18.024 PRACTICE QUIZ II

Here are some formulas (for reference) for velocity $\vec{v}$ and acceleration $\vec{a}$.

In terms of $\vec{T}$ and $\vec{N}$:

$$\begin{aligned}\vec{v} &= \left(\frac{ds}{dt}\right)\vec{T},\\ \vec{a} &= \left(\frac{d^2s}{dt^2}\right)\vec{T} + \kappa\left(\frac{ds}{dt}\right)^2\vec{N}.\end{aligned}$$

In polar coordinates, in terms of $\vec{u}_r$ and $\vec{u}_\theta$:

$$\begin{aligned}\vec{v} &= \left(\frac{dr}{dt}\right)\vec{u}_r + \left(r\frac{d\theta}{dt}\right)\vec{u}_\theta,\\ \vec{a} &= \left(\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right)\vec{u}_r + \left(r\frac{d^2\theta}{dt^2} + 2\frac{dr}{dt}\frac{d\theta}{dt}\right)\vec{u}_\theta.\end{aligned}$$

1. (16 points) Consider the curve given in polar coordinates by $r = e^{-t}$, $\theta = t$ for $0 \le t \le 2M\pi$ (M a positive integer).

(a) Sketch this curve when $M = 2$.
(b) Find the length of this curve for general M. What happens as M becomes large?

2. (16 points) A particle moves along a curve C in space. Its acceleration vector has constant length 3 and its speed at time $t \ge 0$ is $1/(1+2t)$. Find the curvature of the curve in terms of t.

3. (20 points)

(a) Complete the definition. A vector-valued function

$$\vec{f} : S \to \mathbb{R}^3$$

where $S \subset \mathbb{R}^2$ contains a ball $B(\vec{a}; r)$ of radius r around $\vec{a} \in \mathbb{R}^2$ is said to be *differentiable at* $\vec{a}$ if for all $\vec{v} \in$ [blank],

$$\vec{f}(\vec{a} + \vec{v}) = \vec{f}(\vec{a}) + T_a\vec{v} + |\vec{v}|E_{\vec{a}}(\vec{v})$$

where [blank]. (Hint: the second blank is a fact about the function $E_{\vec{a}}$.)

(b) Let $f(x, y)$ be a function defined in $\mathbb{R}^2$ (the plane). Answer the following questions "yes" or "no" (+3 points for each correct answer, −3 for each incorrect answer).

Date: Spring 2001.

(i) Suppose that $D_1 f$ and $D_2 f$ exist at $(0,0)$. Does it follow that f is continuous at $(0,0)$? **Y/N** Does it follow that the functions $g(t) = f(t,0)$ and $h(t) = f(0,t)$ are continuous at $t = 0$? **Y/N**

(ii) Suppose that $D_1 f$ and $D_2 f$ exist in a neighborhood of $(0,0)$ and are continuous at $(0,0)$. Does it follow that f is continuous at $(0,0)$? **Y/N** Does it follow that $f'(\vec{0}; \vec{y})$ exists for all $\vec{y}$? **Y/N**

4. (16 points)

(a) Suppose $f(x,y,z)$ is a differentiable scalar-valued function such that $f(1,1,1) = 2$, and the $\vec{\nabla} f(1,1,1) = (3,4,5)$. Find the (Cartesian) equation of the tangent plane to the level surface $f(x,y,z) = 2$ at $(x,y,z) = (1,1,1)$.

(b) Suppose $f(x,y)$ is a differentiable scalar-valued function such that $f(1,1) = 2$, and $\vec{\nabla} f(1,1) = (3,4)$. Find the (Cartesian) equation of the tangent plane to the graph of f (i.e. $z = f(x,y)$) when $(x,y) = (1,1)$.

(c) Show that $f(x,y) = (\sin x)(\sin y)$ has a critical point (i.e. the gradient is $\vec{0}$) at $(x,y) = (0,0)$. Does f have a minimum, maximum, or saddle point here?

5. (16 points) Given a differentiable function $F(u,v)$, consider the composite function $f(x,y) = F(3x-y, 2x-y)$. Find $\frac{\partial f}{\partial x}(1,1)$ if $D_1 F(1,1) = -4$, $D_1 F(2,1) = 7$, $D_2 F(1,1) = 3$, $D_2 F(2,1) = -3$.

6. (16 points) The equation $x^2 + z^3 + yz = 3$ defines z implicitly as a function of x and y, say $z = f(x,y)$. Find $\frac{\partial f}{\partial x}$ in terms of x, y, and z. Find $\frac{\partial^2 f}{\partial x^2}$ in terms of x, y, z, and $\frac{\partial f}{\partial x}$.

Office hours this week: Wednesday 3-5.