SKETCHES OF SOLUTIONS TO 18.024 PRACTICE QUIZ III

Solutions will be discussed in recitation on Thursday. Sketches of solutions will also appear on the web page on Thursday.

1. (16 points) Let C be the curve $\vec{\alpha}(t) = (t^2, 2t, -3t)$ from $(0,0,0)$ to $(1,2,-3)$ in $\mathbb{R}^3$.

(a) Evaluate $\int_C \vec{F} \cdot d\vec{\alpha}$ if $F(x,y,z) = 3x\vec{i} + xy\vec{j} + yz\vec{k}$.
(b) Evaluate $\int_C \vec{\nabla}\phi \cdot d\vec{\alpha}$ if $\phi(x,y,z) = xy^2 \sin z$.

Answer. $\vec{\alpha}' = (2t, 2, -3)$. (a)

$$\int_0^1 (3t^2, 2t^3, -6t^2)\cdot(2t,2,-3)dt = \int_0^1 (7t^3+18t^2)dt = \left(7t^4/4 + 6t^3\right)_0^1 = 7/4+6 = 31/4.$$

(b) $\phi(1,2,-3) - \phi(0,0,0) = 4\sin(-3)$.

2. (16 points) For each of the following vector fields, either find a function ϕ such that $\vec{\nabla}\phi = \vec{F}$ or explain how you know that no such function exists.

(a) $\vec{F}(x,y,z) = (y^2, 2xy+2, yz)$
(b) $\vec{F}(x,y,z) = (y^2, 2xy+2, z)$
(c) $\vec{F}(x,y,z) = (y^2, 2xy+2, xz)$

Answer. (a) None exists as $\frac{\partial F_3}{\partial y} \neq \frac{\partial F_2}{dz}$. (b) $xy^2 + 2y + z^2/2$. (c) None exists as $\frac{\partial F_3}{\partial x} \neq \frac{\partial F_1}{dz}$.

3. (16 points) Set up a triple integral for the volume of the solid consisting of those points for which $x \geq 0$, $y \geq 0$, $z \geq 0$, and $x + y^2 + z \leq 1$,

(a) in which the first integration (the one on the inside) is with respect to z. Your answer should be of the form

$$\int_?^? \int_?^? \int_?^? 1\ dz\ dy\ dx \quad \text{or} \quad \int_?^? \int_?^? \int_?^? 1\ dz\ dx\ dy.$$

(b) in which the first integration is with respect to y.

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Answer. (a)

$$\int_0^1 \int_0^{\sqrt{1-x}} \int_0^{1-x-y^2} 1\,dz\,dy\,dx \quad \text{or} \quad \int_0^1 \int_0^{1-y^2} \int_0^{1-x-y^2} 1\,dz\,dx\,dy.$$

(b)

$$\int_0^1 \int_0^{1-x} \int_0^{\sqrt{1-x-z}} 1\,dy\,dz\,dx \quad \text{or} \quad \int_0^1 \int_0^{1-z} \int_0^{\sqrt{1-x-z}} 1\,dy\,dx\,dz.$$

4. (16 points) Find the y-coordinate $\overline{y}$ of the centroid of the region in the plane bounded by $y = x^4$ and $y = 1$.

$$\begin{aligned}
\overline{y} &= \frac{\int_{-1}^{1}\int_{x^4}^{1} y\,dy\,dx}{\int_{-1}^{1}\int_{x^4}^{1} 1\,dy\,dx} \\
&= \frac{\int_{-1}^{1} \left(y^2/2\right)_{x^4}^{1}\,dx}{\int_{-1}^{1}(1-x^4)\,dx} \\
&= \frac{\int_{-1}^{1}(1/2 - x^8/2)\,dx}{\int_{-1}^{1}(1-x^4)\,dx} \\
&= \frac{\left(x/2 - x^9/18\right)_{-1}^{1}}{\left(x - x^5/5\right)_{-1}^{1}} \\
&= \frac{2(1/2 - 1/18)}{2(4/5)} \\
&= 5/9
\end{aligned}$$

5. (16 points) $\vec{f}$ be a continuously differentiable vector field defined on an open set U in V_m. Consider the following conditions on $\vec{f}$:

(a) $\int_C \vec{f} \cdot d\vec{\alpha} = 0$ for every closed piecewise-smooth curve C in U.
(b) $\vec{f} = \vec{\nabla}\phi$ for some function ϕ defined on U
(c) $D_i f_j = D_j f_i$ in U (where $\vec{f}(\vec{x}) = (f_1(\vec{x}), \ldots, f_n(\vec{x}))$.

- Does (a) imply (b)? **Y/N**
- Does (b) imply (a)? **Y/N**
- Does (b) imply (c)? **Y/N**
- Does (c) imply (b)? **Y/N**

(+4 for each correct answer, −4 for each incorrect answer)

Answer. YYYN (Do you know what the counterexample is to the last statement?)

6. (20 points) Suppose $\vec{\alpha}_1$, $\vec{\alpha}_2$, $\vec{\alpha}_3$ are paths $[0,1] \to \mathbb{R}^2$ given by

$$\vec{\alpha}_1(t) = (0, 1-t) \qquad \text{for } 0 \leq t \leq 1,$$
$$\vec{\alpha}_2(t) = \begin{cases} (0, 2t) & \text{for } 0 \leq t \leq 1/2, \\ (2(t-1/2), 1) & \text{for } 1/2 \leq t \leq 1, \end{cases}$$
$$\vec{\alpha}_3(t) = \begin{cases} (2t, 1) & \text{for } 0 \leq t \leq 1/2, \\ (2-2t, 2-2t) & \text{for } 1/2 \leq t \leq 1. \end{cases}$$

Suppose $\vec{F}$ is a vector field on $\mathbb{R}^2$, and $\int \vec{F} \cdot d\vec{\alpha_1} = 3$, $\int \vec{F} \cdot d\vec{\alpha_2} = e$, and $\int \vec{F} \cdot d\vec{\alpha_3} = \pi$.

(a) Suppose the path $\vec{\alpha}_4 : [0, \pi/2] \to \mathbb{R}^2$ is given by $\alpha_4(t) = (\sin t, \sin t)$. Calculate $\int \vec{F} \cdot d\vec{\alpha_4}$.

(b) Is $\vec{F}$ conservative? (Explain.)

Answer. (a) $e + 3 - \pi$. (b) No, as the paths $\vec{\alpha_1}$ and $\vec{\alpha_3}$ have the same start and end points, but $\int \vec{F} \cdot d\vec{\alpha_1} \neq \int \vec{F} \cdot d\vec{\alpha_3}$.