MODERN ALGEBRA (MATH 210) PROBLEM SETS 1 AND 2

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Problem Set 1. (due Friday, Oct. 8 at noon at Jarod Alper's door, 380–J)

1. Prove that A_4 is not simple, by producing a 4-element subgroup V (called the *Klein 4-group*) normal in A_4. (We will soon see that A_n is simple for all other n.)

2. If H is a subgroup of index 2 in a group G, show that (i) $g^2 \in H$ for every $g \in G$, and (ii) $H \lhd G$.

3.

(a) If G is a group in which $x^2 = 1$ for every $x \in G$, prove that G is abelian.
(b) Let G be a group of order 4. Prove that either G is cyclic or $x^2 = 1$ for every $x \in G$.
(c) Show that every group of order ≤ 5 is abelian.

(Lang Ex. I.1)

4. Let G be a finite group with $K \lhd G$. If $|K|$ and $[G : K]$ are relatively prime, prove that K is the unique subgroup of G having order K.

5. If S is a subgroup of a group G, let N_S be the set of all elements $x \in G$ such that $xSx^{-1} = S$. N_S is called the *normalizer* of S. (Lang p. 14–15)

(a) Show that N_S is a subgroup.
(b) If K is any subgroup of G containing H and such that H is normal in K, then show that $K \subset N_H$.
(c) If K is a subgroup of N_H, show that KH is a group and H is normal in KH.
(d) Show that N_H is the largest subgroup of G in which H is normal. In other words, any other subgroup of G containing H, in which H is normal, must be contained in N_H.

6. If G is a group, then the automorphisms of G themselves form a group, denoted $\mathrm{Aut}(G)$. The conjugations (those of the form $g \mapsto x^{-1}gx$) are called the *inner automorphisms* $\mathrm{Inn}(G)$. Show that $\mathrm{Inn}(G) \lhd \mathrm{Aut}(G)$.

Problem Set 2. (due Friday, Oct. 15 at noon at Jarod Alper's door, 380–J)

1. The normalizer (see problem 5 on set 1) of a one-element set a is called the *centralizer* of a. Let $\sigma = [123 \cdots n]$ in S_n. Show that the conjugacy class of σ has $(n-1)!$ elements. Show that the centralizer of σ is the cyclic group generated by σ. (Lang Ex. I.36)

Date: Monday, October 4, 2004.

2.

(a) Show that S_n is generated by the transpositions $(12), (13), \ldots, (1n)$.
(b) Show that S_n is generated by the transpositions $(12), (23), \ldots, ((n-1)n)$.
(c) Show that S_n is generated by the cycles (12) and $(123 \cdots n)$.
(d) Assume that n is prime. Let $\sigma = (123 \cdots n)$ and $\tau = (rs)$ be any transposition. Show that σ and τ generate S_n.

(Lang Ex. 1.38)

3. If H and K are subgroups of a group G, prove that HK is a subgroup of G if and only if $HK = KH$.

4. Show that the center of $GL(2, \mathbb{R})$ (the invertible 2×2 matrices with real entries) is the set of all *scalar matrices* aI with $a \neq 0$.

5. If H and K are normal subgroups of a group G with $HK = G$, prove that

$$G/(H \cap K) \cong (G/H) \times (G/K).$$

(*Hint.* What can you say about the kernel and image of $\phi : G \to (G/H) \times (G/K)$?)

6. Let G be a group. A *commutator* in G is an element of the form $aba^{-1}b^{-1}$ with $a, b \in G$. Let G^c be the subgroup generated by the commutators. Then G^c is called the *commutator subgroup*. Show that $G^c \lhd G$. Show that any homomorphism of G into an abelian group factors through G/G^c. (Lang Ex. I.3)

7. *(Goursat's Lemma)* Let G, G' be groups, and let H be a subgroup of $G \times G'$ such that the two projections $p_1 : H \to G$ and $p_2 : H \to G'$ are surjective. Let N be the kernel of p_2 and N' be the kernel of p_1. Show that the image of H in $G/N \times G'/N'$ is the graph of an isomorphism $G/N \cong G'/N'$. (Lang Ex. I.5)

8. A sequence of groups $G_{i-1} \xrightarrow{d_{i-1}} G_i \xrightarrow{d_i} G_{i+1}$ is *exact* at G_i if $\ker d_i = \operatorname{im} d_{i-1}$. Suppose S is a finite simple group. If G is a finite group, let $\nu_S(G)$ be the number of times in which S appears in a Jordan-Hölder filtration of G. Suppose that

$$0 \longrightarrow G_1 \longrightarrow G_2 \longrightarrow \cdots \longrightarrow G_{n-1} \longrightarrow G_n \longrightarrow 0 \qquad (1)$$

is an exact sequence of finite groups (i.e. it is exact at every step). Show that $\sum_{i=1}^{n} (-1)^i \nu_S(G) = 0$. (*Hint:* Show it first for the case of a *short exact sequence* $0 \longrightarrow H_1 \longrightarrow H_2 \longrightarrow H_3 \longrightarrow 0$. Then show that you can "break up" the long exact sequence (1) into short exact sequences $0 \longrightarrow \operatorname{im} d_{i-1} \longrightarrow G_i \longrightarrow \operatorname{im} d_i \longrightarrow 0$.) Note that your argument will work for exact sequences of finite-dimensional vector spaces as well, in which case ν is $\dim$.

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