SKETCHES OF SOLUTIONS TO 18.024 PRACTICE QUIZ I

There are many ways to approach all of these problems.

1. (20 points) Let L_1 be the line through the point $P = (a, 0, 0)$ on the x-axis with direction vector $(-3, 1, -1)$. Let L_2 be the line $X = (1, 2, 0) + t(1, -1, 2)$. If L_1 and L_2 intersect, find the point P.

Solution. We seek an a such that

$$(a, 0, 0) + s(-3, 1, -1) = (1, 2, 0) + t(1, -1, -2),$$

for some s and t, i.e. $a - 3s = 1 + t$, $s = 2 - t$, $-s = -2t$. Solving this system of three linear equations in three unknowns, we find $a = 11$ (and $s = 4$, $t = -2$).

Another approach: Let P be the plane parametrized by

$$(1, 2, 0) + s(-3, 1, -1) + t(1, -1, -2).$$

Then it intersects the x-axis at $(a, 0, 0)$. A normal vector to P is given by $(-3, 1, -1) \times (1, -1, 2) = (1, 5, 2)$, so P is given by the Cartesian equation

$$1(x - 1) + 5(y - 2) + 2(z - 0) = 0.$$

Substituting $(x, y, z) = (a, 0, 0)$, we find $a = 11$.

2. (24 points) Let A be a k by n matrix; let r be the rank of A. Answer the following questions in terms of n, k, and r. (Give answers only.)

(a) What can you say about the dimension of the row space of A?
(b) What can you say about the dimension of the solution space of the equation $AX = 0$?
(c) What can you say if the system $AX = C$ fails to have a solution for some C?
(d) What can you say if you know A has an inverse?

Solution. You should think of A as giving a map from V_n to V_k.

(a) It is r (by definition of rank). Note that this shows that $r \leq k$. (Similarly, $V \leq n$.)

(b) The dimension of the solution space is $n - r$.

(c) The image of A can't be all of V_k, so $r < k$. (Conversely, if $r = k$, then $AX = C$ has a solution for al C.)

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(d) Then $n = k = r$.

3. (20 points) Find conditions on a, b, c that are both necessary and sufficient for the following system to have a solution.

$$\begin{array}{rcrcrcl} & & y & - & 2z & = & a \\ x & - & y & + & z & = & b \\ x & + & y & - & 3z & = & c \end{array}$$

Solution. By manipulating the equations, we find that they are equivalent to:

$$\begin{array}{rcrcrcl} x & & & - & z & = & b+a \\ & & y & - & 2z & = & a \\ & & & & 0 & = & c-b-2a. \end{array}$$

A good way of doing this is by putting

$$\begin{pmatrix} 0 & 1 & -2 & a \\ 1 & -1 & 1 & b \\ 1 & 1 & -3 & c \end{pmatrix}$$

into reduced row echelon form using the Gauss-Jordan elimination method. From the equations we immediately see that the condition $c = b + 2a$ is necessary for the system to have a solution. Conversely, if $c = b+2a$, then the system has a solution, in fact a one-parameter family given by

$$(x, y, z) = (t + b + a, 2t + a, t).$$

(Note that to get full marks on a question like this, you have to show both necessity *and* sufficiency.)

4. (20 points) Find the inverse of the matrix

$$A = \begin{pmatrix} 2 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Solution. There are many ways of working this out. The ugliest is to use the formula involving determinants and cofactors. Better (at least in this case) is to use something like the "augmented matrix", i.e. apply Gauss-Jordan to

$$\left(\begin{array}{cccc|cccc} 2 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{cccc|cccc} 2 & 0 & 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right)$$

$$\to \left(\begin{array}{cccc|cccc} 2 & 0 & 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right)$$

$$\to \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1/2 & 0 & 0 & -1/2 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right)$$

Then the inverse is the 4×4 matrix on the right, i.e.

$$A^{-1} = \begin{pmatrix} 1/2 & 0 & 0 & -1/2 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

(You can quickly check by multiplying this by A.)

Notice that this inverse was easy to calculate, partially because the matrix was "upper triangular". For the same reason, if you knew that $a = 2w + z$, $b = x - y$, $c = y + z$, and $d = z$, you would have no trouble figuring out w, x, y, and z in terms of a, b, c, and d: $w = a/2 - d/2$, $x = b + c - d$, $y = c - d$, $z = d$. In fact, this is really the same calculation — do you see why?

5. (16 points) Let A be a 5 by 5 matrix. Show that if A^3 has rank less than 5, then A has rank less than 5.

Solution. $\det(A^3) = (\det A)^3$. If A^3 has rank less than 5 (i.e. if A^3 is singular), then $\det(A^3) = 0$, so $\det A = 0$, so A has rank less than 5 (i.e. A is singular).

(Follow-up question, a bit too tricky to appear on the quiz: show that if A^2 has rank less than 3, then A has rank less than 4.)

Another tricky question: Suppose A, B, and C are three vectors in V_5. Can $3A+2B+4C$, $A+4B+2C$, $9A+4B+3C$, and $A+2B+5C$ be linearly independent?

Solution. No. (There's a one-line solution — do you see it?)