Math 108 Combinatorics  
Fall 2005

Homework 5 Solutions

Problem 1.

(a) Let $G$ be the graph on 5 vertices $a, b, c, d,$ and $e$ with edge set $(a, b), (a, c), (a, d),
(b, c), (b, d)$ and $(c, e)$. Compute the chromatic polynomial $\chi_G(x)$ of $G$. (You may use
the contraction-deletion recurrence or the Mobiüs function on the bond lattice, or a
combination of the two.)

(b) For any graph $G$, what is the degree of $\chi_G(x)$? What is the coefficient of $x^{n-1}$ (where
$n$ is the degree of the polynomial)? What is the smallest exponent of $x$ that appears
(with non-zero coefficient)? Show that the coefficients of $\chi_G(x)$ alternate in sign. Prove
your claims; the most likely methods of proof use the contraction-deletion recurrence
or the Mobiüs function – and some statements are more easily proved with one or the
other.

Solution to (a). First, let $H$ be the induced graph on vertices $a, b, c,$ and $d$. Let $\alpha = (c, e)$.
Note that $G/\alpha = H$ and $G \setminus \alpha$ is the disjoint union of $H$ with an isolated vertex. Thus
$$\chi_G(x) = \chi_{G/\alpha}(x) - \chi_{G/\alpha}(x) = x\chi_H(x) - \chi_H(x) = (x - 1)\chi_H(x).$$

(See the proof of part (b) for a comment on the chromatic polynomial of a disjoint union.)

Let $\beta = (b, d)$. Then $I = H \setminus \beta$ is a triangle with an extra edge $\gamma$ and $H/\beta = K_3$.

Claim 1: The degree of $\chi_G(x)$ is $\mid V(G)\mid$.
Claim 2: The coefficient of $x^{n-1}$ is $-\mid E(G)\mid$.
Claim 3: The (non-zero) coefficients of the chromatic polynomial alternate in sign.
Claim 4: The smallest exponent of $x$ that appears in $\chi_G(x)$ is the number of connected
components of $G$.
Proof of Claims 1, 2, 3, and 4 via the contraction-deletion recurrence: First note that if $G$ is a disjoint union of graphs $H_1$ and $H_2$, then clearly $\chi_G(x) = \chi_{H_1}(x)\chi_{H_2}(x)$. So for claim 4, it suffices to show that if $G$ is connected, then the constant coefficient is zero and the coefficient of $x$ is non-zero.

The proofs of all the claims are by double induction on $|V(G)|$ and $|E(G)|$ (we also need to know that the leading coefficient is 1). The base cases are when $G$ is the graph on $n \geq 1$ vertices with no edges. Here, $\chi_G(x) = x^n$ and all the claims hold. Suppose that the claims hold for graphs with less than $n$ vertices and graphs with $n$ vertices and less than $m > 0$ edges. Let $G$ be a graph with $n$ vertices and $m$ edges and let $e$ be any edge of $G$.

By induction, $\chi_{G/e}(x)$ has degree $n$ with leading coefficient 1, the coefficient of $x^{n-1}$ is $-(m-1)$, and the non-zero coefficients alternate in sign. Similarly, $\chi_{G/e}(x)$ has degree $n-1$ with leading coefficient 1, the coefficient of $x^{n-2}$ is $-(m-1)$, and the non-zero coefficients alternate in sign. From this and the contraction-deletion recurrence, it is clear that $\chi_G(x)$ has degree $n$ with leading coefficient 1, the coefficient of $x^{n-1}$ is $-m$, and the coefficients alternate in sign.

For claim 4, by induction the constant coefficients of $\chi_{G/e}(x)$ and $\chi_{G/e}(x)$ are zero, so the same is true of $\chi_G(x)$. If $G$ is connected, then $G/e$ is connected. By induction, the coefficient of $x$ in $\chi_{G/e}(x)$ is non-zero, and has the opposite sign as the coefficient of $x$ in $\chi_{G/e}(x)$. Thus, by contraction-deletion, the coefficient of $x$ in $\chi_G(x)$ is non-zero.

Proof of Claims 1 and 2 via the bond lattice: Let $G$ be a graph with $n$ vertices and $m$ edges. Recall that

$$\chi_G(x) = \sum_{\sigma} \mu(\hat{0}, \sigma)x^{\#\text{blocks in } \sigma},$$

where the sum is over all partitions $\sigma$ of $V(G)$ into connected components. The largest possible number of blocks in $\sigma$ is $n$, corresponding to the partition $\sigma = \hat{0}$ of $V(G)$ into $n$ one-vertex components. In this case, $\mu(\hat{0}, \sigma) = \mu(\hat{0}, \hat{0}) = 1$, so the leading term of $\chi_G(x)$ is $x^n$.

There are $m$ partitions $\sigma$ with $n-1$ blocks, obtained by choosing an edge $e = (v, w)$ and partitioning $V(G)$ into the connected component $\{v, w\}$ and $n-2$ one-vertex components. Since $\mu(\hat{0}, \sigma) = -1$ for each such $\sigma$, it follows that the coefficient of $x^{n-1}$ is $-m$.

Proving that the coefficients of the chromatic polynomial alternate in sign via the bond lattice is possible (it is done in the textbook) but more difficult. Proving that the smallest exponent of $x$ that appears in $\chi_G(x)$ is the number of connected components of $G$ requires showing that $\mu(\hat{0}, \hat{1}) \neq 0$, which also requires some work. \hfill \square

Problem 2. Choose four of the items from the first 10 pages of Richard Stanley’s “Catalan addendum” and construct bijective maps between them.

Sample Solution. We will construct bijections between

(h): Lattice paths from $(0, 0)$ to $(n, n)$ with steps $(0, 1)$ or $(1, 0)$, never rising above the line $y = x$. 
Write each lattice path in \((h)\) as the sequence of lattice points \((x_0, y_0), (x_1, y_1), \ldots, (x_{2n}, y_{2n})\), where necessarily
\[
(x_0, y_0) = (0, 0), \ (x_1, y_1) = (1, 0), \ (x_{2n-1}, y_{2n-1}) = (n, n - 1), \text{ and } (x_{2n}, y_{2n}) = (n, n).
\]

\((e^4)\): Lattice paths from \((0, 0)\) to \((n - 1, n - 1)\) with steps \((0, 1), (1, 0),\) and \((1, 1)\), never going above the line \(y = x\), such that the steps \((1, 1)\) only appear on the line \(y = x\). Equivalently, lattice paths from \((1, 0)\) to \((n, n - 1)\) with steps \((0, 1), (1, 0),\) and \((1, 1)\), never going above the line \(y = x - 1\), such that the steps \((1, 1)\) only appear on the line \(y = x - 1\).

\((h) \Leftrightarrow (e^4)\): Given a lattice path \(P\) in \((h)\), construct a lattice path \(P'\) from \((1, 0)\) to \((n, n - 1)\) by removing any lattice point \((i, i)\) in \(P\). That \(P'\) is in \((e^4)\) follows from two observations: first, any point \((i, j)\) in \(P'\) have \(i < j\), so \(P'\) never goes above \(y = x - 1\); and second, any step \((1, 1)\) occurs where a point \((i, i)\) was removed, and hence is necessarily a step from \((i, i - 1)\) to \((i + 1, i)\) for some \(i\).

This is a bijection, because given any path \(P'\) in \((e^4)\), we can reconstruct \(P\) in \((h)\): namely, append \((0, 0)\) and \((1, 1)\) to the beginning and end, and insert \((i, i)\) whenever a step \((1, 1)\) occurs from \((i, i - 1)\) to \((i + 1, i)\).

\((r^4)\): Sequences \(a_1, \ldots, a_n\) of nonnegative integers satisfying \(a_1 + \cdots + a_i \geq i\) and \(\sum a_j = n\).

\((h) \Leftrightarrow (r^4)\): Given a lattice path \(P\) in \((h)\), let \(a_i\) be the number of \((1, 0)\) (horizontal) steps between the \((i - 1)\)-st and \(i\)-th \((0, 1)\) (vertical) steps. Then the sequence \(a_1, \ldots, a_n\) must be in \((r^4)\): the total number of horizontal steps is \(n\), so \(\sum a_j = n\), and after \(i\) vertical steps, we cannot be above the line \(y = x\), so the number of horizontal steps, \(a_1 + \cdots + a_i\), must be at least \(i\).

This is a bijection since we can reconstruct \(P\) from any sequence in \((r^4)\): take \(a_1\) horizontal steps, then one vertical step, then \(a_2\) horizontal steps, etc. The conditions on sequences in \((r^4)\) ensure the resulting path is in \((h)\).

\((t^4)\): Sequences \(a_1, \ldots, a_{2n}\) of nonnegative integers with \(a_1 = 1, a_{2n} = 0\) and \(a_i - a_{i-1} = \pm 1\).

\((h) \Leftrightarrow (t^4)\): Given a lattice path \(P\) in \((h)\), let \(a_i = a_{i-1} + 1\) if the \(i\)-th step was horizontal and \(a_i = a_{i-1} - 1\) if the \(i\)-th step was vertical (let \(a_0 = 0\)). Then \(a_1 = 1, a_{2n} = 0\) (since there are equal numbers of horizontal and vertical steps), \(a_i - a_{i-1} = \pm 1\), and the \(a_i\) are all nonnegative, because at any point, there have been at least as many horizontal steps as vertical steps. So we get a sequence in \((t^4)\).

This is a bijection since we can reconstruct \(P\) from any sequence in \((t^4)\), where the \(i\)-th step is horizontal if \(a_i - a_{i-1} = 1\) and vertical otherwise.

\((u^4)\): Sequences of \(n - 1\) 1’s and any number of -1’s such that every partial sum is nonnegative.
$(t^4) \Rightarrow (u^4)$: To each sequence in $u^4$, append a 1 followed by enough $-1$’s to form a sequence with $n$ 1’s and $n$ -1’s. This defines a bijection between $(u^4)$ and $(U^4)$: sequences of $n$ 1’s and $n$ -1’s in which every partial sum is nonnegative.

Obviously the partial sums of a sequence in $(U^4)$ give a sequence in $(t^4)$ and the successive differences of a sequence in $(t^4)$ give a sequence in $(U^4)$. \hfill$\blacksquare$