18.024 PRACTICE QUIZ IV SOLUTIONS

Quiz 4 will be on Wednesday from 10 to 11:30 am. On Tuesday, I will have office hours from 4 to 7 pm, and Benoit will have office hours 7 to 9 pm. (Benoit's office hours were officially listed as being from 10 am to 1 pm.)

1. Find the mass of the hemisphere $x^2 + y^2 + z^2 = a^2$, $z \geq 0$, if the density, in mass per unit area, is $\delta = z^2$. (This is a surface integral, not a volume integral!)

Solution.

$$\int_0^{2\pi} \int_0^{\pi/2} a^2 \cos^2 \phi (a^2 \sin \phi) \, d\phi d\theta = 2\pi a^4 \left(-\cos^3 \phi / 3 \right)_0^{\pi/2} = 2\pi a^4/3.$$

2. Let C be a simple closed curve in the xy-plane, traversed in the *clockwise* direction, and let I_z denote the moment of inertia about the z-axis of the region enclosed by C. Show that

$$I_z = q \oint_C x^3 \, dy - y^3 \, dx$$

for some q (and find q).

Solution. Let R be the region enclosed by C. By Green's theorem,

$$\oint_C -y^3 \, dx + x^3 \, dy = -\iint_R 3(x^2 + y^2) \, dxdy.$$

Hence

$$I_z = \iint_R (x^2 + y^2) \, dxdy = -\frac{1}{3} \oint_C x^3 \, dy - y^3 \, dx,$$

and $q = -1/3$.

3. The set of points satisfying the equation $4x^2 + 4xy + 2y^2 - 2y = 3$ is an ellipse. Find the area of the region bounded by the ellipse. (A suitable linear transformation will carry this ellipse to a circle.)

Solution. Let $u = 2x + y$ and $v = y$, and let S be the region

$$\{(x, y) | x^2 + 4xy + 2y^2 - 2y \leq 3\}$$

in the (x, y)-plane, corresponding to the region

$$T = \{(u, v) | u^2 + v^2 - 2v \leq 3\} = \{(u, v) | u^2 + (v - 1)^2 \leq 2^2\}$$

in the (u, v)-plane. T is a circle of radius 2 and hence area 4π. Then $x = (u - v)/2$, $y = v$, so

$$\frac{\partial(x, y)}{\partial(u, v)} = 1/2.$$

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Thus the area of S is

$$\iint_S 1\,dxdy = \iint_T \frac{\partial(x,y)}{\partial(u,v)}\,dudv = \frac{1}{2}\iint_T 1\,dudv = 2\pi.$$

4. Suppose S is the portion of the surface $z = 1 - x^2 - y^2$ above the xy-plane, and $\vec{F} = (e^{x+y+z}, -e^{x+y+z}, x^2+y^2)$. Calculate $\iint_S \vec{F}\cdot\vec{n}\,dA$, where $\vec{n}$ is in the upwards direction. Hint: Use Stokes' theorem (note that $\vec{F} = \vec{\nabla}\times\vec{G}$, where $\vec{G}(x,y,z) = (-y^3/3, x^3/3, e^{x+y+z})$) or the Divergence theorem to show that it is equal to $\iint_{S'} \vec{F}\cdot\vec{n}\,dA$, where S' is the unit disc in the xy-plane given by $x^2+y^2 \le 1$, $z = 0$, and $\vec{n} = \vec{k}$ is the upward normal, and then calculate this integral.

Note: there was a typo in the hint in the handout, in the formula for $\vec{G}$. It doesn't change the soluton, but is relevant for the aside below.

Solution 1: Divergence theorem (Compare to the solution to # 6.) Let V be the 3-dimensional region above S' and below S. As $\vec{\nabla}\cdot\vec{F} = 0$ (by direct computation),

$$\begin{aligned}\iint_S \vec{F}\cdot\vec{n}\,dA &= \iint_{S'} \vec{F}\cdot\vec{k}\,dA \\ &= \iint_{S'} \left(x^2+y^2\right)\,dxdy \\ &= \int_0^{2\pi}\int_0^1 \left(r^2\right) r\,drd\theta \\ &= \pi/2\end{aligned}$$

Solution 2: Stokes' Theorem. Let C be the circle $x^2+y^2 = 1$, $z = 0$, traversed counterclockwise (viewed from above). Then by Stokes' Theorem (twice),

$$\iint_S \vec{F}\cdot\vec{n}\,dA = \oint_C \vec{G}\cdot\,d\vec{\alpha} = \iint_{S'} \vec{F}\cdot\vec{n}\,dA.$$

Then proceed as in Solution 1.

Solution 2 Aside. Hence $\pi/2 = \oint_C \vec{G}\cdot\,d\vec{\alpha}$. Parametrize $\vec{\alpha}$ by $(\cos\theta, \sin\theta, 0)$ $(0 \le \theta \le 2\pi)$. Then we have shown that

$$\begin{aligned}\pi/2 &= \int_0^{2\pi} \left(-\frac{1}{3}\sin^3\theta, \frac{1}{3}\cos^3\theta, e^{\cos\theta+\sin\theta}\right)\cdot(-\sin\theta, \cos\theta, 0)\,d\theta \\ &= \frac{1}{3}\int_0^{2\pi} \left(\sin^4\theta + \cos^4\theta\right)\,d\theta.\end{aligned}$$

A short argument (can you see it?) shows that

$$\int_0^{2\pi} \sin^4\theta\,d\theta = \int_0^{2\pi} \cos^4\theta\,d\theta,$$

so we've shown (with relatively little effort) that

$$\int_0^{2\pi} \sin^4\theta\,d\theta = \int_0^{2\pi} \cos^4\theta\,d\theta = 3\pi/4.$$

5. You put a perfectly spherical egg is through an egg slicer, resulting in n slices of identical height. But you forgot to peel it first! Show that the amount of eggshell in each slice is the same.

Solution. Suppose the egg has radius R, so each slice has height $2R/n$. A slice is parametrized by $0 \leq \theta \leq 2\pi$, $\phi_1 \leq \phi \leq \phi_2$, where $R\cos\phi_1 - R\cos\phi_2 = 2R/n$. Hence the area of the part f the surface of the sphere in slice is

$$\int_0^{2\pi}\int_{\phi_1}^{\phi_2} R^2 \sin\phi \, d\phi d\theta = 2\pi R^2 \left(\cos\phi_1 - \cos\phi_2\right) = 4\pi R^2/n.$$

Hence each of the n slices has area $4\pi R^2/n$ (and not surprisingly, their total is $4\pi R^2$).

6. Let S_1 be the unit disc $z = 0$, $x^2 + y^2 \leq a^2$ in the xy-plane; let S_2 be the upper hemisphere of radius a; let $\vec{n}_i$ be the unit upward normal to S_i. Evaluate (for $i = 1, 2$) $\iint_{S_i} (\vec{F} \cdot \vec{n}_i) \, dA$, if $\vec{F}$ is the vector field

$$\vec{F} = (y^2 + z^2)\vec{i} + (x^2 + 2z^2)\vec{j} + (3z + 2)\vec{k}.$$

Solution. First,

$$\iint_{S_!} \vec{F} \cdot \vec{n}_1 \, dA = \iint_{S_1} 2 \, dA = 2 \, \text{Area}(S_1) = 2\pi a^2.$$

Note that $\vec{\nabla} \cdot \vec{F} = 3$. Let V be the region below S_2 and above S_1 (i.e. the inside of the hemisphere). By the Divergence Theorem,

$$\begin{aligned} \iint_{S_2} \vec{F} \cdot \vec{n} \, dA - \iint_{S_1} \vec{F} \cdot \vec{n} \, dA &= \iiint_V \vec{\nabla} \cdot \vec{F} \, dV \\ &= 3 \, \text{vol}(V) \\ &= 3 \cdot \frac{1}{2} \cdot \frac{4}{3} \pi a^3 \\ &= 2\pi a^3 \end{aligned}$$

Hence $\iint_{S_2} \vec{F} \cdot \vec{n}_2 \, dA = 2\pi a^3 + 2\pi a^2$.

Extra practice question. Suppose X is a convex region in the plane that is "sufficiently big", bounded by a piecewise-differentiable curve C (oriented counter-clockwise). A chord of length 1 slides around X, and its midpoint sweeps out a smaller curve C'. *Theorem.* The area between C and C' is $\pi/4$

(a) Prove the theorem in the case when C is a circle of large radius R. Prove the theorem in the case when C is a large rectangle. (Hint: see the figures.)

(b) Prove the theorem in general as follows. Suppose the chord moves around C as $0 \leq t \leq 1$, counterclockwise. Let $t \mapsto (\alpha(t), \beta(t))$ $(0 \leq t \leq 1)$ be the "counterclockwise (leading) endpoint" of the chord, and let $t \mapsto (\gamma(t), \beta(t))$ be the "clockwise endpoint". Explain why, as $0 \leq t \leq 1$, $(\alpha(t), \beta(t)) - (\gamma(t), \delta(t))$ describes a circle of radius 1, counterclockwise. Show that the area inside C is

$\int \alpha\, d\beta$ (i.e. $\int_0^1 \alpha(t)\beta'(t)\, dt$), and also $\int \gamma\, d\delta$. Show that $\int(\alpha-\gamma)\, d(\beta-\gamma) = \pi$. Show that the area inside C' is

$$\int \left(\frac{\alpha+\gamma}{2}\right) d \left(\frac{\beta+\delta}{2}\right).$$

(Assume everything in sight is a Green's region.) Prove the theorem.

Solution to (b).

$$\begin{aligned}
\text{Area}(C) - \text{Area}(C') &= \left(\frac{1}{2}\int \alpha\, d\beta + \frac{1}{2}\int \gamma\, d\delta\right) - \int \left(\frac{\alpha+\gamma}{2}\right) d\left(\frac{\beta+\delta}{2}\right) \\
&= \int \left(\frac{\alpha-\gamma}{2}\right) d\left(\frac{\beta-\delta}{2}\right) \\
&= \frac{1}{4}\int (\alpha-\gamma)\, d(\beta-\delta) \\
&= \pi/4
\end{aligned}$$