the form $m_1 \otimes \cdots \otimes m_n - m_1' \otimes \cdots \otimes m_n'$ where $(m_1', \dots, m_n')$ is a rearrangement of $(m_1, \dots, m_n)$. The *exterior algebra* $\wedge^* M$ is defined to be the quotient of $T^* M$ by the (two-sided) ideal generated by all elements of the form $x \otimes y + y \otimes x$ for all $x, y \in M$. Thus $\wedge^n M$ is the quotient of $M \otimes \cdots \otimes M$ by the relations of the form $m_1 \otimes \cdots \otimes m_n - (-1)^{\operatorname{sgn}} m_1' \otimes \cdots \otimes m_n'$ where $(m_1', \dots, m_n')$ is a rearrangement of $(m_1, \dots, m_n)$, and the sgn is even if the rearrangement is an even permutation, and odd if the rearrangement is an odd permutation. (It is a "skew-commutative" A-algebra.) It is most correct to write $T^*_A(M)$, $\operatorname{Sym}^*_A(M)$, and $\wedge^*_A(M)$, but the "base ring" is usually omitted for convenience.

5.4. *Exercise.* If $\mathcal{F}$ is a quasicoherent sheaf, then define the quasicoherent sheaves $T^n \mathcal{F}$, $\operatorname{Sym}^n \mathcal{F}$, and $\wedge^n \mathcal{F}$. If $\mathcal{F}$ is locally free of rank m, show that $T^n \mathcal{F}$, $\operatorname{Sym}^n \mathcal{F}$, and $\wedge^n \mathcal{F}$ are locally free, and find their ranks.

You can also define the sheaf of non-commutative algebras $T^* \mathcal{F}$, the sheaf of algebras $\operatorname{Sym}^* \mathcal{F}$, and the sheaf of skew-commutative algebras $\wedge^* \mathcal{F}$.

5.5. Important exercise. If $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ is an exact sequence of locally free sheaves, then for any r, there is a filtration of $\operatorname{Sym}^r \mathcal{F}$:

$$\operatorname{Sym}^r \mathcal{F} = F^0 \supseteq F^1 \supseteq \cdots \supseteq F^r \supset F^{r+1} = 0$$

with quotients

$$F^p / F^{p+1} \cong (\operatorname{Sym}^p \mathcal{F}') \otimes (\operatorname{Sym}^{r-p} \mathcal{F}'')$$

for each p.

5.6. Exercise. Suppose $\mathcal{F}$ is locally free of rank n. Then $\wedge^n \mathcal{F}$ is called the *determinant (line) bundle*. Show that $\wedge^r \mathcal{F} \times \wedge^{n-r} \mathcal{F} \to \wedge^n \mathcal{F}$ is a perfect pairing for all r.

5.7. Exercise. If $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ is an exact sequence of locally free sheaves, then for any r, there is a filtration of $\wedge^r \mathcal{F}$:

$$\wedge^r \mathcal{F} = F^0 \supseteq F^1 \supseteq \cdots \supseteq F^r \supset F^{r+1} = 0$$

with quotients

$$F^p / F^{p+1} \cong (\wedge^p \mathcal{F}') \otimes (\wedge^{r-p} \mathcal{F}'')$$

for each p. In particular, $\det \mathcal{F} = (\det \mathcal{F}') \otimes (\det \mathcal{F}'')$.

5.8. Exercise (torsion-free sheaves). An R-module M is torsion-free if $rm = 0$ implies $r = 0$ or $m = 0$. Show that this satisfies the hypotheses of the affine communication lemma. Hence we make a definition: a quasicoherent sheaf is *torsion-free* if for one (or by the affine communication lemma, for any) affine cover, the sections over each affine open are torsion-free. By definition, "torsion-freeness is affine-local". Show that a quasicoherent sheaf is torsion-free if all its stalks are torsion-free. Hence "torsion-freeness" is "stalk-local." [This exercise is wrong! "Torsion-freeness" is should be defined as "torsion-free stalks" — it is (defined as) a "stalk-local" condition. Here is a better exercise. Show that if M is torsion-free, then so is any localization of M. In particular, M_f is torsion-free, so this notion satisfies half the hypotheses of the affine communication lemma. Also, $M_{\mathfrak{p}}$ is torsion-free, so this implies that $\tilde{M}$ is torsion-free. Find an example on a two-point space showing that R might not be torsion-free even though $\mathcal{O}_{\operatorname{Spec} R} = \tilde{R}$ is torsion-free.]

6. Quasicoherent sheaves of ideals, and closed subschemes

I then defined quasicoherent sheaves of ideals, and closed subschemes. But I'm happier with the definition I gave in class 15, so I'll leave the discussion until then.

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FOUNDATIONS OF ALGEBRAIC GEOMETRY CLASS 15

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CONTENTS

Last day: quasicoherence is affine-local, (locally) free sheaves and vector bundles, invertible sheaves and line bundles, torsion-free sheaves, quasicoherent sheaves of ideals and closed subschemes.

Today: Quasicoherent sheaves form an abelian category; finite type and coherent sheaves; support; rank; quasicoherent sheaves of ideals and closed subschemes.

I'd like to start by restating some of the definitions and arguments from last day.

Suppose X is a scheme. Recall that $\mathcal{O}_X$-module $\mathcal{F}$ is a quasicoherent sheaf if one of two equivalent things is true.
(i) For every affine open $\operatorname{Spec} R$ and distinguished affine open $\operatorname{Spec} R_f$ thereof, the restriction map $\phi : \Gamma(\operatorname{Spec} R, \mathcal{F}) \to \Gamma(\operatorname{Spec} R_f, \mathcal{F})$ factors as:

$$\phi : \Gamma(\operatorname{Spec} R, \mathcal{F}) \to \Gamma(\operatorname{Spec} R, \mathcal{F})_f \cong \Gamma(\operatorname{Spec} R_f, \mathcal{F}).$$

(ii) For any affine open set $\operatorname{Spec} R$, $\mathcal{F}|_{\operatorname{Spec} R} \cong \tilde{M}$ for some R-module M.

I will use both definitions today.

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1. Quasicoherent sheaves form an abelian category

Last day, I showed that the quasicoherent sheaves on X form an abelian category, and in fact an abelian subcategory of $\mathcal{O}_X$-modules. I restated the argument in a better way today. I've moved this exposition back into the Class 14 notes.

2. Finiteness conditions on quasicoherent sheaves: finitely generated quasicoherent sheaves, and coherent sheaves

There are some natural finiteness conditions on an A-module M. I will tell you three. In the case when A is a Noetherian ring, which is the case that almost all of you will ever care about, they are all the same.

The first is the most naive: a module could be *finitely generated*. In other words, there is a surjection $A^p \to M \to 0$.

The second is reasonable too: it could be finitely presented. In other words, it could have a finite number of generators with a finite number of relations: there exists a *finite presentation*

$$A^q \to A^p \to M \to 0.$$

The third is frankly a bit surprising, and I'll justify it soon. We say that an A-module M is *coherent* if (i) it is finitely generated, and (ii) whenenver we have a map $A^p \to M$ (not necessarily surjective!), the kernel is finitely generated.

Clearly coherent implies finitely presented, which in turn implies finitely generated.

2.1. *Proposition. — If A is Noetherian, then these three definitions are the same.*

Preparatory facts. If R is any ring, not necessarily Noetherian, we say an R-module is Noetherian if it satisfies the ascending chain condition for submodules. **Exercise.** M Noetherian implies that any submodule of M is a finitely generated R-module. Hence for example if R is a Noetherian ring then finitely generated = Noetherian. **Exercise.** If $0 \to M' \to M \to M'' \to 0$ is exact, then M' and M'' are Noetherian if and only if M is Noetherian. (Hint: Given an ascending chain in M, we get two simultaneous ascending chains in M' and M''.) **Exercise.** A Noetherian as an A-module implies A^n is a Noetherian A-module. **Exercise.** If A is a Noetherian ring and M is a finitely generated A-module, then any submodule of M is finitely generated. (Hint: suppose $M' \hookrightarrow M$ and $A^n \twoheadrightarrow M$. Construct N with
$$\begin{tikzcd} N \arrow[r, hook] \arrow[d, two heads] & A^n \arrow[d, two heads] \\ M' \arrow[r, hook] & M \end{tikzcd}$$
.)

Proof. Clearly both finitely presented and coherent imply finitely generated.

Suppose M is finitely generated. Then take any $A^p \xrightarrow{\alpha} M$. $\ker \alpha$ is a submodule of a finitely generated module over A, and is thus finitely generated. (Here's why submodules of finitely generated modules over Noetherian rings are also finitely generated: Show it is true for $M = A^n$ — this takes some inspiration. Then given $N \subset N$, consider $A^n \twoheadrightarrow M$, and take the submodule corresponding to N.) Thus we have shown coherence. By choosing a surjective $A^p \to M$, we get finite presentation. $\square$

Hence almost all of you can think of these three notions as the same thing.

2.2. *Lemma. — The coherent A-modules form an abelian subcategory of the category of A-modules.*

I will prove this in the case where A is Noetherian, but I'll include a series of short exercises in the notes that will show it in general.

Proof if A is Noetherian. Recall that we have four things to check (see our discussion earlier today). We quickly check that 0 is finitely generated (=coherent), and that if M and N are finitely generated, then $M \oplus N$ is finitely generated. Suppose now that $f : M \to N$ is a map of finitely generated modules. Then $\operatorname{coker} f$ is finitely generated (it is the image of N), and $\ker f$ is too (it is a submodule of a finitely generated module over a Noetherian ring). $\square$

Easy Exercise (only important for non-Noetherian people). Show A is coherent (as an A-module) if and only if the notion of finitely presented agrees with the notion of coherent.

I want to say a few words on the notion of coherence. There is a good reason for this definition — because of this lemma. There are two sorts of people who should care. Complex geometers should care. They consider complex-analytic spaces with the classical topology. One can define the notion of coherent $\mathcal{O}_X$-module in a way analogous to this. You can then show that the structure sheaf is coherent, and this is very hard. (It is called Oka's theorem, and takes a lot of work to prove.) I believe the notion of coherence may have come originally from complex geometry.

The second sort of people who should care are the sort of arithmetic people who sometimes are forced to consider non-Noetherian rings. (For example, for people who know what they are, the ring of adeles is non-Noetherian.)

Warning: it is common in the later literature to define coherent as finitely generated. It's possible that Hartshorne does this. Please don't do this, as it will only cause confusion. (In fact, if you google the notion of coherent sheaf, you'll get this faulty definition repeatedly.) I will try to be scrupulous about this. Besides doing this for the reason of honesty, it will also help you see what hypotheses are actually necessary to prove things — and that always helps me remember what the proofs are.

2.3. Exercise. If $f \in A$, show that if M is a finitely generated (resp. finitely presented, coherent) A-module, then M_f is a finitely generated (resp. finitely presented, coherent) A_f-module.

Exercise. If $(f_1, \ldots, f_n) = A$, and M_{f_i} is finitely generated (resp. coherent) A_{f_i}-module for all i, then M is a finitely generated (resp. coherent) A-module.

I'm not sure if that exercise is even true for finitely presented. That's one of several reasons why I think that "finitely presented" is a worse notion than coherence.

Definition. A quasicoherent sheaf $\mathcal{F}$ is *finite type* (resp. *coherent*) if for every affine open $\operatorname{Spec} R$, $\Gamma(\operatorname{Spec} R, \mathcal{F})$ is a finitely generated (resp. coherent) R-module.

Thanks to the affine communication lemma, and the two previous exercises, it suffices to check this on the opens in a single affine cover.

3. Coherent modules over non-Noetherian rings

Here are some notes on coherent modules over a general ring. Read this only if you really want to! I did not discuss this in class, but promised it in the notes.

Suppose A is a ring. We say an A-module M is *finitely generated* if there is a surjection $A^n \to M \to 0$. We say it is *finitely presented* if there is a presentation $A^m \to A^n \to M \to 0$. We say M is *coherent* if (i) M is finitely generated, and (ii) every map $A^n \to M$ has a finitely generated kernel. The reason we like this third definition is that coherent modules form an abelian category.

Here are some quite accessible problems working out why these notions behave well.

1. Show that coherent implies finitely presented implies finitely generated.

2. Show that 0 is coherent.

Suppose for problems 3–9 that

$$0 \to M \to N \to P \to 0 \tag{1}$$

is an exact sequence of A-modules.

Hint $\star$. Here is a *hint* which applies to several of the problems: try to write

$$\begin{tikzcd} 0 \arrow[r] & A^p \arrow[r] \arrow[d] & A^{p+q} \arrow[r] \arrow[d] & A^q \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & M \arrow[r] & N \arrow[r] & P \arrow[r] & 0 \end{tikzcd}$$

and possibly use the snake lemma.

3. Show that N finitely generated implies P finitely generated. (You will only need right-exactness of (1).)

4. Show that M, P finitely generated implies N finitely generated. (Possible hint: $\star$.) (You will only need right-exactness of (1).)

5. Show that N, P finitely generated need not imply M finitely generated. (Hint: if I is an ideal, we have $0 \to I \to A \to A/I \to 0$.)

6. Show that N coherent, M finitely generated implies M coherent. (You will only need left-exactness of (1).)

7. Show that N, P coherent implies M coherent. Hint for (i):

$$\begin{tikzcd}
& A^q \arrow[d] \arrow[dr] & & & \\
& & A^p \arrow[d] \arrow[dr] & & \\
0 \arrow[r] & M \arrow[r] \arrow[d] & N \arrow[r] \arrow[d] & P \arrow[r] \arrow[dr] & 0 \\
& 0 & 0 & & 0
\end{tikzcd}$$

(You will only need left-exactness of (1).)

8. Show that M finitely generated and N coherent implies P coherent. (Hint for (ii): $\star$.)

9. Show that M, P coherent implies N coherent. (Hint: $\star$.) We don't need exactness on the left for this.

At this point, we have shown that if two of (1) are coherent, the third is as well.

10. Show that a finite direct sum of coherent modules is coherent.

11. Suppose M is finitely generated, N coherent. Then if $\phi : M \to N$ is any map, then show that $\mathrm{Im}\,\phi$ is coherent.

12. Show that the kernel and cokernel of maps of coherent modules are coherent.

At this point, we have verified that coherent A-modules form an abelian subcategory of the category of A-modules. (Things you have to check: 0 should be in this set; it should be closed under finite sums; and it should be closed under taking kernels and cokernels.)

13. Suppose M and N are coherent submodules of the coherent module P. Show that $M + N$ and $M \cap N$ are coherent. (Hint: consider the right map $M \oplus N \to P$.)

14. Show that if A is coherent (as an A-module) then finitely presented modules are coherent. (Of course, if finitely presented modules are coherent, then A is coherent, as A is finitely presented!)

15. If M is finitely presented and N is coherent, show that $\mathrm{Hom}(M, N)$ is coherent. (Hint: Hom is left-exact in its first entry.)

16. If M is finitely presented, and N is coherent, show that $M \otimes N$ is coherent.

17. If $f \in A$, show that if M is a finitely generated (resp. finitely presented, coherent) A-module, then M_f is a finitely generated (resp. finitely presented, coherent) A_f-module. Hint: localization is exact. (This problem appears earlier as well, as Exercise 2.3.)

18. Suppose $(f_1, \dots, f_n) = A$. Show that if M_{f_i} is finitely generated for all i, then M is too. (Hint: Say M_{f_i} is generated by $m_{ij} \in M$ as an A_{f_i}-module. Show that the m_{ij} generate M. To check surjectivity $\oplus_{i,j} A \to M$, it suffices to check "on $D(f_i)$" for all i.)

19. Suppose $(f_1, \dots, f_n) = A$. Show that if M_{f_i} is coherent for all i, then M is too. (Hint from Rob Easton: if $\phi : A^2 \to M$, then $(\ker \phi)_{f_i} = \ker(\phi_{f_i})$, which is finitely generated for all i. Then apply the previous exercise.)

20. Show that the ring $A := k[x_1, x_2, \dots]$ is not coherent over itself. (Hint: consider $A \to A$ with $x_1, x_2, \dots \mapsto 0$.) Thus we have an example of a finitely presented module that is not coherent; a surjection of finitely presented modules whose kernel is not even finitely generated; hence an example showing that finitely presented modules don't form an abelian category.

4. Support of a sheaf

Suppose $\mathcal{F}$ is a sheaf of abelian groups (resp. sheaf of $\mathcal{O}_X$-modules) on a topological space X (resp. ringed space $(X, \mathcal{O}_X)$). Define the *support* of a section s of $\mathcal{F}$ to be

$$\operatorname{Supp} s = \{p \in X : s_p \neq 0 \text{ in } \mathcal{F}_p\}.$$

I think of this as saying where s "lives". Define the *support* of $\mathcal{F}$ as

$$\operatorname{Supp} \mathcal{F} = \{p \in X : \mathcal{F}_p \neq 0\}.$$

It is the union of "all the supports of sections on various open sets". I think of this as saying where $\mathcal{F}$ "lives".

4.1. *Exercise.* The support of a finite type quasicoherent sheaf on a scheme is a closed subset. (Hint: Reduce to an affine open set. Choose a finite set of generators of the corresponding module.) Show that the support of a quasicoherent sheaf need not be closed. (Hint: If $A = \mathbb{C}[t]$, then $\mathbb{C}[t]/(t-a)$ is an A-module supported at a. Consider $\oplus_{a \in \mathbb{C}} \mathbb{C}[t]/(t-a)$.)

5. Rank of a finite type sheaf at a point

The *rank* $\mathcal{F}$ of a finite type sheaf at a point p is $\dim \mathcal{F}_p / \mathfrak{m}\mathcal{F}_p$ where $\mathfrak{m}$ is the maximal ideal corresponding to p. More explicitly, on any affine set $\operatorname{Spec} A$ where $p = [\mathfrak{p}]$ and $\mathcal{F}(\operatorname{Spec} A) = M$, then the rank is $\dim_{A/\mathfrak{p}} M_{\mathfrak{p}}/\mathfrak{p}M_{\mathfrak{p}}$. By Nakayama's lemma, this is the minimal number of generators of $M_{\mathfrak{p}}$ as an $A_{\mathfrak{p}}$-module.

Note that this definition is consistent with the notion of rank of a locally free sheaf. In that case, the rank is a (locally) constant function of the point. The converse is sometimes true, as is shown in Exercise 5.2 below.

If $\mathcal{F}$ is quasicoherent (not necessarily finite type), then $\mathcal{F}_p/\mathfrak{m}\mathcal{F}_p$ can be interpreted as the fiber of the sheaf at the point. A section of $\mathcal{F}$ over an open set containing p can be said to take on a value at that point, which is an element of $\mathcal{F}_p/\mathfrak{m}\mathcal{F}_p$.

5.1. *Exercise.*

(a) If $m_1, \dots, m_n$ are generators at P, they are generators in an open neighborhood of P. (Hint: Consider $\operatorname{coker} A^n \xrightarrow{(f_1,\dots,f_n)} M$ and Exercise 4.1.)
(b) Show that at any point, $\operatorname{rank}(\mathcal{F} \oplus \mathcal{G}) = \operatorname{rank}(\mathcal{F}) + \operatorname{rank}(\mathcal{G})$ and $\operatorname{rank}(\mathcal{F} \otimes \mathcal{G}) = \operatorname{rank} \mathcal{F} \operatorname{rank} \mathcal{G}$ at any point. (Hint: Show that direct sums and tensor products commute with ring quotients and localizations, i.e. $(M \oplus N) \otimes_R (R/I) \cong M/IM \oplus N/IN$, $(M \otimes_R N) \otimes_R (R/I) \cong (M \otimes_R R/I) \otimes_{R/I} (N \otimes_R R/I) \cong M/IM \otimes_{R/I} N/IM$, etc.) Thanks to Jack Hall for improving this problem.
(c) Show that rank is an upper semicontinuous function on X. (Hint: Generators at P are generators nearby.)

5.2. *Important Exercise.* If X is reduced, $\mathcal{F}$ is coherent, and the rank is constant, show that $\mathcal{F}$ is locally free. (Hint: choose a point $p \in X$, and choose generators of the stalk $\mathcal{F}_p$. Let U be an open set where the generators are sections, so we have a map $\phi : \mathcal{O}_U^{\oplus n} \to \mathcal{F}|_U$. The cokernel and kernel of ϕ are supported on closed sets by Exercise 4.1. Show that these closed subsets don't include p. Make sure you use the reduced hypothesis!) Thus coherent sheaves are locally free on a dense open set. Show that this can be false if X is not reduced. (Hint: $\operatorname{Spec} k[x]/x^2$, $M = k$.)

You can use the notion of rank to help visualize finite type sheaves, or even quasicoherent sheaves. (We discussed first finite type sheaves on reduced schemes. We then generalized to quasicoherent sheaves, and to nonreduced schemes.)

5.3. Exercise: Geometric Nakayama. Suppose X is a scheme, and $\mathcal{F}$ is a finite type quasicoherent sheaf. Show that if $\mathcal{F}_x \otimes k(x) = 0$, then there exists V such that $\mathcal{F}|_V = 0$. Better: if I have a set that generates the fiber, it generates the stalk.

5.4. *Less important Exercise.* Suppose $\mathcal{F}$ and $\mathcal{G}$ are finite type sheaves such that $\mathcal{F} \otimes \mathcal{G} \cong \mathcal{O}_X$. Then $\mathcal{F}$ and $\mathcal{G}$ are both invertible (Hint: Nakayama.) This is the reason for the adjective "invertible" these sheaves are the invertible elements of the "monoid of finite type sheaves".

6. Quasicoherent sheaves of ideals, and closed subschemes

This section is important, and short only because we have built up some machinery.

We now define closed subschemes, and what it means for some functions on a scheme to "cut out" another scheme. The intuition we want to make precise is that a closed subscheme of X is something that on each affine looks like $\operatorname{Spec} R/I$ "$\hookrightarrow$" $\operatorname{Spec} R$.

Suppose $\mathcal{I} \hookrightarrow \mathcal{O}_X$ is a *quasicoherent sheaf of ideals*. (Quasicoherent sheaves of ideals are, not suprisingly, sheaves of ideals that are quasicoherent.) Not all sheaves of ideals are quasicoherent.

6.1. Exercise. *(A non-quasicoherent sheaf of ideals)* Let $X = \operatorname{Spec} k[x]_{(x)}$, the germ of the affine line at the origin, which has two points, the closed point and the generic point η. Define $\mathcal{I}(X) = \{0\} \subset \mathcal{O}_X(X) = k[x]_{(x)}$, and $\mathcal{I}(\eta) = k(x) = \mathcal{O}_X(\eta)$. Show that $\mathcal{I}$ is not a quasicoherent sheaf of ideals.

The cokernel of $\mathcal{I} \to \mathcal{O}_X$ is also quasicoherent, so we have an exact sequence of quasicoherent sheaves

$$0 \to \mathcal{I} \to \mathcal{O}_X \to \mathcal{O}_X/\mathcal{I} \to 0. \tag{2}$$

(This exact sequence will come up repeatedly. We could call it the *closed subscheme exact sequence*.) Now $\mathcal{O}_X/\mathcal{I}$ is finite tyupe (as over any affine open set, the corresponding module is generated by a single element), so $\operatorname{Supp} \mathcal{O}_X/\mathcal{I}$ is a closed subset. Also, $\mathcal{O}_X/\mathcal{I}$ is a sheaf of rings. Thus we have a topological space $\operatorname{Supp} \mathcal{O}_X/\mathcal{I}$ with a sheaf of rings. I claim this is a scheme. To see this, we look over an affine open set $\operatorname{Spec} R$. Here $\Gamma(\operatorname{Spec} R, \mathcal{I})$ is an ideal I of R. Then $\Gamma(\operatorname{Spec} R, \mathcal{O}_X/\mathcal{I}) = R/I$ (because quotients behave well on affine open sets).

I claim that on this open set, $\operatorname{Supp} \mathcal{O}_X/\mathcal{I}$ is the closed subset $V(I)$, which I can identify with the topological space $\operatorname{Spec} R/I$. Reason: $[\mathfrak{p}] \in \operatorname{Supp}(\mathcal{O}_X/\mathcal{I})$ if and only if $(R/I)_\mathfrak{p} \neq 0$ if and only if $\mathfrak{p}$ contains I if and only if $[\mathfrak{p}] \in \operatorname{Spec} R/I$.

(Remark for experts: when you have a sheaf supported in a closed subset, you can interpret it as a sheaf *on* that closed subset. More precisely, suppose X is a topological space, $i : Z \hookrightarrow X$ is an inclusion of a closed subset, and $\mathcal{F}$ is a sheaf on X with $\operatorname{Supp} \mathcal{F} \subset Z$. Then we have a natural map $\mathcal{F} \to i_* i^{-1} \mathcal{F}$ (corresponding to the map $i^{-1}\mathcal{F} \to i^{-1}\mathcal{F}$, using the adjointness of i^{-1} and i_*). You can check that this is an isomorphism on stalks, and hence an isomorphism, so $\mathcal{F}$ can be interpreted as the pushforward of a sheaf on the closed subset. Thanks to Jarod and Joe for this comment.)

I next claim that on the distinguished open set $D(f)$ of $\operatorname{Spec} R$, the sections of $\mathcal{O}_X/\mathcal{I}$ are precisely $(R/I)_f \cong R_f/I_f$. (Reason that $(R/I)_f \cong R_f/I_f$: take the exact sequence $0 \to I \to R \to R/I \to 0$ and tensor with R_f, which preserves exactness.) Reason: On $\operatorname{Spec} R$, the sections of $\mathcal{O}_X/\mathcal{I}$ are R/I, and $\mathcal{O}_X/\mathcal{I}$ is quasicoherent, hence the sections over $D(f)$ are $(R/I)_f$.

That's it!

We say that a *closed subscheme* of X is anything arising in this way from a quasicoherent sheaf of ideals. In other words, there is a tautological correspondence between quasicoherent sheaves of ideals and closed subschemes.

Important remark. Note that closed subschemes of affine schemes are affine. (This is tautological using our definition, but trickier using other definitions.)

Exercise. Suppose $\mathcal{F}$ is a locally free sheaf on a scheme X, and s is a section of $\mathcal{F}$. Describe how $s = 0$ "cuts out" a closed subscheme. (A picture is very useful here!)

6.2. Reduction of a scheme. The *reduction* of a scheme is the "reduced version" of the scheme. If R is a ring, then the nilradical behaves well with respect to localization with respect to an element of the ring: $\mathfrak{N}(R)_f$ is naturally isomorphic to $\mathfrak{N}(R_f)$ (check this!). Thus on any scheme, we have an ideal sheaf of nilpotents, and the corresponding closed subscheme is called the *reduction* of X, and is denoted X^{red}. We will soon see that X^{red} satisfies a universal property; we will need the notion of a morphism of schemes to say what this universal property is.

6.3. *Unimportant exercise.*

(a) X^{red} has the same underlying topological space as X: there is a natural homeomorphism of the underlying topological spaces $X^{\text{red}} \cong X$. Picture: taking the reduction may be interpreted as shearing off the fuzz on the space.
(b) Give an example to show that it is *not* true that $\Gamma(X^{\text{red}}, \mathcal{O}_{X^{\text{red}}}) = \Gamma(X, \mathcal{O}_X)/\sqrt{\Gamma(X, \mathcal{O}_X)}$. (Hint: $\coprod_{n>0} \operatorname{Spec} k[t]/(t^n)$ with global section $(t, t, t, \dots)$.) Motivation for this exercise: this *is* true on each affine open set.

By Exercise 4.1, we have that the reduced locus of a locally Noetherian scheme is open. More precisely: Let $\mathcal{I}$ be the ideal sheaf of X^{red}, so on X we have an exact sequence

$$0 \to \mathcal{I} \to \mathcal{O}_X \to \mathcal{O}_{X^{\text{red}}} \to 0$$

of quasicoherent sheaves on X. Then $\mathcal{I}$ is coherent as X is locally Noetherian. Hence the support of I is closed. The complement of the support of I is the **reduced locus**. Geometrically, this says that "the fuzz is on a closed subset". (A picture is really useful here!)

6.4. Important exericse (the reduced subscheme induced by a closed subset). Suppose X is a scheme, and K is a closed subset of X. Show that the following construction determines a closed subscheme Y: on any affine open subset $\operatorname{Spec} R$ of X, consider the ideal $I(K \cap \operatorname{Spec} R)$. This is called the *reduced subscheme induced by* K. Show that Y is reduced.

7. Discussion of future topics

I then discussed the notion of when a sheaf is *generated by global sections*, and gave a preview of quasicoherent sheaves on projective A-schemes. These ideas will appear in the notes for class 16.

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FOUNDATIONS OF ALGEBRAIC GEOMETRY CLASS 16

CONTENTS

Last day: much more on quasicoherence. Quasicoherent sheaves form an abelian category. Finite type (= "locally finitely generated") and coherent sheaves. Support of a sheaf. Rank of a finite type sheaf at a point. Closed subschemes at a point.

Today: effective Cartier divisors; quasicoherent sheaves on projective A-schemes corresponding to graded modules, line bundles on projective A-schemes, O(n), generated by global sections, Serre's theorem, the adjoint functors $\sim$ and Γ_*.

1. YET MORE ON CLOSED SUBSCHEMES

Here are few more notions about closed subschemes.

In analogy with closed subsets of a topological space, we can define finite unions and arbitrary intersections of closed subschemes. On affine open set $\operatorname{Spec} R$, if for each i in an index set, I_i corresponds to a closed subscheme, the scheme-theoretic intersection of the closed subschemes corresponds to the ideal generated by the I_i (here the index set may be infinite), and the scheme-theoretic union corresponds to the intersection of by all I_i (here the index should be finite).

Exercise: Describe the scheme-theoretic intersection of $(y - x^2)$ and y in $\mathbb{A}^2$. Describe the scheme-theoretic union. Draw a picture.

Exercise: Prove some fact of your choosing showing that closed subschemes behave similarly to closed subsets. For example, if X, Y, and Z are closed subschemes of W, show that $(X \cap Z) \cup (Y \cap Z) = (X \cup Y) \cap Z$.

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1.1. From closed subschemes to effective Cartier divisors. There is a special name for a closed subscheme locally cut out by one equation that is not a zero-divisor. More precisely, it is a closed subscheme such that there exists an affine cover such that on each one it is cut out by a single equation, not a zero-divisor. (This does not mean that on *any* affine it is cut out by a single equation — this notion doesn't satisfy the "gluability" hypothesis of the Affine Communication Theorem. If $I \subset R$ is generated by a non-zero divisor, then $I_f \subset R_f$ is too. But "not conversely". I might give an example later.) We call this an *effective Cartier divisor*. (This admittedly unwieldy terminology! But there is a reason for it.) By Krull, it is pure codimension 1.

Remark: if $I = (u) = (v)$, and u is not a zero-divisor, then u and v differ by a unit in R. Proof: $u \in (v)$ implies $u = av$. Similarly $v = bu$. Thus $u = abu$, from which $u(1-ab) = 0$. As u is not a zero-divisor, $1 = ab$, so a and b are units.

Reason we care: effective Cartier divisors give invertible sheaves. If $\mathcal{I}$ is an effective Cartier divisor on X, then $\mathcal{I}$ is an invertible sheaf. Reason: locally, sections are multiples of a single generator u, and there are no "relations".

The invertible sheaf corresponding to an effective Cartier divisor is for various reasons defined to be the dual of the ideal sheaf. This line bundle has a canonical section: Dualizing $0 \to \mathcal{I} \to \mathcal{O}$ gives us $\mathcal{O} \to \mathcal{I}^*$. **Exercise.** This section vanishes along our actual effective Cartier divisor.

1.2. Exercise. Describe the invertible sheaf in terms of transition functions. More precisely, on any affine open set where the effective Cartier divisor is cut out by a single equation, the invertible sheaf is trivial. Determine the transition functions between two such overlapping affine open sets. Verify that there is indeed a canonical section of this invertible sheaf, by describing it.

To describe the tensor product of such invertible sheaves: if $I = (u)$ (locally) and $J = (v)$, then the tensor product corresponds to (uv).

We get a monoid of effective Cartier divisors, with unit $\mathcal{I} = \mathcal{O}$. Notation: D is an effective Cartier divisor. $\mathcal{O}(D)$ is the corresponding line bundle. $\mathcal{O}(-D)$ is the ideal sheaf.

$$0 \to \mathcal{O}(-D) \to \mathcal{O} \to \mathcal{O}_D \to 0.$$

D is associated to the closed subscheme.

Hence we can get a bunch of invertible sheaves, by taking differences of these two. Surprising fact: we almost get them all! (True for all nonsingular schemes. This is true for all projective schemes. It is very hard to describe an invertible sheaf that is not describable in such a way.)

We now describe quasicoherent sheaves on a projective A-scheme. Recall that a projective A-scheme is produced from the data of $\mathbb{Z}^{\geq 0}$-graded ring S_*, with $S_0 = A$, and S_+ finitely generated as an A-module. The resulting scheme is denoted $\operatorname{Proj} S_*$.

Let $X = \operatorname{Proj} S_*$. Suppose M_* is a graded S_* module, *graded by* $\mathbb{Z}$. We define the quasicoherent sheaf $\widetilde{M_*}$ as follows. (I will avoid calling it $\tilde{M}$, as this might cause confusion with the affine case.) On the distinguished open $D(f)$, we let

$$\widetilde{M_*}(D(f)) \cong \widetilde{(M_f)_0}.$$

(More correctly: we define a sheaf $\widetilde{M_*(f)}$ on $D(f)$ for each f. We give identifications of the restriction of $\widetilde{M_*(f)}$ and $\widetilde{M_*(g)}$ to $D(fg)$. Then by an earlier problem set problem telling how to glue sheaves, these sheaves glue together to a single sheaf on $\widetilde{M_*}$ on X. We then discard the temporary notation $\widetilde{M_*(f)}$.)

This is clearly quasicoherent, because it is quasicoherent on each $D(f)$. If M_* is finitely generated over S_*, then so $\tilde{M}_*$ is a finite type sheaf.

I will now give some straightforward facts.

If M_* and M_*' agree in high enough degrees, then $\widetilde{M_*} \cong \widetilde{M_*'}$. Thus the map from graded S_*-modules to quasicoherent sheaves on $\operatorname{Proj} S_*$ is not a bijection.

Given a map of graded modules $\phi : M_* \to N_*$, we we get an induced map of sheaves $\widetilde{M_*} \to \widetilde{N_*}$. Explicitly, over $D(f)$, the map $M_* \to N_*$ induces $M_*[1/f] \to N_*[1/f]$ which induces $\phi_f : (M_*[1/f])_0 \to (N_*[1/f])_0$; and this behaves well with respect to restriction to smaller distinguished open sets, i.e. the following diagram commutes.

$$\begin{tikzcd}
(M_*[1/f])_0 \arrow[r, "\phi_f"] \arrow[d] & (N_*[1/f])_0 \arrow[d] \\
(M_*[1/(fg)])_0 \arrow[r, "\phi_{fg}"] & (N_*[1/(fg)])_0.
\end{tikzcd}$$

In fact $\sim$ is a functor from the category of graded S_*-modules to the category of quasicoherent sheaves on $\operatorname{Proj} S_*$. This isn't quite an isomorphism, but it is close. The relationship is akin to that between presheaves and sheaves, and the sheafification functor, as we will see before long.

2.1. Exercise. Show that $\widetilde{M_*} \otimes \widetilde{N_*} \cong \widetilde{M_* \otimes_{S_*} N_*}$. (Hint: describe the isomorphism of sections over each $D(f)$, and show that this isomorphism behaves well with respect to smaller distinguished opens.)

If $I_* \subset S_*$ is a graded ideal, we get a closed subscheme. Example: Suppose $S_* = k[w, x, y, z]$, so $\operatorname{Proj} S_* \cong \mathbb{P}^3$. Suppose $I_* = (wx - yz, x^2 - wy, y^2 - xz)$. Then we get the exact sequence of graded S_*-modules

$$0 \to I_* \to S_* \to S_*/I_* \to 0.$$

Which closed subscheme of $\mathbb{P}^3$ do we get? The twisted cubic!

2.2. Exercise. Show that if I_* is a graded ideal of S_*, show that we get a closed immersion $\operatorname{Proj} S_*/I_* \hookrightarrow \operatorname{Proj} S_*$.

3. Invertible sheaves (line bundles) on projective A-schemes

We now come to one of the most fundamental concepts in projective geometry.

First, I want to mention something that I should have mentioned some time ago.

3.1. Exercise. Suppose S_* is generated over S_0 by $f_1, \dots, f_n$. Suppose $d = \operatorname{lcm}(\deg f_1, \dots, \deg f_n)$. Show that S_{d*} is generated in "new" degree 1 (= "old" degree d). (Hint: I like to show this by induction on the size of the set $\{\deg f_1, \dots, \deg f_n\}$.) This is handy, because we can stick every Proj in some projective space via the construction of Exercise 2.2.

Suppose that S_* is generated in degree 1. By the previous exercise, this is not a huge assumption. Suppose M_* is a graded S_*-module. Define the graded module $M(n)_*$ so that $M(n)_m := M_{n+m}$. Thus the quasicoherent sheaf $\widetilde{M(n)}_*$ is given by

$$\Gamma(D(f), \widetilde{M(n)_*}) = \widetilde{(M_f)}_n$$

where here the subscript means we take the nth graded piece. (These subscripts are admittedly confusing!)

As an incredibly important special case, define $\mathcal{O}_{\operatorname{Proj} S_*}(n) := \widetilde{S(n)}_*$. When the space is implicit, it can be omitted from the notation: $\mathcal{O}(n)$ (pronounced "oh of n").

3.2. Important exercise. If S_* is generated in degree 1, show that $\mathcal{O}_{\operatorname{Proj} S_*}(n)$ is an invertible sheaf.

3.3. Essential exercise. Calculate $\dim_k \Gamma(\mathbb{P}^m_k, \mathcal{O}_{\mathbb{P}^m_k}(n))$.

I'll get you started on this. As always, consider the "usual" affine cover. Consider the $n = 1$ case. Say $S_* = k[x_0, \dots, x_m]$. Suppose we have a global section of $\mathcal{O}(1)$. On $D(x_0)$, the sections are of the form $f(x_0, \dots, x_n)/x_0^{\deg f - 1}$. On $D(x_1)$, the sections are of the form $g(x_0, \dots, x_n)/x_1^{\deg g - 1}$. They are supposed to agree on the overlap, so

$$x_0^{\deg f - 1} g(x_0, \dots, x_n) = x_1^{\deg g - 1} f(x_0, \dots, x_n).$$

How is this possible? Well, we must have that $g = x_1^{\deg g - 1} \times$ some linear factor, and $f = x_0^{\deg f - 1} \times$ the same linear factor. Thus on $D(x_0)$, this section must be some linear form. On $D(x_1)$, this section must be the same linear form. By the same argument, on each

$D(x_i)$, the section must be the same linear form. Hence (with some argumentation), the global sections of $\mathcal{O}(1)$ correspond to the linear forms in $x_0, \ldots, x_m$, of which there are $m+1$.

Thus $x+y+2z$ is a section of $\mathcal{O}(1)$ on $\mathbb{P}^2$. It isn't a function, but I can say when this section vanishes — precisely where $x+y+2z=0$.

3.4. Exercise. Show that $\mathcal{F}(n) \cong \mathcal{F} \otimes \mathcal{O}(n)$.

3.5. Exercise. Show that $\mathcal{O}(m+n) \cong \mathcal{O}(m) \otimes \mathcal{O}(n)$.

3.6. Exercise. Show that if $m \neq n$, then $\mathcal{O}_{\mathbb{P}^l_k}(m)$ is not isomorphic to $\mathcal{O}_{\mathbb{P}^l_k}(n)$ if $l > 0$. (Hence we have described a countable number of invertible sheaves (line bundles) that are non-isomorphic. We will see later that these are *all* the line bundles on projective space $\mathbb{P}^n_k$.)

4. Generation by global sections, and Serre's Theorem

(I discussed this in class 15, but should have discussed them here. Hence they are in the class 16 notes.)

Suppose $\mathcal{F}$ is a sheaf on X. We say that $\mathcal{F}$ is *generated by global sections at a point* p if we can find $\phi : \mathcal{O}^{\oplus I} \to \mathcal{F}$ that is surjective at the stalk of p: $\phi_p : \mathcal{O}_p^{\oplus I} \to \mathcal{F}_p$ is surjective. (Some what more precisely, the stalk of $\mathcal{F}$ at p is generated by global sections of $\mathcal{F}$. The global sections in question are the images of the $|I|$ factors of $\mathcal{O}_p^{\oplus I}$.) We say that $\mathcal{F}$ is *generated by global sections* if it is generated by global sections at all p, or equivalently, if we can we can find $\mathcal{O}^{\oplus I} \to \mathcal{F}$ that is surjective. (By our earlier result that we can check surjectivity at stalks, this is the same as saying that it is surjective at all stalks.) If I can be taken to be finite, we say that $\mathcal{F}$ is generated by a finite number of global sections. We'll see soon why we care.

4.1. *Exercise.* If quasicoherent sheaves $\mathcal{L}$ and $\mathcal{M}$ are generated by global sections at a point p, then so is $\mathcal{L} \otimes \mathcal{M}$. (This exercise is less important, but is good practice for the concept.)

4.2. *Easy exercise.* An invertible sheaf $\mathcal{L}$ on X is generated by global sections if and only if for any point $x \in X$, there is a section of $\mathcal{L}$ not vanishing at x. (Hint: Nakayama.)

4.3. *Lemma. — Suppose $\mathcal{F}$ is a finite type sheaf on* X. *Then the set of points where $\mathcal{F}$ is generated by global sections is an open set.*

Proof. Suppose $\mathcal{F}$ is generated by global sections at a point p. Then it is generated by a finite number of global sections, say m. This gives a morphism $\phi : \mathcal{O}^{\oplus m} \to \mathcal{F}$, hence

$\operatorname{im}\phi \hookrightarrow \mathcal{F}$. The support of the (finite type) cokernel sheaf is a closed subset not containing p. $\square$

(Back to class 16!)

4.4. Important Exercise (an important theorem of Serre). Suppose S_0 is a Noetherian ring, and S_* is generated in degree 1. Let $\mathcal{F}$ be any finite type sheaf on $\operatorname{Proj} S_*$. Then for some integer n_0, for all $n \geq n_0$, $\mathcal{F}(n)$ can be generated by a finite number of global sections.

I'm going to sketch how you should tackle this exercise, after first telling you the reason we will care.

4.5. *Corollary.* — Thus any coherent sheaf $\mathcal{F}$ on $\operatorname{Proj} S_*$ can be presented as:

$$\oplus_{\text{finite}} \mathcal{O}(-n) \to \mathcal{F} \to 0.$$

We're going to use this a lot!

Proof. Suppose we have m global sections $s_1, \ldots, s_m$ of $\mathcal{F}(n)$ that generate $\mathcal{F}(n)$. This gives a map

$$\oplus_m \mathcal{O} \longrightarrow \mathcal{F}(n)$$

given by $(f_1, \ldots, f_m) \mapsto f_1 s_1 + \cdots + f_m s_m$ on any open set. Because these global sections generate $\mathcal{F}$, this is a surjection. Tensoring with $\mathcal{O}(-n)$ (which is exact, as tensoring with any locally free is exact) gives the desired result. $\square$

Here is now a hint/sketch for the Serre exercise 4.4.

We can assume that S_* is generated in degree 1; we can do this thanks to Exercise 3.1. Suppose $\deg f = 1$. Say $\mathcal{F}|_{D(f)} = \tilde{M}$, where M is a $(S_*[1/f])_0$-module, generated by $m_1, \ldots, m_n$. These elements generate all the stalks over all the points of $D(f)$. They are sections over this big distinguished open set. It would be wonderful if we knew that they had to be restrictions of global sections, i.e. that there was a global section m_i' that restricted to m_i on $D(f)$. If that were always true, then we would cover X with a finite number of each of these $D(f)$'s, and for each of them, we would take the finite number of generators of the corresponding module. Sadly this is not true.

However, we will see that $f^N m$ "extends", where m is any of the m_i's, and N is sufficiently large. We will see this by (easily) checking first that $f^N m$ extends over another distinguished open $D(g)$ (i.e. that there is a section of $\mathcal{F}(N)$ over $D(g)$ that restricts to $f^n m$ on $D(g) \cap D(f) = D(fg)$). But we're still not done, because we don't know that the extension over $D(g)$ and over some other $D(g')$ agree on the overlap $D(g) \cap D(g') = D(gg')$ — in fact, they need not agree! But after multiplying both extensions by $f^{N'}$ for large enough N', we will see that they agree on the overlap. By quasicompactness, we need to to extend over only a finite number of $D(g)$'s, and to make sure extensions agree over the finite number of pairs of such $D(g)$'s, so we will be done.

Great, let's make this work. Let's investigate this on $D(g) = \operatorname{Spec} A$, where the degree of g is also 1. Say $\mathcal{F}|D(g) \cong \tilde{N}$. Let $f' = f/g$ be "the function corresponding to f on $D(g)$". We have a section over $D(f')$ on the affine scheme $D(g)$, i.e. an element of $N_{f'}$, i.e. something of the form $n/(f')^N$ for some $n \in N$. So then if we multiply it by f'^N, we can certainly extend it! So if we multiply by a big enough power of f, m certainly extends over any $D(g)$.

As described earlier, the only problem is, we can't guarantee that the extensions over $D(g)$ and $D(g')$ agree on the overlap (and hence glue to a single extensions). Let's check on the intersection $D(g) \cap D(g') = D(gg')$. Say $m = n/(f')^N = n'/(f')^{N'}$ where we can take $N = N'$ (by increasing N or N' if necessary). We certainly may not have $n = n'$, but by the (concrete) definition of localization, after multiplying with enough f''s, they become the same.

In conclusion after multiplying with enough f's, our sections over $D(f)$ extend over each $D(g)$. After multiplying by even more, they will all agree on the overlaps of any two such distinguished affine. Exercise 4.4 is to make this precise.

5. Every quasicoherent sheaf on a projective A-scheme arises from a graded module

We have gotten lots of quasicoherent sheaves on $\operatorname{Proj} S_*$ from graded S_*-modules. We'll now see that we can get them all in this way.

We want to figure out how to "undo" the $\tilde{M}$ construction. When you do the essential exercise 3.3, you'll suspect that in good situations,

$$M_n \cong \Gamma(\operatorname{Proj} S_*, \tilde{M}(n)).$$

Motivated by this, we define

$$\Gamma_n(\mathcal{F}) := \Gamma(\operatorname{Proj} S_*, \mathcal{F}_n).$$

Then $\Gamma_*(\mathcal{F})$ is a graded S_*-module, and we can dream that $\Gamma_*(\mathcal{F})^\sim \cong \mathcal{F}$. We will see that this is indeed the case!

5.1. Exercise. Show that Γ_* gives a functor from the category of quasicoherent sheaves on $\operatorname{Proj} S_*$ to the category of graded S_*-modules. (In other words, show that if $\mathcal{F} \to \mathcal{G}$ is a morphism of quasicoherent sheaves on $\operatorname{Proj} S_*$, describe the natural map $\Gamma_*(\mathcal{F}) \to \Gamma_*(\mathcal{G})$, and show that such natural maps respect the identity and composition.)

Note that $\sim$ and Γ_* cannot be inverses, as $\sim$ can turn two different graded modules into the same quasicoherent sheaf.

Our initial goal is to show that there is a natural isomorphism $\widetilde{\Gamma_*(\mathcal{F})} \to \mathcal{F}$, and that there is a natural map $M_* \to \Gamma_*(\widetilde{M_*})$. We will show something better: that $\sim$ and Γ_* are adjoint.

We start by describing the natural map $M_* \to \Gamma_* \widetilde{M_*}$. We describe it in degree n. Given an element m_n, we seek an element of $\Gamma(\operatorname{Proj} S_*, \widetilde{M_*}(n)) = \Gamma(\operatorname{Proj} S_*, \widetilde{M_{(n+*)}})$. By shifting the grading of M_* by n, we can assume $n = 0$. For each $D(f)$, we certainly have an element of $(M[1/f])_0$ (namely m), and they agree on overlaps, so the map is clear.

5.2. Exercise. Show that this canonical map need not be injective, nor need it be surjective. (Hint: $S_* = k[x]$, $M_* = k[x]/x^2$ or $M_* = \{$ polynomials with no constant terms $\}$.)

The natural map $\widetilde{\Gamma_* \mathcal{F}} \to \mathcal{F}$ is more subtle (although it will have the advantage of being an isomorphism).

5.3. Exercise. Describe the natural map $\widetilde{\Gamma_* \mathcal{F}} \to \mathcal{F}$ as follows. First describe it over $D(f)$. Note that sections of the left side are of the form m/f^n where $m \in \Gamma_{n \deg f} \mathcal{F}$, and $m/f^n = m'/f^{n'}$ if there is some N with $f^N(f^{n'}m - f^n m') = 0$. Show that your map behaves well on overlaps $D(f) \cap D(g) = D(fg)$.

5.4. Longer Exercise. Show that the natural map $\widetilde{\Gamma_* \mathcal{F}} \to \mathcal{F}$ is an isomorphism, by showing that it is an isomorphism over $D(f)$ for any f. Do this by first showing that it is surjective. This will require following some of the steps of the proof of Serre's theorem (Exercise 4.4). Then show that it is injective.

5.5. *Corollary. — Every quasicoherent sheaf arises from this tilde construction. Each closed subscheme of* $\operatorname{Proj} S_*$ *arises from a graded ideal* $I_* \subset S_*$.

In particular, let $x_0, \ldots, x_n$ be generators of S_1 as an A-module. Then we have a surjection of graded rings

$$A[t_0, \ldots, t_n] \to S_*$$

where $t_i \mapsto x_i$. Then this describes $\operatorname{Proj} S_*$ as a closed subscheme of $\mathbb{P}^n_A$.

5.6. Exercise (Γ_* and $\sim$ are adjoint functors). Prove part of the statement that Γ_* and $\sim$ are adjoint functors, by describing a natural bijection $\operatorname{Hom}(M_*, \Gamma_*(\mathcal{F})) \cong \operatorname{Hom}(\widetilde{M_*}, \mathcal{F})$. For the map from left to right, start with a morphism $M_* \to \Gamma_*(\mathcal{F})$. Apply $\sim$, and postcompose with the isomorphism $\widetilde{\Gamma_* \mathcal{F}} \to \mathcal{F}$, to obtain

$$\widetilde{M_*} \to \widetilde{\Gamma_* \mathcal{F}} \to \mathcal{F}.$$

Do something similar to get from right to left. Show that "both compositions are the identity in the appropriate category". (Is there a clever way to do that?)

5.7. Saturated S_*-modules. We end with a remark: different graded S_*-modules give the same quasicoherent sheaf on $\operatorname{Proj} S_*$, but the results of this section show that there is a "best" graded module for each quasicoherent sheaf, and there is a map form each graded module to its "best" version, $M_* \to \Gamma_*(\widetilde{M_*})$. A module for which this is an isomorphism (a "best" module) is called *saturated*. I don't think we'll use this notation, but other people do.

This "saturation" map $M_* \to \Gamma_*(\widetilde{M_*})$ should be seen as analogous to the sheafification map, taking presheaves to sheaves.

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FOUNDATIONS OF ALGEBRAIC GEOMETRY CLASS 17

CONTENTS

Last day: effective Cartier divisors; quasicoherent sheaves on projective A-schemes corresponding to graded modules, line bundles on projective A-schemes, O(n), generated by global sections, Serre's theorem, the adjoint functors $\sim$ and Γ_*.

Today: Associated points; more on normality; invertible sheaves and divisors take 1.

Our goal for today and part of next day is to develop tools to understand what invertible sheaves there can be on a scheme. As a key motivating example, we will show (by next day) that the only invertible sheaves on $\mathbb{P}^n_k$ are $\mathcal{O}(m)$.

But first, I want to tell you about *associated points* and the *ring of fractions* of a scheme. This topic isn't logically needed, but it is a description of the "most interesting points" of a scheme, where "all the action is".

1. ASSOCIATED POINTS

Recall that for an integral (= irreducible + reduced, by an earlier homework problem) scheme X, we have the notion of the *function field*, which is the stalk at the generic point. For any affine open subset $\operatorname{Spec} R$, we have that R is a subring of the function field.

It would be nice to generalize this to more general schemes, with possibly many components, and with nonreduced behavior.

The answer to this question is that on a "nice" (Noetherian) scheme, there are a finite number of points that will have similar information. (On a locally Noetherian scheme, we'll also have the notion of associated points, but there could be an infinite number of them.) I then drew a picture of a scheme with two components, one of which looked like a (reduced) line, and one of which was a plane, with some nonreduced behavior ("fuzz") along a line of it, and even more nonreduced behavior ("more fuzz") at a point of the line.

Date: Wednesday, November 30, 2005. Very minor update January 31, 2007.

I stated that the associated points are the generic points of the two components, plus the generic point of the line where this is fuzz, and the point where there is more fuzz.

We will define associated points of locally Noetherian schemes, and show the following important properties. You can skip the proofs if you want, but you should remember these properties.

(1) The generic points of the irreducible components are associated points. The other associated points are called *embedded points*.

(2) If X is reduced, then X has no embedded points. (This jibes with the intuition of the picture of associated points I described earlier.)

(3) If X is affine, say $X = \operatorname{Spec} R$ affine, then the natural map

$$R \to \prod_{\text{associated } \mathfrak{p}} R_{\mathfrak{p}} \tag{1}$$

is an injection. The primes corresponding to the associated points of R will be called *associated primes*. (In fact this is backwards; we will define associated primes first, and then define associated points.) The ring on the right of (1) is called the *ring of fractions*. If X is a locally Noetherian scheme, then the products of the stalks at the associated points will be called the *ring of fractions* of X. Note that if X is integral, this is the function field.

We define a *rational function* on a locally Noetherian scheme: it is an equivalence class. Any function defined on an open set containing all associated points is a rational function. Two such are considered the same if they agree on an open subset containing all associated points. If X is reduced, this is the same as requiring that they are defined on an open set of each of the irreducible components. A rational function has a maximal domain of definition, because any two actual functions on an open set (i.e. sections of the structure sheaf over that open set) that agree as "rational functions" (i.e. on small enough open sets containing associated points) must be the same function, by this fact **(3)**. We say that a rational function f is *regular* at a point p if p is contained in this maximal domain of definition (or equivalently, if there is some open set containing p where f is defined).

We similarly define *rational and regular sections of an invertible sheaf $\mathcal{L}$* on a scheme X.

(4) A function is a zero divisor if and only if it vanishes at an associated point of $\operatorname{Spec} R$.

Okay, let's get down to business.

An ideal $I \subset A$ is *primary* if $I \neq A$ and if $xy \in I$ implies either $x \in I$ or $y^n \in I$ for some $n > 0$.

I like to interpret maximal ideals as "the quotient is a field", and prime ideals as "the quotient is an integral domain". We can interpret primary ideals similarly as "the quotient is not 0, and every zero-divisor is nilpotent".

1.1. Exercise. Show that if $\mathfrak{q}$ is primary, then $\sqrt{\mathfrak{q}}$ is prime. If $\mathfrak{p} = \sqrt{\mathfrak{q}}$, we say that $\mathfrak{q}$ is $\mathfrak{p}$-*primary*.

1.2. Exercise. Show that if $\mathfrak{q}$ and $\mathfrak{q}'$ are $\mathfrak{p}$-primary, then so is $\mathfrak{q} \cap \mathfrak{q}'$.

1.3. Exercise (reality check). Find all the primary ideals in $\mathbb{Z}$. (Answer: (0) and (p^n).)

(Here is an unimportant side remark for experts; everyone else should skip this. Warning: a prime power need not be primary. An example is given in Atiyah-Macdonald, p. 51. $A = k[x,y,z]/(xy - z^2)$. then $\mathfrak{p} = (x,y)$ is prime but $\mathfrak{p}^2$ is not primary. Geometric hint that there is something going on: this is a ruling of a cone.)

A *primary decomposition* of an ideal $I \subset A$ is an expression of the ideal as a finite intersection of primary ideals.

$$I = \cap_{i=1}^n \mathfrak{q}_i$$

If there are "no redundant elements" (i.e. the $\sqrt{\mathfrak{q}_i}$ are all distinct, and for no i is $\mathfrak{q}_i \supset \cap_{j \neq i} \mathfrak{q}_j$), we say that the decomposition is *minimal*. Clearly any ideal with a primary decomposition has a minimal primary decomposition (using Exercise 1.2).

1.4. Exercise. Suppose A is a Noetherian ring. Show that every proper ideal $I \neq A$ has a primary decomposition. (Hint: Noetherian induction.)

1.5. Important Example. Find a minimal primary decomposition of (x^2, xy). (Answer: $(x) \cap (x^2, xy, y^n)$.)

In order to study these objects, we'll need a definition and a useful fact.

If $I \subset A$ is an ideal, and $x \in A$, then $(I : x) := \{a \in A : ax \in I\}$. (We will use this terminology only for this section.) For example, x is a *zero-divisor* if $(0 : x) \neq 0$.

1.6. Useful Exercise. (a) If $\mathfrak{p}$, $\mathfrak{p}_1$, …, $\mathfrak{p}_n$ are prime ideals, and $\mathfrak{p} = \cap \mathfrak{p}_i$, show that $\mathfrak{p} = \mathfrak{p}_i$ for some i. (Hint: assume otherwise, choose $f_i \in \mathfrak{p}_i - \mathfrak{p}$, and consider $\prod f_i$.)
(b) If $\mathfrak{p} \supset \cap \mathfrak{p}_i$, then $\mathfrak{p} \supset \mathfrak{p}_i$ for some i.
(c) Suppose $I \subseteq \cup^n \mathfrak{p}_i$. Show that $I \subset \mathfrak{p}_i$ for some i. (Hint: by induction on n.)

1.7. *Theorem (Uniqueness of primary decomposition). — Suppose I has a minimal primary decomposition*

$$I = \cap_{i=1}^n \mathfrak{q}_i.$$

Then the $\sqrt{\mathfrak{q}_i}$ are precisely the prime ideals that are of the form

$$\sqrt{(I : x)}$$

for some $x \in A$. Hence this list of primes is independent of the decomposition.

These primes are called the *associated primes* of the ideal.

Proof. We make a very useful observation: for any $x \in A$,

$$(I : x) = (\cap \mathfrak{q}_i : x) = \cap(\mathfrak{q}_i : x),$$

from which

$$\sqrt{(I : x)} = \cap \sqrt{(\mathfrak{q}_i : x)} = \cap_{x \notin \mathfrak{q}_j} \mathfrak{p}_j. \tag{2}$$

Now we prove the result.

Suppose first that $\sqrt{(I : x)}$ is prime, say $\mathfrak{p}$. Then $\mathfrak{p} = \cap_{x \notin \mathfrak{q}_j} \mathfrak{p}_j$ by (2), and by Exercise 1.6(a), $\mathfrak{p} = \mathfrak{p}_j$ for some j.

Conversely, we find an x such that $\sqrt{(I : x)} = \sqrt{\mathfrak{q}_i}$ $(= \mathfrak{p}_i)$. Take $x \in \cap_{j \neq i} \mathfrak{q}_j - \mathfrak{q}_i$ (which is possible by minimality of the primary decomposition). Then by (2), we're done. □

If A is a ring, the *associated primes* of A are the associated primes of 0.

1.8. Exercise. Show that these associated primes behave well with respect to localization. In other words if A is a Noetherian ring, and S is a multiplicative subset (so, as we've seen, there is an inclusion-preserving correspondence between the primes of $S^{-1}A$ and those primes of A not meeting S), then the associated primes of $S^{-1}A$ are just the associated primes of A not meeting S.

We then define the *associated points* of a locally Noetherian scheme X to be those points $p \in X$ such that, on any affine open set $\operatorname{Spec} A$ containing p, p corresponds to an associated prime of A. If furthermore X is quasicompact (i.e. X is a Noetherian scheme), then there are a finite number of associated points.

1.9. Exercise. Show that the minimal primes of 0 are associated primes. (We have now proved important fact **(1)**.) (Hint: suppose $\mathfrak{p} \supset \cap_{i=1}^n \mathfrak{q}_i$. Then $\mathfrak{p} = \sqrt{\mathfrak{p}} \supset \sqrt{\cap_{i=1}^n \mathfrak{q}_i} = \cap_{i=1}^n \sqrt{\mathfrak{q}_i} = \cap_{i=1}^n \mathfrak{p}_i$, so by Exercise 1.6(b), $\mathfrak{p} \supset \mathfrak{p}_i$ for some i. If $\mathfrak{p}$ is minimal, then as $\mathfrak{p} \supset \mathfrak{p}_i \supset (0)$, we must have $\mathfrak{p} = \mathfrak{p}_i$.) Show that there can be other associated primes that are not minimal. (Hint: Exercise 1.5.)

1.10. Exercise. Show that if A is reduced, then the only associated primes are the minimal primes. (This establishes **(2)**.)

The $\mathfrak{q}_i$ corresponding to minimal primes are unique, but the $\mathfrak{q}_i$ corresponding to other associated primes are not unique, but we will not need this fact, and hence won't prove it.

1.11. *Proposition. — The natural map* $R \to \prod R_{\mathfrak{p}}$ *is an inclusion.*

This establishes **(3)**.

Proof. Suppose $r \mapsto 0$. Thus there are $s_i \in R - \mathfrak{p}$ with $s_i r = 0$. Then $I := (s_1, \dots, s_n)$ is an ideal consisting only of zero-divisors. Hence $I \subseteq \cap \mathfrak{p}_i$. Then $I \subset \mathfrak{p}_i$ for some i by Exercise 1.6(c), contradicting $s_i \notin \mathfrak{p}_i$. $\square$

1.12. *Proposition. — The set of zero-divisors is precisely the union of the associated primes.*

This establishes **(4)**: a function is a zero-divisor if and only if it vanishes at an associated point. Thus (for a Noetherian scheme) a function is a zero divisor if and only if its zero locus contains one of a finite set of points.

You may wish to try this out on the example of Exercise 1.5.

Proof. If $\mathfrak{p}_i$ is an associated prime, then $\mathfrak{p}_i = \sqrt{(0 : x)}$ from the proof of Theorem 1.7, so $\cup \mathfrak{p}_i$ is certainly contained in the set D of zero-divisors.

For the converse, verify the inclusions and equalities (**Exercise**)

$$D = \cup_{x \neq 0} (0 : x) \subseteq \cup_{x \neq 0} \sqrt{(0 : x)} \subseteq D.$$

Hence

$$D = \cup_{x \neq 0} \sqrt{(0 : x)} = \cup_x \left(\cap_{x \notin \mathfrak{q}_j} \mathfrak{p}_j \right) \subseteq \cup \mathfrak{p}_j$$

using (2). $\square$

(Note for experts from Kirsten and Joe: Let X be a locally Noetherian scheme, $x \in X$. Then x is an associated point of X if and only if every nonunit of $\mathcal{O}_{X,x}$ is a zero-divisor. Proof: We must show that a prime ideal $\mathfrak{p}$ of a Noetherian ring A is associated if and only if every nonunit of $A_\mathfrak{p}$ is a zero-divisor, i.e., if and only if $\mathfrak{p}A_\mathfrak{p}$ is an associated prime in $A_\mathfrak{p}$. But this is obvious since primary decompositions respect localization.)

2. Invertible sheaves and divisors

We want to understand invertible sheaves (line bundles) on a given sheaf X. How can we describe many of them? How can we describe them all?

In order to answer this question, I should tell you a bit more about normality.

2.1. **A bit more on normality.** I earlier defined normality in the wrong way, only for integral schemes: I said that an integral scheme X is normal if and only if for every affine open set $\operatorname{Spec} R$, R is integrally closed in its fraction field.

Here is the right definition: we say a scheme X is normal if all of its stalks $\mathcal{O}_{X,x}$ are normal. (In particular, all stalks are necessarily domains.) This is clearly a local property: if $\cup U_i$ is an open cover of X, then X is normal if and only if each U_i is normal.

Note that for Noetherian schemes, normality can be checked at closed points, as integral closure behaves well under localization (we've checked that), and every open set contains closed points of the scheme (we've checked that), so any point is a generization of a closed point.

As reducedness is a stalk-local property (we've checked that X is reduced if and only if all its stalks are reduced), a normal scheme is necessarily reduced. It is not true however that normal schemes are integral. For example, the disjoint union of two normal schemes is normal. So for example $\operatorname{Spec} k \coprod \operatorname{Spec} k \cong \operatorname{Spec}(k \times k) \cong \operatorname{Spec} k[x]/(x(x-1))$ is normal, but its ring of global sections is not a domain.

Unimportant remark. Normality satisfies the hypotheses of the Affine Covering Lemma, fairly tautologically, because it is a stalk-local property. We can say more explicitly and ring-theoretically what it means for $\operatorname{Spec} A$ to be normal, at least when A is Noetherian. It is that $\operatorname{Spec} A$ is normal if and only if A is reduced, and it is integrally closed in its ring of fractions. (The ring of fractions was defined earlier today in the discussion on associated points. It is the product of the localizations at the associated points. In this case, as A is reduced, it is the product of the localizations at the minimal primes.) Basically, most constructions that make sense for domains and involve function fields should be generalized to Noetherian rings in general, and the role of "function field" should be replaced by "ring of fractions".

I should finally state "Hartogs' theorem" explicitly and rigorously. (Caution: No one else calls this Hartogs' Theorem. I've called it this because of the metaphor to complex geometry.)

2.2. *"Hartogs' theorem". — Suppose A is a Noetherian normal domain. Then in $\operatorname{Frac}(A)$,*

$$A = \cap_{\mathfrak{p} \text{ height } 1} A_{\mathfrak{p}}.$$

More generally, if A is a product of Noetherian normal domains (i.e. $\operatorname{Spec} A$ is Noetherian normal scheme), then in the ring of fractions of A,

$$A = \cap_{\mathfrak{p} \text{ height } 1} A_{\mathfrak{p}}.$$

I stated the special case first so as to convince you that this isn't scary.

To show you the power of this result, let me prove Krull's Principal Ideal Theorem in the case of Noetherian normal domains. (Eventually, I hope to add to the notes a proof of Krull's Principal Ideal Theorem in general, as well as "Hartogs' Theorem".)

2.3. *Theorem (Krull's Principal Ideal Theorem for Noetherian normal domains). — Suppose A is a Noetherian normal domains, and $f \in A$. Then the minimal primes containing f are all of height precisely 1.*

Proof. The first statement implies the second: because A is a domain, the associated primes of $\operatorname{Spec} A$ are precisely the minimal (i.e. height 0) primes. If f is a not a zero-divisor, then f is not an element of any of these primes, by Proposition 1.12.

So we will now prove the first statement.

Suppose $f \in \mathrm{Frac}(A)$. We wish to show that the minimal primes containing f are all height 1. If there is one which is height greater than 1, then after localizing at this prime, we may assume that A is a local ring with maximal ideal $\mathfrak{m}$ of height at least 2, and that the only prime containing f is $\mathfrak{m}$. Let $g = 1/f \in \mathrm{Frac}(A)$. Then $g \in A_{\mathfrak{p}}$ for all height 1 primes $\mathfrak{p}$, so by "Hartogs' Theorem", $g \in A$. Thus $gf = 1$. But $g, f \in A$, and $f \in \mathfrak{m}$, so we have a contradiction.

Exercise. Suppose f and g are two global sections of a Noetherian normal scheme with the same poles and zeros. Show that each is a unit times the other.

I spent the rest of the class discussing Cartier divisors. I've put these notes with the class 18 notes.

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FOUNDATIONS OF ALGEBRAIC GEOMETRY CLASS 18

CONTENTS

Last day: Associated points; more on normality; invertible sheaves and divisors take 1.

Today: Invertible sheaves and divisors. Morphisms of schemes.

1. INVERTIBLE SHEAVES AND DIVISORS

We next develop some mechanism of understanding invertible sheaves (line bundles) on a given scheme X. Define $\operatorname{Pic} X$ to be the group of invertible sheaves on X. How can we describe many of them? How can we describe them all? Our goal for the first part of today will be to partially address this question. As an important example, we'll show that we have already found all the invertible sheaves on projective space $\mathbb{P}^n_k$ — they are the $\mathcal{O}(m)$.

One moral of this story will be that invertible sheaves will correspond to "codimension 1 information".

Recall one way of getting invertible sheaves, by way of *effective Cartier divisors*. Recall that an effective Cartier divisor is a closed subscheme such that there exists an affine cover such that on each one it is cut out by a single equation, not a zero-divisor. (This does not mean that on *any* affine it is cut out by a single equation — this notion doesn't satisfy the "gluability" hypothesis of the Affine Communication Lemma. If $I \subset R$ is generated by a non-zero divisor, then $I_f \subset R_f$ is too. But "not conversely". I might give an example later, involving an elliptic curve.) By Krull's Principal Ideal Theorem, it is pure codimension 1.

Remark: if $I = (u) = (v)$, and u is not a zero-divisor, then u and v differ multiplicatively by a unit in R. Proof: $u \in (v)$ implies $u = av$. Similarly $v = bu$. Thus $u = abu$, from which $u(1 - ab) = 0$. As u is not a zero-divisor, $1 = ab$, so a and b are units. In other words, the generator of such an ideal is well-defined up to a unit.

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The reason we care: effective Cartier divisors give invertible sheaves. If $\mathcal{I}$ is an effective Cartier divisor on X, then $\mathcal{I}$ is an invertible sheaf. Reason: locally, sections are multiples of a single generator u, and there are no "relations".

Recall that the invertible sheaf $\mathcal{O}(D)$ corresponding to an effective Cartier divisor is defined to be the *dual* of the ideal sheaf $\mathcal{I}_D$. The ideal sheaf itself is sometimes denoted $\mathcal{O}(-D)$. We have an exact sequence

$$0 \to \mathcal{O}(-D) \to \mathcal{O} \to \mathcal{O}_D \to 0.$$

The invertible sheaf $\mathcal{O}(D)$ has a canonical section: Dualizing $0 \to \mathcal{I} \to \mathcal{O}$ gives us $\mathcal{O} \to \mathcal{I}^*$.

Exercise. This section vanishes along our actual effective Cartier divisor.

Exercise. Conversely, if $\mathcal{L}$ is an invertible sheaf, and s is a section that is not locally a zero divisor (make sense of this!), then $s = 0$ cuts out an effective Cartier divisor D, and $\mathcal{O}(D) \cong \mathcal{L}$. (If X is locally Noetherian, "not locally a zero divisor" translate to "does not vanish at an associated point".)

Define the *sum* of two effective Cartier divisors as follows: if $I = (u)$ (locally) and $J = (v)$, then the sum corresponds to (uv) locally. (Verify that this is well-defined!)

Exercise. Show that $\mathcal{O}(D + E) \cong \mathcal{O}(D) \otimes \mathcal{O}(E)$.

Thus we have a map of semigroups, from effective Cartier divisors to invertible sheaves with sections not locally zero-divisors (and hence also to the Picard group of invertible sheaves).

Hence we can get a bunch of invertible sheaves, by taking differences of these two. The surprising fact: we "usually get them all"! In fact it is very hard to describe an invertible sheaf on a finite type k-scheme that is not describable in such a way (we will see later today that there are none if the scheme is nonsingular or even factorial; and we might see later in the year that there are none if the scheme is quasiprojective).

Instead, I want to take another tack. Some of what we do will generalize to the non-normal case, which is certainly important, and experts are invited to think about this.

Define a *Weil divisor* as a formal sum of height 1 irreducible closed subsets of X. (This makes sense more generally on any pure dimensional, or even locally equidimensional, scheme.) In other words, a Weil divisor is defined to be an object of the form

$$\sum_{Y \subset X \text{ height } 1} n_Y [Y]$$

the n_Y are integers, all but a finite number of which are zero. Weil divisors obviously form an abelian group, denoted $\operatorname{Weil} X$.

A Weil divisor is said to be *effective* if $n_Y \geq 0$ for all Y. In this case we say $D \geq 0$, and by $D_1 \geq D_2$ we mean $D_1 - D_2 \geq 0$. The *support* of a Weil divisor D is the subset

$\cup_{n_Y \neq 0} Y$. If $U \subset X$ is an open set, there is a natural restriction map $\operatorname{Weil} X \to \operatorname{Weil} U$, where $\sum n_Y[Y] \mapsto \sum_{Y \cap U \neq \emptyset} n_Y[Y \cap U]$.

Suppose now that X is a Noetherian scheme, regular in codimension 1. We add this hypothesis because we will use properties of discrete valuation rings. Suppose that $\mathcal{L}$ is an invertible sheaf, and s a rational section not vanishing on any irreducible component of X. Then s determines a Weil divisor

$$\operatorname{div}(s) := \sum_Y \operatorname{val}_Y(s)[Y].$$

(Recall that $\operatorname{val}_Y(s) = 0$ for all but finitely many Y, by problem 46 on problem set 5.) This is the "divisor of poles and zeros of s". (To determine the valuation $\operatorname{val}_Y(s)$ of s along Y, take any open set U containing the generic point of Y where $\mathcal{L}$ is trivializable, along with any trivialization over U; under this trivialization, s is a function on U, which thus has a valuation. Any two such trivializations differ by a unit, so this valuation is well-defined.) This map gives a group homomorphism
(1)
$$\operatorname{div} : \{(\text{invertible sheaf } \mathcal{L}, \text{rational section } s \text{ not vanishing at any minimal prime})\}/\Gamma(X, \mathcal{O}_X^*) \to \operatorname{Weil} X.$$

1.1. *Exercise.* (a) (divisors of rational functions) Verify that on $\mathbb{A}^1_k$, $\operatorname{div}(x^3/(x+1)) = 3[(x)] - [(x+1)] = 3[0] - [-1]$.
(b) (divisor of a rational sections of a nontrivial invertible sheaf) Verify that on $\mathbb{P}^1_k$, there is a rational section of $\mathcal{O}(1)$ "corresponding to" x^2/y. Calculate $\operatorname{div}(x^2/y)$.

We want to classify all invertible sheaves on X, and this homomorphism (1) will be the key. Note that any invertible sheaf will have such a rational section (for each irreducible component, take a non-empty open set not meeting any other irreducible component; then shrink it so that $\mathcal{L}$ is trivial; choose a trivialization; then take the union of all these open sets, and choose the section on this union corresponding to 1 under the trivialization). We will see that in reasonable situations, this map div will be injective, and often even an isomorphism. Thus by forgetting the rational section (taking an appropriate quotient), we will have described the Picard group. Let's put this strategy into action. *Suppose from now on that X is normal.*

1.2. *Proposition. — The map* div *is injective.*

Proof. Suppose $\operatorname{div}(\mathcal{L}, s) = 0$. Then s has no poles. Hence by Hartogs' theorem, s is a regular section. Now s vanishes nowhere, so s gives an isomorphism $\mathcal{O}_X \to \mathcal{L}$ (given by $1 \mapsto s$). $\square$

Motivated by this, we try to find the inverse map to div.

Definition. Suppose D is a Weil divisor. If $U \subset X$ is an open subscheme, define $\operatorname{Frac}(U)$ to be the field of total fractions of U, i.e. the product of the stalks at the minimal primes of U. (As described earlier, if U is irreducible, this is the function field.) Define $\operatorname{Frac}(U)^*$ to be those rational functions not vanishing at any generic point of U (i.e. not vanishing on any irreducible component of U). Define the sheaf $\mathcal{O}(D)$ by

$$\Gamma(U, \mathcal{O}(D)) := \{s \in \mathrm{Frac}(U)^* : \operatorname{div} s + D|_U \geq 0\}.$$

Note that the sheaf $\mathcal{O}(D)$ has a canonical rational section, corresponding to $1 \in \mathrm{Frac}(U)^*$.

1.3. *Proposition. — Suppose $\mathcal{L}$ is an invertible sheaf, and s is a rational section not vanishing on any irreducible component of X. Then there is an isomorphism $(\mathcal{L}, s) \cong (\mathcal{O}(\operatorname{div} s), t)$, where t is the canonical section described above.*

Proof. We first describe the isomorphism $\mathcal{O}(\operatorname{div} s) \cong \mathcal{L}$. Over open subscheme $U \subset X$, we have a bijection $\Gamma(U, \mathcal{L}) \to \Gamma(U, \mathcal{O}(\operatorname{div} s))$ given by $s' \mapsto s'/s$, with inverse obviously given by $t' \mapsto st'$. Clearly under this bijection, s corresponds to the section 1 in $\mathrm{Frac}(U)^*$; this is the section we are calling t. $\square$

We denote the subgroup of $\operatorname{Weil} X$ corresponding to divisors of rational *functions* the subgroup of *principal divisors*, which we denote $\operatorname{Prin} X$. Define the *class group* of X, $\operatorname{Cl} X$, by $\operatorname{Weil} X / \operatorname{Prin} X$. By taking the quotient of the inclusion (1) by $\operatorname{Prin} X$, we have the inclusion

$$\operatorname{Pic} X \hookrightarrow \operatorname{Cl} X.$$

We're now ready to get a hold of $\operatorname{Pic} X$ rather explicitly!

First, some algebraic preliminaries.

1.4. *Exercise.* Suppose that A is a Noetherian domain. Show that A is a Unique Factorization Domain if and only if all height 1 primes are principal. You can use this to answer that homework problem, about showing that $k[w, x, y, z]/(wz - xy)$ is not a Unique Factorization Domain.

1.5. *Exercise.* Suppose that A is a Noetherian domain. Show that A is a Unique Factorization Domain if and only if A is integrally closed and $\operatorname{Cl} \operatorname{Spec} A = 0$. (One direction is easy: we have already shown that Unique Factorization Domains are integrally closed in their fraction fields. Also, the previous exercise shows that all height 1 primes are principal, so that implies that $\operatorname{Cl} \operatorname{Spec} A = 0$. It remains to show that if A is integrally closed and $\operatorname{Cl} X = 0$, then all height 1 prime ideals are principal. "Hartogs" may arise in your argument.)

Hence $\operatorname{Cl}(\mathbb{A}^n_k) = 0$, so $\boxed{\operatorname{Pic}(\mathbb{A}^n_k) = 0}$. (Geometers will find this believable: "$\mathbb{C}^n$ is a contractible manifold, and hence should have no nontrivial line bundles".)

Another handy trick is the following. Suppose Z is an irreducible codimension 1 subset of X. Then we clearly have an exact sequence:

$$0 \longrightarrow \mathbb{Z} \xrightarrow{1 \mapsto [Z]} \operatorname{Weil} X \longrightarrow \operatorname{Weil}(X - Z) \longrightarrow 0.$$

When we take the quotient by principal divisors, we get:

$$\mathbb{Z} \xrightarrow{1 \mapsto [Z]} \operatorname{Cl} X \longrightarrow \operatorname{Cl}(X - Z) \longrightarrow 0.$$

For example, let $X = \mathbb{P}^n_k$, and Z be the hyperplane $x_0 = 0$. We have

$$\mathbb{Z} \to \operatorname{Cl} \mathbb{P}^n_k \to \operatorname{Cl} \mathbb{A}^n_k \to 0$$

from which $\operatorname{Cl} \mathbb{P}^n_k = \mathbb{Z}[Z]$ (which is $\mathbb{Z}$ or 0), and $\operatorname{Pic} \mathbb{P}^n_k$ is a subgroup of this.

1.6. *Important exercise.* Verify that $[Z] \to \mathcal{O}(1)$. In other words, let Z be the Cartier divisor $x_0 = 0$. Show that $\mathcal{O}(Z) \cong \mathcal{O}(1)$. (For this reason, people sometimes call $\mathcal{O}(1)$ the *hyperplane class* in $\operatorname{Pic} X$.)

Hence $\operatorname{Pic} \mathbb{P}^n_k \hookrightarrow \operatorname{Cl} \mathbb{P}^n_k$ is an isomorphism, and $\boxed{\operatorname{Pic} \mathbb{P}^n_k \cong \mathbb{Z}}$, with generator $\mathcal{O}(1)$. The *degree* of an invertible sheaf on $\mathbb{P}^n$ is defined using this: the degree of $\mathcal{O}(d)$ is of course d.

More generally, if X is *factorial* — all stalks are Unique Factorization Domains — then for any Weil divisor D, $\mathcal{O}(D)$ is invertible, and hence the map $\operatorname{Pic} X \to \operatorname{Cl} X$ is an isomorphism. (Proof: It will suffice to show that $[Y]$ is Cartier if Y is any irreducible codimension 1 set. Our goal is to cover X by open sets so that on each open set U there is a function whose divisor is $[Y \cap U]$. One open set will be $X - Y$, where we take the function 1. Next, suppose $x \in Y$; we will find an open set $U \subset X$ containing x, and a function on it. As $\mathcal{O}_{X,x}$ is a unique factorization domain, the prime corresponding to 1 is height 1 and hence principal (by Exercise 1.4). Let $f \in \operatorname{Frac} A$ be a generator. Then f is regular at x. f has a finite number of zeros and poles, and through x there is only one 0, notably $[Y]$. Let U be X minus all the others zeros and poles.)

I will now mention a bunch of other examples of class groups and Picard groups you can calculate.

For the first, I want to note that you can restrict invertible sheaves on X to any subscheme Y, and this can be a handy way of checking that an invertible sheaf is not trivial. For example, if X is something crazy, and $Y \cong \mathbb{P}^1$, then we're happy, because we understand invertible sheaves on $\mathbb{P}^1$. Effective Cartier divisors sometimes restrict too: if you have effective Cartier divisor on X, then it restricts to a closed subscheme on Y, locally cut out by one equation. If you are fortunate that this equation doesn't vanish on any associated point of Y, then you get an effective Cartier divisor on Y. You can check that the restriction of effective Cartier divisors corresponds to restriction of invertible sheaves.

1.7. *Exercise: a torsion Picard group.* Show that Y is an irreducible degree d hypersurface of $\mathbb{P}^n$. Show that $\operatorname{Pic}(\mathbb{P}^n - Y) \cong \mathbb{Z}/d$. (For differential geometers: this is related to the fact that $\pi_1(\mathbb{P}^n - Y) \cong \mathbb{Z}/d$.)

1.8. *Exercise.* Let $X = \operatorname{Proj} k[w, x, y, z]/(wz - xy)$, a smooth quadric surface. Show that $\operatorname{Pic} X \cong \mathbb{Z} \oplus \mathbb{Z}$ as follows: Show that if L and M are two lines in different rulings (e.g. $L = V(w, x)$ and $M = V(w, y)$), then $X - L - M \cong \mathbb{A}^2$. This will give you a surjection

$\mathbb{Z} \oplus \mathbb{Z} \twoheadrightarrow \operatorname{Cl} X$. Show that $\mathcal{O}(L)$ restricts to $\mathcal{O}$ on L and $\mathcal{O}(1)$ on M. Show that $\mathcal{O}(M)$ restricts to $\mathcal{O}$ on M and $\mathcal{O}(1)$ on L. (This is a bit longer to do, but enlightening.)

1.9. *Exercise.* Let $X = \operatorname{Spec} k[w, x, y, z]/(xy - z^2)$, a cone. show that $\operatorname{Pic} X = 1$, and $\operatorname{Cl} X \cong \mathbb{Z}/2$. (Hint: show that the ruling $Z = \{x = z = 0\}$ generates $\operatorname{Cl} X$ by showing that its complement is isomorphic to $\mathbb{A}^2_k$. Show that $2[Z] = \operatorname{div}(x)$ (and hence principal), and that Z is not principal (an example we did when discovering the power of the Zariski tangent space).

Note: on curves, the invertible sheaves correspond to formal sums of points, modulo equivalence relation.

Number theorists should note that we have recovered a common description of the class group: formal sums of primes, modulo an equivalence relation.

Remark: Much of this discussion works without the hypothesis of normality, and indeed because non-normal schemes come up all the time, we need this additional generality. Think through this if you like.

2. MORPHISMS OF SCHEMES

Here are two motivations that will "glue together".

(a) We'll want morphisms of affine schemes $\operatorname{Spec} R \to \operatorname{Spec} S$ to be precisely the ring maps $S \to R$. Then we'll want maps of schemes to be things that "look like this". "the category of affine schemes is opposite to the category of rings". More correctly there is an equivalence of categories...

(b) We are also motivated by the theory of differentiable manifolds. We'll want a continuous maps from the underlying topological spaces $f : X \to Y$, along with a "pullback morphism" $f^{\#} : \mathcal{O}_S \to f_* \mathcal{O}_X$. There are many things we'll want to be true, that seem make a tall order; a clever idea will give us all of this for free. (i) Certainly values at points should map. They can't be the same: $\operatorname{Spec} \mathbb{C} \to \operatorname{Spec} \mathbb{R}$. (ii) $\operatorname{Spec} k[\epsilon]/\epsilon^2 \to \operatorname{Spec} k[\delta]/\delta^2$ is given by a map $\delta \mapsto q\epsilon$. These aren't distinguished by maps on points. (iii) Suppose you have a function σ on Y (i.e. $\sigma \in \Gamma(Y, \mathcal{O}_Y)$. Then it will pull back to a function $f^{-1}(\sigma)$ on X. However we make sense of pullbacks of functions (i) and (ii), certainly the locus where $f^{-1}(\sigma)$ vanishes on X should be the pullback of the locus where σ vanishes on Y. This will imply that the maps on stalks will be a local map (if $f(p) = q$ then $f^{\#} : \mathcal{O}_{Y,q} \to \mathcal{O}_{X,p}$ sends the maximal ideal. translating to: then germs of functions vanishing at q pullback to germs of functions vanishing at p). This last thing does it for us.

3. RINGED SPACES AND THEIR MORPHISMS

A ringed space is a topological space X along with a sheaf $\mathcal{O}_X$ of rings (called the *structure sheaf*. Our central example is a scheme. Another example is a differentiable manifold with the analytic topology and the sheaf of differentiable functions.

A **morphism of ringed spaces** $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a continuous map $f : X \to Y$ (also sloppily denoted by the same name "f") along with a morphism $f^\# : \mathcal{O}_Y \to f_*\mathcal{O}_X$ of sheaves on Y (or equivalently but less usefully $f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$ of sheaves on X, by adjointness). The morphism is often denoted $X \to Y$ when the structure sheaves and morphisms "between" them are clear from the context. There is an obvious notion of composition of morphisms; hence there is a category of ringed spaces. Hence we have notion of automorphisms and isomorphisms.

Slightly unfortunate notation: $f : X \to Y$ often denotes everything. Also used for maps of underlying sets, or underlying topological spaces. Usually clear from context.

3.1. *Exercise.* If $W \subset X$ and $Y \subset Z$ are both open immersions of ringed spaces, show that any morphism of ringed spaces $X \to Y$ induces a morphism of ringed spaces $W \to Z$.

3.2. *Exercise.* Show that morphisms of ringed spaces glue. In other words, suppose X and Y are ringed spaces, $X = \cup_i U_i$ is an open cover of X, and we have morphisms of ringed spaces $f_i : U_i \to Y$ that "agree on the overlaps", i.e. $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$. Show that there is a unique morphism of ringed spaces $f : X \to Y$ such that $f|_{U_i} = f_i$. (Long ago we had an exercise proving this for topological spaces.)

3.3. *Easy important exercise.* Given a morphism of ringed spaces $f : X \to Y$ with $f(p) = q$, show that there is a map of stalks $(\mathcal{O}_Y)_q \to (\mathcal{O}_X)_p$.

3.4. Important Example. Suppose $f^\# : S \to R$ is a morphism of rings. Define a morphism of ringed spaces as follows. $f : sp(\operatorname{Spec} R) \to sp(\operatorname{Spec} S)$. First as sets. $\mathfrak{p}$ prime in S, then ${f^\#}^{-1}(\mathfrak{p})$ is prime in R.

We interrupt this definition for a picture: $R = \operatorname{Spec} k[x, y]$, $S = \operatorname{Spec} k[t]$, $t \mapsto x$. Draw picture. Look at primes $(x - 2, y - 3)$. Look at (0). Look at $(x - 3)$. $(y - x^2)$.

It's a continuous map of topological spaces: $D(s)$ pulls back to $D(f^\# s)$. Now the map on sheaves. If $s \in S$, then show that $\Gamma(D(s), f_*\mathcal{O}_R) = R_{f^\# s} \cong R \otimes_S S_s$. (**Exercise.** Verify that $R_{f^\# s} \cong R \otimes_S S_s$ if you haven't seen this before.) Show that $f_* : \Gamma(D(s), \mathcal{O}_S) = S_s \to \Gamma(D(s), f_*\mathcal{O}_R) = R \otimes_S S_s$ given by $s' \mapsto 1 \otimes s'$ is a morphism of sheaves on the distinguished base of S, and hence defines a morphism of sheaves $f_*\mathcal{O}_R \to \mathcal{O}_S$.

4. Definition of morphisms of schemes

A morphism $f : X \to Y$ of schemes is a morphism of ringed spaces. Sadly, if X and Y are schemes, then there are morphisms $X \to Y$ *as ringed spaces* that are not morphisms as schemes. (See Example II.2.3.2 in Hartshorne for an example.)

The idea behind definition of morphisms is as follows. We define morphisms of affine schemes as in Important Example 3.4. (Note that the category of affine schemes is "opposite to the category of rings": given a morphisms of schemes, we get a map of rings in the opposite direction, and vice versa.)

4.1. *Definition/Proposition.* — A morphism of schemes $f : X \to Y$ is a morphism of ringed spaces that looks locally like morphisms of affines. In other words, if $\operatorname{Spec} A$ is an affine open subset of X and $\operatorname{Spec} B$ is an affine open subset of Y, and $f(\operatorname{Spec} A) \subset \operatorname{Spec} B$, then the induced morphism of ringed spaces (Exercise 3.1) is a morphism of affine schemes. It suffices to check on a set $(\operatorname{Spec} A_i, \operatorname{Spec} B_i)$ where the $\operatorname{Spec} A_i$ form an open cover X.

We could prove the proposition using the affine communication theorem, but there's a clever trick. For this we need a digression on locally ringed spaces. They will not be used hereafter.

A *locally ringed space* is a ringed space $(X, \mathcal{O}_X)$ such that the stalks $\mathcal{O}_{X,x}$ are all local rings. A *morphism of locally ringed spaces* $f : X \to Y$ is a morphism of ringed spaces such that the induced map of stalks (Exercise 3.3) $\mathcal{O}_{Y,q} \to \mathcal{O}_{X,p}$ sends the maximal ideal of the former to the maximal ideal of the latter. (This is sometimes called a "local morphism of local rings".) This means something rather concrete and intuitive: "if $p \mapsto q$, and g is a function vanishing at q, then it will pull back to a function vanishing at p." Note that locally ringed spaces form a category.

4.2. *Exercise.* Show that morphisms of locally ringed spaces glue (cf. Exercise 3.2). (Hint: Basically, the proof of Exercise 3.2 works.)

4.3. *Easy important exercise.* (a) Show that $\operatorname{Spec} R$ is a locally ringed space. (b) The morphism of ringed spaces $f : \operatorname{Spec} R \to \operatorname{Spec} S$ defined by a ring morphism $f^\# : S \to R$ (Exercise 3.4) is a morphism of locally ringed spaces.

Proposition 4.1 now follows from:

4.4. *Key Proposition.* — *If* $f : \operatorname{Spec} R \to \operatorname{Spec} S$ *is a morphism of locally ringed spaces then it is the morphism of locally ringed spaces induced by the map* $f^\# : S = \Gamma(\operatorname{Spec} S, \mathcal{O}_{\operatorname{Spec} S}) \to \Gamma(\operatorname{Spec} R, \mathcal{O}_{\operatorname{Spec} R}) = R.$.

Proof. Suppose $f : \operatorname{Spec} R \to \operatorname{Spec} S$ is a morphism of locally ringed spaces. Then we wish to show that $f^\# : \mathcal{O}_{\operatorname{Spec} S} \to f_* \mathcal{O}_{\operatorname{Spec} R}$ is the morphism of sheaves given by Exercise 3.4 (cf. Exercise 4.3(b)). It suffices to checked this on the distinguished base.

Note that if $s \in S$, $f^{-1}(D(s)) = D(f^\# s)$; this is where we use the hypothesis that f is a morphism of locally ringed spaces.

The commutative diagram

$$\begin{tikzcd} \Gamma(\operatorname{Spec} S, \mathcal{O}_{\operatorname{Spec} S}) \arrow[r, "f^{\#}_{\operatorname{Spec} S}"] \arrow[d] & \Gamma(\operatorname{Spec} R, \mathcal{O}_{\operatorname{Spec} R}) \arrow[d, "\otimes_S S_s"] \\ \Gamma(D(s), \mathcal{O}_{\operatorname{Spec} S}) \arrow[r, "f^{\#}_{D(s)}"] & \Gamma(D(f^{\#}s), \mathcal{O}_{\operatorname{Spec} R}) \end{tikzcd}$$

may be written as

$$\begin{tikzcd} S \arrow[r, "f^{\#}_{\operatorname{Spec} S}"] \arrow[d] & R \arrow[d, "\otimes_S S_s"] \\ S_s \arrow[r, "f^{\#}_{D(s)}"] & R_{f^{\#}s}. \end{tikzcd}$$

We want that $f^{\#}_{D(s)} = (f^{\#}_{\operatorname{Spec} S})_s$. This is clear from the commutativity of that last diagram. □

In particular, we can check on an affine cover, and then we'll have it on all affines. Also, morphisms glue (Exercise 4.2). And: the composition of two morphisms is a morphism.

4.5. *Exercise.* Make sense of the following sentence: "$\mathbb{A}^{n+1} - \vec{0} \to \mathbb{P}^n$ given by

$$(x_0, x_1, \ldots, x_{n+1}) \mapsto [x_0; x_1; \ldots; x_n]$$

is a morphism of schemes." Caution: you can't just say where points go; you have to say where functions go. So you'll have to divide these up into affines, and describe the maps, and check that they glue.

4.6. The category of schemes (or k-schemes, or R-schemes, or Z-schemes).

We have thus defined a *category* of schemes. We then have notions of **isomorphism** and **automorphism**. It is often convenient to consider subcategories. For example, the category of k-schemes (where k is a field) is defined as follows. The objects are morphisms of the form $\begin{tikzcd} X \arrow[d] \\ \operatorname{Spec} k \end{tikzcd}$. (This is definition is identical to the one we gave earlier, but in a more satisfactory form.) The morphism (in the category of schemes, not in the category of k-schemes) $X \to \operatorname{Spec} k$ is called the **structure morphism**. The morphisms in the category of k-schemes are commutative diagrams

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[d] \\ \operatorname{Spec} k \arrow[r, "="] & \operatorname{Spec} k \end{tikzcd}$$

which is more conveniently written as a commutative diagram

$$\begin{tikzcd} X \arrow[rr] \arrow[dr] & & Y \arrow[dl] \\ & \operatorname{Spec} k. & \end{tikzcd}$$

For example, complex geometers may consider the category of $\mathbb{C}$-schemes.

When there is no confusion, simply the top row of the diagram is given. More generally, if R is a ring, the category of R-schemes is defined in the same way, with R replacing k. And if Z is a scheme, the category of Z-schemes is defined in the same way, with Z replacing $\operatorname{Spec} k$.

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FOUNDATIONS OF ALGEBRAIC GEOMETRY CLASS 19

CONTENTS

Last day: Associated points; more on normality; invertible sheaves and divisors take 1.

Today: Maps to affine schemes; surjective, open immersion, closed immersion, quasicompact, locally of finite type, finite type, affine morphism, finite, quasifinite. Images of morphisms: constructible sets, and Chevalley's theorem (finite type morphism of Noetherian schemes sends constructibles to constructibles).

Last day, I defined a morphism of schemes $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ as follows.

I first defined the notion of a morphism of ringed spaces $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$, which is a continuous map of topological spaces $f : X \to Y$ along with a map of sheaves of rings (on Y) $\mathcal{O}_Y \to f_*\mathcal{O}_X$, or equivalently (by adjointness of inverse image and pushfoward) $f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$ (a map of sheaves of rings on X). This should be seen as a description of how to pull back functions on Y to get functions on X.

An example is a morphism of affine schemes $\operatorname{Spec} A \to \operatorname{Spec} B$. These correspond to morphisms of rings $B \to A$.

Then a morphism of schemes $X \to Y$ can be defined as a morphism of these ringed spaces, that locally looks like a morphism of affine schemes. In other words, X can be covered by affine open sets, such that for each such $\operatorname{Spec} R$, there is an affine open set $\operatorname{Spec} S$ of Y containing its image, such that the map $\operatorname{Spec} R \to \operatorname{Spec} S$ is of the form described in the primordial example.

We proved this by temporarily introducing a new concept, that of a *locally ringed space*. Then a morphism of schemes $X \to Y$ is just the same as a morphism of locally ringed spaces; we showed this by showing this for affine schemes.

I encouraged you to get practice with this in the following exercise, to make sense of the map $\mathbb{A}^{n+1} - 0 \to \mathbb{P}^n$ "given by" $(x_0, \ldots, x_n) \mapsto [x_0; \ldots; x_n]$.

We thus have described the *category of schemes*. The notion of an isomorphism of schemes subsumes our earlier definition.

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I described the category of k-schemes, or more generally A-schemes where A is a ring. More generally, if S is a scheme, we have the category of S-schemes. The objects are diagrams of the form

$$\begin{tikzcd} X \arrow[d] \\ S \end{tikzcd}$$

and morphisms are commutative diagrams of the form

$$\begin{tikzcd} X \arrow[rr] \arrow[dr] & & Y \arrow[dl] \\ & S & \end{tikzcd}$$

The category of k-schemes corresponds to the case $S = \operatorname{Spec} k$, and the category of A-schemes correspond to the case $S = \operatorname{Spec} A$.

We now give some examples.

0.1. *Exercise.* Show that morphisms $X \to \operatorname{Spec} A$ are in natural bijection with ring morphisms $A \to \Gamma(X, \mathcal{O}_X)$. (Hint: Show that this is true when X is affine. Use the fact that morphisms glue.)

In particular, there is a canonical morphism from a scheme to Spec of its space of global sections. (Warning: Even if X is a finite-type k-scheme, the ring of global sections might be nasty! In particular, it might not be finitely generated.)

Example: Suppose S_* is a graded ring, with $S_0 = A$. Then we get a natural morphism $\operatorname{Proj} S_* \to \operatorname{Spec} A$. For example, we have a natural map $\mathbb{P}^n_A \to \operatorname{Spec} A$

0.2. *Exercise.* Show that $\operatorname{Spec} \mathbb{Z}$ is the final object in the category of schemes. In other words, if X is any scheme, there exists a unique morphism to $\operatorname{Spec} \mathbb{Z}$. (Hence the category of schemes is isomorphic to the category of $\mathbb{Z}$-schemes.)

0.3. *Exercise.* Show that morphisms $X \to \operatorname{Spec} \mathbb{Z}[t]$ correspond to global sections of the structure sheaf.

This is one of our first explicit examples of an important idea, that of representable functors! This is a very useful idea, whose utility isn't immediately apparent. We have a contravariant functor from schemes to sets, taking a scheme to its set of global sections. We have another contravariant functor from schemes to sets, taking X to $\operatorname{Hom}(X, \operatorname{Spec} \mathbb{Z}[t])$. This is describing an "isomorphism" of the functors. More precisely, we are describing an isomorphism $\Gamma(X, \mathcal{O}_X) \cong \operatorname{Hom}(X, \operatorname{Spec} \mathbb{Z}[t])$ that behaves well with respect to morphisms of schemes: given any morphism $f : X \to Y$, the diagram

$$\begin{tikzcd} \Gamma(Y, \mathcal{O}_Y) \arrow[r, "\sim"] \arrow[d, "f^*"] & \mathrm{Hom}(Y, \mathrm{Spec}\, \mathbb{Z}[t]) \arrow[d, "f\circ"] \\ \Gamma(X, \mathcal{O}_X) \arrow[r, "\sim"] & \mathrm{Hom}(X, \mathrm{Spec}\, \mathbb{Z}[t]) \end{tikzcd}$$

commutes. *Given a contravariant functor from schemes to sets, by Yoneda's lemma, there is only one possible scheme* Y*, up to isomorphism, such that there is a natural isomorphism between this functor and* $\mathrm{Hom}(\cdot, Y)$. If there is such a Y, we say that the functor is *representable*.

Here are a couple of examples of something you've seen to put it in context. (i) The contravariant functor $\mathrm{Hom}(\cdot, Y)$ (i.e. $X \mapsto \mathrm{Hom}(X, Y)$) is representable by Y. (ii) Suppose we have morphisms $X, Y \to Z$. The contravariant functor $\mathrm{Hom}(\cdot, X) \times_{\mathrm{Hom}(\cdot, Z)} \mathrm{Hom}(\cdot, Y)$ is representable if and only if the fibered product $X \times_Z Y$ exists (and indeed then the contravariant functor is represented by $\mathrm{Hom}(\cdot, X \times_Z Y)$). This is purely a translation of the definition of the fibered product — you should verify this yourself.

Remark for experts: The global sections form something better than a set — they form a scheme. You can define the notion of ring scheme, and show that if a contravariant functor from schemes to rings is representable (as a contravariant functor from schemes to sets) by a scheme Y, then Y is guaranteed to be a ring scheme. The same is true for group schemes.

0.4. *Related Exercise.* Show that global sections of $\mathcal{O}_X^*$ correspond naturally to maps to $\mathrm{Spec}\, \mathbb{Z}[t, t^{-1}]$. ($\mathrm{Spec}\, \mathbb{Z}[t, t^{-1}]$ is a *group scheme*. We will discuss group schemes more in class 36.)

Morphisms and tangent spaces. Suppose $f : X \to Y$, and $f(p) = q$. Then if we were in the category of manifolds, we would expect a tangent map, from the tangent space of p to the tangent space at q. Indeed that is the case; we have a map of stalks $\mathcal{O}_{Y,q} \to \mathcal{O}_{X,p}$, which sends the maximal ideal of the former $\mathfrak{n}$ to the maximal ideal of the latter $\mathfrak{m}$ (we have checked that this is a "local morphism" when we briefly discussed locally ringed spaces). Thus $\mathfrak{n}^2 \to \mathfrak{m}^2$, from which $\mathfrak{n}/\mathfrak{n}^2 \to \mathfrak{m}/\mathfrak{m}^2$, from which we have a natural map $(\mathfrak{m}/\mathfrak{m}^2)^\vee \to (\mathfrak{n}/\mathfrak{n}^2)^\vee$. This is the map from the tangent space of p to the tangent space at q that we sought.

0.5. *Exercise.* Suppose X is a finite type k-scheme. Describe a natural bijection $\mathrm{Hom}(\mathrm{Spec}\, k[\epsilon]/\epsilon^2, X)$ to the data of a k-valued point (a point whose residue field is k, necessarily closed) and a tangent vector at that point.

1. Properties of morphisms of schemes

I'm going to define a lot of useful notions.

The notion of **surjective** will have the same meaning as always: $X \to Y$ is surjective if the map of sets is surjective.

1.1. *Unimportant Exercise.* Show that integral ring extensions induces a surjective map of spectra. (Hint: Recall the Cohen-Seidenberg Going-up Theorem: Suppose $B \subset A$ is an inclusion of rings, with A integrally dependent on B. For any prime $\mathfrak{q} \subset B$, there is a prime $\mathfrak{p} \subset A$ such that $\mathfrak{p} \cap B = \mathfrak{q}$.)

Definition. If U is an open subscheme of Y, then there is a natural morphism $U \to Y$. We say that $f : X \to Y$ is an *open immersion* if f gives an isomorphism from X to an open subscheme of Y. (Really, we want to say that X "is" an open subscheme of Y.) Observe that if f is an open immersion, then $f^{-1}\mathcal{O}_Y \cong \mathcal{O}_X$.

1.2. *Exercise.* Suppose $i : U \to Z$ is an open immersion, and $f : Y \to Z$ is any morphism. Show that $U \times_Z Y$ exists. (Hint: I'll even tell you what it is: $(f^{-1}(U), \mathcal{O}_Y|_{f^{-1}(U)})$.)

1.3. *Easy exercise.* Show that open immersions are monomorphisms.

Suppose X is a closed subscheme of Y. Then there is a natural morphism $i : X \to Y$: on the affine open set $\operatorname{Spec} R$ of Y, where X is "cut out" by the ideal $I \subset R$, this corresponds to the ring map $R \to R/I$. A morphism $f : W \to Y$ is a **closed immersion** if it can be factored as

$$\begin{tikzcd} W \arrow[rr, "f"] \arrow[dr, "\sim"'] & & Y \\ & X \arrow[ur, "i"'] & \end{tikzcd}$$

where $i : X \to Y$ is a closed subscheme. (Really, we want to say that W "is" a closed subscheme of Y.)

(Example: If X is a scheme and X^{red} is its reduction, then there is a natural closed immersion $X^{\text{red}} \to X$.)

1.4. *Proposition (the property of being a closed immersion is affine-local on the target). — Suppose* $f : X \to Y$ *is a morphism of schemes. It suffices to check that* f *is a closed immersion on an affine open cover of* Y.

Reason: The way in which closed subschemes are defined is local on the target.

(In particular, a morphism of affine schemes is a closed immersion if and only if it corresponds to a surjection of rings.)

1.5. Exercise. Suppose $Y \to Z$ is a closed immersion, and $X \to Z$ is any morphism. Show that the fibered product $X \times_Z X$ exists, by explicitly describing it. Show that the projection $X \times_Z Y \to Y$ is a closed immersion. We say that "closed immersions are preserved by base change" or "closed immersions are preserved by fibered product". (Base change is another word for fibered products.)

1.6. Less important exercise. Show that closed immersions are monomorphisms.

Definition. A morphism $X \to Y$ is a *locally closed immersion* if it factors into $X \xrightarrow{f} Z \xrightarrow{g} Y$ where f is a closed immersion and g is an open immersion. Example: $\operatorname{Spec} k[t, t^{-1}] \to \operatorname{Spec} k[x, y]$ where $x \mapsto t$, $y \mapsto 0$. (Unimportant fact: as the composition of monomorphisms are monomorphisms, so locally closed immersions are monomorphisms. Clearly open immersions and closed immersions are locally closed immersions.)

(Interesting question: is this the same as defining locally closed immersions as open immersions of closed immersions? In other words, can the roles of open and closed immersions in the definition be reversed?)

A morphism $f : X \to Y$ is **quasicompact** if for every open affine subset U of Y, $f^{-1}(U)$ is quasicompact.

1.7. *Exercise (quasicompactness is affine-local on the target).* Show that a morphism $f : X \to Y$ is quasicompact if there is cover of Y by open affine sets U_i such that $f^{-1}(U_i)$ is quasicompact. (Hint: easy application of the affine communication lemma!)

1.8. *Exercise.* Show that the composition of two quasicompact morphisms is quasicompact.

A morphism $f : X \to Y$ is **locally of finite type** if for every affine open set $\operatorname{Spec} B$ of Y, $f^{-1}(\operatorname{Spec} B)$ can be covered with open sets $\operatorname{Spec} A_i$ so that the induced morphism $B \to A_i$ expresses A_i as a finitely generated B-algebra.

A morphism is **of finite type** if it is locally of finite type and quasicompact. Translation: for every affine open set $\operatorname{Spec} B$ of Y, $f^{-1}(\operatorname{Spec} B)$ can be covered with *a finite number of* open sets $\operatorname{Spec} A_i$ so that the induced morphism $B \to A_i$ expresses A_i as a finitely generated B-algebra.

1.9. *Exercise (the notions "locally of finite type" and "finite type" are affine-local on the target).* Show that a morphism $f : X \to Y$ is locally of finite type if there is a cover of Y by open affine sets $\operatorname{Spec} R_i$ such that $f^{-1}(\operatorname{Spec} R_i)$ is locally of finite type over R_i.

1.10. *Exercise.* Show that a morphism $f : X \to Y$ is locally of finite type if for *every* affine open subsets $\operatorname{Spec} A \subset X$, $\operatorname{Spec} B \subset Y$, with $f(\operatorname{Spec} A) \subset \operatorname{Spec} B$, A is a finitely generated B-algebra. (Hint: use the affine communication lemma on $f^{-1}(\operatorname{Spec} B)$.)

Example: the "structure morphism" $\mathbb{P}^n_A \to \operatorname{Spec} A$ is of finite type, as $\mathbb{P}^n_A$ is covered by $n + 1$ open sets of the form $\operatorname{Spec} A[x_1, \dots, x_n]$. More generally, $\operatorname{Proj} S_* \to \operatorname{Spec} A$ (where $S_0 = A$) is of finite type.

More generally still: our earlier definition of schemes of "finite type over k" (or "finite type k-schemes") is now a special case of this more general notion: a scheme X is of finite type over k means that we are given a morphism $X \to \operatorname{Spec} k$ (the "structure morphism") that is of finite type.

Here are some properties enjoyed by morphisms of finite type.

1.11. Exercises. These exercises are important and not hard.

- Show that a closed immersion is a morphism of finite type.
- Show that an open immersion is locally of finite type. Show that an open immersion into a Noetherian scheme is of finite type. More generally, show that a quasicompact open immersion is of finite type.
- Show that a composition of two morphisms of finite type is of finite type.
- Suppose we have a composition of morphisms $X \xrightarrow{f} Y \xrightarrow{g} Z$, where f is quasicompact, and $g \circ f$ is finite type. Show that f is finite type.
- Suppose $f : X \to Y$ is finite type, and Y is Noetherian. Show that X is also Noetherian.

A morphism $f : X \to Y$ is **affine** if for every affine U of Y, $f^{-1}(U)$ is an affine scheme. Clearly affine morphisms are quasicompact. Also, clearly closed immersions are affine: if $X \to Y$ is a closed immersion, then the preimage of an affine open set $\operatorname{Spec} R$ of Y is (isomorphic to) some $\operatorname{Spec} R/I$, by the definition of closed subscheme.

1.12. *Proposition (the property of "affineness" is affine-local on the target).* A morphism $f : X \to Y$ is affine if there is a cover of Y by open affine sets U such that $f^{-1}(U)$ is affine.

Proof. As usual, we use the Affine Communication Theorem. We check our two criteria. First, suppose $f : X \to Y$ is affine over $\operatorname{Spec} S$, i.e. $f^{-1}(\operatorname{Spec} S) = \operatorname{Spec} R$. Then $f^{-1}(\operatorname{Spec} S_s) = \operatorname{Spec} R_{f\# s}$.

Second, suppose we are given $f : X \to \operatorname{Spec} S$ and $(f_1, \dots, f_n) = S$ with X_{f_i} affine ($\operatorname{Spec} R_i$, say). We wish to show that X is affine too. Define R as the kernel of S-modules

$$R_1 \times \cdots \times R_n \to R_{12} \times \cdots R_{(n-1)n}$$

where $X_{f_i f_j} = \operatorname{Spec} R_{ij}$. Then R is clearly an S-module, and has a ring structure. We define a morphism $\operatorname{Spec} R \to \operatorname{Spec} S$. Note that $R_{f_i} = R_i$. Then we define $\operatorname{Spec} R \to \operatorname{Spec} S$ via $\operatorname{Spec} R_i \to \operatorname{Spec} R_{f_i} \hookrightarrow \operatorname{Spec} S$. The morphisms glue. □

This has some non-obvious consequences, as shown in the next exercise.

1.13. *Exercise.* Suppose X is an affine scheme, and Y is a closed subscheme locally cut out by one equation (e.g. if Y is an effective Cartier divisor). Show that $X - Y$ is affine. (This is clear if Y is globally cut out by one equation f; then if $X = \operatorname{Spec} R$ then $Y = \operatorname{Spec} R_f$. However, Y is not always of this form.)

1.14. Example. Here is an explicit consequence. We showed earlier that on the cone over the smooth quadric surface $\operatorname{Spec} k[w, x, y, z]/(wz - xy)$, the cone over a ruling $w = x = 0$ is not cut out scheme-theoretically by a single equation, by considering Zariski-tangent spaces. We now show that it isn't even cut out set-theoretically by a single equation.

For if it were, its complement would be affine. But then the closed subscheme of the complement cut out by $y = z = 0$ would be affine. But this is the scheme $y = z = 0$ (also known as the wx-plane) minus the point $w = x = 0$, which we've seen is non-affine. (For comparison, on the cone $\operatorname{Spec} k[x, y, z]/(xy - z^2)$, the ruling $x = z = 0$ is not cut out scheme-theoretically by a single equation, but it *is* cut out set-theoretically by $x = 0$.) Verify all this!

We remark here that we have shown that if $f : X \to Y$ is an affine morphism, then $f_*\mathcal{O}_X$ is a quasicoherent sheaf of algebras (a quasicoherent sheaf with the structure of an algebra "over $\mathcal{O}_X$"). We'll soon reverse this process to obtain Spec of a quasicoherent sheaf of algebras.

A morphism $f : X \to Y$ is **finite** if for every affine $\operatorname{Spec} R$ of Y, $f^{-1}(\operatorname{Spec} R)$ is the spectrum of an R-algebra that is a finitely-generated R-*module*. Clearly finite morphisms are affine. Note that $f_*\mathcal{O}_X$ is a finite type quasicoherent sheaf of algebras (= coherent if X is locally Noetherian).

1.15. *Exercise (the property of finiteness is affine-local on the target).* Show that a morphism $f : X \to Y$ is finite if there is a cover of Y by open affine sets $\operatorname{Spec} R$ such that $f^{-1}(\operatorname{Spec} R)$ is the spectrum of a finite R-algebra.

(Hint: Use Exercise 1.12, and that $f_*\mathcal{O}_X$ is finite type.)

1.16. *Easy exercise.* Show that closed immersions are finite morphisms.

Degree of a finite morphism at a point. Suppose $f : X \to Y$ is a finite morphism. $f_*\mathcal{O}_X$ is a finite type (quasicoherent) sheaf on Y, and the rank of this sheaf at a point p is called the *degree* of the finite morphism at p. This is a upper semicontinuous function (we've shown that the rank of a finite type sheaf is uppersemicontinuous in an exercise when we discussed rank).

1.17. *Exercise.* Show that the rank at p is non-zero if and only if $f^{-1}(p)$ is non-empty.

1.18. *Exercise.* Show that finite morphisms are *closed*, i.e. the image of any closed subset is closed.

A morphism is **quasifinite** if it is of finite type, and for all $y \in Y$, the scheme $X_y = f^{-1}(y)$ is finite over y.

1.19. *Exercise.* (a) Show that if a morphism is finite then it is quasifinite. (b) Show that the converse is not true. (Hint: $\mathbb{A}^1 - \{0\} \to \mathbb{A}^1$.)

1.20. Images of morphisms. I want to go back to the point that the image of a finite morphism is closed. Something more general is true. We answer the question: what can the image of a morphism look like? We know it can be open (open immersion), and closed

(closed immersions), locally closed (locally closed immersions). But it can be weirder still: Consider $\mathbb{A}^2 \to \mathbb{A}^2$ given by $(x, y) \mapsto (x, xy)$. then the image is the plane, minus the y-axis, plus the origin. It can be stranger still, and indeed if S is *any* subset of a scheme Y, it can be the image of a morphism: let X be the disjoint union of spectra of the residue fields of all the points of S, and let $f : X \to Y$ be the natural map. This is quite pathological (e.g. likely horribly noncompact), and we will show that if we are in any reasonable situation, the image is essentially no worse than arose in the previous example.

We define a *constructible subset* of a scheme to be a subset which belongs to the smallest family of subsets such that (i) every open set is in the family, (ii) a finite intersection of family members is in the family, and (iii) the complement of a family member is also in the family. So for example the image of $(x, y) \mapsto (x, xy)$ is constructible.

Note that if $X \to Y$ is a morphism of schemes, then the preimage of a constructible set is a constructible set.

1.21. Exercise. Suppose X is a Noetherian scheme. Show that a subset of X is constructible if and only if it is the finite disjoint union of locally closed subsets.

Chevalley's Theorem. Suppose $f : X \to Y$ is a morphism of finite type of Noetherian schemes. Then the image of any constructible set is constructible.

I might give a proof in the notes eventually. See Atiyah-Macdonald, Exercise 7.25 for the key algebraic argument. Next quarter, we will see that in good situations (e.g. if the source is projective over k and the target is quasiprojective) then the image is closed.

We end with a useful fact about images of schemes that didn't naturally fit in anywhere in the previous exposition.

1.22. *Fast important exercise.* Show that the image of an irreducible scheme is irreducible.

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FOUNDATIONS OF ALGEBRAIC GEOMETRY CLASS 20

CONTENTS

Last day: Maps to affine schemes; surjective, open immersion, closed immersion, quasicompact, locally of finite type, finite type, affine morphism, finite, quasifinite. Images of morphisms: constructible sets, and Chevalley's theorem (finite type morphism of Noetherian schemes sends constructibles to constructibles).

Today: Pushforwards and pullbacks of quasicoherent sheaves.

This is the last class of the first quarter of this three-quarter sequence. Last day, I defined a large number of classes of morphisms. Today, I will talk about how quasicoherent sheaves push forward or pullback. I'll then sum up what's happened in this class, and give you some idea of what will be coming in the next quarter.

1. PUSHFORWARDS AND PULLBACKS OF QUASICOHERENT SHEAVES

There are two things you can do with modules and a ring homomorphism $B \to A$. If M is an A-module, you can create an B-module M_B by simply treating it as an B-module. If N is an B-module, you can create an A-module $N \otimes_B A$.

These notions behave well with respect to localization (in a way that we will soon make precise), and hence work (often) in the category of quasicoherent sheaves. The two functors are adjoint:

$$\mathrm{Hom}_A(N \otimes_B A, M) \cong \mathrm{Hom}_B(N, M_B)$$

(where this isomorphism of groups is functorial in both arguments), and we will see that this remains true on the scheme level.

One of these constructions will turn into our old friend pushforward. The other will be a relative of pullback, whom I'm reluctant to call an "old friend".

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The main message of this section is that in "reasonable" situations, the pushforward of a quasicoherent sheaf is quasicoherent, and that this can be understood in terms of one of the module constructions defined above. We begin with a motivating example:

2.1. *Exercise.* Let $f : \operatorname{Spec} A \to \operatorname{Spec} B$ be a morphism of affine schemes, and suppose M is an A-module, so $\tilde{M}$ is a quasicoherent sheaf on $\operatorname{Spec} A$. Show that $f_* \tilde{M} \cong \widetilde{M_B}$. (Hint: There is only one reasonable way to proceed: look at distinguished opens!)

In particular, $f_* \tilde{M}$ is quasicoherent. Perhaps more important, this implies that the pushforward of a quasicoherent sheaf under an affine morphism is also quasicoherent. The following result doesn't quite generalize this statement, but the argument does.

2.2. *Theorem. — Suppose $f : X \to Y$ is a morphism, and X is a Noetherian scheme. Suppose $\mathcal{F}$ is a quasicoherent sheaf on X. Then $f_* \mathcal{F}$ is a quasicoherent sheaf on Y.*

The fact about f that we will use is that the preimage of any affine open subset of Y is a finite union of affine sets (f is quasicompact), and the intersection of any two of these affine sets is also a finite union of affine sets (this is a definition of the notion of a *quasiseparated morphism*). Thus the "correct" hypothesis here is that f is quasicompact and quasiseparated.

Proof. By the first definition of quasicoherent sheaves, it suffices to show the following: if $\mathcal{F}$ is a quasicoherent sheaf on X, and $f : X \to \operatorname{Spec} R$, then the following diagram commutes:

$$\begin{tikzcd}
\Gamma(X, \mathcal{F}) \arrow[rr, "\mathrm{res}_{D(g) \subset \operatorname{Spec} R}"] \arrow[dr, "\otimes_R R_g"'] & & \Gamma(X_g, \mathcal{F}) \\
& \Gamma(X, \mathcal{F})_g \arrow[ur, "\sim"'] &
\end{tikzcd}$$

This was a homework problem (# 18 on problem set 6)! $\square$

2.3. *Exercise.* Give an example of a morphism of schemes $\pi : X \to Y$ and a quasicoherent sheaf $\mathcal{F}$ on X such that $\pi_* \mathcal{F}$ is not quasicoherent. (Answer: $Y = \mathbb{A}^1$, $X =$ countably many copies of $\mathbb{A}^1$. Then let $f = t$. X_t has a global section $(1/t, 1/t^2, 1/t^3, \dots)$. The key point here is that infinite direct sums do not commute with localization.)

Coherent sheaves don't always push forward to coherent sheaves. For example, consider the structure morphism $f : \mathbb{A}^1_k \to \operatorname{Spec} k$, given by $k \mapsto k[t]$. Then $f_* \mathcal{O}_{\mathbb{A}^1_k}$ is the $k[t]$, which is not a finitely generated k-module. Under especially good situations, coherent sheaves do push forward. For example:

2.4. *Exercise.* Suppose $f : X \to Y$ is a finite morphism of Noetherian schemes. If $\mathcal{F}$ is a coherent sheaf on X, show that $f_*\mathcal{F}$ is a coherent sheaf. (Hint: Show first that $f_*\mathcal{O}_X$ is finite type = locally finitely generated.)

Once we define cohomology of quasicoherent sheaves, we will quickly prove that if $\mathcal{F}$ is a coherent sheaf on $\mathbb{P}^n_k$, then $\Gamma(\mathbb{P}^n_k)$ is a finite-dimensional k-module, and more generally if $\mathcal{F}$ is a coherent sheaf on $\operatorname{Proj} S_*$, then $\Gamma(\operatorname{Proj} S_*)$ is a coherent A-module (where $S_0 = A$). This is a special case of the fact the "pushforwards of coherent sheaves by projective morphisms are also coherent sheaves". We will first need to define "projective morphism"! This notion is a generalization of $\operatorname{Proj} S_* \to \operatorname{Spec} A$.

3. Pullback of quasicoherent sheaves

(Note added in February: I will try to reserve the phrase "pullback of a sheaf" for pullbacks of quasicoherent sheaves f^*, and "inverse image sheaf" for f^{-1}, which applies in a more general situation, in the category of sheaves on topological spaces.)

I will give four definitions of the pullback of a quasicoherent sheaf. The first one is the most useful in practice, and is in keeping with our emphasis of quasicoherent sheaves as just "modules glued together". The second is the "correct" definition, as an adjoint of pushforward. The third, which we mention only briefly, is *more* correct, as adjoint in the category of $\mathcal{O}_X$-modules. And we end with a fourth definition.

We note here that pullback to a closed subscheme or an open subscheme is often called **restriction**.

3.1. Construction/description of the pullback. Let us now define the pullback functor precisely. Suppose $X \to Y$ is a morphism of schemes, and $\mathcal{G}$ is a quasicoherent sheaf on Y. We will describe the pullback quasicoherent sheaf $f^*\mathcal{G}$ on X by describing it as a sheaf on the distinguished affine base. In our base, we will permit only those affine open sets $U \subset X$ such that $f(U)$ is contained in an affine open set of Y. The distinguished restriction map will force this sheaf to be quasicoherent.

Suppose $U \subset X$, $V \subset Y$ are affine open sets, with $f(U) \subset V$, $U \cong \operatorname{Spec} A$, $V \cong \operatorname{Spec} B$. Suppose $\mathcal{F}|_V \cong \tilde{N}$. Then define $\Gamma(f^*_V\mathcal{F}, U) := N \otimes_B A$. Our main goal will be to show that this is independent of our choice of V.

We begin as follows: we fix an affine open subset $V \subset Y$, and use it to define sections over any affine open subset $U \subset f^{-1}(V)$. We show that this gives us a quasicoherent sheaf $f^*_V\mathcal{G}$ on $f^{-1}(V)$, by showing that these sections behave well with respect to distinguished restrictions. First, note that if $D(f) \subset U$ is a distinguished open set, then

$$\Gamma(f^*_V\mathcal{F}, D(f)) = N \otimes_B A_f \cong (N \otimes_B A) \otimes_A A_f = \Gamma(f^*_V\mathcal{F}, U) \otimes_A A_f.$$

Define the restriction map $\Gamma(f^*_V\mathcal{F}, U) \to \Gamma(f^*_V\mathcal{F}, D(f))$ by

$$\Gamma(f^*_V\mathcal{F}, U) \to \Gamma(f^*_V\mathcal{F}, U) \otimes_A A_f \tag{1}$$

(with $\alpha \mapsto \alpha \otimes 1$ of course). Thus on the *distinguished affine topology of* $\operatorname{Spec} A$ we have defined a quasicoherent sheaf.

Finally, we show that if $f(U)$ is contained in *two* affine open sets V_1 and V_2, then the alleged sections of the pullback we have described do not depend on whether we use V_1 or V_2. More precisely, we wish to show that

$$\Gamma(f^*_{V_1}\mathcal{F}, U) \quad \text{and} \quad \Gamma(f^*_{V_2}\mathcal{F}, U)$$

have a canonical isomorphism, which commutes with the restriction map (1).

Let $\{W_i\}_{i \in I}$ be an affine cover of $V_1 \cap V_2$ by sets that are distinguished in *both* V_1 and V_2 (possible by the Proposition we used in the proof of the Affine Communication Lemma). Then by the previous paragraph, as $f^*_{V_1}\mathcal{F}$ is a sheaf on the distinguished base of V_1,

$$\Gamma(f^*_{V_1}\mathcal{F}, U) = \ker\left(\oplus_i \Gamma(f^*_{V_1}\mathcal{F}, f^{-1}(W_i)) \to \oplus_{i,j}\Gamma(f^*_{V_1}\mathcal{F}, f^{-1}(W_i \cap W_j))\right).$$

If $V_1 = \operatorname{Spec} B_1$ and $W_i = D(g_i)$, then

$$\Gamma(f^*_{V_1}\mathcal{F}, f^{-1}(W_i)) = N \otimes_{B_1} A_{f^\# g_i} \cong N \otimes_{(B_1)_{g_i}} A_{f^\# g_i} = \Gamma(f^*_{W_i}\mathcal{F}, f^{-1}(W_i)),$$

so

$$\Gamma(f^*_{V_1}\mathcal{F}, U) = \ker\left(\oplus_i \Gamma(f^*_{W_i}\mathcal{F}, f^{-1}(W_i)) \to \oplus_{i,j}\Gamma(f^*_{W_i}\mathcal{F}, f^{-1}(W_i \cap W_j))\right). \tag{2}$$

The same argument for V_2 yields

$$\Gamma(f^*_{V_2}\mathcal{F}, U) = \ker\left(\oplus_i \Gamma(f^*_{W_i}\mathcal{F}, f^{-1}(W_i)) \to \oplus_{i,j}\Gamma(f^*_{W_i}\mathcal{F}, f^{-1}(W_i \cap W_j))\right). \tag{3}$$

But the right sides of (2) and (3) are the same, so the left sides are too. Moreover, (2) and (3) behave well with respect to restricting to a distinguished open $D(g)$ of $\operatorname{Spec} A$ (just apply $\otimes_A A_g$ to the the right side) so we are done.

Hence we have described a quasicoherent sheaf $f^*\mathcal{G}$ on X whose behavior on affines mapping to affines was as promised.

3.2. *Theorem.* —

1. *The pullback of the structure sheaf is the structure sheaf.*
2. *The pullback of a finite type (=locally finitely generated) sheaf is finite type.*
3. *The pullback of a finitely presented sheaf is finitely presented. Hence if $f : X \to Y$ is a morphism of locally Noetherian schemes, then the pullback of a coherent sheaf is coherent. (It is not always true that the pullback of a coherent sheaf is coherent, and the interested reader can think of a counterexample.)*
4. *The pullback of a locally free sheaf of rank r is another such. (In particular, the pullback of an invertible sheaf is invertible.)*
5. *(functoriality in the morphism) $\pi_1^*\pi_2^*\mathcal{F} \cong (\pi_2 \circ \pi_1)^*\mathcal{F}$*
6. *(functoriality in the quasicoherent sheaf) $\mathcal{F}_1 \to \mathcal{F}_2$ induces $\pi^*\mathcal{F}_1 \to \pi^*\mathcal{F}_2$*
7. *If s is a section of $\mathcal{F}$ then there is a natural section π^*s that is a section of $\pi^*\mathcal{F}$.*
8. *(stalks) If $\pi : X \to Y$, $\pi(x) = y$, then $(\pi^*\mathcal{F})_x \cong \mathcal{F}_y \otimes_{\mathcal{O}_{Y,y}} \mathcal{O}_{X,x}$. The previous map, restricted to the stalks, is $f \mapsto f \otimes 1$. (In particular, the locus where the section on the target vanishes pulls back to the locus on the source where the pulled back section vanishes.)*

(9) *(fibers) Pullbacks of fibers are given as follows: if* $\pi : X \to Y$*, where* $\pi(x) = y$*, then* $\pi^*\mathcal{F}/\mathfrak{m}_{X,x}\pi^*\mathcal{F} \cong (\mathcal{F}/\mathfrak{m}_{Y,y}\mathcal{F}) \otimes_{\mathcal{O}_{Y,y}/\mathfrak{m}_{Y,y}} \mathcal{O}_{X,x}/\mathfrak{m}_{X,x}$

(10) *(tensor product)* $\pi^*(\mathcal{F} \otimes \mathcal{G}) = \pi^*\mathcal{F} \otimes \pi^*\mathcal{G}$

(11) *pullback is a right-exact functor*

All of the above are interconnected in obvious ways. For example, given $\mathcal{F}_1 \to \mathcal{F}_2$ and a section s of $\mathcal{F}_1$, then we can pull back the section and then send it to $\pi^*\mathcal{F}_2$, or vice versa, and we get the same thing.

I used some of these results to help give an intuitive picture of the pullback.

Proof. Most of these are left to the reader. It is convenient to do right-exactness early (e.g. before showing that finitely presented sheaves pull back to finitely presented sheaves). For the tensor product fact, show that $(M \otimes_S R) \otimes (N \otimes_S R) \cong (M \otimes N) \otimes_S R$, and that this behaves well with respect to localization. The proof of the fiber fact is as follows. $(S, \mathfrak{n}) \to (R, \mathfrak{m})$.

$$\begin{tikzcd} S \arrow[r] \arrow[d] & R \arrow[d] \\ S/\mathfrak{n} \arrow[r] & R/\mathfrak{m} \end{tikzcd}$$

$(N \otimes_S R) \otimes_R (R/\mathfrak{m}) \cong (N \otimes_S (S/\mathfrak{n})) \otimes_{S/\mathfrak{n}} (R/\mathfrak{m})$ as both sides are isomorphic to $N \otimes_S (R/\mathfrak{m})$. $\square$

3.3. *Unimportant Exercise.* Verify that the following is an example showing that pullback is not left-exact: consider the exact sequence of sheaves on $\mathbb{A}^1$, where p is the origin:

$$0 \to \mathcal{O}_{\mathbb{A}^1}(-p) \to \mathcal{O}_{\mathbb{A}^1} \to \mathcal{O}_p \to 0.$$

(This is a closed subscheme exact sequence; also an effective Cartier exact sequence. Algebraically, we have $k[t]$-modules $0 \to tk[t] \to k[t] \to k \to 0$.) Restrict to p.

3.4. Pulling back closed subschemes. Suppose $Z \hookrightarrow Y$ is a closed immersion, and $X \to Y$ is any morphism. Then we define the pullback of the closed subscheme Z to X as follows. We pullback the quasicoherent sheaf of ideals on Y defining Z to get a quasicoherent sheaf of ideals on X (which we take to define W). Equivalently, on any affine open Y, Z is cut out by some functions; we pull back those functions to X, and denote the scheme cut out by them by W.

Exercise. Let W be the pullback of the closed subscheme Z to X. Show that $W \cong Z \times_Y X$. In other words, the fibered product with a closed immersion always exists, and closed immersions are preserved by fibered product (or by pullback), i.e. if

$$\begin{tikzcd} W \arrow[r, "g'"] \arrow[d] & X \arrow[d] \\ Z \arrow[r, "g"] & Y \end{tikzcd}$$

is a fiber diagram, and g is a closed immersion, then so is g'. (This is actually a repeat of an exercise in class 19 — sorry!)

3.5. Three more "definitions". Pullback is left-adjoint of the pushforward. This is a theorem (which we'll soon prove), but it is actually a pretty good definition. If it exists, then it is unique up to unique isomorphism by Yoneda nonsense.

The problem is this: pushforwards don't always exist (in the category of quasicoherent sheaves); we need the quasicompact and quasiseparated hypotheses. However, pullbacks always exist. So we need to motivate our definition of pullback even without the quasicompact and quasiseparated hypothesis. (One possible motivation will be given in Remark 3.7.)

3.6. *Theorem. — Suppose* $\pi : X \to Y$ *is a quasicompact, quasiseparated morphism. Then pullback is left-adjoint to pushforward. More precisely,* $\operatorname{Hom}(f^*\mathcal{G}, \mathcal{F}) \cong \operatorname{Hom}(\mathcal{G}, f_*\mathcal{F})$.

(The quasicompact and quasiseparated hypothesis is required to ensure that the pushforward exists, not because it is needed in the proof.)

More precisely still, we describe natural homomorphisms that are functorial in both arguments. We show that it is a bijection of sets, but it is fairly straightforward to verify that it is an isomorphism of groups. Not surprisingly, we will use adjointness for modules.

Proof. Let's unpack the right side. What's an element of $\operatorname{Hom}(\mathcal{G}, f_*\mathcal{F})$? For every affine V in Y, we get an element of $\operatorname{Hom}(\mathcal{G}(V), \mathcal{F}(f^{-1}(V)))$, and this behaves well with respect to distinguished opens. Equivalently, for every affine V in Y and U in $f^{-1}(V) \subset X$, we have an element $\operatorname{Hom}(\mathcal{G}(V), \mathcal{F}(U))$, that behaves well with respect to localization to distinguished opens on both affines. By the adjoint property, this corresponds to elements of $\operatorname{Hom}(\mathcal{G}(V) \otimes_{\mathcal{O}_Y(V)} \mathcal{O}_X(U), \mathcal{F}(U))$, which behave well with respect to localization. And that's the left side. $\square$

3.7. *Pullback for ringed spaces.* (This is actually conceptually important but distracting for our exposition; we encourage the reader to skip this, at least on the first reading.) Pullbacks and pushforwards may be defined in the category of modules over ringed spaces. We define pushforward in the usual way, and then define the pullback of an $\mathcal{O}_Y$-module using the adjoint property. Then one must show that (i) it exists, and (ii) the pullback of a quasicoherent sheaf is quasicoherent. The fourth definition is as follows: suppose we have a morphism of ringed spaces $\pi : X \to Y$, and an $\mathcal{O}_Y$-module $\mathcal{G}$. Then we define $f^*\mathcal{G} = f^{-1}\mathcal{G} \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X$. We will not show that this definition is equivalent to ours, but the interested reader is welcome to try this as an exercise. There is probably a proof in Hartshorne. The statements of Theorem 3.6 apply in this more general setting. (Really the third definition "requires" the fourth.)

Here is a hint as to why this definition is equivalent to ours (a hint for the exercise if you will). We need to show that $f^{-1}\mathcal{F} \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X$ ("definition 4") and $f^*\mathcal{F}$ ("definition 1") are isomorphic. You should (1) find a natural morphism from one to the other, and (2) show that it is an isomorphism at the level of stalks. The difficulty of (1) shows the disadvantages of our definition of quasicoherent sheaves.

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FOUNDATIONS OF ALGEBRAIC GEOMETRY BONUS HANDOUT: PROOFS OF "HARTOGS" AND KRULL

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I said earlier that I hoped to give you proofs of (i) "Hartogs' Theorem" for normal Noetherian schemes, (ii) Krull's Principal Ideal Theorem, and (iii) the fact that if $(R, \mathfrak{m})$ is a Noetherian ring, then $\cap \mathfrak{m}^i = 0$ (corresponding to the fact that a function that is analytically zero at a point is zero in a neighborhood of that point).

You needn't read these; but you may appreciate the fact that the proofs aren't that long. Thus there are very few statements in this class (beyond Math 210) that we actually used, but didn't justify.

I am going to repeat the Nakayama statements, so the entire argument is in one place.

0.1. *Nakayama's Lemma version 1. — Suppose* R *is a ring,* I *an ideal of* R*, and* M *is a finitely-generated* R*-module. Suppose* $M = IM$*. Then there exists an* $a \in R$ *with* $a \equiv 1 \pmod{I}$ *with* $aM = 0$*.*

Proof. Say M is generated by $m_1, \dots, m_n$. Then as $M = IM$, we have $m_i = \sum_j a_{ij} m_j$ for some $a_{ij} \in I$. Thus

$$(\mathrm{Id}_n - A) \begin{pmatrix} m_1 \\ \vdots \\ m_n \end{pmatrix} = 0 \tag{1}$$

where Id_n is the $n \times n$ identity matrix in R, and $A = (a_{ij})$. We can't quite invert this matrix, but we almost can. Recall that any $n \times n$ matrix M has an adjoint matrix $\operatorname{adj}(M)$ such that $\operatorname{adj}(M)M = \det(M)\mathrm{Id}_n$. The coefficients of $\operatorname{adj}(M)$ are polynomials in the coefficients of M. (You've likely seen this in the form of a formula for M^{-1} when there *is* an inverse.) Multiplying both sides of (1) on the left by $\operatorname{adj}(M)$, we obtain

$$\det(\mathrm{Id}_n - A) \begin{pmatrix} m_1 \\ \vdots \\ m_n \end{pmatrix} = 0.$$

But when you expand out $\det(\mathrm{Id}_n - A)$, you get something that is $1 \pmod{I}$. $\square$

Here is why you care: Suppose I is contained in all maximal ideals of R. (The intersection of all the maximal ideals is called the *Jacobson radical*, but I won't use this phrase. For comparison, recall that the nilradical was the intersection of the *prime ideals* of R.) Then I claim that any $a \equiv 1 \pmod{I}$ is invertible. For otherwise $(a) \neq R$, so the ideal (a) is

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contained in some maximal ideal $\mathfrak{m}$ — but $a \equiv 1 \pmod{\mathfrak{m}}$, contradiction. Then as a is invertible, we have the following.

0.2. *Nakayama's Lemma version 2.* — *Suppose R is a ring, I an ideal of R contained in all maximal ideals, and M is a finitely-generated R-module. (The most interesting case is when R is a local ring, and I is the maximal ideal.) Suppose $M = IM$. Then $M = 0$.*

0.3. Important exercise (Nakayama's lemma version 3). Suppose R is a ring, and I is an ideal of R contained in all maximal ideals. Suppose M is a *finitely generated* R-module, and $N \subset M$ is a submodule. If $N/IN \hookrightarrow M/IM$ an isomorphism, then $M = N$.

0.4. Important exercise (Nakayama's lemma version 4). Suppose $(R, \mathfrak{m})$ is a local ring. Suppose M is a finitely-generated R-module, and $f_1, \dots, f_n \in M$, with (the images of) $f_1, \dots, f_n$ generating $M/\mathfrak{m}M$. Then $f_1, \dots, f_n$ generate M. (In particular, taking $M = \mathfrak{m}$, if we have generators of $\mathfrak{m}/\mathfrak{m}^2$, they also generate $\mathfrak{m}$.)

0.5. *Important Exercise that we will use soon.* Suppose S is a subring of a ring R, and $r \in R$. Suppose there is a faithful $S[r]$-module M that is finitely generated as an S-module. Show that r is integral over S. (Hint: look carefully at the proof of Nakayama's Lemma version 1, and change a few words.)

We are ready to prove "Hartogs' Theorem".

0.6. *"Hartogs' theorem".* — *Suppose A is a Noetherian normal domain. Then in $\operatorname{Frac}(A)$,*

$$A = \cap_{\mathfrak{p} \text{ height } 1} A_{\mathfrak{p}}.$$

More generally, if A is a product of Noetherian normal domains (i.e. $\operatorname{Spec} A$ is Noetherian normal scheme), then in the ring of fractions of A,

$$A = \cap_{\mathfrak{p} \text{ height } 1} A_{\mathfrak{p}}.$$

I stated the special case first so as to convince you that this isn't scary.

Proof. Obviously the right side is contained in the left. Assume we have some x in all $A_{\mathfrak{p}}$ but not in A. Let I be the "ideal of denominators":

$$I := \{r \in A : rx \in A\}.$$

(The ideal of denominators arose in an earlier discussion about normality.) We know that $I \neq A$, so choose $\mathfrak{q}$ a minimal prime containing I.

Observe that this construction behaves well with respect to localization (i.e. if $\mathfrak{p}$ is any prime, then the ideal of denominators x in $A_{\mathfrak{p}}$ is the $I_{\mathfrak{p}}$, and it again measures the failure of "'Hartogs' Theorem" for x,' this time in $A_{\mathfrak{p}}$). But Hartogs' Theorem is vacuously true for dimension 1 rings, so hence no height 1 prime contains I. Thus $\mathfrak{q}$ has height at least 2. By localizing at $\mathfrak{q}$, we can assume that A is a local ring with maximal ideal $\mathfrak{q}$, and that $\mathfrak{q}$ is the only prime containing I. Thus $\sqrt{I} = \mathfrak{q}$, so there is some n with $I \subset \mathfrak{q}^n$. Take a minimal such n, so $I \not\subset \mathfrak{q}^{n-1}$, and choose any $y \in \mathfrak{q}^{n-1} - \mathfrak{q}^n$. Let $z = yx$. Then $z \notin A$ (so $\mathfrak{q}z \notin \mathfrak{q}$), but $\mathfrak{q}z \subset A$: $\mathfrak{q}z$ is an ideal of A.

I claim $\mathfrak{q}z$ is not contained in $\mathfrak{q}$. Otherwise, we would have a finitely-generated A-module (namely $\mathfrak{q}$) with a faithful $A[z]$-action, forcing z to be integral over A (and hence in A) by Exercise 0.5.

Thus $\mathfrak{q}z$ is an ideal of A not contained in $\mathfrak{q}$, so it must be A! Thus $\mathfrak{q}z = A$ from which $\mathfrak{q} = A(1/z)$, from which $\mathfrak{q}$ is principal. But then $\operatorname{ht} Q = \dim A \leq \dim_{A/Q} Q/Q^2 \leq 1$ by Nakayama's lemma 0.4, contradicting the fact that $\mathfrak{q}$ has height at least 2. $\square$

We now prove:

0.7. *Krull's Principal Ideal Theorem. — Suppose A is a Noetherian ring, and $f \in A$. Then every minimal prime $\mathfrak{p}$ containing f has height at most 1. If furthermore f is not a zero-divisor, then every minimal prime $\mathfrak{p}$ containing f has height precisely 1.*

0.8. *Lemma. — If R is a Noetherian ring with one prime ideal. Then R is Artinian, i.e., it satisfies the descending chain condition for ideals.*

The notion of Artinian rings is very important, but we will get away without discussing it much.

Proof. If R is a ring, we define more generally an *Artinian R-module*, which is an R-module satisfying the descending chain condition for submodules. Thus R is an Artinian ring if it is Artinian over itself as a module.

If $\mathfrak{m}$ is a maximal ideal of R, then any finite-dimensional $(R/\mathfrak{m})$-vector space (interpreted as an R-module) is clearly Artinian, as any descending chain

$$M_1 \supset M_2 \supset \cdots$$

must eventually stabilize (as $\dim_{R/\mathfrak{m}} M_i$ is a non-increasing sequence of non-negative integers).

Exercise. Show that for any n, $\mathfrak{m}^n/\mathfrak{m}^{n+1}$ is a finitely-dimensional $R/\mathfrak{m}$-vector space. (Hint: show it for $n = 0$ and $n = 1$. Use the dimension for $n = 1$ to bound the dimension for general n.) Hence $\mathfrak{m}^n/\mathfrak{m}^{n+1}$ is an Artinian R-module.

As $\sqrt{0}$ is prime, it must be $\mathfrak{m}$. As $\mathfrak{m}$ is finitely generated, $\mathfrak{m}^n = 0$ for some n. **Exercise.** Prove this. (Hint: suppose $\mathfrak{m}$ can be generated by m elements, each of which has kth power 0, and show that $\mathfrak{m}^{m(k-1)+1} = 0$.)

Exercise. Show that if $0 \to M' \to M \to M'' \to 0$ is an exact sequence of modules. then M is Artinian if and only if M' and M'' are Artinian.

Thus as we have a finite filtration

$$R \supset \mathfrak{m} \supset \cdots \supset \mathfrak{m}^n = 0$$

all of whose quotients are Artinian, so R is Artinian as well. $\square$

Proof of Krull's principal ideal theorem 0.7. Suppose we are given $x \in A$, with $\mathfrak{p}$ a minimal prime containing x. By localizing at $\mathfrak{p}$, we may assume that A is a local ring, with maximal ideal $\mathfrak{p}$. Suppose $\mathfrak{q}$ is another prime strictly containing $\mathfrak{p}$.

$$\begin{tikzcd}
x \arrow[dr, hook] & & \\
& \mathfrak{p} \arrow[r, hook] & A \\
\mathfrak{q} \arrow[ur, hook] & &
\end{tikzcd}$$

For the first part of the theorem, we must show that $A_{\mathfrak{q}}$ has dimension 0. The second part follows from our earlier work: if any minimal primes are height 0, f is a zero-divisor, by our identification of the associated primes of a ring as the union of zero-divisors.

Now $\mathfrak{p}$ is the only prime ideal containing (x), so $A/(x)$ has one prime ideal. By Lemma 0.8, $A/(x)$ is Artinian.

We invoke a useful construction, the n*th symbolic power of a prime ideal*: if R is a ring, and $\mathfrak{q}$ is a prime ideal, then define

$$\mathfrak{q}^{(n)} := \{r \in R : rs \in \mathfrak{q}^n \text{ for some } s \in R - \mathfrak{q}\}.$$

We have a descending chain of ideals in A

$$\mathfrak{q}^{(1)} \supset \mathfrak{q}^{(2)} \supset \cdots,$$

so we have a descending chain of ideals in $A/(x)$

$$\mathfrak{q}^{(1)} + (x) \supset \mathfrak{q}^{(2)} + (x) \supset \cdots$$

which stabilizes, as $A/(x)$ is Artinian. Say $\mathfrak{q}^{(n)} + (x) = \mathfrak{q}^{(n+1)} + (x)$, so

$$\mathfrak{q}^{(n)} \subset \mathfrak{q}^{(n+1)} + (x).$$

Hence for any $f \in \mathfrak{q}^{(n)}$, we can write $f = ax + g$ with $g \in \mathfrak{q}^{(n+1)}$. Hence $ax \in \mathfrak{q}^{(n)}$. As $\mathfrak{p}$ is minimal over x, $x \notin \mathfrak{q}$, so $a \in \mathfrak{q}^{(n)}$. Thus

$$\mathfrak{q}^{(n)} = (x)\mathfrak{q}^{(n)} + \mathfrak{q}^{(n+1)}.$$

As x is in the maximal ideal $\mathfrak{p}$, the second version of Nakayama's lemma 0.2 gives $\mathfrak{q}^{(n)} = \mathfrak{q}^{(n+1)}$.

We now shift attention to the local ring $A_{\mathfrak{q}}$, which we are hoping is dimension 0. We have $\mathfrak{q}^{(n)}A_{\mathfrak{q}} = \mathfrak{q}^{(n+1)}A_{\mathfrak{q}}$ (the symbolic power construction clearly construction commutes with respect to localization). For any $r \in \mathfrak{q}^n A_{\mathfrak{q}} \subset \mathfrak{q}^{(n)}A_{\mathfrak{q}}$, there is some $s \in A_{\mathfrak{q}} - \mathfrak{q}A_{\mathfrak{q}}$ such that $rs \in \mathfrak{q}^{n+1}A_{\mathfrak{q}}$. As s is invertible, $r \in \mathfrak{q}^{n+1}A_{\mathfrak{q}}$ as well. Thus $\mathfrak{q}^n A_{\mathfrak{q}} \subset \mathfrak{q}^{n+1}A_{\mathfrak{q}}$, but as $\mathfrak{q}^{n+1}A_{\mathfrak{q}} \subset \mathfrak{q}^n A_{\mathfrak{q}}$, we have $\mathfrak{q}^n A_{\mathfrak{q}} = \mathfrak{q}^{n+1}A_{\mathfrak{q}}$. By Nakayama's Lemma version 4 (Exercise 0.4),

$$\mathfrak{q}^n A_{\mathfrak{q}} = 0.$$

Finally, any local ring $(R, \mathfrak{m})$ such that $\mathfrak{m}^n = 0$ has dimension 0, as $\operatorname{Spec} R$ consists of only one point: $[\mathfrak{m}] = V(\mathfrak{m}) = V(\mathfrak{m}^n) = V(0) = \operatorname{Spec} R$. □

Finally:

0.9. *Proposition. — If $(A, \mathfrak{m})$ is a Noetherian local ring, then $\cap_i \mathfrak{m}^i = 0$.*

It is tempting to argue that $\mathfrak{m}(\cap_i \mathfrak{m}^i) = \cap_i \mathfrak{m}^i$, and then to use Nakayama's lemma 0.4 to argue that $\cap_i \mathfrak{m}^i = 0$. Unfortunately, it is not obvious that this first equality is true: product does not commute with infinite intersections in general. I heard this argument from Kirsten Wickelgren, who I think heard it from Greg Brumfiel. We used it in showing an equivalence in that big chain of equivalent characterizations of discrete valuation rings.

Proof. Let $I = \cap_i \mathfrak{m}^i$. We wish to show that $I \subset \mathfrak{m}I$; then as $\mathfrak{m}I \subset I$, we have $I = \mathfrak{m}I$, and hence by Nakayama's Lemma 0.4, $I = 0$. Fix a primary decomposition of $\mathfrak{m}I$. It suffices to show that $\mathfrak{p}$ contains I for any $\mathfrak{p}$ in this primary decomposition, as then I is contained in all the primary ideals in the decomposition of $\mathfrak{m}I$, and hence $\mathfrak{m}I$.

Let $\mathfrak{q} = \sqrt{\mathfrak{p}}$. If $\mathfrak{q} \neq \mathfrak{m}$, then choose $x \in \mathfrak{m} - \mathfrak{q}$. Now x is not nilpotent in $R/\mathfrak{p}$, and hence is not a zero-divisor. But $xI \subset \mathfrak{p}$, so $I \subset \mathfrak{p}$.

On the other hand, if $\mathfrak{q} = \mathfrak{m}$, then as $\mathfrak{m}$ is finitely generated, and each generator is in $\sqrt{\mathfrak{p}}$, there is some a such that $\mathfrak{m}^a \subset \mathfrak{p}$. But $I \subset \mathfrak{m}^a$, so we are done. □

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