\[
\frac{dx}{dt} = -3 \cos^2 t \sin t \\
\frac{dy}{dt} = 3 \sin^2 t \cos t
\]

\[
L = \int_0^{\pi/2} \sqrt{9 \cos^4 t \sin^2 t + 9 \sin^4 t \cos^2 t} \, dt
\]

\[
= \int_0^{\pi/2} \sqrt{9 \cos^2 t \sin^2 t (\cos^2 t + \sin^2 t)} \, dt
\]

\[
= \int_0^{\pi/2} \sqrt{9 \cos^2 t \sin^2 t} \, dt
\]

\[
= 3 \int_0^{\pi/2} \cos t \sin t \, dt \quad \text{(cost and sint are both positive for } 0 < t < \frac{\pi}{2})
\]

\[
u = \sin t \quad du = \cos t \, dt
\]

\[
u = 0 \text{ when } t = 0, \quad u = 1 \text{ when } t = \frac{\pi}{2}
\]

\[
= 3 \left[ \int_0^1 u \, du \right] = 3 \left[ \frac{3}{2} u^2 \right]_0^1 = \frac{3}{2}
\]

\[\text{2) }\]

Chop up the infinite interval \([r, \infty)\) into pieces of length \(\Delta x\). The force at distance \(x\) is \(\frac{GM}{x^2}\) \((m=1)\), so the work to move through one little piece is \(\Delta W = \frac{GM}{x^2} \Delta x\).
So the total work in moving from $x=r$ to $x\to\infty$ is

$$W = \int_r^\infty \frac{GM}{x^2} \, dx = GM \int_r^\infty x^{-2} \, dx$$

$$= GM \lim_{t \to \infty} \left[-x^{-1}\right]_r^t$$

$$= GM \left[-x^{-1}\right]_r^\infty + \frac{1}{r} = \frac{GM}{r}.$$ 

Cut up the sieve into rings of width $\Delta r$. 

The area of one ring is about $2\pi r \Delta r$ (snip it open and unroll it into a rectangle, like we did with cylinders when calculating volumes). 

So the ring lets through about

$$\left(\frac{r^2}{10} + 1\right) 2\pi r \Delta r$$

grams of sand/ls.

So the total amount of sand the sieve lets through in 1 s is
\[ \int_0^5 (\frac{r^2}{10} + 1) 2\pi r \, dr = 2\pi \int_0^5 (\frac{1}{10} r^3 + r) \, dr \]

\[ = 2\pi \left[ \frac{1}{40} r^4 + \frac{1}{2} r^2 \right]_0^5 = 2\pi \left[ \frac{5^4}{40} + \frac{25}{2} \right] \]

grams of sand /s

I think that's simplified enough, don't you?

4. If \( y = \frac{\ln t + 3}{t} \), then \( y(1) = \frac{\ln 1 + 3}{1} = 3 \)

and \( y' = \frac{t(\frac{1}{t})-(\ln t+3)}{t^2} = -\frac{2-\ln t}{t^2} \),

so \( t^2 y' + ty = (-2-\ln t) + (\ln t + 3) = 1 \)

5a) The natural growth gives us \( \frac{1}{10} P \) millions of people added each year, but also \( \frac{1}{10} \sqrt{P} \) million people are leaving each year.

So

\[ p' = \frac{1}{10} P - \frac{1}{10} \sqrt{P} = \frac{1}{10} \sqrt{P} (\sqrt{P} - 1). \]
b) Equilibrium is when $P$ is constant so $P' = 0$.
\[ 0 = \frac{1}{10} \sqrt{P} (\sqrt{P} - 1) \] if $P = 0$ or $P = 1$.
So $0$ or $1$ million are equilibria.

c) Thinking about (but not taking the time to draw) the direction field,
slopes are positive if $P > 1$ and negative if $0 < P < 1$.
If $P_0 = \frac{1}{2}$, the population will go down to $0$.
If $P_0 = 1$, the population will hold steady at $1$ million.
If $P_0 = 4$, the population will increase forever.

d) $t_0 = 0 \quad P_0 = 4$.
\[ t_1 = 5 \quad P_1 = 4 + 5 \cdot \frac{1}{10} \sqrt{4} (\sqrt{4} - 1) \]
\[ = 4 + 1 = 5. \]
\[ t_2 = 10 \quad P_2 = 5 + 5 \cdot \frac{1}{10} \sqrt{5} (\sqrt{5} - 1) \]
\[ = 5 + \frac{1}{2} (5 - \sqrt{5}) = \frac{5 - \sqrt{5}}{2} \text{ million people.} \]
\( \frac{dP}{dt} = \frac{1}{10} \sqrt{P} (\sqrt{P} - 1) \)

\[
\int \frac{dP}{\sqrt{P} (\sqrt{P} - 1)} = \int \frac{1}{10} \, dt = \frac{1}{10} \, t + C
\]

\[
\begin{align*}
|u &= \sqrt{P} \\
\, \, du &= \frac{1}{2} P^{-\frac{1}{2}} \, dp = \frac{1}{2\sqrt{P}} \, dp
\end{align*}
\]

\[
\int \frac{2 \, du}{u - 1} = 2 \ln |u - 1| = 2 \ln |\sqrt{P} - 1|
\]

So \( \ln |\sqrt{P} - 1| = \frac{1}{5} \, t + C \) \( \text{(different } C \text{ from before)} \)

\[
|\sqrt{P} - 1| = e^{\frac{1}{5} \, t + C} = e^C \cdot e^{\frac{1}{5} \, t}
\]

\[
|\sqrt{P} - 1| = \pm e^C \cdot e^{\frac{1}{5} \, t} = A e^{\frac{1}{5} \, t}, \quad A = \pm e^C.
\]

Now use \( P(0) = 4 \):

\[
|\sqrt{4} - 1| = Ae^0 \quad \Rightarrow \quad A = 1.
\]

\[
|\sqrt{P} - 1| = e^{\frac{1}{5} \, t}
\]

\[
\sqrt{P} = 1 + e^{\frac{1}{5} \, t}
\]

\[
P = (1 + e^{\frac{1}{5} \, t})^2.
\]
\( a_1 = 4, \ a_2 = 4 - \frac{3}{4} < 4. \)

So \( \{a_n\} \) decreases at the first step.

Suppose \( a_k \geq a_{k+1} \). Then

\[
\frac{3}{a_k} < \frac{3}{a_{k+1}}
\]

\[
-\frac{3}{a_k} > -\frac{3}{a_{k+1}}
\]

\[
4 - \frac{3}{a_k} > 4 - \frac{3}{a_{k+1}}
\]

\( a_{k+1} > a_{k+2} \).

So \( \{a_n\} \) decreases at the next step as well. Therefore the sequence is always decreasing.

Since \( a_1 = 4 \) and \( \{a_n\} \) is decreasing, \( a_n \leq 4 \) always. We know \( a_1 = 4 \geq 3 \).

Suppose \( a_k \geq 3 \). Then

\[
\frac{3}{a_k} \leq \frac{3}{3} = 1
\]

\[
-\frac{3}{a_k} \geq -1
\]

\[
4 - \frac{3}{a_k} \geq 4 - 1 = 3
\]

\( a_{k+1} \geq 3 \).
So $a_n \geq 3$ always.

We now know $\{a_n\}$ is a bounded and monotonic sequence, and therefore it must converge.

Let $L = \lim_{n \to \infty} a_n$, which we now know exists.

Then

$$L = \lim_{n \to \infty} a_{n+1} = \lim_{n \to \infty} \left(4 - \frac{3}{a_n}\right) = 4 - \frac{3}{L}$$

$$\Rightarrow L^2 = 4L - 3 \Rightarrow L^2 - 4L + 3 = 0$$

$$\Rightarrow (L-3)(L-1) = 0 \Rightarrow L = 3 \text{ or } L = 1.$$  

But $a_n \geq 3$ for all $n$, so we can't have $L = 1$. So $L = 3$.

\(\square\)

1) \(\frac{1}{n(n+2)} = \frac{A}{n} + \frac{B}{n+2}\)

1 = A(n+2) + Bn

\(n=0: 2A=1\) \hspace{1cm} \(\frac{1}{n(n+2)} = \frac{1}{2} \left[ \frac{1}{n} - \frac{1}{n+2} \right]\)

\(n=-2: -2B=1\)

\(S_1 = \frac{1}{2} \left[ 1 - \frac{1}{3} \right]\)

\(S_2 = \frac{1}{2} \left[ 1 - \frac{1}{3} + \frac{1}{2} - \frac{1}{4} \right]\)
\( S_3 = \frac{1}{2} \left[ 1 - \frac{1}{3} + \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{5} \right] \)

\( S_4 = \frac{1}{2} \left[ 1 - \frac{1}{3} + \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{5} + \frac{1}{4} - \frac{1}{6} \right] \)

\( S_5 = \frac{1}{2} \left[ 1 - \frac{1}{3} + \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{5} + \frac{1}{4} - \frac{1}{6} + \frac{1}{5} - \frac{1}{7} \right] \)

Now we can start to see the pattern of what’s left.

\( S_n = \frac{1}{2} \left[ 1 + \frac{1}{2} - \frac{1}{n+1} - \frac{1}{n+2} \right]. \)

\( \lim_{n \to \infty} \frac{1}{n(n+2)} = \lim_{n \to \infty} S_n = \lim_{n \to \infty} \frac{1}{2} \left[ 1 + \frac{1}{2} - \frac{1}{n+1} - \frac{1}{n+2} \right] \)

\( = \frac{1}{2} \left( 1 + \frac{1}{2} \right) = \frac{3}{4}. \)

6) \( \sum_{k=1}^{\infty} \frac{2^{k-1}}{5^{4k+1}} - 3 \sum_{k=1}^{\infty} \frac{1}{5^{4k+1}} \)

\( = \left( \frac{1}{25} + \frac{2}{5^3} + \frac{4}{5^4} + \ldots \right) - 3 \left( \frac{1}{25} + \frac{1}{5^3} + \frac{1}{5^4} + \ldots \right) \)

\( a = \frac{1}{25}, \ r = \frac{2}{5} \)

\( = \frac{\frac{1}{25}}{1 - \frac{2}{5}} - 3 \cdot \frac{\frac{1}{25}}{1 - \frac{2}{5}} = \frac{5}{3} \cdot \frac{1}{25} - 3 \cdot \frac{5}{4} \cdot \frac{1}{25} \)

\( = \frac{1}{15} - \frac{3}{20} = -\frac{5}{60} = -\frac{1}{12}. \)
a) \( \lim_{n \to \infty} e^{-\frac{1}{n}} = e^0 = 1. \)

So \( \Sigma e^{-\frac{1}{n}} \) diverges by the Divergence Test.

b) \( \frac{2n}{(n+3)^{3/2}} \) looks similar to \( \frac{2n}{n^{3/2}} = \frac{2}{\sqrt{n}}. \)

\[
\lim_{n \to \infty} \frac{\frac{2n}{(n+3)^{3/2}}}{\frac{2}{\sqrt{n}}} = \lim_{n \to \infty} \frac{n^{3/2}}{(n+3)^{3/2}} = \left( \lim_{n \to \infty} \frac{n}{n+3} \right)^{3/2} = 1.
\]

So by the Limit Comparison Test, \( \Sigma_{n=1}^{\infty} \frac{2n}{(n+3)^{3/2}} \) does the same thing as \( \Sigma_{n=1}^{\infty} \frac{2}{\sqrt{n}}. \)

Since \( \Sigma_{n=1}^{\infty} \frac{1}{n^{1/2}} \) is a divergent \( p \)-series,

the series \( \Sigma_{n=1}^{\infty} \frac{2n}{(n+3)^{3/2}} \) diverges.

c) Let \( f(x) = xe^{-x}. \)
\[
f'(x) = e^{-x} - xe^{-x} = (1-x)e^{-x} < 0
\]
if \( x > 0.1. \)

So \( f \) is a continuous, decreasing, positive function. By the Integral Test, \( \Sigma_{n=1}^{\infty} ne^{-n} \) does the same thing as \( \int xe^{-x} dx. \)
\[ \int x e^{-x} \, dx \quad u = x \quad du = dx \\
\quad \frac{du}{dv} = e^{-x} \quad v = -e^{-x} \]

\[ = -xe^{-x} + \int e^{-x} \, dx = -xe^{-x} - e^{-x} + C. \]

So

\[ \int_{1}^{\infty} xe^{-x} \, dx = \lim_{t \to \infty} \left[ -xe^{-x} - e^{-x} \right] \]

\[ = \lim_{t \to \infty} \left( \frac{t}{e^t} + \frac{1}{e^t} - e^{-1} \right) \]

\[ = - \lim_{t \to \infty} \frac{t}{et} + 2e^{-1} \quad \infty \quad \infty \quad : \text{use l'Hôpital} \]

\[ = - \lim_{t \to \infty} \frac{1}{e^t 2e^{-1}} = 2e^{-1}. \]

Since the integral converges, so does the series.

\[ \frac{3}{k^2 + 7} < \frac{3}{k^2}, \quad \text{and} \quad 3 \sum_{k=1}^{\infty} \frac{1}{k^2} \]

a convergent p-series. So the series converges by the Comparison Test.
a) **TRUE**

b) **TRUE**: since $y' < 0$.

c) **FALSE**: consider $a_n = b_n = (-1)^n$.

d) **TRUE**: since $a_n \leq q$, for a decreasing sequence, $\{a_n\}$ must be a bounded monotonic sequence.

e) **TRUE**: $P = K$ is the only non-zero equilibrium.