Technion lecture notes

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September 19, 2011

1 Traveling fronts in a homogeneous medium

We will consider reaction-diffusion equations of the form

$$\frac{\partial T}{\partial t} + u(x) \cdot \nabla T = \frac{\partial}{\partial x_n}\left(a_{nm}(x)\frac{\partial T}{\partial x_m}\right) + f(T).$$

The function $f(T)$, called the reaction rate, will always be Lipschitz, and satisfy

$$f(T) \geq 0 \text{ for } 0 \leq T \leq 1,\ f(T) = 0 \text{ for } T \notin (0,1). \tag{1.1}$$

We will distinguish two types of the reaction rate: we say that f is of the KPP-type if

$$f(T) > 0 \text{ for } 0 < T < 1 \tag{1.2}$$
$$f(T) \leq f'(0)T \tag{1.3}$$

while f is of the ignition type if

$$\exists\theta_0 \in (0,1) \text{ so that } f(T) = 0 \text{ for } 0 \leq T \leq \theta_0 \text{ and } f(T) > 0 \text{ for } T \in (\theta_0, 1). \tag{1.4}$$

1.1 Existence of fronts in the KPP case: an ODE proof

Following the historical order of development we begin with showing that traveling wave solutions of a reaction-diffusion equation

$$T_t = \Delta T + \frac{1}{4}f(T) \tag{1.5}$$

exist when the nonlinearity $f(T)$ is of the KPP type. Here the factor $1/4$ is a simple normalization. We also assume that f is normalized so that $f'(0) = 1$, which, in particular, implies that $f(T) \leq T$ because of (1.3).

Traveling waves are solutions of (1.5) of the form $T(t,x) = U(x - ct)$ with the positive function $U(x)$ satisfying the boundary conditions

$$U(x) \to 1 \text{ as } x \to -\infty,\ U(x) \to 0 \text{ as } x \to +\infty,\ U(x) > 0 \text{ for all } x \in \mathbb{R}. \tag{1.6}$$

The function $U(x)$ satisfies an ODE

$$-cU' = U'' + \frac{1}{4}f(U), \quad U(-\infty) = 1, \quad U(+\infty) = 0. \tag{1.7}$$

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We introduce $V = -U'$ so that (1.7) becomes a system

$$\begin{aligned}\frac{dU}{dx} &= -V \\ \frac{dV}{dx} &= -cV + \frac{1}{4}f(U).\end{aligned} \tag{1.8}$$

This system has two equilibria: $(U, V) = (0, 0)$ and $(U, V) = (1, 0)$. A traveling wave with the boundary conditions (1.6) corresponds to a heteroclinic orbit of (1.8) that connects the second equilibrium $(1, 0)$ as $x \to -\infty$ to the first, $(0, 0)$, as $x \to +\infty$, with $U > 0$ along the trajectory. This means that $(0, 0)$ has to be a stable equilibrium point, and, in addition, trajectories can not spiral around $(0, 0)$ since $U > 0$ along the heteroclinic orbit. Therefore, both eigenvalues of the linearized problem at $(0, 0)$

$$\frac{d}{dx}\begin{pmatrix} U \\ V \end{pmatrix} = A_0 \begin{pmatrix} U \\ V \end{pmatrix}, \quad A_0 = \begin{pmatrix} 0 & -1 \\ \frac{1}{4}f'(0) & -c \end{pmatrix}$$

have to be negative. The eigenvalues of the matrix A_0 satisfy

$$\lambda^2 + c\lambda + \frac{1}{4}f'(0) = 0$$

and are given by

$$\lambda_{1,2} = \frac{-c \pm \sqrt{c^2 - 1}}{2}.$$

They are negative if and only if $c \geq 1$. The linearization around $(1, 0)$ gives

$$\frac{d}{dx}\begin{pmatrix} \tilde{U} \\ \tilde{V} \end{pmatrix} = A_1 \begin{pmatrix} \tilde{U} \\ \tilde{V} \end{pmatrix}, \quad A_1 = \begin{pmatrix} 0 & -1 \\ \frac{1}{4}f'(1) & -c \end{pmatrix}.$$

The eigenvalues of A_1 satisfy

$$\lambda^2 + c\lambda + \frac{1}{4}f'(1) = 0$$

so that they have a different sign: $\lambda_1 > 0$, $\lambda_2 < 0$. Thus, the point $(1, 0)$ is a saddle. Note that the unstable direction $(1, -\lambda_1)$ corresponding to $\lambda_1 > 0$ lies in the second and fourth quadrants.

Let us look at the the triangle D formed by the lines $l_1 = \{V = \gamma U\}$, $l_2 = \{V = \alpha(1 - U)\}$ and the interval $l_3 = \{[0, 1]\}$ on the U-axis. We check that with an appropriate choice of γ and α all trajectories of (1.8) point into D on the boundary ∂D if $c \geq 1$. That means that the unstable manifold of $(1, 0)$ has to end at $(0, 0)$ since it may not cross the boundary of the triangle. That is, U and V stay positive along a heteroclinic orbit that starts at $(1, 0)$ and ends at $(0, 0)$ – this is a monotonic positive traveling wave we want to exist (it is monotonic since $V > 0$ along the trajectory). In particular that will show that traveling waves exist for all $c \geq c_* = 1$.

Now, we show that, indeed, all trajectories point into D on the boundary ∂D. Along the segment l_3 we have $V = 0$ and

$$\frac{dU}{dx} = 0, \quad \frac{dV}{dx} = \frac{1}{4}f(U) > 0$$

so that trajectories point upward, that is, into D. Along l_1 we have $\dfrac{dU}{dx} = -V < 0$ and

$$\frac{dV}{dU} = c - \frac{f(U)}{4V} = c - \frac{f(U)}{4\gamma U}.$$

That means that the trajectory points into D if the slope $\frac{dV}{dU} \geq \gamma$ along l_1. This is true if

$$c - \frac{f(U)}{4\gamma U} \geq \gamma$$

for all $U \in [0,1]$. This is equivalent to

$$c\gamma - \gamma^2 \geq \frac{f(U)}{4U}. \tag{1.9}$$

We have $\frac{f(U)}{U} \leq 1$ and hence (1.9) holds provided that

$$c\gamma - \gamma^2 \geq \frac{1}{4}. \tag{1.10}$$

Such $\gamma > 0$ exists if $c \geq 1$. Let us check that with this choice of c and γ all trajectories point into D also along the segment l_2. Indeed we have along l_2: $\frac{dU}{dx} = -V < 0$ and

$$\frac{dV}{dU} = c - \frac{f(U)}{4V} = c - \frac{f(U)}{4\alpha(1-U)}.$$

That means that the trajectory points into D if the slope $\frac{dV}{dU} \geq -\alpha$ along l_1. This is true if

$$c - \frac{f(U)}{4\alpha(1-U)} \geq -\alpha$$

for all $U \in [0,1]$, or

$$c\alpha + \alpha^2 \geq \frac{f(U)}{4(1-U)}.$$

This is true, for instance, if $\alpha \geq \inf_{0\leq U\leq 1} \frac{f(U)}{4(1-U)}$.

Therefore a traveling front exists provided that $c \geq 1$. However, we have also shown that no traveling front exists for $c < 1$ because the eigenvalues of the linearized problem at $(0,0)$ are no longer real, meaning that the heteroclinic orbit would have to pass into the region $U < 0$. Thus we have proved the following theorem.

Theorem 1.1 *A traveling front solution of (1.7) with a KPP-type nonlinearity exists for all $c \geq 1$, and no traveling front can exist for $c < 1$.*

Remark 1.2 We note that the traveling waves that propagate with the speeds $c > c_* = 1$ are in a sense not "reaction-diffusion" waves. More precisely, their existence is unrelated to diffusion: let $U_0(x)$ be solution of

$$\frac{dU_0}{dx} = -\frac{1}{4}f(U_0), \quad U_0(0) = 1/2. \tag{1.11}$$

Such solution exists and satisfies the boundary conditions

$$U_0(x) \to 1 \text{ as } x \to -\infty \text{ and } U_0(x) \to 0 \text{ as } x \to +\infty. \tag{1.12}$$

Then given any $c > 0$ the function $T(t,x) = U_0\left(\frac{x}{c} - t\right)$ is a traveling wave solution of

$$\frac{\partial T}{\partial t} = 0 \cdot \frac{\partial^2 T}{\partial x^2} + \frac{1}{4}f(T).$$

Thus these traveling waves exist even at zero diffusion coefficient and are therefore not quite "reaction-diffusion" waves.

1.2 Traveling waves in the ignition case

The phenomenon of having a family of traveling waves with various speeds is absent in the case of an ignition nonlinearity.

Theorem 1.3 *Let $f(T)$ be of ignition type, that is, it satisfies (1.4). Then there exists a unique $c = c_*$ so that a traveling wave solution of*

$$T_t = T_{xx} + f(T), \tag{1.13}$$

of the form $T(t,x) = U(x - ct)$, with $U(x) > 0$ satisfying the boundary conditions

$$U(-\infty) = 1, \quad U(+\infty) = 0. \tag{1.14}$$

exists. The traveling front is unique up to translation: given any two traveling waves $U_1(x)$, $U_2(x)$ there exists $\xi \in \mathbb{R}$ so that $U_1(x) = U_2(x + \xi)$ for all $x \in \mathbb{R}$.

One may prove this result using the ODE methods similar to those in the KPP case – this is left as an exercise for the reader. We present below a different, "PDE-like" proof that, while quite a bit longer and less elementary, can be generalized to many other problems, in particular, those involving inhomogeneous coefficients and higher dimensions.

Before we present the proof we note that in the ignition case waves at zero diffusivity do not exist: solutions of (1.11) do not satisfy the boundary conditions (1.12) if f is an ignition-type nonlinearity because then $U(x) \to \theta_0$ as $x \to +\infty$. This explains qualitatively uniqueness of the traveling front speed. We also remark that if we fix the traveling wave by the normalization $U(0) = \theta_0$ then the traveling wave is given by $U(x) = \theta_0 \exp(-cx)$ for $x \geq 0$. Hence we have to find c so that in the variables $(U, V = -U')$ the stable manifold of the point $(1,0)$ in (1.8) would pass through the point $(\theta_0, c\theta_0)$. Not surprisingly such c is unique.

Proof of Theorem 1.3: existence

We will prove existence of traveling waves following a method introduced in [3]. Consider a family of traveling wave problems on a (large) finite interval $(-a, a)$:

$$-cU_a' = U_a'' + \tau f(U_a), \tag{1.15}$$

supplemented by the boundary conditions

$$U_a(-a) = 1, \quad U_a(a) = 0. \tag{1.16}$$

Here $\tau \in [0,1]$ is a parameter such that at $\tau = 1$ we have the problem we are interested in, and at $\tau = 0$ we have a simple problem we can solve. The basic procedure would be to construct solutions of (1.15)-(1.16) for $\tau = 1$ on the finite interval $(-a, a)$ and then pass to the limit $a \to +\infty$ hoping that we may choose a subsequence $a_n \to +\infty$ so that the limit

$$U(x) = \lim_{n \to +\infty} U_{a_n}(x) \tag{1.17}$$

exists and satisfies the traveling wave problem

$$\begin{aligned} &-cU' = U'' + f(U), \\ &U(-\infty) = 1, \quad U(+\infty) = 0. \end{aligned} \tag{1.18}$$

There are several difficulties here: first, we need to construct a solution to (1.18), next we should show that the limit (1.17), indeed, exists, and finally, that it satisfies the boundary conditions in (1.18). Even assuming for the moment that (1.18) has a solution, if we were to try solving (1.15) with some arbitrary $c \in \mathbb{R}$ we would likely obtain a solution U_a that would either stay very close to 1 all the way between $x = -a$ and some x_1 close to a, and then transition to 0 close to $x = a$, or would stay very close to 0 between $x = a$ and some x_2 close to $x = -a$ and then transition to 1 close to $x = -a$. In both cases, if we were to pass to the limit $a \to +\infty$ we would violate the boundary conditions: in the former case we would obtain the limit $U \equiv 1$, and in the latter $U \equiv 0$. To avoid either of these scenarios, we must find solutions of (1.15) that have the transition from "U close to 1" to "U close to 0" in a region on the x-axis that is, roughly, fixed, independent of a. In order to accomplish this, we allow the speed c in (1.15) to depend on a also, and add a normalization condition

$$\max_{x \geq 0} U_a(x) = \theta_0. \tag{1.19}$$

Now the unknowns of the problem are both the function U_a and the speed c_a. It follows from (1.19) that $f(U_a) \equiv 0$ for $x \geq 0$, whence an application of the maximum principle implies that (1.19) can be rephrased as

$$U_a(0) = \theta_0, \quad U_a(x) < \theta_0 \text{ for all } x > 0. \tag{1.20}$$

As the speed c_a now varies with a we need to ensure that, in addition to (1.17), the limit

$$c^* = \lim_{an \to +\infty} c_{a_n}$$

exists, along some subsequence $a_n \to +\infty$. This is yet another technical difficulty.

A priori bounds on the speed

In order to establish existence of solutions to the system (1.15)-(1.19) we need to prove a priori bounds on the solution (assuming first that it exists). We begin with bounds on the speed c_a that will, in particular, be necessary to be able to pass to the limit $a \to +\infty$.

Lemma 1.4 *Let (c_a, U_a) be a solution of*

$$\begin{aligned} &-c_a U_a' = U_a'' + \tau f(U_a), \quad -a < x < a, \qquad (1.21)\\ &U_a(-a) = 1, \quad U_a(a) = 0,\\ &\max_{x \geq 0} U_a(x) = \theta_0. \end{aligned}$$

(1.22)

There exist a constant C and $a_0 > 0$ so that $|c_a| \leq C$ for all $a > a_0$ and all $\tau \in [0, 1]$.

Proof. The proof uses the sliding method. First, choose $m > 0$ so that $f(u) \leq mu$ for all $u > 0$. Next, consider a function

$$v(x) = Ae^{-x},$$

with $A > 0$. We would like to ensure that $v(x)$ is a super-solution to (1.21):

$$-c_a v_x \geq v_{xx} + \tau f(v), \tag{1.23}$$

which would, indeed, hold, if

$$-c_a v_x \geq v_{xx} + mv. \tag{1.24}$$

The latter inequality holds provided that

$$c_a \geq 1 + m. \tag{1.25}$$

Assume that (1.25) holds and set $\bar{A} = 2e^a$. Then for all x we have

$$U_a(x) \leq 1 < \bar{A}e^{-a} \leq v(x; \bar{A}).$$

Let us now decrease A (this is the idea of the sliding method) until there exists a point x_0 so that $v(x_0) = U_a(x_0)$. More precisely, we set

$$A_0 = \sup\{A \leq \bar{A} : \quad \text{there exists } x \in [-a, a] \text{ such that } v(x; A) = U_a(x)\}.$$

Since both $v(x)$ and $U_a(x)$ are continuous, there exists x_0 such that $v(x_0; A_0) = U_a(x_0)$ and $v(x; A_0) \geq U_a(x)$ for all $x \in [-a, a]$. We claim that $x_0 = -a$. Assume this is false. It is impossible that $x_0 = a$ since $v(a) > 0$ and $U_a(a) = 0$, thus x_0 is an interior point of $(-a, a)$ where the difference $z(x) = v(x) - U_a(x)$ attains a local minimum $z(x_0) = 0$. However, $z(x)$ satisfies

$$-c_a z_x - z_{xx} \geq mv(x) - \tau f(U_a) \geq mU_a - f(U_a) \geq 0 \quad \text{for all } x \in [-a, a].$$

Therefore, $z(x)$ can not attain an interior minimum on $[-a, a]$. This is a contradiction that shows that $x_0 = -a$. In that case, however, we have

$$A_0 e^a = 1,$$

hence $A_0 = e^{-a}$. As a consequence, $U_a(0) \leq v(0) = e^{-a}$. This, however, contradicts the normalization condition $U_a(0) = \theta_0$, provided that $a > \ln(1/\theta_0)$. Therefore, c_a satisfies

$$c_a < 1 + m, \tag{1.26}$$

for all $\tau \in [0, 1]$. Next, we prove a lower bound for c_a. the idea is very similar except that now we consider a sub-solution of the form

$$w(x) = 1 - Ae^x.$$

In order for it to be a sub-solution it suffices to require that

$$-c_a w_x - w_{xx} \leq 0, \tag{1.27}$$

which is equivalent to $c_a < -1$. Let us assume that $c_a < -1$ and, once again, choose $\bar{A} = 2^a$. Then we have

$$w(x, \bar{A}) \leq w(-a, \bar{A}) < 0 < U_a(x), \quad \text{for all } x \in [-a, a].$$

We start decreasing A until we get to the first A_1 so that $w(x; A_1) = U_a(x)$ at some $x \in [-a, a]$:

$$A_1 = \sup\{A < \bar{A} : \quad \text{there exists } x \in [-a, a] \text{ such that } w(x; A) = U_a(x)\}.$$

We proceed now as in the proof of the upper bound for c_a: there exists x_1 such that $w(x_1; A_1) = U_a(x_1)$ and $w(x; A_0) \geq U_a(x)$ for all $x \in [-a, a]$. Now, we claim that $x_0 = a$. If this is false, it is impossible that $x_0 = -a$ since $w(-a) < 1$ and $U_a(-a) = 1$. Hence, x_1 must be an interior point of $(-a, a)$ where the difference $p(x) = U_a(x) - w(x)$ attains a local minimum $p(x_0) = 0$. However, $p(x)$ satisfies

$$-c_a p_x - p_{xx} \geq \tau f(U_a) \geq 0 \quad \text{for all } x \in [-a, a].$$

Therefore, $p(x)$ can not attain an interior minimum on $[-a,a]$. This is a contradiction that shows that $x_0 = -a$. In that case, however, we have

$$A_0 e^a = 1,$$

hence $A_0 = e^{-a}$. As a consequence, $U_a(0) \geq v(0) = 1 - e^{-a}$. This, however, contradicts the normalization condition $U_a(0) = \theta_0$, provided that $a > \ln(1/\theta_0)$. This contradiction shows that $c_a > -1$, whence we have shown that

$$-1 < c_a < 1 + m, \tag{1.28}$$

and Lemma 1.4 is proved. □

A priori bounds on the solution

Next, we prove a priori bounds for U_a.

Lemma 1.5 *Let (c_a, U_a) be a solution of*

$$\begin{aligned} &-c_a U_a' = U_a'' + \tau f(U_a), \quad -a < x < a, \\ &U_a(-a) = 1, \quad U_a(a) = 0, \\ &\max_{x \geq 0} U_a(x) = \theta_0. \end{aligned} \tag{1.29}$$

Then we have

$$0 < U_a(x) < 1 \textit{ for all } x \in (-a,a). \tag{1.30}$$

In addition, there exists a constant $C > 0$ that depends only on the function f but nothing else, and $a_0 > 0$ so that we have the following bound:

$$\tau \int_{-a}^{a} f(U_a(x))dx + \int_{-a}^{a} |U_{a,x}(x)|^2 dx \leq C, \textit{ for all } a > a_0 \textit{ and all } \tau \in [0,1]. \tag{1.31}$$

Informally, the bound on the integral of $f(U_a)$ says that U_a can not stay strictly inside the interval $(\theta_0, 1)$ for too long as that would violate the bound. This indicates that U_a will have to approach 1 not just as $x \to -a$ but rather in a finite region. The L^2-bound on $U_{a,x}$, roughly, shows that U_a can not oscillate to much. These two bounds will be very useful in identifying the limit $a \to +\infty$.
Proof. The first statement, (1.30) follows immediately from the maximum principle and the fact that $f(u) = 0$ outside of the interval $u \in (0,1)$. Hence, we only need to prove (1.31). Let us integrate (1.29):

$$c_a = U_a'(a) - U_a'(-a) + \tau \int_{-a}^{a} f(U_a(x))dx. \tag{1.32}$$

The uniform bounds on c_a that we have obtained in Lemma 1.4 and the parabolic regularity up to the boundary imply that, as $0 \leq U_a \leq 1$ there exists a constant $K > 0$ such that

$$|U_a'(x)| \leq K \text{ for all } x \in (-a,a). \tag{1.33}$$

It follows now from (1.32) that

$$\tau \int_{-a}^{a} f(U_a(x))dx \leq 1 + m + 2K. \tag{1.34}$$

Similarly, if we multiply (1.29) by U_a and integrate, we obtain

$$\frac{c_a}{2} + \int_{-a}^{a} |U_{a,x}(x)|^2 = -U_a'(-a) + \tau \int_{-a}^{a} f(U_a(x))U_a(x)dx. \tag{1.35}$$

Since we have bounded already all terms in (1.32) except for the L^2-norm of $U_{a,x}$, and $0 \le U_a \le 1$, the bound on this term also follows:

$$\int_{-a}^{a} |U_{a,x}|^2 dx \le \frac{1}{2} + K + 1 + m + 2K.$$

The proof of Lemma 1.5 is now complete. □

Monotonicity of U_a

In order to see that $U_a(x)$ is monotonic in x, let us recall that $f(u) \ge 0$ for $u \in (0,1)$. Therefore, the function $U_a(x)$ can not attain a local minimum at any point inside the interval $(-a,a)$. As it attains its maximum at $x = -a$, in order for U_a to attain a local maximum at a point $x \in (-a,a)$, it should have also to attain a local minimum on the interval $(-a,x)$, which is impossible. Hence, it can attain neither a local minimum nor a local maximum on $(-a,a)$, and it follows that $U_a(x)$ is monotonically decreasing.

Existence of a solution

With the a priori bounds we can now prove existence of the solution to the system (1.15)-(1.19). Consider a family of maps $\mathcal{K}_\tau$, of the space $\mathbb{R} \times C^{1,\alpha}(\mathbb{R})$ onto itself defined by

$$\mathcal{K}_\tau(c,v) = (r_\tau, Z_\tau).$$

Here the function $Z_\tau(x)$ is the solution of the linear boundary value problem

$$-cZ_\tau' = Z_\tau'' + \tau f(v), \quad -a < x < a,$$

with the boundary condition

$$Z_\tau(-a) = 1, \quad Z_\tau(a) = 0,$$

and the number

$$r_\tau = \theta_0 - \max_{x \ge 0} Z(x) + c.$$

Note that the $(c,U) = \mathcal{K}_\tau(c,U)$ for some $\tau \in [0,1]$ if and only if U satisfies the system (1.15)-(1.19).

The standard elliptic regularity estimates imply that the family $\mathcal{K}_\tau$ is continuous in τ and each $\mathcal{K}_\tau$ is a compact map of the Banach space $X = \mathbb{R} \times C^{1,\alpha}(\mathbb{R})$ onto itself that leaves the convex set $S_0 = \{0 \le v(x) \le 1\}$ invariant. Lemmas 1.4 and 1.5 imply any fixed point of $\mathcal{K}_\tau$ that lies in S_0 satisfies an a priori estimate $\|(c,v)\|_X \le R$ with a constant R that is independent of $\tau \in [0,1]$. Moreover, the range of $\mathcal{K}_0$ consists of one point in S_0, that is, $\mathcal{K}_0$ is a constant map: it is the solution of

$$-cZ_0' - Z_0'' = 0,$$
$$Z(-a) = 1, \quad Z(0) = \theta_0, \quad Z(a) = 0,$$

and is given explicitly by

$$Z_0^c(x) = \frac{e^{-cx} - e^{-ca}}{e^{ca} - e^{-ca}}.$$

The speed c here is the unique solution of

$$1-e^{-ca}=\theta_0(e^{ca}-e^{-ca}),$$

or

$$e^{ca}=\frac{1}{\theta_0}-1.$$

Now, the Leray-Schauder fixed point theorem implies that $\mathcal{K}_1$ has a fixed point, and such fixed point (c_a,U_a) is a solution of the system (1.15)-(1.19) by the definition of $\mathcal{K}_\tau$.

Passing to the limit $a\to+\infty$

The a priori bounds on c_a and U_a in Lemmas 1.4 and 1.5 allow us to find a sequence $a_n\to+\infty$ such that the limits

$$c^*=\lim_{n\to+\infty}c_{a_n}$$

and

$$U(x)=\lim_{n\to+\infty}U_{a_n}(x) \tag{1.36}$$

exist. Convergence in (1.36) is locally uniform, and, due to parabolic regularity estimates, we also have convergence of $U_{a_n}(x)$ to $U(x)$ in $C^{2,\alpha}_{loc}(\mathbb{R})$. Therefore, the limit satisfies

$$-c^*U_x=U_{xx}+f(U), \tag{1.37}$$

and, in addition, the normalization

$$U(0)=\theta_0,\quad U(x)\le\theta_0 \text{ for all } x>0, \tag{1.38}$$

as well as the a priori bounds

$$0\le U(x)\le 1,$$

and

$$\int_{-\infty}^{\infty}f(U(x))dx\le C,\quad \int_{-\infty}^{\infty}|U_x|^2dx\le C. \tag{1.39}$$

It remains to show that $U(x)$ satisfies the correct boundary conditions at infinity: a particularly obvious danger to have in mind is that it could conceivably happen that $U(x)\equiv\theta_0$. As $U_a(x)$ are monotonically decreasing, so is $U(x)$. Therefore, the limits

$$u_l=\lim_{x\to-\infty}U(x),\quad u_r=\lim_{x\to+\infty}U(x),$$

exist, and $u_l\ge\theta_0\ge u_r$. The first integral bound in (1.42) implies that $f(u_l)=f(u_r)=0$, hence $u_l=1$ or $u_l=\theta_0$, and $u_r\in[0,\theta_0]$. The key to the boundary conditions is positivity of the speed.

Lemma 1.6 *We have $c^*>0$.*

Proof. Let us go back to (1.32):

$$c_a=U_a'(a)-U_a'(-a)+\int_{-a}^{a}f(U_a(x))dx\ge U_a'(a)+\int_{-a}^{a}f(U_a(x))dx. \tag{1.40}$$

We used monotonicity of U_a in the last step above. The last term in the right side above is bounded from below; there exists a constant $C>0$ so that

$$\int_{-a}^{a}f(U_a(x))dx\ge C. \tag{1.41}$$

This can be seen as follows: let x_1 and x_2 be two points such that $x_1 < x_2$, and

$$U_a(x_1) = \frac{3}{4} + \frac{\theta_0}{4}, \quad U_a(x_2) = \frac{1}{4} + \frac{3}{4}\theta_0, \text{ and } U_a(x_2) \le U_a(x) \le U_a(x_1) \text{ for all } x \in (x_1, x_2).$$

Then $f(U_a(x)) \ge c_0$ for all $x \in (x_1, x_2)$, with some constant $c_0 > 0$. We also have, using Lemma 1.5:

$$\frac{\theta_0}{2} = \int_{x_1}^{x_2} U_{a,x}(x)dx \le |x_2 - x_1|^{1/2} \left(\int_{-a}^{a} |U_{a,x}|^2 dx \right)^{1/2} \le C|x_2 - x_1|^{1/2}.$$

Therefore, $|x_1 - x_2| \ge c_0'$ for some constant c_0'. As the function $f(U_a(x))$ is uniformly positive on this interval, (1.41) follows.

Next, we show that there exists a sequence $a_n \to +\infty$ such that

$$U_{a_n,x}(a_n) \to 0 \text{ as } n \to +\infty. \tag{1.42}$$

This is accomplished by a different family of shifts: consider

$$\Phi_a(x) = U_a(a + x), \quad x \in (-2a, 0).$$

Then for the same reason as for U_{a_n} we can choose a subsequence $a_n \to +\infty$ such that the limit

$$\Phi(x) = \lim_{n\to+\infty} \Phi_{a_n}(x)$$

exists for all $x < 0$. As $f(U_a) = 0$ for all $x \in (0, a)$, the function Φ satisfies the linear problem

$$-c^*\Phi_x = \Phi_{xx}, \quad \Phi(0) = 0.$$

As, in addition, we know that $0 \le \Phi(x) \le 1$, it follows that $\Phi(x) \equiv 0$, which, in turn, implies (1.42). Returning to (1.40) we may now conclude that $c^* > 0$. □

We will now deduce from Lemma 1.6 that $u_r = 0$. The functions $U_a(x)$ are explicit on the interval $(0, a)$:

$$U_a(x) = \theta_0 e^{-c_a x}, \quad \text{for } x > 0.$$

Lemma 1.6 implies that $U_a(x) \le \theta_0 e^{-Cx}$ for all $x > 0$, with a fixed constant $C > 0$. Therefore, the limit $U(x)$ obeys the same upper bound, hence $u_r = 0$. It remains only to rule out the possibility that $u_l = \theta_0$. We have, however,

$$c^* u_l = \int_{-\infty}^{\infty} f(U(x))dx,$$

hence $f(U(x)) \not\equiv 0$. On the other hand, if $u_l = \theta_0$ then we would have $0 < U(x) < \theta_0$ which would imply $f(U(x)) \equiv 0$. Hence, $u_l = \theta_0$ is impossible, and thus $u_l = 1$. This finishes the existence part of Theorem 1.3.

Uniqueness of traveling waves

Uniqueness of the traveling front speed c^* follows from Theorem 1.7 below (this result is independent of the uniqueness of the speed, as will be seen from the proof of Theorem 1.7!). On the other hand, if c^* is unique, uniqueness of the traveling wave profile is simple. We shift the wave so that the normalization

$$\max_{x\ge0} U(x) = \theta_0$$

holds. Then the wave has an explicit expression $U(x) = \theta_0 e^{-c^* x}$ for $x > 0$. The uniqueness of the profile for $x < 0$ follows immediately since both $U(0)$ and $U'(0)$ are now prescribed.

1.3 The front-like initial data

We look now at the behavior of solutions of the Cauchy problem

$$T_t = T_{xx} + fT), \tag{1.43}$$

with general initial data $T_0(x) = T(0,x)$ such that $T_0(x) = 1$ for $x \le x_0$, $T_0(x) = 0$ for $x \ge x_1$ and $0 \le T_0(x) \le 1$. The main result is that such initial data propagates with the speed $c_* = 1$ of the slowest traveling front in the KPP case and with the speed of the unique traveling wave in the ignition case. More precisely, we have the following.

Theorem 1.7 *Let $T(t,x)$ be solution of (1.43) with the initial data $T_0(x)$ as above. Then given any $x \in \mathbb{R}$ we have*

$$\lim_{t\to\infty} T(t, x+ct) = \begin{cases} 0, & \text{if } c > c_*, \\ 1, & \text{if } c < c_*. \end{cases} \tag{1.44}$$

Here $c_* = 1$ is the minimal speed in the KPP case and the unique traveling front speed in the ignition case.

This means qualitatively that T moves with the speed c_*. More precise statements on the convergence to a traveling front in the ignition case may be obtained by spectral methods, or in the KPP case as in the original paper [5], but we will not go into details – see [7] for detailed references.

We will prove (1.44) only in the ignition case. The idea is to use the traveling wave solution $U(x-ct)$ to construct a super-solution and a sub-solution. Let $U(x)$ be the traveling wave, solution of

$$-c_*U' = U'' + f(U), \quad U(0) = \theta_0.$$

We look for a sub-solution for T of the form of a "slightly corrected shift of a traveling wave"

$$\psi_l(t,x) = U(x - c_*t + x_1 + \xi_1(t)) - q_1(t,x).$$

The functions $\xi_1(t)$ and $q_1(t,x,z)$ are to be chosen so as to make ψ_l be a sub-solution. The shift x_1 and the "downward shift" $q_1(0,x)$ will be then used to make sure that initially we have $\psi_l(0,x) \le T_0(x)$. In order for ψ_l to be a sub-solution we need

$$G[\psi_l] = \frac{\partial \psi_l}{\partial t} - \frac{\partial^2 \psi_l}{\partial x^2} - f(\psi_l) \le 0.$$

We have

$$G[\psi_l] = \dot{\xi}_1 U' - \frac{\partial q_1}{\partial t} + \frac{\partial^2 q_1}{\partial x^2} + f(U) - f(U - q_1).$$

With an appropriate choice of x_1, that is, by shifting U sufficiently to the left we may ensure that $T_0(x) \ge U(x) - q_{10}(x)$ with $0 \le q_{10}(x) \le (1-\theta_0)/2$ and $q_{10}(x) \in L^1(\mathbb{R})$. This is because the traveling wave profile $U(x)$ approaches its limits as $x \to \pm\infty$ exponentially fast.

First, we choose $q_1(t,x)$ to be the solution of

$$\frac{\partial q_1}{\partial t} = \frac{\partial^2 q_1}{\partial x^2}, \quad q_1(0,x) = q_{10}(x) \tag{1.45}$$

so that we have

$$\|q_1(t)\|_\infty \le \frac{C}{\sqrt{t}} \|q_{10}\|_{L^1(\mathbb{R})} \tag{1.46}$$

for $t \ge 1$. This makes

$$G[\psi_l] = \dot{\xi}_1 U' + f(U) - f(U - q_1).$$

We may find $\delta > 0$ so that if $U \in (1-\delta, 1)$ and $q_1 \in (0, (1-\theta_0)/2)$ then $f(U) \leq f(U - q_1)$. Hence we have in this range of U:

$$G[\psi_l] \leq \dot{\xi}_1 U' \leq 0 \tag{1.47}$$

provided that $\dot{\xi}_1 \geq 0$. Furthermore, if δ is sufficiently small and $U \in (0, \delta)$ then $f(U) = f(U - \delta) = 0$ and hence in this range of U we have (1.47) with the equality sign on the left. Finally, if $U \in (\delta, 1-\delta)$ then $|f(U) - f(U - q_1)| \leq K|q_1|$ and $U' \leq -\beta$ with positive constants K and β that depend on $\delta > 0$. Therefore, in this region we have

$$G[\psi_l] = \dot{\xi}_1 U' + f(U) - f(U - q_1) \leq -\beta\dot{\xi}_1 + K\|q_1(t)\|_\infty.$$

Hence, we have $G[\psi_l] \leq 0$ everywhere provided that

$$\dot{\xi}_1(t) \geq \frac{K\|q_1(t)\|_\infty}{\beta}. \tag{1.48}$$

Thus we may choose

$$\xi_1(t) = C\sqrt{t}. \tag{1.49}$$

Therefore we obtain a lower bound for T:

$$T(t, x) \geq U(x - c_* t + C\sqrt{t}) - q_1(t, x, z). \tag{1.50}$$

In order to obtain an upper bound we set $\psi_u = U(x - c_* t - x_2 - \xi_2(t)) + q_2(t, x)$ and look for $\xi_2(t)$ and $q_2(t, x)$ so that $G[\psi_u] \geq 0$. The constant x_2 is chosen so that

$$T_0(x) \leq U(x - x_2) + q_2(0, x)$$

with $q_2(0, x) \in L^1(\mathbb{R})$ and $0 \leq q_2(0, x) \leq \theta_0/2$, as with $q_1(0, x)$. The function $q_2(t, x)$ is then chosen to satisfy the same heat equation (1.45) as q_1. Hence it obeys the same time decay bounds as q_1. With the above choice of q_2 we have

$$G(\psi_u) = -\dot{\xi}_2 U' + f(U) - f(U + q_2).$$

Once again, we consider three regions of values for U. First, if $1 - \delta \leq U \leq 1$ with a sufficiently small $\delta > 0$ then $f(U) - f(U + q_2) \geq 0$, as $q_2 \geq 0$. Hence $G[\psi_u] \geq 0$ in this region provided that $\dot{\xi}_2 \geq 0$. Second, as $q_2 \leq \theta_0/2$ we have $f(U) = f(U + q_2) = 0$ if $0 \leq U \leq \delta$ with a sufficiently small $\delta > 0$. Hence $G[\psi_u] \geq 0$ in that region under the same condition $\dot{\xi}_2 \geq 0$. Finally, if $U \in (\delta, 1-\delta)$ then $U' \leq -\beta$ with $\beta > 0$ and $|f(U) - f(U + q_2)| \leq K\|q_2\|_\infty$. That means that $G[\psi_u] \geq 0$ if we choose ξ_2 so that

$$\dot{\xi}_2 \geq \frac{K\|q_2\|_\infty}{\beta}.$$

Therefore we may choose

$$\xi_2(t) = C\sqrt{t},$$

as with $\xi_1(t)$. Thus we obtain upper and lower bounds

$$U(x - c_* t + \xi_1(t) + x_1) - q_1(t, x) \leq T(t, x) \leq U(x - c_* t - \xi_2(t) - x_2) + q_2(t, x) \tag{1.51}$$

that imply in particular that

$$U(x - c_* t + C_0[1 + \sqrt{t}]) - \frac{C_0}{\sqrt{t}} \leq T(t, x) \leq U(x - c_* t - C_0[1 + \sqrt{t}]) + \frac{C_0}{\sqrt{t}} \tag{1.52}$$

with a constant C_0 determined by the initial conditions. Now, if we take $x = x_0 + ct$ with $c < c_*$ and use the lower bound in (1.52) we get $T(t, x_0 + ct) \to 1$ as $t \to \infty$. On the other hand, if we take $x = x_0 + ct$ with $c > c_*$ and use the upper bound we obtain $T(t, x_0 + ct) \to 0$ as $t \to \infty$.

One may obtain the KPP result from the ignition case by cutting of a KPP-type $f(T)$ at a point $\theta_0 > 0$ and then letting $\theta_0 \to 0$. We omit the details. □

We confess that a better effort using the spectral methods and functional analysis shows that in the ignition case solution actually converges to a traveling wave exponentially fast in time. However, this requires a different technique that we do not go into here.

1.4 Generalizations of the notion of a traveling front

There are many generalizations of the notion of a traveling front. For example, one can define non-planar fronts in a shear flow as follows. Consider the problem

$$T_t + u(y)T_x = T_{xx} + f(T). \tag{1.53}$$

This problem is posed in a cylinder $x \in \mathbb{R}$, $y \in \Omega$, where Ω is a bounded set, with the Neumann boundary condition at $\partial\Omega$:

$$\frac{\partial T}{\partial n} = 0 \text{ on } \mathbb{R} \times \partial\Omega. \tag{1.54}$$

Of course, this problem can not admit solutions of the form $U(x-ct)$ simply because the coefficients depend on the transverse variable y. We can, however, define a non-planar traveling front as a solution of (1.53)-(1.54) of the form $T(t, x, y) = U(x - ct, y)$, with the function $U(x, y)$ that has the limits at infinity:

$$\lim_{x\to-\infty} U(x, y) = 1, \quad \lim_{x\to+\infty} U(x, y) = 0,$$

uniformly in y. Their existence can be proved nearly exactly as in the homogeneous case: see [4] for details. The result is also very similar to the homogeneous case: in the KPP case non-planar fronts exist for all speeds $c \geq c_0$ with some c_0 that now appears from an eigenvalue problem. On the other hand, non-planar front is unique in the case of an ignition nonlinearity.

Another generalization is possible for the periodic media: consider a reaction-diffusion equation of the form

$$T_t + u(x) \cdot \nabla T = \nabla \cdot (a(x)\nabla T) + f(x, T), \quad x \in \mathbb{R}^d, \tag{1.55}$$

with the functions $u(x)$, $a(x)$ and $f(x, T)$ periodic in x:

$$u(x + L_j e_j) = u(x), \quad a(x + L_j e_j) = a(x), \quad f(x + L_j e_j, u) = f(x, u).$$

Here L_j is the period in the direction e_j, $j = 1, \ldots, d$. Then we can define a pulsating traveling front in a direction e_j as a solution of (1.55) of the form $T(t, x) = U(x_j - ct, x)$ with a function $U(\xi, x)$, $\xi \in \mathbb{R}$, $x \in \mathbb{R}^d$ that is periodic in the second variable, and satisfies

$$\lim_{\xi\to-\infty} U(\xi, x) = 1, \quad \lim_{\xi\to+\infty} U(\xi, x) = 0,$$

uniformly in x. A basic reference for the existence of the pulsating traveling fronts is [1] where other references can be found.

2 Spreading in advection-diffusion problems

2.1 The method of characteristics for the first order equations

Let us first recall the method of characteristics for a linear first order equation

$$\frac{\partial \phi}{\partial t} + u(x) \cdot \nabla \phi = 0, \quad x \in \mathbb{R}^d, \tag{2.1}$$

where $u(x)$ is a prescribed vector field. We also prescribe the initial data $\phi(0, x) = \phi_0(x)$. Define the characteristic curves as the trajectories of

$$\frac{dX}{dt} = u(X(t)), \quad X(0) = x, \tag{2.2}$$

and consider the function $U(t) = \phi(t, X(t))$. Let us differentiate $U(t)$ with respect to t:

$$\frac{dU}{dt} = \frac{\partial \phi(t, X(t)}{\partial t} + \nabla \phi(t, X(t)) \cdot \frac{dX}{dt} = 0. \tag{2.3}$$

Therefore, we have $U(t) = U(0)$ for all t, hence $\phi(t, X(t)) = \phi_0(x)$. Alternatively, we can express this as $\phi(t, x) = \phi_0(Y(t; x))$, where $Y(t; x)$ is the point such that if we start the trajectory of (2.2) at the point $X(0) = Y(t; x)$ then $X(t) = x$.

2.2 Random walks and the method of characteristics for elliptic and parabolic equations

2.2.1 The Laplace equation

How can we generalize the method of characteristics to second order problems?

A simple way to understand this is in terms of discrete equations. Consider the finite difference analog of the Laplace equation: let U be a domain of the two-dimensional square lattice $\mathbb{Z}^2$, and let $u(x)$ solve the difference equation

$$u(x + 1, y) + u(x - 1, y) + u(x, y + 1) + u(x, y - 1) - 4u(x, y) = 0, \tag{2.4}$$

with the boundary condition $u(x, y) = g(x, y)$ on the boundary ∂U. Here $g(x, y)$ is a prescribed non-negative function, which is positive somewhere. We claim that the solution of this problem has the following probabilistic interpretation. Let $(X(t), Y(t))$ be the standard random walk on the lattice $\mathbb{Z}^2$ – a particle starts at a position $(X(0), Y(0)) = (x, y)$ and the probability for it to go up down, left of right is equal to $1/4$. At a new site it decides again to go to one of the neighboring sites with equal probabilities. Let $(\bar{x}, \bar{y})$ be the first point where $(X(t), Y(t))$ reaches the boundary ∂U of the domain. The point $(\bar{x}, \bar{y})$ is, of course, random. The beautiful observation is that the function $v(x, y) = \mathbb{E}(g(\bar{x}, \bar{y})$ gives a solution of (2.4), connecting this discrete problem to the random walk. Why? First, it is immediate that if (x, y) is on the boundary then, of course, $\bar{x} = x$ and $\bar{y} = y$ with probability one, so $v(x, y) = g(x, y)$ in that case. On the other hand, if (x, y) is inside U then

$$v(x, y) = \frac{1}{4}(v(x + 1, y) + v(x - 1, y) + v(x, y + 1) + v(x, y - 1))$$

simply from the definition of the random walk, the definition of $v(x, y)$ and the Markov property of the random walk. The latter means that the jump at each site is independent of the past.

Now, if we let the mesh size be not 1 but $h > 0$ and let $h \downarrow 0$, the discrete equation (2.4) becomes the Laplace equation, while the random walk becomes the Brownian motion. More precisely, for a bounded domain $\Omega \subset \mathbb{R}^d$, solution of a boundary value problem

$$\Delta u = 0 \text{ in } \Omega, \quad u = g \text{ on } \partial\Omega, \tag{2.5}$$

has the following probabilistic interpretation: let U be a domain in $\mathbb{R}^n$ and let $g(x)$ be a continuous function on the boundary ∂U. Consider a Brownian motion $B(t;x)$ that starts at a point $x \in U$ and let $\bar{x}$ be a (random) point where $B(t;x)$ hits the boundary ∂U for the first time. Then solution of (2.5) is $u(x) = \mathbb{E}(g(\bar{x}))$. Now, if $g(x)$ is positive at some point $x_0 \in \partial U$ (and thus in a neighborhood of x_0 as well) then with a positive probability we have $g(\bar{x}) > 0$, which means that $u(x) = \mathbb{E}(g(\bar{x})) > 0$ as well.

This gives a simple "physical" explanation of the maximum principle: it is easy to see that $\mathbb{E}(g(\bar{x})) \leq \sup_{z \in \partial U} g(z)$ – expected value of a function can not exceed its maximum.

2.2.2 The heat equation

We will now consider the heat (or diffusion) equation

$$\frac{\partial u}{\partial t} - \Delta u = 0. \tag{2.6}$$

Usually it is obtained from a balance of heat or concentration that assumes that the flux of heat is $F = -\nabla u$, where u is the temperature – heat flows from hot to cold. Here, we derive it informally starting with a probabilistic model.

Consider a lattice on the real line of mesh size h: $x_n = nh$. Let $X(t)$ be a random walk on this lattice that starts at some point x, and after a delay τ jumps to the left or right with probability $1/2$: $P(X(\tau) = x + h) = P(X(\tau) = x - h) = 1/2$. Then it waits again for time τ, and again jumps to the left or right with probability $1/2$, and so on. Let S be a subset of the real line and define $u(t,x) = P(X(t) \in S|X(0) = x)$ – this is the probability that at a time $t > 0$ the particle is inside the set S given that it started at the point x at time $t = 0$.

Let us derive an equation for $u(t,x)$. Since the process "starts anew" after every jump we have the relation

$$P(X(t) \in S|X(0) = x) = \frac{1}{2}(X(t-\tau) \in S|X(0) = x + h) + \frac{1}{2}(X(t-\tau) \in S|X(0) = x - h),$$

which is

$$u(t,x) = \frac{1}{2}u(t-\tau, x+h) + \frac{1}{2}u(t-\tau, x-h). \tag{2.7}$$

Let use assume that τ and h are small and use Taylor's formula in the right side above. Then (2.7) becomes:

$$\begin{aligned} u(t,x) = \frac{1}{2}\left[u(t,x) - \tau\frac{\partial u(t,x)}{\partial t} + h\frac{\partial u(t,x)}{\partial x} + \frac{h^2}{2}\frac{\partial^2 u}{\partial x^2} + \frac{\tau^2}{2}\frac{\partial^2 u(t,x)}{\partial t^2} - \tau h\frac{\partial^2 u(t,x)}{\partial x \partial t}\right] \\ + \frac{1}{2}\left[u(t,x) - \tau\frac{\partial u(t,x)}{\partial t} - h\frac{\partial u(t,x)}{\partial x} + \frac{h^2}{2}\frac{\partial^2 u}{\partial x^2} + \frac{\tau^2}{2}\frac{\partial^2 u(t,x)}{\partial t^2} + \tau h\frac{\partial^2 u(t,x)}{\partial x \partial t}\right] + \dots, \end{aligned}$$

which is

$$\tau\frac{\partial u}{\partial t} = \frac{h^2}{2}\frac{\partial^2 u}{\partial x^2} + \frac{\tau^2}{2}\frac{\partial^2 u(t,x)}{\partial t^2} + \dots$$

In order to get a non-trivial balance we set $\tau = h^2$. Then the term involving u_{tt} in the right side is smaller than the rest and in the leading order we obtain

$$\frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^2 u}{\partial x^2}, \tag{2.8}$$

which is the diffusion equation (we could get rid of the factor of $1/2$ if we took $\tau = h^2/2$ but probabilists do not like that). It is supplemented by the initial condition

$$u(0,x) = \begin{cases} 1, & \text{if } x \in S \\ 0 & \text{if } x \notin S. \end{cases}$$

More generally, we can take a bounded function $f(x)$ defined on the real line and set

$$v(t,x) = \mathbb{E}\{f(X(t))|\ X(0) = x\}.$$

Essentially an identical argument shows that if $\tau = h^2$ then in the limit $h \to 0$ we get the following Cauchy problem for $v(t,x)$:

$$\begin{aligned} \frac{\partial v}{\partial t} &= \frac{1}{2}\frac{\partial^2 v}{\partial x^2} \\ v(0,x) &= f(x). \end{aligned} \tag{2.9}$$

What should we expect for the solutions of the Cauchy problem given this informal probabilistic representation? First, it should preserve positivity: if $f(x) \geq 0$ for all $x \in \mathbb{R}$, we should have $u(t,x) \geq 0$ for all $t > 0$ and $x \in \mathbb{R}$. Second, the maximum principle should hold: if $f(x) \leq M$ for all $x \in \mathbb{R}$, then we should have $u(t,x) \leq M$ for all $t > 0$ and $x \in \mathbb{R}$ because the expected value of a quantity can not exceed its maximum. We should also expect $\max_{x\in\mathbb{R}} v(t,x)$ to decay in time, at least if $f(x)$ is compactly supported – this is because the random walk will tend to spread around and at large times the probability to find it on the set where $f(x)$ does not vanish, is small.

2.3 Advection-diffusion equations

More generally, solutions of an advection-diffusion problem

$$\phi_t + u(x) \cdot \nabla\phi = \frac{\sigma}{2}\Delta\phi, \quad \phi(0,x) = \phi_0(x), \tag{2.10}$$

can be represented in terms of solutions of stochastic differential equations as follows. Let $X(t)$ be the solution of the stochastic differential equation

$$dX = -u(X)dt + \sqrt{\sigma}dB, \quad X(0) = x. \tag{2.11}$$

Here $B(t)$ is the standard Brownian motion, and (2.11) may be written in the integral form as

$$X(t) = x + \int_0^t u(X(s))ds + B(t). \tag{2.12}$$

Then solution of (2.10) is given by

$$\phi(t,x) = \mathbb{E}(\phi_0(X(t)). \tag{2.13}$$

This is the direct generalization of the random walks expression for the heat equation that we have obtained before, combined with the method of characteristics for the first order equations.

The trajectories of the solutions of (2.12) depend very strongly on the geometric character of the vector field $u(x)$. We would like to translate these properties into the properties of the solutions of PDEs (2.10), and develop analytical techniques for this. Contrary to intuition, probabilistic techniques, while very efficient in many such problems, do not always provide an easier approach than the analytical methods.

3 The spreading rate in a shear flow: the KPP case

The simplest flow (other than a constant drift) that we can analyze in the context of reaction-diffusion equations. A shear flow in $\mathbb{R}^n$ is a unidirectional flow of the form $u = (u(x), 0)$. Here we have decomposed $x = (x, y)$, $x \in \mathbb{R}^{n-1}$, and $u(y)$ is a scalar function. A reaction-diffusion-advection equation in such a flow has the form

$$\frac{\partial T}{\partial t} + u(y)\frac{\partial T}{\partial x} = \Delta T + f(T). \tag{3.1}$$

We will consider this problem in a cylinder $D = \mathbb{R}_x \times \Omega_y$, where Ω is a nice bounded domain. We will assume that

$$\int_\Omega u(y)dy = 0, \tag{3.2}$$

this is simply a normalization condition. The boundary conditions along $\partial\Omega$ are homogeneous Neumann ("adiabatic" in the physical language):

$$\frac{\partial T}{\partial n} = 0 \text{ on } \mathbb{R} \times \partial\Omega. \tag{3.3}$$

The nonlinearity $f(T)$ will be usually taken of KPP type, though many results are applicable to the ignition nonlinearity as well. Hence we will assume that

$$f(T) > 0 \text{ for } 0 < T < 1,\ f(0) = f(1) = 0,\ f(T) \le f'(0)T.$$

We first recall the no-flow case when solutions propagate with the minimal speed $c_* = 2\sqrt{f'(0)}$. A quick formal way to see why this happens is through an observation that the behavior of the KPP solutions is determined by the linearization at small T. More precisely, if we start with

$$\frac{\partial T}{\partial t} = \Delta T + f(T) \tag{3.4}$$

and linearize around $T = 0$ we get

$$\frac{\partial q}{\partial t} = \Delta q + f'(0)q.$$

Let us look for the solutions of this equation of the form

$$q(t, x) = e^{-\lambda(x-ct)} \tag{3.5}$$

with $\lambda > 0$. A direct calculation shows that c and λ are related by

$$\lambda^2 - \lambda c + f'(0) = 0. \tag{3.6}$$

A middle-school algebra leads to the conclusion that such a $\lambda > 0$ exists if and only if $c \ge c_* = 2\sqrt{f'(0)}$. This is exactly the correct range of speeds for the one-dimensional KPP equation.

The above naive idea of linearization actually applies in the presence of a shear and other flows. However, the ansatz (3.5) should be modified (depending on the flow type). In the shear case the correct ansatz is

$$q(t, x) = e^{-\lambda(x-ct)}\psi(y) \tag{3.7}$$

with $\lambda > 0$ and $\psi > 0$. We insert this ansatz into the linearization

$$\frac{\partial q}{\partial t} + u(y)\frac{\partial q}{\partial x} = \Delta q + f'(0)q \tag{3.8}$$

and obtain an eigenvalue problem for ψ in Ω:

$$-\Delta\psi - \lambda u(y)\psi = (\lambda^2 - c\lambda + f'(0))\psi \text{ in } \Omega \qquad (3.9)$$
$$\frac{\partial\psi}{\partial n} = 0 \text{ on } \partial\Omega.$$

As the quadratic equation (3.6) in the homogeneous case, the eigenvalue problem (3.9) is the key point to understanding the propagation phenomena. In order to understand for which c an eigenvalue $\lambda > 0$ exists we consider an auxiliary problem

$$-\Delta\psi - \lambda u(y)\psi = \mu(\lambda)\psi \text{ in } \Omega \qquad (3.10)$$
$$\frac{\partial\psi}{\partial n} = 0 \text{ on } \partial\Omega,$$
$$\|\psi\|_{L^\infty(\Omega)} = 1.$$

Here $\mu(\lambda)$ is the leading eigenvalue that corresponds to a positive eigenfunction ψ. Its existence is guaranteed by the general Sturm-Liouville theory. The direct analog of (3.6) is

$$\lambda^2 - c\lambda + f'(0) = \mu(\lambda). \qquad (3.11)$$

Proposition 3.1 *There exists a unique c_* so that (3.11) has two real solutions $\lambda_{1,2}$ for $c > c_*$, no real solutions for $c < c_*$ and one solution for $c = c_*$.*

Not surprisingly, c_* will turn out to be the minimal speed of a non-planar traveling front solution of (3.1): a non-planar traveling front is a solution of (3.1) of the form $U(x - ct, y)$ with the Neumann boundary conditions at $\mathbb{R}_x \times \partial\Omega_y$ that satisfies

$$U(x, y) \to 1 \text{ as } x \to -\infty,\ U(x, y) \to 0 \text{ as } x \to +\infty, \text{ both uniformly in } \Omega. \qquad (3.12)$$

It turns out that such fronts exist for all $c \geq c_*$ so that c_* is the minimal front speed.

Theorem 3.2 *Non-planar traveling front solutions of (3.1) exist for all $c \geq c_*$ and do not exist for all $c < c_*$.*

We will not present the proof here. Let us just mention that the proof for $c = c_*$ follows very closely the "PDE" proof of existence of the traveling waves for an ignition nonlinearity that we have presented above. Existence of traveling waves for $c > c_*$ uses some appropriately chosen sub- and super-solutions, that guarantee that a non-trivial traveling wave has to exist.

Proof of Proposition 3.1. First, we observe that $\mu(\lambda)$ has a variational characterization:

$$\mu(\lambda) = \inf \frac{\int |\nabla\phi|^2 - \lambda \int u(y)\phi^2(y)}{\int \phi^2}.$$

Thus $\mu(t)$ is concave as a minimum of a family of affine functions of λ. Furthermore, $\mu(0) = 0$ (and $\psi|_{\lambda=0} = \text{const}$), and we may compute formally

$$-\Delta\psi' - \lambda u(y)\psi' - u(y)\psi = \mu'(\lambda)\psi + \mu(\lambda)\psi'. \qquad (3.13)$$

Here prime denotes derivative with respect to λ and ψ is normalized so that $\int_\Omega \psi^2 = 1$. Multiplying (3.13) by ψ and integrating, using the normalization of ψ, we obtain

$$\mu'(\lambda) = -\int u(y)\psi^2(y)dy. \qquad (3.14)$$

In particular $\psi = 1$ at $\lambda = 0$ so that (3.2) implies that $\mu'(0) = 0$. Let us now justify our formal differentiation with respect to λ. Let $\psi_\lambda(x)$ be the principal eigenfunction of (3.10). Note that $\psi_\lambda(x) \to 1$ as $\lambda \to 0$. Integrating (3.10) over Ω we obtain

$$-\lambda \int u(y)\psi_\lambda(y)dy = \mu(\lambda) \int \psi_\lambda(y)dy.$$

Therefore, as $\mu(0) = 0$, we have

$$\frac{\mu(\lambda) - \mu(0)}{\lambda} = -\frac{\int u(y)\psi_\lambda(y)dy}{\int \psi_\lambda(y)dy} \to \int u(y)dy = 0 \text{ as } \lambda \to 0.$$

Now the conclusion of Proposition 3.1 follows from (3.11) and the facts that $\mu(\lambda)$ is concave, $\mu(0) = 0$ and $\mu'(0) = 0$. Indeed,

$$\mu(\lambda) = \lambda^2 - c\lambda + f'(0).$$

Thus

$$c = \frac{\lambda^2 - \mu(\lambda) + f'(0)}{\lambda}.$$

Since $\mu(\lambda)$ is concave we have that $\lambda^2 - \mu(\lambda) + f'(0)$ is convex. Moreover $(\lambda^2 - \mu(\lambda) + f'(0))/\lambda \to +\infty$ as $\lambda \to 0$ and $\lambda \to +\infty$. We conclude that there exists

$$c_* = \min_\lambda \frac{\lambda^2 - \mu(\lambda) + f'(0)}{\lambda},$$

and for any $c > c_*$ we can find t_1 and t_2 such that

$$c = \frac{\lambda_1^2 - \mu(\lambda_1) + f'(0)}{\lambda_1}, c = \frac{\lambda_2^2 - \mu(\lambda_2) + f'(0)}{\lambda_2}.$$

This finishes the proof of Proposition 3.1. □

We now show that the speed c_* defined above defines the asymptotic speed of spreading.

Theorem 3.3 *Let the initial data $T_0(x)$ for (3.1) be continuous and satisfy $T_0(x, y) = 0$ for $|x| > x_1$ and $T_0(x, y) > 0$ on an open set. Then we have $T(t, x+ct, y) \to 0$ as $t \to +\infty$, uniformly on compact sets, provided that $c > c_*$. On the other hand, if $0 \le c < c_*$ then $T(t, x + ct, y) \to 1$ as $t \to +\infty$, uniformly on compact sets.*

Proof. First we consider the case $c' > c_*$ which is much easier. We simply introduce the function

$$q(t, x, y) = Ae^{-\lambda(x-ct)}\psi(y)$$

with λ, c and ψ as in (3.9), and with $c_* < c < c'$. Then q satisfies

$$\frac{\partial q}{\partial t} + u(y)\frac{\partial q}{\partial x} = \Delta q + f'(0)q \ge \Delta q + f(q).$$

Moreover, $q(0, x, y) \ge T_0(x, y)$ provided that the factor A is large enough. Hence the comparison principle implies that

$$T(t, x, y) \le q(t, x, y)$$

and thus $T(t, x + c't, y) \to 0$ as $t \to +\infty$.

The second part of the theorem that deals with the case $0 \le c < c_*$ is more difficult. The argument we present is from a paper by Mallordy and Roquejoffre [6]. The idea is to construct a function described in the following lemma.

Lemma 3.4 *Given $c < c_*$ but sufficiently close to c_*, and a real number $\theta > 0$ there exists a non-negative function $\phi(x,y) \le \theta$ that has compact support, is everywhere less than θ, satisfies the Neumann boundary conditions at $\partial\Omega$, and satisfies*

$$-c\frac{\partial\phi}{\partial x} + u(y)\frac{\partial\phi}{\partial x} = \Delta\phi + (f'(0) - \delta)\phi \le \Delta\phi + f(\phi) \text{ on supp } \phi.$$

Moreover, the number θ is independent of the support of ϕ. The constant $\delta > 0$ above is chosen so that $f(T) \ge (f'(0) - \delta)T$ for $0 \le T \le \theta$.

We refer to [6] for the proof of this lemma (that is quite a bit more technical than one would expect) and finish the proof of Theorem 3.3.

Given a function ϕ as in Lemma 3.4 with $c < c_*$ but close to c_*, we consider the function $\Phi(t,x,y)$ that solves (3.1) with the initial data $\Phi(t,x,y) = \phi(x,y)$. Then, the function $\Psi(t,x,y) = \Phi(t, x+ct, y)$ is monotonic in t and thus has a limit Ψ_∞. It has to satisfy

$$-c\frac{\partial\Psi_\infty}{\partial x} + u(y)\frac{\partial\Psi_\infty}{\partial x} = \Delta\Psi_\infty + f(\Psi_\infty)$$

with the Neumann boundary conditions. One then argues that Ψ_∞ has to converge to a constant as $x \to \pm\infty$, that is, either to 0 or 1. However, as $0 < c < c_*$, the left and right limits have to coincide. Thus $\Psi_\infty = 0$ or 1 everywhere. However, $\Psi_\infty > 0$ where $\phi > 0$ so that $\Psi_\infty = 1$.

To finish the proof of Theorem 3.3 we observe that given the initial data T_0 that is positive somewhere, $T(t=1) > 0$ everywhere. Hence we may choose $\theta > 0$ small and a corresponding $\phi(x,y)$ so that $T(t=1) > \phi$. Then $T(t,x,y) \ge \Phi(t,x,y)$ and thus $T(t,x,y) \to 1$ uniformly on compact set in the reference frame moving with the speed c to the right, that is, $T(t, x+ct, y) \to 1$. Thus we are done for the case when $c < c_*$ is close to c_*. Similarly we show that $T(t, x-ct, y) \to 1$ for $c < -c_*$ but close to $-c_*$.

The above facts imply that $T(t, x+ct, y) \to 1$ for all $-c_*^l < c < c_*$. Indeed, let $-c_*^l < c < c_*$ and choose $X_1(t) = -c_1 t$, $X_2(t) = c_2 t$ with c_1 close to c_*^l and c_2 close to c_*. Choose a time t_0 so that $T(t, X_1(t), z) > 1 - \varepsilon$, $T(t, X_1(t), z) > 1 - \varepsilon$ for all $t \ge t_0$. Define $D(t) = [X_1(t), X_2(t)] \times \Omega$ and let $h_0 = \inf_{D(t_0)} T(t_0, x, y)$. Consider the function $h(t)$ that solves $\dot{h} = f_\varepsilon(h)$ with with $f_\varepsilon(h) = mh(1 - 2\varepsilon - h)$. Here the constant m is chosen so that $f(h) > f_\varepsilon(h)$. The initial data for $h(t)$ is given by $h(t_0) = h_0$ so that $T(t_0, x, y) \ge h_0$ in $D(t_0)$. The funcion $h(t)$ is then a sub-solution for $T(t,x,y)$ in $D(t)$ for $t \ge t_0$, that is, $T(t,x,y) \ge h(t)$ for all $(x,y) \in D(t)$. Thus, $T(t,x,y) \ge 1 - 3\varepsilon$ for a sufficiently large t in all of $D(t)$. □

4 The fast shear flow asymptotics: the KPP case

Let us now consider a chemical reaction occurring in a fast shear flow, that is, we consider

$$\frac{\partial T}{\partial t} + Au(y)\frac{\partial T}{\partial x} = \Delta T + f(T) \tag{4.1}$$

with a large flow amplitude $A \gg 1$. We are interested in the behavior of the spreading rate $c_*(A)$ as $A \to \infty$. The mean-zero condition (3.2) is still kept so that any increase in $c_*(A)$ would not be due to the simple translation of the front by a large mean flow. Rather, the speed up should be due to front wrinkling: as A increase the front gets more and more curved (wrinkled) that results in the increase of the area of interaction of the hot and cold material. As a result the cold material is converted into hot at a faster rate and the spreading rate increases. The purpose of the next proposition is to quantify this phenomenon.

Theorem 4.1 *Assume $u(y) \not\equiv 0$ satisfies (3.2). Then the spreading rate $c_*(A)$ satisfies the following properties: (i) $c_*(A)$ is increasing A, (ii) $c_*(A)/A$ is decreasing in A, and (iii) $c_*(A)/A$ converges as $A \to \infty$ to a positive limit $\rho > 0$.*

Proof. The eigenvalue problem (3.10) with $u(y)$ replaced by $Au(y)$ takes the form

$$-\Delta\psi - A\lambda u(y)\psi = \mu_A(At)\psi \text{ in } \Omega \qquad (4.2)$$
$$\frac{\partial\psi}{\partial n} = 0 \text{ on } \partial\Omega.$$

Hence $\mu_A(t) = \mu(At)$, and (compare to (3.11)) the spreading rate $c_*(A)$ is the smallest number c so that the equation

$$\lambda^2 - c\lambda + f'(0) = \mu(A\lambda) \qquad (4.3)$$

has a real non-negative solution λ. Geometrically, $c = c_*(A)$ is such that the graph of the convex function $g(\lambda) = \lambda^2 - c\lambda + f'(0)$ is tangent to the graph of the function $s_A(\lambda) = \mu(A\lambda)$. Recall (see the proof of Proposition 3.1) that the function $\mu(t)$ is concave, $\mu(0) = 0$ and $\mu'(0) = 0$. This implies that $c_*(A)$ is an increasing function of A.

On the other hand, introducing $t = A\lambda$ we see that $\alpha_*(A) = c_*(A)/A$ may be characterized as the number α so that equation

$$\mu(t) = \frac{t^2}{A^2} + f'(0) - \alpha t$$

has a unique solution. That means that the graphs of $\mu(t)$ and $\tilde{g}(t) = \frac{t^2}{A^2} + f'(0) - \alpha(A)t$ are tangent to each other. It is geometrically clear that $\alpha(A)$ is a decreasing function of A and that $\alpha(A)$ tends to the slope ρ of the tangent line to the graph of $\mu(t)$ that passes through the point $(0, f'(0))$. □

5 Fast shear flow asymptotics: the ignition case

We now obtain the estimates for the traveling front speed in a fast shear flow in the ignition case.

Theorem 5.1 *Let (c_A, U^A) be the traveling wave solution of*

$$-c^A\frac{\partial U^A}{\partial x} + Au(y)\frac{\partial U^A}{\partial x} = \Delta U^A + f(U^A), \qquad (5.1)$$
$$\frac{\partial U^A}{\partial n} = 0, \ \ on\ \partial\Omega,$$
$$U^A(-\infty, y) = 1, \ \ U^A(+\infty, y) = 0$$

in a cylinder $D = \{x \in \mathbb{R}, y \in \times\Omega\}$. The nonlinearity $f(s)$ is either of the KPP or of the iginition type. Then there exist two constants $C_{1,2}$ that depend on the flow profile $u(y)$ and the nonlinearity f so that

$$C_1 A \le c^A \le C_2 A. \qquad (5.2)$$

Proof. We present the proof due to S. Heinze. The upper bound in (5.2) follows from the results on the spreading rate of general solutions to

$$\frac{\partial T}{\partial t} + Au(y)\frac{\partial T}{\partial x} = \Delta T + f(T) \qquad (5.3)$$

with rapidly decaying initial data. We observe that the function $\psi(t,x) = e^{-\lambda(x-ct)}$ is a super-solution of (5.3) provided that

$$c = \lambda + \frac{M}{\lambda} + A\|u\|_{L^\infty}.$$

The constant M is chosen so that $f(T) \le MT$. Hence solutions may not propagate faster than the above speed and the upper bound in (5.2) follows.

The lower bound is slightly more intricate. Let

$$F(s) = \int_0^s f(\xi) d\xi$$

be the antiderivative of f. Define the corrector $\chi(y)$ as

$$\Delta\chi = u(y), \quad \frac{\partial \chi}{\partial n} = 0 \text{ on } \partial\Omega, \quad \int_\Omega \chi(y) dy = 0. \tag{5.4}$$

First, we integrate (5.1) over the cylinder $D = \mathbb{R} \times \Omega$ to obtain

$$c^A = \int f(U^A) dx dy. \tag{5.5}$$

Next, we multiply (5.1) by U^A and integrate to obtain

$$\frac{c^A}{2} + \int |\nabla U^A|^2 = \int U^A f(U^A) \le \int f(U^A) = c^A.$$

Hence we have

$$\int |\nabla U^A|^2 \le \frac{c^A}{2}. \tag{5.6}$$

Multiply (5.1) by $f(U^A)\chi(y)$ and integrate over the whole cylinder to obtain

$$\begin{aligned} &-c^A \int f(U^A)\chi(y)U_x^A + A\int u(y)\chi(y)f(U^A)U_x^A \\ &= -\int f'(U^A)\chi(y)|\nabla U^A|^2 - \int f(U^A)\frac{\partial \chi}{\partial y}U_y^A + \int \chi(y) f^2(U^A). \end{aligned}$$

The first integral on the left side is equal to (after evaluating the integral in x) $F(1)\int \chi(y)dy$ and hence vanishes. Combining the other terms we obtain

$$-F(1)A\int u(y)\chi(y)dy = -\int f'(U^A)\chi(y)|\nabla U^A|^2 - \int f(U^A)\frac{\partial \chi}{\partial y}\frac{\partial U^A}{\partial y} + \chi\int f^2(U^A).$$

Now, using the definition (5.4) of the function χ, and the fact that the function f is bounded, we obtain

$$F(1)A\int |\nabla\chi|^2 dy \le C\int |\nabla U^A|^2 + C\sqrt{\int f^2(U^A)}\sqrt{\int |\nabla U^A|^2} + C\int f^2(U^A).$$

This inequality together with the bounds (5.5) and (5.6) imply that

$$CA \le c^A.$$

Hence the lower bound in (5.2) holds. □

6 Strong flow asymptotics for the principal Dirichlet eigenvalue

We consider the Dirichlet eigenvalue problem

$$\begin{aligned} &-\Delta\phi + Au\cdot\nabla\phi = \lambda(A)\phi \ x\in\Omega, \\ &\phi = 0 \text{ on } \partial\Omega. \end{aligned} \tag{6.1}$$

Here $u(x)$ is a smooth incompresible flow, and Ω is a bounded domain. We assume that $u\cdot n = 0$ on the boundary $\partial\Omega$.

Theorem 6.1 *([2]) Let $\phi(x) > 0$ be the principal eigenfunction of (6.1) and $\lambda(A)$ the corresponding principal eigenvalue, then*

$$\liminf_{A\to+\infty} \lambda(A) = +\infty \tag{6.2}$$

if and only if there exists not function $\psi \in H_0^1(\Omega)$ such that

$$u\cdot\nabla\psi = 0 \ \textit{a.e. in } \Omega. \tag{6.3}$$

Moreover, if such ψ exists then

$$\lim_{A\to+\infty} \lambda(A) = \inf_{\psi} \int_\Omega |\nabla\psi|^2 dx, \tag{6.4}$$

with infimum taken over all functions ψ such that (6.3) holds and $\|\psi\|_{L^2(\Omega)} = 1$.

Proof. First, assume that

$$\liminf_{A\to+\infty} \lambda(A) < +\infty. \tag{6.5}$$

Let ϕ_A be the corresponding eigenfunctions normalized so that $\|\phi\|_A = 1$. Then, as

$$\int_\Omega |\nabla\phi_A|^2 dx = \lambda(A), \tag{6.6}$$

there exists a sequence $A_n \to \infty$ so that the corresponding sequence $\phi_n := \phi_{A_n}$ is uniformly bounded in $H_0^1(\Omega)$. Therefore, after possibly passing to a subsequence, the sequence ϕ_n converges weakly to a limit $\bar\phi \in H_0^1(\Omega)$. It also converges strongly to $\bar\phi$ in $L^2(\Omega)$, whence $\|\bar\phi\|_2 = 1$. Then, dividing (6.1) by A and letting $A \to +\infty$ we obtain

$$u\cdot\nabla\bar\phi = 0,$$

in the weak sense, that is, (6.3) holds. Therefore, if the flow u has no first integrals in $H_0^1(\Omega)$ in the sense of (6.3) then (6.2) holds. In addition, passing to the limit in (6.6) we obtain

$$\liminf_{A\to+\infty} \lambda(A) \ge \int_\Omega |\nabla\bar\phi|^2 dx. \tag{6.7}$$

Now, let us assume that u does have some first integrals in $H_0^1(\Omega)$. Given any test function $w \in H_0^1(D)$, and a number $\alpha > 0$, we multiply (6.1) by $w^2/(\varphi+\alpha)$ and integrate over D to obtain

$$\lambda\int_D \frac{w^2\varphi}{\varphi+\alpha} = -\int_D \frac{w^2\Delta\varphi}{\varphi+\alpha} + A\int_D \frac{w^2}{\varphi+\alpha} v\cdot\nabla\varphi. \tag{6.8}$$

For the first term on the right, we have

$$-\int_D \frac{w^2\Delta\varphi}{\varphi+\alpha} = \int_D \nabla\varphi\cdot\left(\frac{2w(\varphi+\alpha)\nabla w - w^2\nabla\varphi}{(\varphi+\alpha)^2}\right)$$
$$= \int_D |\nabla w|^2 - \int_D \frac{|w\nabla\varphi| - (\varphi+\alpha)\nabla w^2}{(\varphi+\alpha)^2} \le \int_D |\nabla w|^2.$$

For the second term on the right of (6.8) we have, since u is incompressible,

$$\int_D \frac{w^2}{\varphi+\alpha} v\cdot\nabla\varphi = \int_D w^2 v\cdot\nabla\ln(\varphi+\alpha) = -2\int_D \ln(\varphi+\alpha)w(v\cdot\nabla w).$$

Hence, equation (6.8) reduces to

$$\lambda\int_D \frac{w^2\varphi}{\varphi+\alpha} \le \int_D |\nabla w|^2 - 2A\int_D \ln(\varphi+\alpha)w(v\cdot\nabla w). \tag{6.9}$$

Now, choose w to be any $H_0^1(D)$ first integral of v (that is, $v\cdot\nabla w = 0$ a.e.). Then, equation (6.9) reduces to

$$\lambda\int_D \frac{w^2\varphi}{\varphi+\alpha} \le \int_D |\nabla w|^2.$$

Upon sending $\alpha \to 0$, the Monotone Convergence Theorem shows

$$\lambda\int_D w^2 \le \int_D |\nabla w|^2$$

for any $H_0^1(D)$ first integral of v.

References

[1] H. Berestycki and F. Hamel, Front propagation in periodic excitable media, Comm. Pure Appl. Math. **55**, 2002, 949–1032.

[2] H. Berestycki, F. Hamel and N. Nadirashvili, Elliptic eigenvalue problems with large drift and applications to nonlinear propagation phenomena, Comm. Math. Phys., **253**, 2005, 451–480.

[3] H. Berestycki, B. Nicolaenko, and B. Scheurer, Traveling wave solutions to combustion models and their singular limits. SIAM J. Math. Anal. **16**, 1985, 1207–1242.

[4] H. Berestycki and L. Nirenberg, Traveling fronts in cylinders, Annales de lIHP, Analyse nonlinéare, **9**, 1992, 497–572.

[5] A. N. Kolmogorov, I. G. Petrovsky, and N. S. Piskunov, Etude de l'eq́uation de la diffusion avec croissance de la quantité de matiére et son application á un probléme biologique, Moskow Univ. Math. Bull., **1**, 1937, 1–25.

[6] J.-F. Mallordy and J.-M. Roquejoffre, A parabolic equation of the KPP type in higher dimensions, SIAM J. Math. Anal.**26**, 1995, 1–20.

[7] J. Xin, Front propagation in heterogeneous media. SIAM Rev. **42**, 2000, 161–230.