Math 42 Winter 2006 Practice Final Exam

1. Evaluate each of the following integrals.
   (a) \[ \int \sin \sqrt{t} \, dt. \]
   (b) \[ \int \frac{1}{x^3 \sqrt{x^2 - 1}} \, dx \]
   (c) \[ \int_0^\pi \sin^2 x \cos^2 x \, dx \]
   (d) \[ \int \frac{y^2 + 1}{y^2 - 2y + 1} \, dy \]

2. The speed of a rocket, in m/s, at \( t \) seconds after launch, is given by the following table.

\[
\begin{array}{cccccccccc}
 t & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\
\hline
\text{Speed} & 0 & 2 & 5 & 9 & 13 & 17 & 20 & 24 & 26 & 28 & 30 \\
\end{array}
\]

Estimate the height of the rocket after 10 seconds using the Midpoint Rule.

3. Consider the region \( R \) in the \( xy \)-plane bounded between \( y = 2x, \ y = x^2, \ x = 1, \) and \( x = 2. \)
   (a) Find the area of \( R. \)
   (b) Find the volume of the solid generated by revolving \( R \) about the \( y \)-axis.

4. Find the arc length of the graph of
   \[ y = x^{3/2}, \quad 0 \leq x \leq 1. \]

5. A large conical tank has a height of 6 m and a radius at the top of 1 m. The tank is filled to a height of 4 m with oil with a density of 100 kg/m\(^3\). How much work does it take to empty the tank by pumping the oil out the top of the tank? You may use the approximation \( g \approx 10 \text{ m/s}^2. \)

6. Find the solution of the initial value problem
   \[ y' = \frac{e^{-\varphi}(t + 1)}{yt^2}, \quad y(1) = 2. \]

7. Suppose the growth of a population is modeled by the modified logistic equation
   \[ \frac{dP}{dt} = \frac{P}{10} \left( 1 - \frac{P}{1000} \right) \left( 1 - \frac{200}{P} \right). \]
   (a) What are the equilibrium values of the population?
(b) If $P(0) = 400$, use Euler’s method with $h = 5$ to estimate the population at time $t = 10$.

8. Determine whether each of the following series converges or diverges.

(a) $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$

(b) $\sum_{n=0}^{\infty} (-1)^n \frac{n^3 + 3n + 2}{n^3 + 6}$

(c) $\sum_{n=1}^{\infty} \frac{n}{2^n(n + 1)}$

9. Find the sums of each of the following series.

(a) $\sum_{n=0}^{\infty} \frac{(-1)^n 2^n}{4^n(2n)!}$

(b) $\sum_{n=1}^{\infty} \frac{1 - 2^n}{4^n}$

10. Find the interval of convergence of

$$\sum_{n=1}^{\infty} \frac{(-2)^n}{\sqrt{n}} (x + 3)^n.$$

11. Find a power series expansion (centered at 0) for

$$f(x) = \frac{x}{2 + x}$$

and its radius of convergence.

12. Isaac Newton showed that

$$(1 - x^2)^{-1/2} = \sum_{n=0}^{\infty} \frac{(2n)!}{4^n(n!)^2} x^{2n}$$

for $-1 < x < 1$.

(a) Using this formula, find a power series expansion for $\sin^{-1} x$.

(b) Use your power series from part (a) with $x = 1/2$ to find an infinite series whose sum is $\pi$.

13. Use power series expansions to compute

$$\lim_{x \to 0} \frac{e^{x^2} - 1}{\cos x - 1}.$$
14. (a) Find the third-degree Taylor polynomial for
\[ f(x) = x^{4/3} \]
about \( a = 27 \).
(b) Estimate the maximum error involved in estimating \( f \) with the Taylor polynomial you found in part (a) for \( 25 \leq x \leq 29 \).

15. True / False: (You do not need to justify your answer.)
(a) \[ \int_1^\infty \frac{1}{x^{\sqrt{2}}} \, dx \] is convergent.
(b) If \( \{a_n\} \) and \( \{b_n\} \) are divergent, then \( \{a_n b_n\} \) is divergent.
(c) If \( \sum c_n 2^n \) is divergent, then \( \sum c_n (-3)^n \) is divergent.
(d) The function \( f(t) = \frac{\ln t}{t} \) is a solution of
\[ t^2 y' + ty = 1. \]
(e) If \( \sum a_n \) converges, then \( \lim_{n \to \infty} a_n = 0 \).