MODERN ALGEBRA (MATH 210) PROBLEM SET 5

1. Suppose $G$ is a finite group. Let $S$ be the subset of $G$ consisting of all elements whose order is a power of $p$. If $S$ is a subgroup, show that $S$ is a $p$-Sylow subgroup, and that it is normal. Conversely, show that if there is a single $p$-Sylow subgroup, then it is $S$.

2. Show that the group of rotations of a tetrahedron is isomorphic to $A_4$.

3. Let $\phi(n)$ be the number of integers smaller than $n$ that are relatively prime to $n$. If $p$ is a prime, and $m$ is positive integer, show that $m \mid \phi(p^m - 1)$.

4. Suppose $G$ is the quotient product of $n$ finite cyclic groups by some subgroup. Show that $G$ can be written as the product of $m$ cyclic groups, where $m \leq n$.

5. Show that $(\mathbb{Z}/2^n)^*$ is not cyclic for $n \geq 3$. (One possible hint: show the result for $n = 3$ and work by induction. A hint for a different proof: find 2 distinct subgroups of order 2.)

6. Show that $(\mathbb{Z}/3^n)^*$ is cyclic for all positive integers $n$.

7. Let $P$ be a 2-Sylow subgroup of $S_8$. Show that $P$ can be written as a semidirect product $D_8 \rtimes D_8 \rtimes \mathbb{Z}/2$. Show that $P$ can be written as a semidirect product $(\mathbb{Z}/2)^4 \rtimes D_8$.

8. Let $F$ be the free group on two generators $x$ and $y$ (whose elements can be written as words with two letters $x$ and $y$). Let $\phi : F \rightarrow D_{2n}$ (where $D_{2n}$ is written interpreted as the symmetries of a regular $n$-gon), where $x$ maps to a rotation by $2\pi/n$ and $y$ maps to a reflection. Show that the kernel of $\phi$ is generated by $x^n$, $y^2$, $xyxy$. (Hence $D_{2n}$ has a presentation $\langle x, y : x^n = y^2 = xyxy = e \rangle$. Hint: remember the related problem about $S_6$.)

This set is due Friday, Nov. 12 at noon at Jarod Alper’s door, 380-J.