18.034 PROBLEM SET 6

Due March 31 in class. No lates will be accepted. Discussion is encouraged, with two caveats: (a) write up your solutions by yourself, and (b) give credit when others came up with ideas (you won't be penalized for this). Give explanations, not just answers. References are to Boyce and DiPrima; the answers to problems from the book are in the back of the book.

1. Problem 3.7.28, p. 178.
2. Problem 3.7.31, p. 179.
3. Problem 4.4.2, p. 224.
4. Let ϕ be a real-valued non-trivial solution (i.e. not the 0-function) of $y'' + \alpha(x)y = 0$ on $a < x < b$, and let ψ be a real-valued non-trivial solution of $y'' + \beta(x)y = 0$ on $a < x < b$. Here α and β are real-valued continuous functions. Suppose that $\beta(x) > \alpha(x)$ for $a < x < b$. Show that if x_1 and x_2 are successive zeros of ϕ on $a < x < b$, then ψ must vanish at some point ξ, $x_1 < \xi < x_2$.

   Hint: Argue by contradiction. Suppose $\psi(x) > 0$ for $x_1 < x < x_2$, and assume $\phi(x) > 0$ for $x_1 < x < x_2$. (Deal with the other cases later.) Show that
   $$(\psi\phi' - \phi\psi')' = \psi\phi'' - \phi\psi'' = (\beta - \alpha)\phi\psi > 0.$$
   Integrate this to get $\psi(x_2)\phi'(x_2) - \psi(x_1)\phi'(x_1) > 0$, using the fact that $\phi(x_1) = \phi(x_2) = 0$. Show that $\phi'(x_2) > 0$ or $\phi'(x_1) < 0$ (or both). Why can't this happen? (Draw a picture of the graph of ϕ near x_1 and x_2 and see what happens.)

   This is an intricate problem, so you should get stuck at various stages. When you do, let me know (by e-mail or by dropping by), and I'll give you a hint. Also, if you're not sure what any of the words or hints mean, let me know.
5. Here is an application for the above problem. Show that any solution ψ of
   $$y'' + xy = 0$$
   on $0 < x < \infty$ has an infinity of zeros there.

   Hint: Consider the equation $y'' + y = 0$, and use the previous problem with $\alpha(x) = 1$, $\beta(x) = x$, $\phi(x) = \cos x$.

   This differential equation is known as the *Airy equation*, and it comes up in physics. This equation arose first in an analysis of the intensity of light in the neighborhood of a caustic (one of those intense lines that forms when light is defracted in a glass of water for example). It also occurs in the analysis of the statics of a standing column.

Date: March 17, 2000.

The Airy equation is a variant of Bessel's equation (and solutions are related to Bessel functions). We might talk more about Bessel functions later in the semester.

If you have seen power series before, you'll quickly be able to work out the power series of two linearly independent solutions $\mathrm{ca}(x)$ and $\mathrm{sa}(x)$, which satisfy initial conditions $\mathrm{ca}(0) = 1$, $\mathrm{ca}'(0) = 0$, $\mathrm{sa}(0) = 0$, $\mathrm{sa}'(0) = 1$. They are analogs of the cosine and sine functions.