Stanford PDE mini-course, Fall 2021

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Abstract

These notes are a compilation of pieces from the notes by Julien Berestycki, [1] Anton Bovier [2] and Ofer Zeitouni [13]. All mistakes are mine, all insights are theirs, even if occasionally misinterpreted.

1 Branching Brownian motion

In these notes, we will be concerned with the time evolution of a large number of particles. Usually, their number will grow in time as well. The simplest such example would be to take N particles $x_1, x_2(t), \dots, x_N(t)$, with each particle performing an independent copy of a fixed random process. For example, each $x_k(t)$ can be a standard Brownian motion in $\mathbb{R}^d$, $d \geq 1$, starting at the point $x_j(0) = 0$. However, if we fix the number N of particles and consider what happens as $t \to +\infty$, the configuration would become really sparse in space. Hence, it may be more interesting to let the number of particles to grow in time, to keep their density from vanishing.

Such models involving an increasing number of particles appear very naturally in the context of biological invasions in ecology, as well as in SIR-type models of epidemics. A simple and common process of this type is the binary branching Brownian motion. It is described as follows. A single particle starts at a position $x \in \mathbb{R}^d$ at $t = 0$ and performs a standard Brownian motion. The particle carries an exponential clock that rings at a random time τ, with

$$\mathbb{P}(\tau > t) = e^{-t}. \tag{1.1}$$

At the time τ the particle splits into two particles that we will refer to as the children, and the original particle is sometimes called the parent. The original particle is removed at the branching event. The two children perform independent standard Brownian motions for $t > \tau$, both of them starting at the position of the branching event. Each of the children carries its own exponential clock, and when the corresponding clock rings, the particle splits into two, and the process continues. Thus, at each time $t > 0$ we have a collection of particles $x_1(t), \dots, x_{N_t}(t)$. Here, $N_t - 1$ is the random number of times all the clocks rang until the time t.

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A way to think of this process in terms of epidemics, is a SIR-type model where nobody is ever removed, but can infect. Each particle meets other particles at the times when the clock rings and infects one other particle at each "infection event".

The above model is usually referred to as a binary Brownian motion since the number of children is limited to two. A simple modification of the above model is a process where the particles may produce a random number of k children at each birth event, with

$$\sum_{k=2}^{\infty} p_k = 1, \tag{1.2}$$

and the average number of children

$$\bar{N} = \sum_{k=2}^{\infty} k p_k. \tag{1.3}$$

We will usually assume that p_k decay sufficiently fast as $k \to +\infty$.

The total number $N(t)$ of particles present at the time $t > 0$ can be thought of as a pure birth process – $N(t)$ can go up but not down. As a warm-up, let us prove the following.

Proposition 1.1 *Let $N(t)$ be the number of particles present in the binary BBM with the exponential clock as in (1.1), then*

$$\mathbb{E}(N(t)) = e^t. \tag{1.4}$$

Proof. Let us consider the first branching time τ_1. If $\tau_1 > t$, then $N(t) = 1$. On the other hand, if $\tau_1 \le t$, then the total number of particles at the time t is the total number of particles coming from the first child born at the time τ_1 plus the total number of particles coming from the second child born at the time τ_1. The time elapsed between τ_1 and t is $t - \tau_1$. This gives the following recursion relation:

$$\begin{aligned}\mathbb{E}(N(t)) &= \mathbb{P}(\tau_1 > t) + 2\mathbb{E}(N(t-\tau_1))\mathbb{P}(\tau_1 < t) = e^{-t} + 2\int_0^t \mathbb{E}(N(t-s))\mathbb{P}(\tau_1 \in ds)\\ &= e^{-t} + 2\int_0^t \mathbb{E}(N(t-s))e^{-s}ds.\end{aligned} \tag{1.5}$$

Let us set $u(t) = \mathbb{E}(N(t))$. Then we can write (1.5) as

$$u(t) = e^{-t} + 2\int_0^t u(t-s)e^{-s}ds. \tag{1.6}$$

Differentiating in t gives

$$\begin{aligned}\frac{du(t)}{dt} &= -e^{-t} + 2u(0)e^{-t} + 2\int_0^t \frac{du(t-s)}{dt}e^{-s}ds = e^{-t} - 2\int_0^t e^{-s}\frac{d}{ds}(u(t-s))ds\\ &= e^{-t} - 2e^{-t}u(0) + 2u(t) - 2\int_0^t e^{-s}u(t-s)ds.\end{aligned} \tag{1.7}$$

We used above the fact that $u(0) = 1$ and integrated by parts the last integral in the first line. Now, we use (1.6) to replace the integral in the very right side of (1.7), and obtain

$$\frac{du(t)}{dt} = -e^{-t} + 2u(t) - (u(t) - e^{-t}) = u(t). \tag{1.8}$$

Since $u(0) = 1$, we deduce that $u(t) = e^t$. □

In the general case, with k children born with probabilities p_k, the analog of (1.5) is

$$\begin{aligned} \mathbb{E}(N(t)) &= \mathbb{P}(\tau > t) + \sum_{k=2}^{\infty} kp_k \mathbb{E}(N(t - \tau_1))\mathbb{P}(\tau_1 < t) \\ &= e^{-t} + \sum_{k=2}^{\infty} kp_k \int_0^t \mathbb{E}(N(t-s))\mathbb{P}(\tau_1 \in ds) = e^{-t} + \bar{N} \int_0^t \mathbb{E}(N(t-s))e^{-s} ds. \end{aligned} \tag{1.9}$$

Hence, the function $u(t)$ satisfies the following analog of (1.6):

$$u(t) = e^{-t} + \bar{N} \int_0^t u(t-s)e^{-s} ds. \tag{1.10}$$

Exercise 1.2 Show that in this case $\mathbb{E}N(t) = \exp((\bar{N} - 1)t)$.

We can actually say more about the total number of particles. Very naively, since the number of particles is very large, one may think that, in the spirit of the law of large numbers, $N(t)$ behaves as $N_\infty e^t$, as $t \to +\infty$, where N_∞ would be a fixed deterministic constant.

Exercise 1.3 Show this is impossible because, for example, the time of the first branching event is random and the above guess would indicate that $N(t)$ should also behave as $2N_\infty e^{t-\tau_1}$, which would be a contradiction.

However, the guess is not very far from the correct picture, except that N_∞ is random.

Proposition 1.4 *Let $N(t)$ be the number of particles in the binary BBM, then*

$$M(t) = e^{-t} N(t) \tag{1.11}$$

is a martingale. Moreover, $M(t)$ converges, as $t \to +\infty$, almost surely and in L^1 to a random variable M_∞.

Proof. First, note that $M(t)$ is a positive random variable with $E(M(t)) = 1$. Hence, it is integrable. Let us first check that $M(t)$ is a martingale. To see this, we first note that for $t > s$ we have

$$\mathbb{E}(M(t)|\mathcal{F}_s) = \mathbb{E}(e^{-t}N(t)|\mathcal{F}_s) = e^{-t}\mathbb{E}(N(t)|\mathcal{F}_s). \tag{1.12}$$

Recall that the exponential clocks have the "lack of memory" property:

$$\mathbb{P}(\tau > s + t|\tau > t) = \mathbb{P}(\tau > s). \tag{1.13}$$

This is because

$$\mathbb{P}(\tau > s+t|\tau > t) = \frac{\mathbb{P}(\tau > s+t \text{ and } \tau > t)}{\mathbb{P}(\tau > t)} = \frac{\mathbb{P}(\tau > s+t)}{\mathbb{P}(\tau > t)} = \frac{e^{-(t+s)}}{e^{-t}} \\ = e^{-s} = \mathbb{P}(\tau > s). \tag{1.14}$$

Note that the lack of memory property is very specific to the exponential clocks, and that is one reason why using an exponential clock is convenient.

The lack of memory property means that we can reset all the clocks at the time s to zero, without changing the law of the process for $t > s$, and deduce from Proposition 1.1 that

$$\mathbb{E}(N(t)|\mathcal{F}_s) = N(s)e^{(t-s)}. \tag{1.15}$$

Using this in (1.12) gives

$$\mathbb{E}(M(t)|\mathcal{F}_s) = e^{-t}\mathbb{E}(N(t)|\mathcal{F}_s) = e^{-t}N(s)e^{(t-s)} = e^{-s}N(s) = M(s). \tag{1.16}$$

Thus, $M(t)$ is a martingale. Doob's theorem implies that, as $M(t)$ is a martingale, it converges almost surely, as $t \to +\infty$ to a random limit M_∞. To prove converges in L^1, it suffices to show that $M(t)$ is uniformly integrable. For that, we can look at

$$\mathbb{E}(M^2(t)) = e^{-2t}\mathbb{E}(N^2(t)). \tag{1.17}$$

We have the following recursion relation for the function $\phi_2(t) = \mathbb{E}(N^2(t))$, analogous to (1.5):

$$\begin{aligned} \phi_2(t) &= \mathbb{P}(\tau_1 < t) + \int_0^t (2\phi_2(t-s) + 2e^{2(t-s)})\mathbb{P}(\tau_1 \in ds) \\ &= e^{-t} + 2\int_0^t (\phi_2(t-s) + e^{2(t-s)})e^{-s}ds. \end{aligned} \tag{1.18}$$

Hence, the function

$$\psi_2(t) = \phi_2(t) + e^{2t} \tag{1.19}$$

satisfies

$$\psi_2(t) = e^{-t} + e^{2t} + 2\int_0^t \psi_2(t-s)e^{-s}ds. \tag{1.20}$$

Differentiating in t gives

$$\begin{aligned} \frac{d\psi_2(t)}{dt} &= -e^{-t} + 2e^{2t} + 2e^{-t}\psi_2(0) + 2\int_0^t \frac{d\psi_2(t-s)}{dt}e^{-s}ds \\ &= 3e^{-t} + 2e^{2t} - 2\int_0^t e^{-s}\frac{d\psi_2(t-s)}{ds}ds = 3e^{-t} + 2e^{2t} - 2e^{-t}\psi_2(0) + 2\psi_2(t) \\ &- 2\int_0^t e^{-s}\psi_2(t-s). \end{aligned} \tag{1.21}$$

Inserting (1.20) into (1.21) and using the fact that $\psi_2(0) = 2$, gives

$$\frac{d\psi_2(t)}{dt} = 2e^{2t} - e^{-t} + 2\psi_2(t) - (\psi_2(t) - e^{-t} - e^{2t}) = 3e^{2t} + \psi_2(t). \tag{1.22}$$

It follows that

$$\psi_2(t) = 3e^{2t} - e^t, \tag{1.23}$$

and

$$\phi_2(t) = 2e^{2t} - e^t. \tag{1.24}$$

This gives

$$\gamma_2(t) := \mathbb{E}(M^2(t)) = e^{-2t}\phi_2(t) = 2 - e^{-t}. \tag{1.25}$$

Thus, $M(t)$ is uniformly integrable, and converges to M_∞ in L^1, as well. □

We can use the above recursive strategy to compute higher moments of $N(t)$. Let us set

$$\phi_k(t) = \mathbb{E}(N^k(t)), \quad \gamma_k(t) = \mathbb{E}(M^k(t)) = e^{-kt}\phi_k(t).$$

These functions satisfy the following analog of (1.18):

$$\begin{aligned}\phi_k(t) &= \mathbb{P}(\tau_1 < t) + \int_0^t \sum_{m=0}^k \binom{k}{m} \phi_m(t-s)\phi_{k-m}(t-s)\mathbb{P}(\tau_1 \in ds)\\ &= e^{-t} + \int_0^t \sum_{m=0}^k \binom{k}{m} \phi_m(t-s)\phi_{k-m}(t-s)e^{-s}ds,\end{aligned} \tag{1.26}$$

and

$$\gamma_k(t)e^{kt} = \mathbb{P}(\tau_1 < t) + \int_0^t \sum_{m=0}^k \binom{k}{m} e^{m(t-s)}\gamma_m(t-s)\gamma_{k-m}(t-s)e^{(k-m)(t-s)}e^{-s}ds. \tag{1.27}$$

The last equation becomes

$$\gamma_k(t) = e^{-(k+1)t} + \int_0^t \sum_{m=0}^k \binom{k}{m} \gamma_m(t-s)\gamma_{k-m}(t-s)e^{-(k+1)s}ds. \tag{1.28}$$

Let us assume that the limits

$$\bar\gamma_k = \lim_{t\to+\infty} \gamma_k(t) \tag{1.29}$$

exist. Then, passing to the limit $t \to +\infty$ in (1.28) leads to a recursive equation

$$\bar\gamma_k = \frac{1}{k+1}\sum_{m=0}^k \binom{k}{m} \bar\gamma_m\bar\gamma_{k-m} = \frac{k!}{k+1}\sum_{m=0}^k \frac{\bar\gamma_m}{m!}\frac{\bar\gamma_{k-m}}{(k-m)!} \tag{1.30}$$

We have already computed that $\bar\gamma_1 = 1$ and $\bar\gamma_2 = 2$. It is immediate to see from (1.30) that

$$\bar\gamma_k = k!. \tag{1.31}$$

Exercise 1.5 Show that M_∞ is an exponential variable with parameter 1. Explain why M_∞ is determined by what happens early in the branching process.

Exercise 1.6 Generalize the claims of Proposition 1.4 and Exercise 1.5 to a general non-binary branching with probabilities p_k to have k children at each branching event, as in (1.2).

2 Maximum of a family of independent particles

A consequence of Proposition 1.4 is that we can think of $x_1(t), \dots, x_{N(t)}(t)$ as a collection of approximately $M_\infty e^t$ random particles, for $t \gg 1$, even though M_∞ itself is random. An important point is that the random variables $x_1(t), \dots, x_{N(t)}(t)$ are correlated, and the nature of these correlations will be crucial to us. In particular, the correlations will be very important when we consider the extrema of such branching processes.

As a first step in this direction, in this section we consider a much simplified model where no correlations are present. Let us think of $N \in \mathbb{N}$ as an analog of the time variable for the BBM and take a large number M of independent particles $X_1, \dots, X_M$, with M that depends on N. In order to mimic the concept that N is the time of the BBM, we assume that X_k are mean zero Gaussian random variables with variance N. Thus, we can think of each X_k as a snapshot of a Brownian motion at the time $t = N$. To make sure that the number M of these variables also mimics BBM, we assume that it grows exponentially in N, as is the case for BBM. We take $M = 2^N$, to ensure that M is an integer. To complete the model we need to prescribe the correlations between X_k. The key difference with the BBM is that here, for an infinitely greater simplicity, we assume that all X_k are independent. This assumption will allow us to calculate many quantities explicitly.

To summarize, one can think of this model as an "uncorrelated BBM", in the sense that at the time N we have 2^N particles and each one has variance N, that of the standard Brownian motion. We are interested in the large N behavior of this system.

2.1 The law of the maximum

The first object we would like to understand is the location of the maximal particle

$$\bar{M}_N = \max(X_1, \dots, X_M). \tag{2.1}$$

In this simple model, since all particles are identical, this can be done by writing

$$\mathbb{P}(\bar{M}_N < y) = \big(\mathbb{P}(X_1 < y)\big)^{2^N} = \big(1 - \mathbb{P}(X_1 > y)\big)^{2^N}. \tag{2.2}$$

The first question is for which y is this probability of order one. In particular, we are interested in the median location m_N such that

$$\mathbb{P}(\bar{M}_N > m_N) = 1/2. \tag{2.3}$$

The second is the width of the transition layer: if we fix $\varepsilon_0 \in (0, 1)$ and consider $Y_N(\varepsilon_0)$ defined by

$$\mathbb{P}(\bar{M}_N > Y_N) = \varepsilon_0, \tag{2.4}$$

the question is if the difference

$$|Y_N(\varepsilon_0) - m_N| \tag{2.5}$$

grows in N or stays of order 1 as $N \to +\infty$, with $\varepsilon_0 \in (0, 1)$ fixed.

We see from (2.2) that for $\mathbb{P}(\bar{M}_N < y)$ to be of order one but not too close to 1, we need $\mathbb{P}(X_1 > y)$ to be of the order 2^{-N}:

$$2^N \mathbb{P}(X_1 > y) \sim O(1). \tag{2.6}$$

Note that for large y we have

$$\begin{aligned}\mathbb{P}(X_1 > y) &= \int_y^\infty e^{-x^2/(2N)} \frac{dx}{\sqrt{2\pi N}} = \int_{y/\sqrt{2N}}^\infty e^{-x^2} \frac{dx}{\sqrt{\pi}} = (1+o(1)) \frac{\sqrt{N}}{y\sqrt{2\pi}} \int_{y/\sqrt{2N}}^\infty 2xe^{-x^2} dx \\ &= (1+o(1)) \frac{\sqrt{N}}{y\sqrt{2\pi}} e^{-y^2/(2N)}, \quad \text{as } N \to \infty.\end{aligned} \tag{2.7}$$

Thus, we are looking for y such that

$$2^N \frac{\sqrt{N}}{y\sqrt{2\pi}} e^{-y^2/(2N)} \sim O(1), \tag{2.8}$$

which means that, to the leading order, we should be looking at y such that

$$\frac{y^2}{2N} \approx N \log 2, \quad y \approx \sqrt{2\log 2} N. \tag{2.9}$$

Note that y is growing linearly in the "time" N, and not diffusively, which would be $\sqrt{N}$:

$$y \approx c_* N, \quad c_* = \sqrt{2 \log 2}. \tag{2.10}$$

Going back to (2.8), we see that the next order correction requires that

$$2^N \sqrt{N} e^{-y^2/(2N)} \sim y \ \approx \sqrt{2\log 2} N. \tag{2.11}$$

This gives

$$N \log 2 - \frac{y^2}{2N} \approx \frac{1}{2} \log N, \tag{2.12}$$

meaning that

$$\begin{aligned} y &\approx \big(2N^2 \log 2 - N \log N\big)^{1/2} = N\sqrt{2\log 2}\Big(1 - \frac{\log N}{2N \log 2}\Big)^{1/2} \\ &\approx N\sqrt{2\log 2}\Big(1 - \frac{\log N}{4N\log 2}\Big) = N\sqrt{2\log 2} - \frac{1}{2\sqrt{2\log 2}} \log N. \end{aligned} \tag{2.13}$$

Let us set

$$c_* = \sqrt{2\log 2}, \quad \lambda_* = \sqrt{2\log 2}, \tag{2.14}$$

then (2.13) can be written as

$$y \approx c_* N - \frac{1}{2\lambda_*} \log N. \tag{2.15}$$

The fact that $\lambda_* = c_*$ in (2.14) is an unfortunate coincidence and should be ignored. They do not always coincide in other models and their roles are different, as we will see. This structure will appear repeatedly later on, and, for reasons to become clear later, the pre-factor $1/2$ in (2.13) indicates that the particles are uncorrelated.

Exercise 2.1 Make the above approximations precise, controlling the errors and show that there exists x_0 so that the median m_N defined in (2.3) satisfies

$$m_N = c_* N - \frac{1}{2\lambda_*}\log N + x_0 + o(1), \quad \text{as } N \to +\infty. \tag{2.16}$$

(ii) More generally, show that for any $\varepsilon_0 > 0$ there exists x_ε so that $Y_N(\varepsilon_0)$ defined in (2.4) satisfies

$$Y_N(\varepsilon_0) = c_* N - \frac{1}{2\lambda_*}\log N + x_\varepsilon + o(1), \quad \text{as } N \to +\infty. \tag{2.17}$$

Now that we know how we should choose the centering for y, let us consider the function

$$u_N(y) = \mathbb{P}(\max_{1\le k\le M} X_k > c_* N - \frac{1}{2\lambda_*}\log N + y). \tag{2.18}$$

Using (2.7), we obtain

$$\begin{aligned}
&2^N \mathbb{P}(X_1 > c_* N - \frac{1}{2\lambda_*}\log N + y) \\
&= \frac{\sqrt{N}2^N}{(c_* N - (1/(2\lambda_*))\log N + y)\sqrt{2\pi}} e^{-(c_* N - (1/(2\lambda_*))\log N + y)^2/(2N)} + o(1) \\
&= \frac{1}{c_*\sqrt{2\pi N}}\Big(1 + \frac{\log N}{2\lambda_*\sqrt{N}} - \frac{y}{\sqrt{N}}\Big)\exp\Big(N\log 2 - \frac{c_*^2}{2}N + \frac{c_*}{2\lambda_*}\log N - c_* y\Big) + o(1) \\
&= \frac{1}{c_*\sqrt{2\pi}} e^{-\lambda_* y} + o(1), \quad \text{as } N \to +\infty.
\end{aligned} \tag{2.19}$$

We used the fact that $c_* = \lambda_*$ above. Going back to (2.2), we see that

$$\begin{aligned}
1 - u_N(y) &= \big(1 - \mathbb{P}(X_1 > c_* N - \frac{1}{2\lambda_*}\log N + y)\big)^{2^N} = \Big(1 - 2^{-N}\frac{1}{c_*\sqrt{2\pi}}e^{-\lambda_* y}\Big)^{2^N} + o(1) \\
&= \exp\Big(-\frac{1}{c_*\sqrt{2\pi}}e^{-\lambda_* y}\Big) + o(1), \quad \text{as } N \to +\infty,
\end{aligned} \tag{2.20}$$

so that

$$u_N(y) = 1 - \exp\Big(-\frac{1}{c_*\sqrt{2\pi}}e^{-\lambda_* y}\Big) + o(1), \quad \text{as } N \to +\infty. \tag{2.21}$$

Note that the leading order term in the right side above is the Gumbel distribution.

Exercise 2.2 We have used both large y asymptotics in (2.7), and also large N asymptotics later on. Check the above approximations, to make sure that y taken in (2.13) is sufficiently large, so that (2.21) does hold for all $y \in \mathbb{R}$.

2.2 An upper bound for the maximum: the first moment method

The above computation for the location of the maximum, and the asymptotics in (2.21) is a little too explicit for an analyst's taste. Let us now, instead, get some bounds on the location of the maximum. They will not be as precise as (2.21) but will allow us to introduce two methods to bound the distribution of the maximum from above and from below that will be useful in the analysis of the maximum of BBM.

For the upper bound, we will use what is known as the first moment method. The total number of particles located to the right of a given $y \in \mathbb{R}$ can be written as follows (recall that $M = 2^N$):

$$M_N(y) = \sum_{k=1}^{M} \mathbb{1}(X_k > y). \tag{2.22}$$

Note that we have, on one hand,

$$\mathbb{P}(\max_{1\le k\le M} X_k > y) = \mathbb{P}(M_N(y) \ge 1), \tag{2.23}$$

and on the other,

$$\mathbb{E}(M_N(y)) = \sum_{k=1}^{M} k\mathbb{P}(M_N(y) = k) \ge \sum_{k=1}^{M} \mathbb{P}(M_N(y) = k) = \mathbb{P}(M_N(y) \ge 1). \tag{2.24}$$

This gives the following ballpark estimate which is the essence of the first moment method:

$$\mathbb{P}(\max_{1\le k\le M} X_k > y) \le \mathbb{E}(M_N(y)) = 2^N \mathbb{P}(X_1 > y). \tag{2.25}$$

We used the fact that the particles are identically distributed in the last step.

This allows us to bound the function $u_N(y)$ introduced in (2.18) using (2.19):

$$\begin{aligned} u_N(y) &= \mathbb{P}(\max_{1\le k\le M} X_k > c_* N - \frac{1}{2\lambda_*}\log N + y) \\ &\le 2^N \mathbb{P}(X_1 > c_* N - \frac{1}{2\lambda_*}\log N + y) = \frac{1}{c_*\sqrt{2\pi}} e^{-\lambda_* y} + o(1), \quad \text{as } N \to +\infty. \end{aligned} \tag{2.26}$$

Thus, we have an upper bound

$$u_N(y) = \mathbb{P}(\max_{1\le k\le M} X_k > c_* N - \frac{1}{2\lambda_*}\log N + y) \le \frac{1}{c_*\sqrt{2\pi}} e^{-\lambda_* y} + o(1), \quad \text{as } N \to +\infty. \tag{2.27}$$

2.3 A lower bound for the maximum: the second moment method

We may also get a lower bound for $u_N(x)$, using what is known as the second moment method. The starting point is, again, (2.23):

$$\mathbb{P}(\max_{1\le k\le M} X_k > y) = \mathbb{P}(M_N(y) \ge 1). \tag{2.28}$$

The second moment method estimates the probability in the right side of (2.28) using the Cauchy-Schwartz inequality

$$\begin{aligned} (\mathbb{E}M_N(y))^2 &= (\mathbb{E}[M_N(y)\mathbb{1}(M_N(y) \ge 1)])^2 \le \mathbb{E}(M_N(y))^2 \mathbb{E}(\mathbb{1}(M_N(y) \ge 1)) \\ &= \mathbb{E}(M_N(y))^2 \mathbb{P}(M_N(y) \ge 1). \end{aligned} \tag{2.29}$$

It follows that

$$\mathbb{P}(\max_{1\le k\le M} X_k > y) = \mathbb{P}(M_N(y) \ge 1) \ge \frac{(\mathbb{E}M_N(y))^2}{\mathbb{E}(M_N(y))^2}. \tag{2.30}$$

This is convenient as it reduces the computation of a probability to a computation of moments. The dominator in (2.30) can be written as

$$\begin{aligned}\mathbb{E}(M_N(y)^2) &= \mathbb{E}\Big(\sum_{k=1}^{M} \mathbb{1}(X_k > y)\Big)^2 = 2^N\mathbb{P}(X_1 > y) + 2^N(2^N-1)[\mathbb{P}(X_1 > y)]^2 \\ &= \mathbb{E}(M_N(y)) + (1-2^{-N})(\mathbb{E}(M_N(y)))^2.\end{aligned} \tag{2.31}$$

Hence, if y in (2.31) is such that

$$\mathbb{E}(M_N(y)) \sim O(1), \tag{2.32}$$

then we have

$$\begin{aligned}\mathbb{P}(\max_{1\le k\le M} X_k > y) &\ge \frac{(\mathbb{E}M_N(y))^2}{\mathbb{E}(M_N(y))^2} = \frac{(\mathbb{E}M_N(y))^2}{\mathbb{E}(M_N(y)) + (1-2^{-N})(\mathbb{E}(M_N(y)))^2} \\ &= \frac{\mathbb{E}M_N(y)}{1+\mathbb{E}(M_N(y))} + o(1), \quad \text{as } N\to+\infty.\end{aligned} \tag{2.33}$$

In order to have (2.32), we take y as in (2.26):

$$\mathbb{E}(M_N(c_*N - \frac{1}{2\lambda_*}\log N + y)) = \frac{1}{c_*\sqrt{2\pi}}e^{-\lambda_* y} + o(1), \quad \text{as } N\to+\infty. \tag{2.34}$$

The following lower bound follows then from (2.33):

$$\mathbb{P}(\max_{1\le k\le M} X_k > c_*N - \frac{1}{2\lambda_*}\log N + y) \ge \frac{e^{-\lambda_* y}}{c_*\sqrt{2\pi} + e^{-\lambda_* y}} + o(1), \quad \text{as } N\to+\infty. \tag{2.35}$$

Summarizing, we have now obtained upper and lower bounds for the function $u_N(y)$:

$$\frac{e^{-\lambda_* y}}{c_*\sqrt{2\pi} + e^{-\lambda_* y}} + o(1) \le u_N(y) \le \frac{1}{c_*\sqrt{2\pi}}e^{-\lambda_* y} + o(1), \quad \text{as } N\to+\infty. \tag{2.36}$$

This tells us that the maximum of $X_1,\dots,X_M$ is located around

$$m_N = c_*N - \frac{1}{2\lambda_*}\log N \tag{2.37}$$

and gives us the limiting distribution when y is centered at that location. Of course, these estimates are less precise than the exact expression (2.21) but they give a very good approximation to it.

Let us also note that the upper and lower bounds in (2.36) match as $y\to+\infty$, so that, in particular, we have

$$e^{\lambda_* y}u_N(y) = \frac{1}{c_*\sqrt{2\pi}} + o(1), \quad \text{as } N\to+\infty \text{ and } y\to+\infty. \tag{2.38}$$

2.4 The extremal process

Let us now try to understand how the collection $X_1, X_2, \ldots, X_M$, $M = 2^N$, looks like as a cloud of points on $\mathbb{R}$. We recenter X_k according to the typical location of the maximum in (2.37): set Z_k by

$$X_k = m_N + Z_k. \tag{2.39}$$

Here, m_N is taken to be

$$m_N = c_* N - \frac{1}{2\lambda_*} \log N, \tag{2.40}$$

with

$$c_* = \lambda_* = \sqrt{2 \log 2}. \tag{2.41}$$

Recall that the median $\bar{m}_N$ of the maximum $\bar{M}_N = \max_{1 \le k \le M} X_k$, defined as,

$$\mathbb{P}(\bar{M}_N > \bar{m}_N) = 1/2, \tag{2.42}$$

satisfies the asymptotics (2.16)

$$\bar{m}_N = c_* N - \frac{1}{2\lambda_*} \log N + x_0 + o(1), \quad \text{as } N \to +\infty. \tag{2.43}$$

Hence, the difference between $\bar{m}_N$ and m_N is just the constant x_0 that is convenient to eliminate, to shorten some computations.

Exercise 2.3 Show that there exists $\varepsilon_0 \in (0, 1)$ so that

$$\mathbb{P}(\bar{M}_N > m_N) = \varepsilon_0 + o(1), \quad \text{as } N \to +\infty. \tag{2.44}$$

Consider the measure, called a random point process

$$\mathcal{E}_N(x) = \sum_{k=1}^{M} \delta(x - Z_k). \tag{2.45}$$

The process $\mathcal{E}_N$ is also known as the extremal process because it is re-centered near the maximum of X_N. One should note the difference between the process $\mathcal{E}_N$ coming from Z_k that are centered at a deterministic location m_N given by (2.40), and the process $\bar{\mathcal{E}}_N$ seen from the tip:

$$\bar{\mathcal{E}}_N(x) = \sum_{k=1}^{M} \delta(x - \bar{Z}_k), \tag{2.46}$$

generated by

$$\bar{Z}_k = X_k - \bar{M}_N, \quad \bar{M}_N = \max_{1 \le k \le M} X_k. \tag{2.47}$$

The points $\bar{Z}_k$ are centered at a random location $\bar{M}_N$. Here, we will focus on $\mathcal{E}_N$.

Note that the sum in (2.45) involves $M = 2^N$ terms, so the total mass of $\mathcal{E}_N$ is 2^N. However, most of the points Z_k are very large: each of Z_k is a Gaussian with mean $(-m_N)$ and variance $\sqrt{N}$. Thus, relatively few of Z_k will lie in any given compact set, hence it is perfectly plausible that the measure $\mathcal{E}_N(x)$ may have a weak limit as $N \to +\infty$, even though its total mass blows up as $N \to +\infty$. We will see how this blowup of the mass is reflected in the nature of the limit.

We will study $\mathcal{E}_N$ via its Laplace transform. The Laplace transform of a random point process $\mathcal{E}_N$ is defined as follows: given a test function $\phi \in C_b^+(\mathbb{R})$ (the set of bounded non-negative continuous functions), set

$$\Psi_N(\phi) = \mathbb{E}\Big(\exp\Big[-\int \phi(x)\mathcal{E}_N(x)dx\Big]\Big) = \mathbb{E}\Big(\exp\Big[-\sum_{k=1}^{M}\phi(Z_k)\Big]\Big). \tag{2.48}$$

A basic result in the theory of point processes is that the Laplace transform determines the law of a point process, and that convergence of the Laplace transforms implies convergence in law of the point processes to some limit – see Appendix to [2].

In the present case, the asymptotics of the Laplace transform of $Z_1, \dots, Z_M$ can be computed essentially explicitly in the limit $N \to +\infty$. First, we write

$$\Psi_N(\phi) = \mathbb{E}\Big(\exp\Big[-\sum_{k=1}^{2^N}\phi(Z_k)\Big]\Big) = \big[\mathbb{E}(\exp(-\phi(Z_1)))\big]^{2^N}, \tag{2.49}$$

and decompose

$$\mathbb{E}(\exp(-\phi(Z_1))) = 1 + \Big[\mathbb{E}(\exp(-\phi(Z_1))) - 1\Big]. \tag{2.50}$$

Let us look at the second term in the parentheses: recall that Z_1 is a Gaussian with the mean $(-m_N)$ and variance N. Hence, we have

$$\begin{aligned}
&\mathbb{E}(\exp(-\phi(Z_1))) - 1 = \frac{1}{\sqrt{2\pi N}}\int e^{-(x+m_N)^2/2N}\Big[e^{-\phi(x)} - 1\Big]dx \\
&= \frac{1}{\sqrt{2\pi N}}\int \exp\Big\{-\frac{x^2}{2N} - \frac{m_N^2}{2N} - \frac{m_N x}{N}\Big\}\Big[e^{-\phi(x)} - 1\Big]dx \\
&= \frac{1}{\sqrt{2\pi N}}\int \exp\Big\{-\frac{c_*^2N^2 - (c_*/\lambda_*)N\log N + 1/(4\lambda_*^2)\log^2 N}{2N} - \frac{(c_*N - 1/(2\lambda_*)\log N)x}{N}\Big\} \\
&\times \exp\Big(-\frac{x^2}{2N}\Big)\Big[e^{-\phi(x)} - 1\Big]dx.
\end{aligned} \tag{2.51}$$

Recall that $c_* = \sqrt{2\log 2}$ and $\lambda_* = c_*$. This gives

$$\mathbb{E}(\exp(-\phi(Z_1))) - 1 = \frac{2^{-N}}{\sqrt{2\pi}}\int e^{-\lambda_* x}\Big[e^{-\phi(x)} - 1\Big]dx + o\big(2^{-N}\big), \quad \text{as } N \to +\infty. \tag{2.52}$$

Thus, the second term in the parentheses in (2.50) is of the order 2^{-N}. Hence we have, combining (2.49), (2.50) and (2.52):

$$\begin{aligned}
\Psi_N(\phi) &= \Big(1 + \Big[\mathbb{E}(\exp(-\phi(Z_1))) - 1\Big]\Big)^{2^N} \\
&= \exp\Big(2^N\Big[\mathbb{E}(\exp(-\phi(Z_1))) - 1\Big]\Big) + o(1) \\
&= \exp\Big(\int e^{-\lambda_* x}\Big[e^{-\phi(x)} - 1\Big]\frac{dx}{\sqrt{2\pi}}\Big) + o(1), \quad \text{as } N \to +\infty.
\end{aligned} \tag{2.53}$$

We deduce that the process $\mathcal{E}_N(x)$ converges as $N \to +\infty$ to a point process with the Laplace transform

$$\Psi(\phi) = \exp\Big(\int e^{-\lambda_* x}\Big[e^{-\phi(x)} - 1\Big]\frac{dx}{\sqrt{2\pi}}\Big). \tag{2.54}$$

In order to identify this limit, let us recall the definition of a Poisson point process with intensity μ. Here, μ is a σ-finite non-negative measure on $\mathbb{R}^d$. The Poisson point process N with intensity μ is characterized by the following properties: first, given any Borel set $B \subset \mathbb{R}^d$, we have, for any $k \geq 0$:

$$\begin{aligned} &P[N(B) = k] = \frac{(\mu(B))^k}{k!}e^{-\mu(B)}, \quad \text{if } \mu(B) < +\infty, \\ &P[N(B) = k] = 0, \quad \text{if } \mu(B) = +\infty. \end{aligned} \tag{2.55}$$

Second, if A and B are two Borel sets such that $A \cap B = \emptyset$, then $N(A)$ and $N(B)$ are independent random variables. The Laplace transform of a Poisson point process is described by the following exercise.

Exercise 2.4 Let N be a Poisson point process with intensity μ, then its Laplace transform is defined, by its action on non-negative continuous bounded functions as

$$\Psi(\phi) = \exp\Big(\int \Big[e^{-\phi(x)} - 1\Big]\mu(dx)\Big). \tag{2.56}$$

It is helpful to start with non-negative simple functions (even though they are not continuous). If in doubt, this is Proposition 4.8 of [2].

Comparing the limit (2.54) of the Laplace transforms Ψ_N of the point processes $\mathcal{E}_N$ and (2.56), we see that we have proved the following.

Theorem 2.5 *The point processes*

$$\mathcal{E}_N = \sum_{k=1}^{2^N} \delta(x - Z_k), \tag{2.57}$$

converges in law as $N \to +\infty$ to a Poisson point process with intensity

$$\mu(x) = \frac{1}{\sqrt{2\pi}}e^{-\lambda_* x}. \tag{2.58}$$

Thus, λ_* is the exponential rate of growth of the density of the random point process associated to $X_1, \dots, X_M$. Note that the limit process has an infinite mass, and that the density of the measure $\mu(x)$ grows exponentially as $x \to -\infty$. The former property reflects the fact that the original extremal point process $\mathcal{E}_N$, before the limit $N \to +\infty$, had the mass 2^N. The latter comes from the fact that most of the particles X_k are located far to the left of the maximum $\bar{M}_N$.

Exercise 2.6 Perform a similar analysis of the extremal process seen from the tip, that is, $\bar{\mathcal{E}}_N$, defined in (2.46), and describe the limit of $\bar{\mathcal{E}}_N$.

Exercise 2.7 Perform the analysis of this section for independent Gaussian random variables with mean zero and variance $\sigma^2 N$. Compute the resulting $c_*(\sigma)$ and $\lambda_*(\sigma)$.

3 The connection of the BBM to partial differential equations

Before continuing with the random processes points of view, let us describe the connection between branching random processes and nonlinear partial differential equations. The starting point is the following fact. Let B_t be the d-dimensional Brownian motion with variance σ, and $g(x)$ be a bounded function. Consider the function

$$u(t,x) = \mathbb{E}_x g(B(t)). \tag{3.1}$$

The notation in the right side means that $B(t)$ starts at the point x at $t = 0$. A key property of the Brownian motion we will be using is that $u(t,x)$ satisfies the heat equation

$$\frac{\partial u}{\partial t} = \frac{\sigma^2}{2}\Delta u, \quad t > 0, \ x \in \mathbb{R}^d, \tag{3.2}$$

with the initial condition $u(0,x) = g(x)$. This result can be found in numerous textbooks on stochastic analysis.

3.1 Discrete linear equations and random walks

Let us first explain informally how the linear partial differential equation (3.2) can be explained in the context of random walks, in a very simple way. The starting point is a discrete time Markov jump process $X_{n\tau}$, with a time step $\tau > 0$, defined on one-dimensional a lattice with mesh size h:

$$h\mathbb{Z} = \{0, \pm h, \pm 2h, \dots\}.$$

The particle position evolves as follows: if the particle is located at a position $x \in h\mathbb{Z}$ at the time $t = n\tau$ then at the time $t = (n+1)\tau$ it jumps to a random position $y \in h\mathbb{Z}$, with the transition probability

$$P(X_{(n+1)\tau} = y |\ X_{n\tau} = x) = k(x-y), \quad x, y \in h\mathbb{Z}. \tag{3.3}$$

Here, $k(x)$ is a prescribed non-negative kernel such that

$$\sum_{y \in h\mathbb{Z}} k(y) = 1. \tag{3.4}$$

The classical symmetric random walk with a spatial step h and a time step τ corresponds to the choice $k(\pm h) = 1/2$, and $k(y) = 0$ otherwise – the particle may only jump to the nearest neighbor on the left and on the right, with equal probabilities.

In order to connect this process to an evolution equation, let us take a function $f : h\mathbb{Z} \to \mathbb{R}$, defined on the lattice, and introduce

$$u(t,x) = \mathbb{E}(f(X_t(x))). \tag{3.5}$$

Here, $X_t(x)$, $t \in \tau\mathbb{N}$, is the above Markov process starting at a position $X_0(x) = x \in h\mathbb{Z}$ at the time $t = 0$. If $f \geq 0$ then one may think of $u(t,x)$ as the expected value of a "prize" to be collected at the time t at a (random) location of $X_t(x)$ given that the process starts at the point x at the time $t = 0$. An important special case is when f is the characteristic function of a set A. Then, $u(t, x)$ is the probability that the jump process $X_t(x)$ that starts at the position $X_0 = x$ is inside the set A at the time t.

As the process $X_t(x)$ is Markov, the function $u(t, x)$ satisfies the following relation

$$u(t+\tau, x) = \mathbb{E}(f(X_{t+\tau}(x))) = \sum_{y\in h\mathbb{Z}} P(X_\tau = y|X_0 = x)\mathbb{E}(f(X_t(y))) = \sum_{y\in h\mathbb{Z}} k(x-y)u(t, y). \tag{3.6}$$

This is because after the initial step when the particle jumps at the time τ from the starting position x to a random position y, the process "starts anew", and runs for time t between the times τ and $t+\tau$ – this is where the second equality in (3.6) comes from. The third equality simply uses the definition (3.3) of $k(x)$. Equation (3.6) can be re-written, using (3.4) as

$$u(t+\tau, x) - u(t, x) = \sum_{y\in h\mathbb{Z}} k(x-y)[u(t, y) - u(t, x)]. \tag{3.7}$$

The key point is that the discrete equation (3.7) leads to various interesting continuous limits as $h \downarrow 0$ and $\tau \downarrow 0$, depending on the choice of the transition kernel $k(y)$, and on the relative size of the spatial mesh size h and the time step τ. In other words, depending on the microscopic model – the particular properties of the random walk – we will end up with different macroscopic continuous models.

The heat equation and random walks

Let us show how this can be done to obtain the heat equation

$$\frac{\partial u}{\partial t} = a\frac{\partial^2 u}{\partial x^2}, \tag{3.8}$$

with a constant diffusivity constant $a > 0$. This gives an informal explanation for the interpretation (3.1)-(3.2) of the heat equation in terms of Brownian motion. We will assume that the transition probability kernel has the form

$$k(x) = \phi\left(\frac{x}{h}\right), \quad x \in h\mathbb{Z}, \tag{3.9}$$

with a non-negative function $\phi(m) \geq 0$ defined on $\mathbb{Z}$, such that

$$\sum_m \phi(m) = 1. \tag{3.10}$$

This form of $k(x)$ allows us to re-write (3.7) as

$$u(t+\tau, x) - u(t, x) = \sum_{y\in h\mathbb{Z}} \phi\left(\frac{x-y}{h}\right)[u(t, y) - u(t, x)], \tag{3.11}$$

or, equivalently,

$$u(t+\tau, x) - u(t, x) = \sum_{m\in\mathbb{Z}} \phi(m)[u(t, x-mh) - u(t, x)]. \tag{3.12}$$

In order to arrive to the heat equation in the limit, we will make the assumption that jumps are symmetric on average:

$$\sum_{m\in\mathbb{Z}} m\phi(m) = 0. \tag{3.13}$$

Then, expanding the right side of (3.12) in h and the left side in τ, we obtain

$$\tau\frac{\partial u(t,x)}{\partial t} = \frac{ah^2}{2}\frac{\partial^2 u}{\partial x^2}(t,x) + \text{lower order terms}, \tag{3.14}$$

with

$$a = \sum_m |m|^2\phi(m). \tag{3.15}$$

To balance the left and the right sides of (3.14), we need to take the time step $\tau = h^2$ – note that the scaling $\tau = O(h^2)$ is essentially forced on us if we want to balance the two sides of this equation. Then, in the limit $\tau = h^2 \downarrow 0$, we obtain the heat equation

$$\frac{\partial u(t,x)}{\partial t} = \frac{a}{2}\frac{\partial^2 u(t,x)}{\partial x^2}. \tag{3.16}$$

The diffusion coefficient a given by (3.15) is the second moment of the jump size – in other words, it measures the "overall jumpiness" of the particles. This is a very simple example of how the microscopic information, the kernel $\phi(m)$, translates into a macroscopic quantity – the overall diffusion coefficient a in the macroscopic equation (3.16).

Exercise 3.1 Show that if (3.13) is violated and

$$b = \sum_{m\in\mathbb{Z}} m\phi(m) \neq 0, \tag{3.17}$$

then one may take $\tau = h$, and the formal limit of (3.12) is the advection equation

$$\frac{\partial u(t,x)}{\partial t} + b\frac{\partial u(t,x)}{\partial x} = 0, \tag{3.18}$$

without any diffusion.

Exercise 3.2 Relate the limit in (3.18) to the law of large numbers and explain the relation $\tau = h$ in these terms. How can (3.16) and the relation $\tau = h^2$ between the temporal and spatial steps be explained in terms of the central limit theorem?

3.2 Linear parabolic equations and branching random walks

Let us now explain how we can obtain a probabilistic interpretation for a parabolic equation with a zero-order term:

$$\frac{\partial u}{\partial t} = \frac{\sigma^2}{2}\frac{\partial^2 u}{\partial x^2} + mu, \tag{3.19}$$

with some $m \in \mathbb{R}$ fixed. Let us consider a branching Brownian motion of variance $\sigma > 0$, with an exponential clock such that

$$\mathbb{P}(\tau > t) = e^{-t}. \tag{3.20}$$

We consider general branching so that a particle branches into k children with probabilities p_k, so that

$$\sum_{k=1}^{\infty} p_k = 1,$$

and the average number of off-spring is

$$\bar{N} = \sum_{k=1}^{\infty} k p_k. \tag{3.21}$$

Let us assume that the BBM starts at $t = 0$ at the position x and denote the locations of the BBM particles at the time $t > 0$ by $X_1(t), \dots, X_{N_t}(t)$. Given a bounded function $g(x)$, consider the function

$$u(t,x) = \mathbb{E}_x \sum_{k=1}^{N_t} g(X_k(t)). \tag{3.22}$$

This is the analog of (3.1) that led to the standard heat equation in the case of the standard Brownian motion, with no branching. In order to get an equation for $u(t,x)$ let us write a recursive relation, very similar to what we have seen in the proofs of Propositions 1.1 and 1.4. Looking at the first branching event gives the renewal relation

$$\begin{aligned} u(t,x) &= \mathbb{E}_x(g(B_t))\mathbb{P}(\tau_1 > t) + \sum_{k=1}^{\infty} k p_k \int_0^t \mathbb{E}_x(u(t-s,B_s))\mathbb{P}(\tau_1 \in ds) \\ &= \mathbb{E}_x(g(B_t))e^{-t} + \sum_{k=1}^{\infty} k p_k \int_0^t \mathbb{E}_x(u(t-s,B_s))e^{-s} ds. \end{aligned} \tag{3.23}$$

Note that (3.1)-(3.2) implies that the function

$$v(t,x) = \mathbb{E}_x(g(B_t))$$

is the solution to the heat equation

$$\frac{\partial v}{\partial t} = \frac{\sigma^2}{2}\Delta v, \tag{3.24}$$

with the initial condition $v(0,x) = g(x)$. Hence, it can be written as

$$v(t,x) = [e^{\sigma^2 t\Delta/2} g(\cdot)](x). \tag{3.25}$$

In addition, for $0 < s < t$ fixed, the function

$$w(\tau,x) = \mathbb{E}_x(u(t-s,B_\tau)) \tag{3.26}$$

is the solution to the heat equation

$$\frac{\partial w}{\partial \tau} = \frac{\sigma^2}{2}\Delta w, \tag{3.27}$$

with the initial condition $w(0,x) = u(t-s,x)$. That is, we have

$$w(\tau,x) = [e^{\sigma^2 \tau\Delta/2} u(t-s,\cdot)](x). \tag{3.28}$$

It follows that

$$\mathbb{E}_x(u(t-s,B_s)) = w(s,x) = [e^{\sigma^2 s\Delta/2}u(t-s,\cdot)](x). \tag{3.29}$$

Hence, (3.23) has the form

$$u(t,x) = e^{-t}[e^{\sigma^2 t\Delta/2}g(\cdot)](x) + \bar{N}\int_0^t [e^{\sigma^2 s\Delta/2}u(t-s,\cdot)](x)e^{-s}ds. \tag{3.30}$$

This is simply a way to write the Duhamel formula for the initial value problem

$$\begin{aligned}&\frac{\partial u}{\partial t} = \frac{\sigma^2}{2}\Delta u + (\bar{N}-1)u,\\ &u(0,x) = g(x).\end{aligned} \tag{3.31}$$

This is exactly (3.19), with $m = \bar{N}-1$.

Exercise 3.3 Show why (3.30) implies (3.31). More generally, consider an evolution equation of the form

$$\frac{du}{dt} = \mathcal{L}u + F(t), \tag{3.32}$$

with an initial condition $u(0) = u_0$. Show that, at least, on the formal level, we have

$$u(t) = e^{\mathcal{L}t}u_0 + \int_0^t e^{\mathcal{L}s}F(t-s)ds, \quad \text{for } t > 0. \tag{3.33}$$

The assumption in Exercise 3.3 is that the exponential $\exp(\mathcal{L}t)$ is well-defined for all $t > 0$. We will often use this result for $\mathcal{L} = \Delta$, for which this condition holds on most reasonable function spaces. A typical example when this is not the case is $\mathcal{L} = -\Delta$.

Exercise 3.4 Note that we have obtained (3.19) with $m = \bar{N}-1 > 0$. Modify the branching process to cover also equations with $m < 0$. Which range of m can be obtained this way? What needs to be done to get $m < -1$?

Exercise 3.5 (i) Given an open set $A \subset \mathbb{R}^n$ with a smooth boundary, let $N_A(t)$ be the number of the binary BBM particles inside A at a time $t > 0$. Use (3.31) wih an appropriate initial condition to find $\mathbb{E}(N_A(t))$.
(ii) Let $N(t)$ be the total number of BBM particles present at a time $t > 0$. Use (3.31) to show that $\mathbb{E}(N(t)) = \exp((\bar{N}-1)t)$. This is the same as the result of Proposition 1.1.
(iii) Use (3.31) and the properties of the standard heat equation to show that $M(t) = N(t)e^{-t}$ is a martingale.
(iv*) Find out if it is possible to use the properties of the standard heat equation to show that $M_\infty = \lim_{t\to+\infty} M(t)$ is exponentially distributed.

3.3 The Fisher-KPP equation and the branching Brownian motion

3.3.1 Derivation of the Fisher-KPP equation

Let us now explain how the Fisher-KPP equation

$$\frac{\partial u}{\partial t} = \frac{\sigma^2}{2}\Delta u + u - u^2 \tag{3.34}$$

arises from very similar considerations, in the context of BBM. This was discovered by Henry McKean in [12].

Given a bounded function $g(x)$, we now consider a functional of the branching Brownian motion not of the additive form (3.22) but multiplicative:

$$v(t,x) = \mathbb{E}_x\Big(\prod_{k=1}^{N_t} g(X_k(t))\Big). \tag{3.35}$$

In order to get an equation for $v(t,x)$ let us write a renewal relation, very similar to what we have seen in (3.23). Looking at the first branching event gives

$$\begin{aligned} v(t,x) &= \mathbb{E}_x(g(B_t))\mathbb{P}(\tau_1 > t) + \sum_{k=1}^{\infty} p_k \int_0^t \mathbb{E}_x\Big(\big[v(t-s, x+B_s)\big]^k\Big)\mathbb{P}(\tau_1 \in ds) \\ &= \mathbb{E}_x(g(B_t))e^{-t} + \sum_{k=1}^{\infty} p_k \int_0^t \mathbb{E}_x\Big(\big[v(t-s, x+B_s)\big]^k\Big)e^{-s}ds. \end{aligned} \tag{3.36}$$

Recall that we consider a branching Brownian motion with a variance $\sigma > 0$. Hence, as in (3.29), we have

$$\mathbb{E}_x\big[v(t-s, x+B_s)\big]^k = [e^{\sigma^2 s\Delta/2} v^k(t-s,\cdot)](x). \tag{3.37}$$

Recall also (3.25):

$$\mathbb{E}_x(g(B_t)) = [e^{\sigma^2 t\Delta/2} g(\cdot)](x). \tag{3.38}$$

Now, (3.36) becomes

$$\begin{aligned} v(t,x) &= [e^{\sigma^2 t\Delta/2} g(\cdot)](x)e^{-t} + \sum_{k=1}^{\infty} p_k \int_0^t [e^{\sigma^2 s\Delta/2} v^k(t-s,\cdot)](x)e^{-s}ds \\ &= [e^{\sigma^2 t\Delta/2} g(\cdot)](x)e^{-t} + \int_0^t [e^{\sigma^2 s\Delta/2} F(v(t-s,\cdot))](x)e^{-s}ds. \end{aligned} \tag{3.39}$$

This is the Duhamel representation for the initial value problem

$$\begin{aligned} &\frac{\partial v}{\partial t} = \frac{\sigma^2}{2}\Delta v - v + F(v), \\ &v(0,x) = g(x). \end{aligned} \tag{3.40}$$

Here, the nonlinearity $F(v)$ is given by the generating function for the branching process:

$$F(v) = \sum_{k=1}^{\infty} p_k v^k. \tag{3.41}$$

From the PDE point of view, it is often convenient to use instead the function

$$u(t,x) = 1 - v(t,x) = 1 - \mathbb{E}_x\Big(\prod_{k=1}^{N_t} g(X_k(t))\Big). \tag{3.42}$$

It satisfies the initial value problem

$$\begin{aligned}&\frac{\partial u}{\partial t}=\frac{\sigma^2}{2}\Delta u+f(u),\\&u(0,x)=1-g(x).\end{aligned}\tag{3.43}$$

Here, we have defined

$$f(u)=1-u-F(1-u)=1-u-\sum_{k=1}^{\infty}p_k(1-u)^k.\tag{3.44}$$

In the case of the purely binary branching, when $p_2=0$ and all other $p_k=0$, the function $f(u)$ takes the form

$$f(u)=1-u-(1-u)^2=u(1-u).$$

Then, (3.43) becomes the classical Fisher-KPP equation

$$\begin{aligned}&\frac{\partial u}{\partial t}=\frac{\sigma^2}{2}\Delta u+u-u^2,\\&u(0,x)=1-g(x).\end{aligned}\tag{3.45}$$

As a small remark, applied mathematicians usually take $\sigma=\sqrt{2}$ and write (3.45) as

$$\begin{aligned}&\frac{\partial u}{\partial t}=\Delta u+u-u^2,\\&u(0,x)=1-g(x).\end{aligned}\tag{3.46}$$

Other symmetric functionals

We have now seen how a partial differential equation can be derived for two functionals of the Brownian motion: the additive functional

$$u(t,x)=\mathbb{E}_x\Big(\sum_{k=1}^{N_t}g(X_k(t))\Big),\tag{3.47}$$

and the multiplicative one

$$v(t,x)=\mathbb{E}_x\Big(\prod_{k=1}^{N_t}g(X_k(t))\Big).\tag{3.48}$$

The function $u(t,x)$ satisfies the linear equation

$$\frac{\partial u}{\partial t}=\frac{\sigma^2}{2}\Delta u+(\bar{N}-1)u,\tag{3.49}$$

and $v(t,x)$ satisfies the nonlinear equation (3.40)

$$\frac{\partial v}{\partial t}=\frac{\sigma^2}{2}\Delta v-v+F(v).\tag{3.50}$$

Both $u(t,x)$ and $v(t,x)$ are expectations of symmetric functions of $g(X_1),\dots,g(X_{N_t})$. The next exercise asks you to consider more general symmetric functionals of $g(X_1),\dots,g(X_{N_t})$.

Exercise 3.6 (i*) Consider the function

$$z(t,x) = \mathbb{E}_x\Big(\sum_{1\le i<j\le N_t} g(X_i(t))g(X_j(t))\Big). \tag{3.51}$$

Write down a renewal relation for $z(t,x)$ and try to describe it as a solution to some partial differential equation.

(ii*) Let F be a symmetric function of its arguments. Investigate if

$$w(t,x) = \mathbb{E}_x\big[F(g(X_1(t)), g(X_2(t)), \dots, g(X_{N_t}(t)))\big] \tag{3.52}$$

satisfies a tractable problem. One should probably first consider the case when F is a symmetric polynomial, which is the case for $u(t,x)$ and $v(t,x)$ defined in (3.47) and (3.48). Can one characterize a class of partial differential equations that have such interpretation in terms of a symmetric functional of the branching Brownian motion?

3.3.2 The law of the maximum of BBM

Let us now make a particular choice of the function $g(x)$ as

$$g(x) = \mathbb{1}(x \ge 0). \tag{3.53}$$

Then, the function $v(t,x)$ defined by (3.35)

$$v(t,x) = \mathbb{E}_x\Big(\prod_{k=1}^{N_t} g(X_k(t))\Big), \tag{3.54}$$

has the meaning

$$v(t,x) = \mathbb{P}_x\big\{\text{all } X_k(t) \ge 0\big\}, \tag{3.55}$$

hence the meaning of $u(t,x)$ is

$$u(t,x) = \mathbb{P}_x\big\{\text{some } X_k(t) \le 0\big\}. \tag{3.56}$$

Therefore, the solution to the initial value problem

$$\begin{aligned} u_t &= \frac{\sigma^2}{2}u_{xx} + f(u), \\ u(0,x) &= \mathbb{1}(x<0), \end{aligned} \tag{3.57}$$

is

$$u(t,x) = \mathbb{P}_x\Big\{\min_{1\le k\le N_t} X_k(t) \le 0\Big\}.$$

The translational and reflection invariance of the process means that $u(t,x)$ can be written also as

$$u(t,x) = \mathbb{P}_0\Big\{\max_{1\le k\le N_t} X_k(t) \ge x\Big\},$$

that is, the solution $u(t,x)$ to the Fisher-KPP equation (3.57) with the (reflected) Heaviside function as the initial condition is the probability distribution function of the maximum of the branching Brownian motion starting at $x = 0$. In particular, the median point $m(t)$ such that

$$\mathbb{P}_0\Big\{\max_{1\le k\le N_t} X_k(t) \ge m(t)\Big\} = \frac{1}{2} \tag{3.58}$$

can be characterized by

$$u(t, m(t)) = \frac{1}{2}. \tag{3.59}$$

Later we will see that there exists $x_0 \in \mathbb{R}$ so that $m(t)$ has the asymptpotics

$$m(t) = c_* t - \frac{3}{2\lambda_*}\log t + x_0 + o(1), \quad \text{as } t \to +\infty. \tag{3.60}$$

Here, the speed c_* and the exponent λ_* are given by

$$c_* = \sqrt{2(\bar{N} - 1)}, \quad \lambda_* = c_*. \tag{3.61}$$

This asymptotics should be contrasted with the asymptotics (2.16)

$$m_N = c_* N - \frac{1}{2\lambda_*}\log N + x_0 + o(1), \quad \text{as } N \to +\infty, \tag{3.62}$$

with

$$c_* = \sqrt{2\log 2}, \quad \lambda_* = c_*, \tag{3.63}$$

for median of the distribution of the maximum of 2^N uncorrelated Gaussian random variables of variance N. This number of particles corresponds to $\bar{N} = 1 + \log 2$, hence the speed c_* and the exponential rate of decay λ_* of the distribution match those for the corresponding BBM. What is different is the pre-factor $3/2$ in front of the $\log t$ term in (3.60) as opposed to $1/2$ in expression (3.62) for the uncorrelated model.

3.3.3 The Laplace transform of the point process for the branching Brownian motion

We now explain how the Laplace transform of the point process of the branching Brownian motion can be connected to the Fisher-KPP equation. Let $X_1(t), \dots, X_{N_t}(t)$ be the locations of the BBM particles at a time $t > 0$, and set

$$\mathcal{E}(t) = \sum_{k=1}^{N_t} \delta(x - X_k(t)). \tag{3.64}$$

For the moment, we assume that the BBM starts at the position $x = 0$ at $t = 0$. Note that at the moment we do not center the locations of the particles near the maximum, as we did for uncorrelated Gaussians in Section 2.4. This will be done later. The Laplace functional of $\mathcal{E}(t)$ is

$$\Psi(\phi)(t) = \mathbb{E}_0 \exp\Big(-\int \phi(x)d\mathcal{E}(t)\Big) = \mathbb{E}_0 \exp\Big(-\sum_{k=1}^{N_t} \phi(X_k(t))\Big). \tag{3.65}$$

Here, $\phi(x)$ is a non-negative bounded test function. The subscript 0 in (3.65) refers to the starting point of the branching Brownian motion. A simple but important observation is that (3.65) can be written as

$$\Psi(\phi)(t) = \mathbb{E}_0 \exp\Big(-\int \phi(x) d\mathcal{E}\Big) = \mathbb{E}_0 \exp\Big(-\sum_{k=1}^{N_t} \phi(X_k(t))\Big) = \mathbb{E}_0\Big(\prod_{k=1}^{N_t} g(X_k(t))\Big), \quad (3.66)$$

with the function $g(x)$ given by

$$g(x) = e^{-\phi(x)}. \quad (3.67)$$

Combining with we have done in Section 3.3.1, we conclude that if we let $u(t,x)$ be the solution to the initial value problem

$$\begin{aligned} &\frac{\partial u}{\partial t} = \frac{\sigma^2}{2}\Delta u + f(u), \\ &u(0,x) = 1 - e^{-\phi(x)}, \end{aligned} \quad (3.68)$$

then

$$\Psi(\phi)(t) = 1 - u(t,0). \quad (3.69)$$

Let us see what happens if $\phi(x) = \lambda \mathbb{1}_{[a,b]}(x)$ is a step function. In that case, the initial condition in (3.68) is a multiple of of $\phi(x)$:

$$u(0,x) = (1 - e^{-\lambda})\mathbb{1}_{[a,b]}(x). \quad (3.70)$$

Exercise 3.7 Let $u(t,x)$ be the solution to (3.68) with the initial condition (3.70), corresponding to $\phi(x) = \lambda \mathbb{1}_{[a,b]}(x)$. Use the definition of the Laplace transform $\Psi(\phi)(t)$ and (3.69) to interpret what it means, in terms of the branching Brownian motion, that $u(t,0)$ is small, or that $u(t,0) \approx 1$.

Thus, the Laplace transform of the point process of the branching Brownian motion can be directly computed in terms of a solution of the Fisher-KPP equation with a suitable initial condition. This provides an occasionally powerful tool to understand the statistics of the branching Brownian motion in terms of solutions of PDEs. For example, one may be interested in finding a location y that would make the limit of the re-centered point process, with

$$Z_j(t) = X_j(t) - y \quad (3.71)$$

be non-trivial. This is what we did in Section 2.4 by re-centering at the locations $y = m_N$. The re-centered point process is

$$\mathcal{E}(t,x,y) = \sum_{k=1}^{N_t} \delta(x + y - X_k(t)). \quad (3.72)$$

The Laplace functional of $\mathcal{E}(t,y)$ is

$$\begin{aligned}\Psi(\phi)(t,y) &= \mathbb{E}_0 \exp\Big(-\int \phi(x)d\mathcal{E}(t,x,y)\Big) = \mathbb{E}_0 \exp\Big(-\sum_{k=1}^{N_t}\phi(X_k(t)-y)\Big)\\ &= \mathbb{E}_0 \exp\Big(-\sum_{k=1}^{N_t}\phi_y(X_k(t))\Big).\end{aligned} \tag{3.73}$$

Here, we have denoted

$$\phi_y(x) = \phi(x-y).$$

It follows that

$$\Psi(\phi(t,y)) = 1 - \tilde{u}(t,0;y), \tag{3.74}$$

and $\tilde{u}(t,x;y)$ is the solution to the Fisher-KPP equation with a shifted initial condition

$$\begin{aligned}&\frac{\partial \tilde{u}}{\partial t} = \frac{\sigma^2}{2}\Delta\tilde{u} + f(\tilde{u}),\\ &\tilde{u}(0,x;y) = 1 - e^{-\phi(x-y)},\end{aligned} \tag{3.75}$$

We see that

$$\tilde{u}(t,x,y) = u(t,x-y), \tag{3.76}$$

with $u(t,x)$ that is the solution to (3.68). In particular, we have

$$\tilde{u}(t,0;y) = u(t,-y), \tag{3.77}$$

and

$$\Psi(\phi)(t,y) = u(t,-y). \tag{3.78}$$

Thus, solution to (3.68) encodes the information about the shifted processes $Z_k(t)$ as well. In particular, to find the shift y that would make the limit of $Z_k(t)$ non-trivial, we need to find the locations where $u(t,y)$ is neither close to 0 nor to 1. A natural question is if this range of points would depend on the initial condition to (3.68), and we will soon see that it does not.

3.3.4 Some basic properties of the Fisher-KPP nonlinearities

To understand what kind of partial differential equations are related to the branching Brownian motion by the above probabilistic interpretation, let us discuss some general properties of the nonlinearities $f(u)$ that can be obtained via (3.44):

$$f(u) = 1 - u - \sum_{k=1}^{\infty} p_k(1-u)^k. \tag{3.79}$$

We claim that $f(u)$ satisfies the following properties: first,

$$f(0) = f(1) = 0, \quad f(u) > 0 \text{ for } 0 < u < 1. \tag{3.80}$$

Second, $f(u)$ is concave, so that, in particular it satisfies the so-called FKPP assumption

$$f(u) \le f'(0)u, \quad \text{for all } u \in (0,1). \tag{3.81}$$

Finally, we have

$$f'(0) = \bar{N} - 1, \tag{3.82}$$

with

$$\bar{N} = \sum_{k=1}^{\infty} k p_k. \tag{3.83}$$

Let us prove these properties. It is immediate to see that $f(1) = 0$ simply from (3.79). We also have

$$f(0) = 1 - \sum_{k=1}^{\infty} p_k = 0,$$

because

$$\sum_{k=1}^{\infty} p_k = 1. \tag{3.84}$$

To finish the proof of (3.80), note that for $0 < u < 1$ we have

$$f(u) = 1 - u - \sum_{k=1}^{\infty} p_k (1-u)^k > 1 - u - \sum_{k=1}^{\infty} p_k (1-u) = 0, \tag{3.85}$$

due to (3.84).

To establish the FKPP property (3.81), we simply note that $f(u)$ of the form (3.79) is concave on $[0, 1]$. This is because

$$f''(u) = -\sum_{k=2}^{\infty} k(k-1)(1-u)^{k-2} \le 0. \tag{3.86}$$

An alternative way to see that (3.81) holds is to first note that

$$f'(0) = -1 + \sum_{k=1}^{\infty} k p_k = \bar{N} - 1, \tag{3.87}$$

which is (3.82). This allows us to write for $0 < u < 1$

$$f(u) = 1 - u - \sum_{k=1}^{\infty} p_k (1-u)^k \le 1 - u - \sum_{k=1}^{\infty} p_k (1 - ku) = (\bar{N} - 1)u = f'(0)u, \tag{3.88}$$

which is (3.81).

As a side remark, if we assume that $p_1 = 0$, to rule out not quite sensical "branching into one particle" events, then any $f(u)$ of the form (3.79) satisfies

$$f'(1) = -1. \tag{3.89}$$

The above conditions are, clearly, not sufficient for a nonlinearity $f(u)$ to be of the BBM origin but give a good idea of the properties of that class.

3.4 Voting models, branching Brownian motion and partial differential equations

It turns out that the connection between the branching Brownian motion and partial differential equations extends beyond the Fisher-KPP type nonlinearities. This is a very interesting observation by Alison Etheridge, Nic Freeman and Sarah Penington [6, 7].

3.4.1 The voting scheme for the Fisher-KPP equation

Let us first discuss how the the McKean's probabilistic interpretation of the standard Fisher-KPP equation

$$\begin{aligned} &\frac{\partial u}{\partial t} = \frac{\sigma^2}{2}\Delta u + u - u^2, \quad t > 0,\ x \in \mathbb{R}^d, \\ &u(0,x) = p(x), \end{aligned} \tag{3.90}$$

can be recast in terms of a voting process. Let us assume that the initial condition $p(x)$ satisfies

$$0 \le p(x) \le 1, \quad \text{for all } x \in \mathbb{R}^d. \tag{3.91}$$

We run a standard binary branching Brownian motion with variance $\sigma > 0$ until a time $t > 0$. Each of the particles $X_1(t), \dots, X_{N_t}(t)$ that are present at the final time t, vote 0 or 1, with the probability $p(x)$:

$$\mathbb{P}(V_j = 1) = p(X_j(t)). \tag{3.92}$$

Next, we propagate the voting decisions back up the genealogy tree, with the rule that the parent votes 1 if at least one of its two children voted 1. Let V_{orig} be the resulting vote of the original ancestral particle that started at $t = 0$ at the position x, and consider the function

$$u(t,x) = \mathbb{P}_x(V_{orig} = 1). \tag{3.93}$$

Let us now use the familiar renewal idea to obtain an equation for the function $u(t,x)$. There are two possible ways in which the original ancestor can vote 1: either both of its children voted 1, or one of them voted 1 and one voted 0. In the latter case, there are two choices of the particle that voted 0. This gives the renewal identity

$$\begin{aligned} u(t,x) &= \mathbb{E}_x p(B_t)\mathbb{P}(\tau_1 > t) \\ &\quad + \int_0^t \mathbb{E}_x(u^2(t-s,B_s) + 2u(t-s,B_s)(1-u(t-s,B_s))\mathbb{P}(\tau_1 \in ds) \\ &= \mathbb{E}_x p(B_t)e^{-t} + \int_0^t \mathbb{E}_x(u^2(t-s,B_s) + 2u(t-s,B_s)(1-u(t-s,B_s))e^{-s}ds. \end{aligned} \tag{3.94}$$

Now, proceeding exactly as in Section 3.3.1, we deduce that $u(t,x)$ satisfies the initial value problem

$$\begin{aligned} &\frac{\partial u}{\partial t} = \frac{\sigma^2}{2}\Delta u - u + u^2 + 2u(1-u), \\ &u(0,x) = p(x). \end{aligned} \tag{3.95}$$

This is equivalent to

$$\begin{aligned}&\frac{\partial u}{\partial t}=\frac{\sigma^2}{2}\Delta u+u-u^2,\\&u(0,x)=p(x),\end{aligned}\tag{3.96}$$

which is the initial value problem for the Fisher-KPP equation.

Of course, this interpretation of the Fisher-KPP equation is not very different from McKean's. The reason is that in this voting scheme the original ancestor votes 0 if and only if all of the particles $X_1(t),\dots,X_{N_t}(t)$ present at the time t vote 0. In other words, we have

$$\begin{aligned}v(t,x)=1-u(t,x)=\mathbb{P}_x\big(V_{orig}=0\big)&=\mathbb{P}\big(V(X_k)=0\ \text{ for all } 1\le k\le N_t\big)\\&=\mathbb{E}_x\Big(\prod_{k=1}^{N_t}(1-p(X_k))\Big),\end{aligned}\tag{3.97}$$

which is exactly McKean's functional. Therefore, $v(t,x)$ satisfies

$$\begin{aligned}&\frac{\partial v}{\partial t}=\frac{\sigma^2}{2}\Delta v-v+v^2,\\&v(0,x)=1-p(x),\end{aligned}\tag{3.98}$$

and (3.97) follows. Nevertheless, we will see that the addition of the voting scheme expands dramatically the class of partial differential equations for which a probabilistic interpretation in terms of the branching Brownian motion is possible.

3.4.2 The voting scheme for the Allen-Cahn equation

To be concrete, consider first a ternary branching Brownian motion starting at the time $t=0$ at a point $x\in\mathbb{R}^n$. That is, each branching event produces three children. Let us run the process until a time $t>0$, with the particles located at $X_1(t),\dots,X_{N_t}(t)$. At the time t, each of these particle votes 1 or 0 with the probabilities

$$\mathbb{P}(V_j=1)=p(X_j(t)),\tag{3.99}$$

as in (3.92). Here, as before, $p(x)$ is a fixed function such that $0\le p(x)\le 1$ for all $x\in\mathbb{R}$, and V_j is the vote of the particle $X_j(t)$. This produces the votes of the youngest generation of particles. So far, except for the ternary branching replacing the binary branching, the process is exactly as when we obtained the Fisher-KPP equation (3.96). The difference with what we have considered above is in how the parents vote. In the present case, we go back up the ternary branching tree, and each parent accepts the vote of the majority of its children. Let V_{orig} be the resulting vote of the original ancestral particle that started at $t=0$ at the position x, and consider the function

$$u(t,x)=\mathbb{P}_x(V_{orig}=1).\tag{3.100}$$

Again, we can use the renewal idea to obtain an equation for the function $u(t,x)$. There are two possible ways in which the original ancestor can vote 1: either all three of its children voted 1, or two of them voted 1 and one voted 0. In the latter case, there are three choices of the particle that voted 0. This gives the renewal identity

$$
\begin{aligned}
u(t,x) &= \mathbb{E}_x p(B_t)\mathbb{P}(\tau_1 > t) \\
&\quad + \int_0^t \mathbb{E}_x(u^3(t-s,B_s) + 3u^2(t-s,B_s)(1-u(t-s,B_s))\mathbb{P}(\tau_1 \in ds) \\
&= \mathbb{E}_x p(B_t)e^{-t} + \int_0^t \mathbb{E}_x(u^3(t-s,B_s) + 3u^2(t-s,B_s)(1-u(t-s,B_s))e^{-s}ds.
\end{aligned}
\tag{3.101}
$$

As before, we deduce that $u(t,x)$ satisfies the initial value problem

$$
\begin{aligned}
&\frac{\partial u}{\partial t} = \frac{\sigma^2}{2}\Delta u - u + u^3 + 3u^2(1-u), \\
&u(0,x) = p(x).
\end{aligned}
\tag{3.102}
$$

A simple computation shows that

$$
u^3 + 3u^2(1-u) - u = 3u^2 - 2u^3 - u = u(3u - 2u^2 - 1) = u(1-u)(2u-1). \tag{3.103}
$$

Thus, $u(t,x)$ satisfies the Allen-Cahn equation

$$
\begin{aligned}
&\frac{\partial u}{\partial t} = \frac{\sigma^2}{2}\Delta u + u(1-u)(2u-1), \\
&u(0,x) = p(x).
\end{aligned}
\tag{3.104}
$$

Note that the nonlinearity

$$
\tilde{f}(u) = u(1-u)(2u-1) \tag{3.105}
$$

does not satisfy the Fisher-KPP properties we have discussed in Section 3.3.4. Indeed, it is not even non-negative for $u \in (0,1)$ but rather changes its sign. Thus, the partial differential equation (3.104) does not have an interpretation in terms of a McKean functional. The voting scheme adds a genuinely new aspect here.

Exercise 3.8 Devise a version of the voting scheme interpretation that would apply to other initial conditions $g(x)$, without the restriction that $0 \le g(x) \le 1$ for all $x \in \mathbb{R}$.

3.4.3 A digression on the Allen-Cahn equation

The Allen-Cahn type equations appear in many applications, ranging from biology and combustion to differential geometry, as a very basic model of a diffusive connection between two stable states. It is worth to discuss them in some detail. The main feature of the nonlinearity is that both $u \equiv 0$ and $u \equiv 1$ are stable steady solutions to the corresponding time-dependent ODE

$$
\frac{du}{dt} = \tilde{f}(u), \quad \tilde{f}(u) = u(1-u)(2u-1). \tag{3.106}
$$

Traditionally, it is common to write (3.104) in terms of the function

$$
w(t,x) = 2u(2t,x) - 1,
$$

and also take $\sigma = 1$, which gives

$$\begin{aligned}
&\frac{\partial w}{\partial t} = \Delta w + w(1-w)(1+w), \\
&w(0,x) = w_0(x),
\end{aligned} \tag{3.107}$$

with $w_0(x) = 2p(x) - 1$. This shifts the steady equilibria to the more symmetric $w = \pm 1$ and the unsteady equilibrium to $w = 0$. Solutions to the partial differential equation (3.107) in one dimension, with the boundary conditions such that

$$w(t,x) \to 1, \quad \text{as } x \to -\infty, \quad w(t,x) \to -1, \quad \text{as } x \to +\infty, \tag{3.108}$$

describe the diffusive transitions between regions in space as $x \to -\infty$ where w is close to the stable equilibrium $w \equiv 1$ and those where w is close to $w \equiv -1$ as $x \to +\infty$. They are often taken as a basic model for many phenomena where such diffusive interfaces come up. There are three obvious steady solutions to the Allen-Cahn equation (3.107) given by constants:

$$w(t,x) \equiv 0, \; w(t,x) \equiv 1 \text{ and } w(t,x) \equiv -1. \tag{3.109}$$

Exercise 3.9 Show that $u = 0$ is an unstable solution to the ODE

$$\frac{dw}{dt} = w - w^3, \tag{3.110}$$

and $u = \pm 1$ are two stable solutions to (3.110).

In addition to the spatially uniform steady solutions to (3.107), there are steady solutions that are not uniform in space. Consider for the moment an ordinary differential equation

$$w_0'' + f(w_0) = 0, \quad x \in \mathbb{R}, \tag{3.111}$$

with the boundary conditions (3.108):

$$w_0(x) \to 1, \quad \text{as } x \to -\infty, \quad w_0(x) \to -1, \quad \text{as } x \to +\infty. \tag{3.112}$$

This equation may be solved explicitly: multiplying (3.111) by w_0' and integrating from $-\infty$ to x, using the boundary conditions, leads to

$$\frac{1}{2}(w_0')^2 + F(w_0) = 0, \quad w_0(\pm\infty) = \pm 1. \tag{3.113}$$

Here, we have defined

$$F(s) = \int_{-1}^{s} f(w)dw. \tag{3.114}$$

Letting $x \to +\infty$ in (3.113) we see that a necessary condition for a solution of (3.113) to exist is that $F(1) = 0$, or

$$\int_{-1}^{1} f(w)dw = 0. \tag{3.115}$$

Exercise 3.10 Show that the solutions of (3.111)-(3.112) are unique, up to a translation in the x-variable – note that if $w_0(x)$ is a solution to (3.111)-(3.112), then so is $\tilde{w}(x) = w_0(x+\xi)$, for any $\xi \in \mathbb{R}$.

Exercise 3.11 Show that if $f(w) = w - w^3$ then $w_0(x)$ has an explicit expression

$$w_0(x) = \tanh\Big(\frac{x}{\sqrt{2}}\Big), \tag{3.116}$$

as well as all its translates $u_0(x + \xi)$, with a fixed $\xi \in \mathbb{R}$.

Exercise 3.9 shows that $u = \pm 1$ are stable solutions to (3.107) with respect to perturbations that are spatially uniform – those are solutions to the ODE (3.110). In order to illustrate their stability with respect to perturbations that are not uniform in space, consider the following exercise.

Exercise 3.12 Let $w(t, x)$ be the solution to (3.107) with an initial condition $w(0, x) = w_0(x)$. Suppose that $w_0(x)$ is smooth and $-1 \le w_0(x) \le 1$ for all $x \in \mathbb{R}$ and there exists $\delta_0 > 0$ so that $|w_0(x)| > \delta_0$ for all $x \in \mathbb{R}$. Show that there exists a constant $\omega > 0$ that does not depend on u_0 and a constant $C_0 > 0$ that may depend on w_0 but only through its L^∞-norm, so that if $w_0(x) > \delta_0$ for all $x \in \mathbb{R}$, then

$$\sup_{x\in\mathbb{R}} |w(t, x) - 1| \le C_0 e^{-\omega t}, \tag{3.117}$$

and if $w_0(x) < -\delta_0$ for all $x \in \mathbb{R}$, then

$$\sup_{x\in\mathbb{R}} |w(t, x) + 1| \le C_0 e^{-\omega t}. \tag{3.118}$$

Does either of the constants ω and C_0 depend on δ_0? First, provide an analytical proof. Second, try to use the probabilistic interpretation of the Allen-Cahn equation in terms of the ternary BBM and voting, to prove the same result.

Let us now discuss the long time behavior of the solutions to (3.107):

$$w_t - w_{xx} = w - w^3, \tag{3.119}$$

with an initial condition $w(0, x) = w_0(x)$ that connects the stable states $w = \pm 1$. That is, $w_0(x)$ satisfies the boundary conditions (3.112):

$$\lim_{x\to-\infty} w_0(x) = -1, \quad \lim_{x\to+\infty} w_0(x) = 1, \tag{3.120}$$

and

$$-1 \le w_0(x) \le 1, \quad \text{for all } x \in \mathbb{R}. \tag{3.121}$$

The simplest picture of the time evolution that one may expect is that, because of the equal stability of the two steady states $w = \pm 1$, neither state wins in the long time limit. It is natural to expect then that the solution to (3.119) with an initial condition $w_0(x)$ satisfying the boundary conditions (3.120) at $t = 0$ would converge to a shift of he steady state of the form (3.116) connecting $w = \pm 1$:

$$\phi(x) = \tanh\Big(\frac{x}{\sqrt{2}}\Big). \tag{3.122}$$

This is the subject of the next theorem, that shows, in addition, that the convergence rate is exponential.

Theorem 3.13 *There exists $\omega > 0$ such that for any smooth and bounded initial condition $w_0(x)$ that satisfies (3.120), we can find $x_0 \in \mathbb{R}$ and $C_0 > 0$ such that the solution $w(t,x)$ to (3.119) with $w(0,x) = w_0(x)$, satisfies*

$$|u(t,x) - \phi(x - x_\infty)| \le C_0 e^{-\omega t}, \quad \text{for all } x \in \mathbb{R} \text{ and } t > 0. \tag{3.123}$$

Since there is a one parameter family of steady solutions, naturally, one may ask how the solution of the initial value problem chooses a particular translation of ϕ in the long time limit. In other words, one would like to know how the shift x_∞ depends on the initial condition u_0. This dependence is quite implicit and there is no simple expression for x_∞.

Exercise 3.14 (**) Prove this theorem using the voting scheme for the Allen-Cahn equation.

Let us recall that the solution to the forced heat equation

$$\begin{aligned} &u_t = \frac{\sigma^2}{2}\Delta u + V(t,x),\\ &u(0,x) = g(x), \end{aligned} \tag{3.124}$$

has the following probabilistic interpretation:

$$u(t,x) = \mathbb{E}_x(g(B_t)) + \mathbb{E}_x \int_0^t V(s, B_{t-s}) ds. \tag{3.125}$$

Here, $V(t,x)$ is a prescribed bounded smooth function, and B_t is the Brownian motion of variance σ that starts at the point $x \in \mathbb{R}^d$.

Exercise 3.15 (**) Consider the forced Allen-Cahn equation:

$$\begin{aligned} &u_t = \frac{\sigma^2}{2}\Delta u + u(1-u)(2u-1) + V(t,x),\\ &u(0,x) = g(x). \end{aligned} \tag{3.126}$$

Here, $V(t,x)$ is a prescribed bounded smooth function. Is there an interpretation of the solution to (3.126) in terms of a modification of the voting scheme for the Allen-Cahn equation?

3.4.4 Generalizations to other equations

We may generalize this voting procedure in infinitely many ways. For example, consider a branching process that has both binary and ternary branching, with $p_2 = \alpha$ and $p_3 = 1 - \alpha$. Once again, at the time t the particles $X_1(t), \dots, X_{N_t}(t)$ vote 1 with the probability (3.99)

$$\mathbb{P}(V_j = 1) = p(X_j(t)), \tag{3.127}$$

with a fixed function $p(x)$ such that $0 \le p(x) \le 1$ for all $x \in \mathbb{R}^d$. The votes of the previous generations are determined as follows. A parent that has three children chooses the majority vote of the children, while the parent with two children votes 1 if at least one of the children votes 1. Then, the renewal (3.101) relation becomes

$$\begin{aligned} u(t,x) &= \mathbb{E}_x p(B_t)\mathbb{P}(\tau_1 > t) \\ &+ (1-\alpha)\int_0^t \mathbb{E}_x(u^3(t-s,B_s) + 3u^2(t-s,B_s)(1-u(t-s,B_s))\mathbb{P}(\tau_1 \in ds) \\ &+ \alpha \int_0^t \mathbb{E}_x(u^2(t-s,B_s) + 2u(t-s,B_s)(1-u(t-s,B_s))\mathbb{P}(\tau_1 \in ds) \\ &= \mathbb{E}_x p(B_t)e^{-t} + \int_0^t \mathbb{E}_x\Big[(1-\alpha)u^3(t-s,B_s) + 3(1-\alpha)u^2(t-s,B_s)(1-u(s,B_s) \\ &+ \alpha u^2(t-s,B_s) + 2\alpha u(t-s,B_s)(1-u(t-s,B_s)\Big]e^{-s}ds. \end{aligned} \tag{3.128}$$

This gives the initial value problem

$$\begin{aligned} &\frac{\partial u}{\partial t} = \frac{\sigma^2}{2}\Delta u - u + (1-\alpha)(u^3 + 3u^2(1-u)) + \alpha(u^2 + 2u(1-u)), \\ &u(0,x) = p(x). \end{aligned} \tag{3.129}$$

Note that

$$\begin{aligned} &(1-\alpha)(u^3 + 3u^2(1-u)) + \alpha(u^2 + 2u(1-u)) - u \\ &= (1-\alpha)(u^3 + 3u^2(1-u) - u) + \alpha(u^2 + 2u(1-u) - u) \\ &= (1-\alpha)u(1-u)(2u-1) + \alpha(u - u^2) = (1-\alpha)u(1-u)(2u-1) + \alpha u(1-u) \\ &= u(1-u)((1-\alpha)(2u-1) + \alpha) = u(1-u)(2(1-\alpha)u + 2\alpha - 1). \end{aligned} \tag{3.130}$$

Let us set

$$k = 2 - 2\alpha,$$

so that

$$2\alpha - 1 = 2 - k - 1 = 1 - k.$$

Hence, (3.129) is

$$\begin{aligned} &\frac{\partial u}{\partial t} = \frac{\sigma^2}{2}\Delta u + u(1-u)(ku + 1 - k), \\ &u(0,x) = p(x). \end{aligned} \tag{3.131}$$

Note that since $\alpha \in (0,1)$, we have $k \in (0,2)$. The case $k = 0$, which is $\alpha = 1$ (purely binary branching) corresponds to the pure FKPP problem, and $k = 2$, which is $\alpha = 0$ (purely ternary branching) to the Allen-Cahn equation.

Exercise 3.16 Assume that $1/2 < \alpha < 1$. Show that there exists $p(\alpha) > 0$ so that for any $t > 0$ there exists a purely binary sub-graph of the tree with the probability at least $p(\alpha)$.

Exercise 3.17 Assume that $k \in (0,1)$. Let the initial condition $p(x)$ for (3.129) be continuous, and such that $0 \le p(x) \le 1$ for all $x \in \mathbb{R}$, and $p(x) \not\equiv 0$. Show that

$$\lim_{t\to+\infty} u(t,x) = 1, \quad \text{for all } x \in \mathbb{R}. \tag{3.132}$$

This is known as the hair trigger effect. The standard PDE proof uses some carefully constructed sub-solutions. Instead, note that $k = 1$ corresponds to $\alpha = 1/2$. This happens to be the threshold value for the existence of a purely binary tree with a positive probability. Use this to prove (3.132).

Let us recall that the nonlinearities that can be obtained by the McKean construction satisfy the Fisher-KPP properties (3.80)-(3.81):

$$f(0) = f(1) = 0, \quad f(u) > 0, \quad f(u) \le f'(0)u, \quad \text{for } 0 < u < 1. \tag{3.133}$$

Exercise 3.18 (i) Show that $f(u) = u(1-u)(ku+1-k)$, with $k \in (0,2)$ satisfies the FKPP property for all $k \in (0,1/2)$. Note that $k = 1/2$ corresponds to $\alpha = 3/4$.
(ii*) Interpret this transition in terms of the voting model.

Exercise 3.19 (*) Let $f(u)$ be a polynomial such that $f(0) = f(1) = 0$. Does there exist a voting scheme for all such $f(u)$ that leads to an equation of the form

$$\begin{aligned} &\frac{\partial u}{\partial t} = \frac{\sigma^2}{2}\Delta u + f(u), \\ &u(0,x) = p(x). \end{aligned} \tag{3.134}$$

This should be contained in the results of [5].

4 Convergence in shape to a traveling wave for the Fisher-KPP equation

In this section, we will first use PDE techniques to prove that solutions to the Fisher-KPP equation converge to a traveling wave in shape as $t \to +\infty$.

4.1 Traveling waves for the Fisher-KPP equation

Let us recall some basic facts about traveling waves for the Fisher-KPP equation

$$u_t = \frac{\sigma^2}{2} u_{xx} + f(u), \ t > 0, \ x \in \mathbb{R}. \tag{4.1}$$

We will assume that $f(u)$ is of the McKean form

$$f(u) = 1 - u - \sum_{k=1}^{\infty} p_k (1-u)^k, \tag{4.2}$$

with

$$\sum_{k=1}^{\infty} p_k = 1, \quad p_k \geq 0. \tag{4.3}$$

Equation (4.1) has special solutions, called traveling waves. These are solutions to (4.1) of the form

$$u(t,x) = U_c(x - ct). \tag{4.4}$$

In order for such $u(t,x)$ to be a solution to (4.1), the function $U_c(x)$ has to satisfy the ODE

$$-cU_c' = U_c'' + f(U_c). \tag{4.5}$$

In addition, we will require that $U_c(x)$ satisfy the following boundary conditions at infinity

$$U_c(x) \to 1, \quad \text{as } x \to -\infty, \quad U_c(x) \to 0, \quad \text{as } x \to -\infty. \tag{4.6}$$

Here is the key result on the existence of traveling waves for the Fisher-KPP type nonlinearities.

Proposition 4.1 *Assume that $f(u)$ has the form (4.2)-(4.3). Then, equation (4.5) has positive solutions $U_c(x) > 0$ that satisfy the boundary conditions (4.6) if and only if*

$$c \geq c_* = \sqrt{2\sigma^2 f'(0)}. \tag{4.7}$$

The key role below will be played by the traveling wave profile $U_*(x) = U_{c_*}(x)$ that corresponds to the minimal speed c_*. Let us recall that for the McKean nonlinearities $f'(0)$ is related to the average number of off-spring at each branching event by

$$f'(0) = \bar{N} - 1, \quad \bar{N} = \sum_{k=1}^{\infty} k p_k, \tag{4.8}$$

so that (4.7) says that

$$c_* = \sqrt{2\sigma^2(\bar{N} - 1)}. \tag{4.9}$$

Exercise 4.2 (i) Prove Proposition 4.1 using the phase plane analysis of the trajectories connecting the equilibria $(U, U') = (0, 0)$ and $(U, U') = (1, 0)$ for a given $c \in \mathbb{R}$. It should help to look for an invariant region that has the form of a triangle in the (U, U')-plane, with the base formed by the interval $(0, 1)$ on the U-axis.
(ii) Show that the claim of Proposition 4.1 holds not only for the McKean nonlinearities of the form (4.2)-(4.3) but for any $f(u)$ that satisfies the Fisher-KPP assumption:

$$f(0) = f(1) = 0, \quad 0 < f(u) \leq f'(0)u \text{ for all } 0 < u < 1. \tag{4.10}$$

(iii) How does the claim of Proposition 4.1 change if you drop the requirement that $U_c(x)$ is positive for all $x \in \mathbb{R}$ but keep the boundary conditions (4.6)? For which $c \in \mathbb{R}$ do such solutions to (4.5) exist?

The next exercise will also be very important. It can also be addressed by ODE methods.

Exercise 4.3 (i) Fix the translation of the traveling wave so that $U_c(0) = 1/2$ and set

$$\lambda_c = \frac{c - \sqrt{c^2 - 2\sigma^2 f'(0)}}{\sigma^2}, \text{for } c > c_*, \tag{4.11}$$

and

$$\lambda_* = \frac{c_*}{\sigma^2}. \tag{4.12}$$

Show that if $c > c_*$ then there exists a constant $A > 0$ so that

$$U_c(x) \sim Ae^{-\lambda_c x}, \quad \text{as } x \to +\infty, \tag{4.13}$$

and that there exists $A > 0$ so that

$$U_*(x) \sim Axe^{-\lambda_* x}, \quad \text{as } x \to +\infty. \tag{4.14}$$

(ii) Either using part (i) or otherwise, show that the traveling wave profile $U_c(x)$ is unique up to translation in x: if both $U_c(x)$ and $\tilde{U}_c(x)$ are solutions to (4.5)-(4.6), with the same $c \geq c_*$, then there exists $x_0 \in \mathbb{R}$ so that $U(x) = \tilde{U}(x + x_0)$ for all $x \in \mathbb{R}$.

The key point of part (i) of Exercise 4.3 is that for $c > c_*$ the traveling waves $U_c(x)$ have a "purely exponential" decay as $x \to +\infty$ but the minimal speed traveling wave $U_*(x)$ has an extra factor of x in front of the exponential.

Note that the coefficient A in (4.14) depends on the translation of the traveling wave: if $U_*(x)$ has asymptotics (4.14), then

$$U_*(x - x_0) \sim A(x - x_0)e^{-\lambda_*(x-x_0)} \sim A[x_0]e^{\lambda_* x_0} xe^{-\lambda_* x}, \quad \text{as } x \to +\infty, \tag{4.15}$$

with

$$A[x_0] = Ae^{\lambda_* x_0}. \tag{4.16}$$

Unless stated otherwise, we will fix the translation of the wave by requiring that $U_*(x)$ has the asymtptotics

$$U_*(x) \sim xe^{-\lambda_* x}, \quad \text{as } x \to +\infty, \tag{4.17}$$

with the pre-factor equal to 1.

4.2 Convergence to a traveling wave in shape

We now consider the long time behavior of the solutions to the Fisher-KPP equation

$$\begin{aligned} &u_t = \frac{\sigma^2}{2} u_{xx} + f(u), \ t > 0, \ x \in \mathbb{R}, \\ &u(0, x) = \mathbb{1}(x \leq 0). \end{aligned} \tag{4.18}$$

Let us recall that if $f(u)$ is of the McKean form (4.2)-(4.3), then the solution to (4.18) is the probability distribution of the maximum of BBM starting at $x = 0$. That is, we have

$$u(t, x) = \mathbb{P}_0\Big(\max_{1\le k\le N_t} X_k > x\Big). \tag{4.19}$$

One may, of course, consider more general initial conditions than in (4.18) with very similar results but it is a little simpler to study convergence to a traveling wave for the step function initial condition.

The study of the large time behavior uses the notion of steepness of the solution. While such arguments date back to the original KPP paper [10], we will use the definition from the recent paper by Giletti and Matano [9]. As we will only need it for smooth functions, we can formulate their notion as follows. Let us denote by $\mathcal{W}$ the class of smooth monotonically decreasing functions $u(x)$, $x \in \mathbb{R}$, such that

$$\lim_{x\to-\infty} u(x) = 1, \quad \lim_{x\to+\infty} u(x) = 0. \tag{4.20}$$

Given two functions $u_1, u_2 \in \mathcal{W}$, we say that u_1 is steeper than u_2 if

$$|u_1'(u_1^{-1}(z))| > |u_2'(u_2^{-1}(z))|, \quad \text{for all } z \in (0, 1). \tag{4.21}$$

In other words, the graph of $u_1(x)$ is steeper than the graph of $u_2(x)$ when compared at each fixed level $z \in (0, 1)$, rather than at a fixed point $x \in \mathbb{R}$. This notion is translation invariant; if u_1 is steeper than u_2, it is also steeper than any translate $u_2(\cdot + h)$, with a fixed $h \in \mathbb{R}$.

Exercise 4.4 Let $U_c(x)$ be a traveling wave solution to (4.18) with $c > c_*$. Show that $U_*(x)$ is steeper than $U_c(x)$.

Equation (4.18) has the following important property.

Proposition 4.5 *Let $u_1(t, x)$ and $u_2(t, x)$ be the solutions to (4.18) with the corresponding initial conditions $u_{10}, u_{20} \in \mathcal{W}$. If u_{10} is steeper than u_{20}, then $u_1(t, \cdot)$ is steeper than $u_2(t, \cdot)$ for all $t > 0$.*

This result was essentially proved for the classical Fisher-KPP equation in the original KPP paper [10]. The PDE proof below does not use the assumption that $f(u)$ is of the McKean form (4.2)-(4.3).

The proof of Proposition 4.5

Let $u_1(t, x)$ and $u_2(t, x)$ be the solutions to (4.18) with the initial conditions $u_{10}, u_{20} \in \mathcal{W}$ such that u_{10} is steeper than u_{20}. First, we note that since the initial conditions are decreasing, both $u_1(t, x)$ and $u_2(t, x)$ are decreasing and have the left and right limits as in (4.20), so that both $u_1(t, \cdot)$ and $u_2(t, \cdot)$ lie in $\mathcal{W}$.

Exercise 4.6 Prove that if the initial condition $u_0(x)$ for (4.18) is monotonically decreasing in x, then $u(t, x)$ is strictly decreasing in x for all $t > 0$. This can be done in two ways.
(i) Differentiate (4.18) in x to get the equation for $v(t, x) = u_x(t, x)$ and use the strong maximum principle to show that if $v(0, x) \le 0$ for all $x \in \mathbb{R}$, then $v(t, x) < 0$ for all $t > 0$ and $x \in \mathbb{R}$, unless $v(t, x) \equiv 0$.
(ii) Assume that $f(u)$ satisfies (4.2)-(4.3) and give a proof using the McKean representation of the solution.

To show that $u_1(t,\cdot)$ is steeper than $u_2(t,\cdot)$ for any $t>0$, consider the difference

$$w(t,x;k_0)=u_1(t,x)-u_2(t,x+k_0),$$

for a fixed $k_0\in\mathbb{R}$. The function $w(t,x;k_0)$ satisfies

$$w_t=w_{xx}+g(t,x)w,\quad g(t,x)=\frac{f(u_1(t,x))-f(u_2(t,x+k_0))}{u_1(t,x)-u_2(t,x+k_0)}, \tag{4.22}$$

with the initial condition

$$w(0,x;k_0)=u_{10}(x)-u_{20}(x+k_0). \tag{4.23}$$

Note that if $f(u)$ is twice differentiable, then the function $g(t,x)$ is differentiable and uniformly bounded.

Since u_{10} is steeper than u_{20}, it is also steeper than $u_{20}(\cdot+k_0)$. Therefore, there exists x_0 so that

$$w(0,x;k_0)>0 \text{ for all } x<x_0,$$

and

$$w(0,x;k_0)<0 \text{ for all } x>x_0.$$

Since $w(t,x;k_0)$ is a solution to the parabolic equation (4.22), the strong maximum principle implies that $w(t,x;k_0)$ has exactly one zero $y(t;k_0)$ for all $t>0$, so that $w(t,x;k_0)>0$ for all $x<y(t;k_0)$ and $w(t,x;k_0)<0$ for all $x>y(t;k_0)$, with $y(0;k_0)=x_0$. In addition, we have $w_x(t,y(t;k_0))<0$, which translates into

$$\partial_x u_1(t,y(t;k_0))<\partial_x u_2(t,y(t;k_0)). \tag{4.24}$$

Since this is true for all $k_0\in\mathbb{R}$, it follows that $u_1(t,\cdot)$ is steeper than $u_2(t,\cdot)$. □

A standard approximation argument shows the following.

Corollary 4.7 *Let $v(t,x)$ and $u(t,x)$ be the solutions to (4.18) with the respective initial conditions $v_{\rm in}\in\mathcal{W}$ and $u_{\rm in}(x)=\mathbb{1}(x\le 0)$. Assume that $v_{\rm in}(x)$ is steeper than the minimal speed traveling wave $U_*(x)$. Then, for any $t>0$ the solution $v(t,\cdot)$ is steeper than $U_*(x)$, and is less steep than $u(t,x)$.*

Convergence in shape

We now establish convergence of the solution in shape to a traveling wave. This result goes back to the original papers of Fisher and KPP. The proof is a slightly simplified version of the KPP argument.

Theorem 4.8 *Let $u(t,x)$ be the solution to (4.18) with the initial condition $u_{\rm in}\in\mathcal{W}$ that is steeper than the minimal speed traveling wave $U_*(x)$, or with $u_{\rm in}(x)=\mathbb{1}(x\le 0)$. Then, there exists a function $m(t)$ such that*

$$\frac{dm(t)}{dt}\to c_*,\quad \textit{as } t\to+\infty, \tag{4.25}$$

and a constant $x_0\in\mathbb{R}$

$$u(t,x+m(t))\to U_*(x+x_0)\quad \textit{as } t\to+\infty, \textit{ uniformly on } \mathbb{R}. \tag{4.26}$$

Here, $U_*(x)$ is normalized via (4.17).

The steepness assumption on the initial condition is really not necessary and is only made to shorten the proof. The result holds for a large class of initial conditions that decay sufficiently fast as $x \to +\infty$. For example, one may assume that $u_{\rm in}(x)$ is non-negative everywhere and is compactly supported on the right: there exists L_0 so that $u_{\rm in}(x) = 0$ for all $x \geq L_0$.

Proof. Corollary 4.7 shows that it suffices to consider the solution $u(t,x)$ to (4.18) with the initial condition $u(0,x) = \mathbb{1}(x \leq 0)$. This is because if we take any solution $v(t,x)$ to (4.18) with the initial condition $v_{\rm in}(x)$ that is steeper than $U_*(x)$, then $v(t,x)$ is steeper than $U_*(x)$ and is less steep than $u(t,x)$, for any $t > 0$. Note that for any $\tau > 0$, the function

$$u^{(\tau)}(t,x) = u(t+\tau,x)$$

is the solution to (4.18) with the initial condition $u^{(\tau)}(0,x) = u(\tau,x)$ that is less steep than $u(0,x) = \mathbb{1}(x \leq 0)$. It follows that for any $t > 0$ and $\tau > 0$ the function $u(t,\cdot)$ is steeper than $u(t+\tau,\cdot)$. In addition, $u(t,\cdot)$ is steeper than the minimal speed traveling wave $U_*(x)$ for all $t > 0$. This is because $u(0,x) = \mathbb{1}(x \leq 0)$ is steeper than $U_*(x)$ and the solution to (4.18) with the initial condition $U_*(x)$ is $U_*(x - c_*t)$, which has the same shape as $U_*(x)$. Hence, if for each $v \in (0,1)$ and $t > 0$, we let $x(t,v)$ be the unique point such that $u(t,x(t,v)) = v$, then, the function

$$E(t,v) = u_x(t,x(t,v)) \leq 0, \tag{4.27}$$

is increasing in t for all $v \in (0,1)$, and

$$E(t,v) \leq \bar{E}(v) := U_*'(U_*^{-1}(v)). \tag{4.28}$$

Let now $m(t)$ be the position such that $u(t,m(t)) = 1/2$ for all $t > 0$, and consider the translate

$$\tilde{u}(t,x) = u(t,x+m(t)),$$

as well as the corresponding inverse $\xi(t,v)$ defined by

$$\tilde{u}(t,\xi(t,v)) = v, \quad \text{for } 0 < v < 1.$$

Observe that $\xi(t,1/2) = 0$ for all $t > 0$, simply because $\tilde{u}(t,0) = 1/2$ by construction. We see from (4.27)-(4.28) that the function $E(t,v)$ is negative and increasing in time. Thus, it has a limit

$$E(t,v) \to E_\infty(v) \leq \bar{E}(v) < 0, \quad \text{as } t \to +\infty. \tag{4.29}$$

Hence

$$\frac{\partial \xi(t,v)}{\partial v} = \frac{1}{E(t,v)} \to \frac{1}{E_\infty(v)}, \quad \text{as } t \to +\infty,$$

and

$$\xi(t,v) = \int_{1/2}^{v} \frac{\partial \xi(t,v')}{\partial v'} dv' \to \int_{1/2}^{v} \frac{dv'}{E_\infty(v')} := \xi_\infty(v). \tag{4.30}$$

As a consequence, the function $\tilde{u}(t,x)$ also converges uniformly on compact sets to a limit $\tilde{u}_\infty(x)$:

$$\tilde{u}(t,x) \to \tilde{u}_\infty(x) \text{ as } t \to +\infty, \tag{4.31}$$

with $\tilde{u}_\infty(x)$ determined by

$$\xi_\infty(\tilde{u}_\infty(x)) = x. \tag{4.32}$$

Moreover, due to (4.29), we have

$$|\xi_\infty(v)| = \int_{1/2}^{v} \frac{dv'}{|E_\infty(v')|} \le \int_{1/2}^{v} \frac{dv'}{|\bar{E}(v')|} := \bar{\xi}(v). \tag{4.33}$$

This yields the correct behavior of the limits $x \to \pm\infty$:

$$\tilde{u}_\infty(-\infty) = 1, \quad \tilde{u}_\infty(+\infty) = 0. \tag{4.34}$$

Furthermore, as $u(t,x)$ is strictly decreasing in x and $u_x(t, m(t)) < 0$, the function $m(t)$ is differentiable in t, with

$$\dot{m}(t) = -\frac{u_t(t, m(t))}{u_x(t, m(t))} = -\frac{u_t(t, m(t))}{\tilde{u}_x(t, 0)}. \tag{4.35}$$

Hence, $\tilde{u}(t,x)$ satisfies

$$\tilde{u}_t - \dot{m}(t)\tilde{u}_x = \tilde{u}_{xx} + f(\tilde{u}). \tag{4.36}$$

By the parabolic regularity theory, the numerator in the very right side of (4.35) is bounded. Moreover, since $\tilde{u}$ converges to $\tilde{u}_\infty(x)$ that is steeper than $U_*(x)$, the denominator is bounded away from zero and converges to $\partial_x \tilde{u}_\infty(0) \neq 0$. It follows that $\dot{m}(t)$ is bounded uniformly in t. In addition, because $u(t', x)$ is less steep than $u(t,x)$ for $t' > t$, we know that the function $\tilde{u}(t,x)$ is decreasing in t and $\tilde{u}_t(t,x) \to 0$ as $t \to +\infty$. Then, passing to the limit $t \to +\infty$ in (4.36), we deduce that there exists $c \in \mathbb{R}$ such that $\dot{m}(t) \to c$ as $t \to +\infty$ and $\tilde{u}_\infty(x)$ satisfies

$$-c\partial_x \tilde{u}_\infty = \partial_x^2 \tilde{u}_\infty + f(\tilde{u}_\infty). \tag{4.37}$$

We see from (4.37) that $\tilde{u}_\infty(x)$ is a traveling wave solution to (4.18) moving with the speed c. It remains to show that $c = c_*$. The key point is that the steepness comparison argument above applies to any traveling wave solution to

$$-cU_c' = U_c'' + f(U_c). \tag{4.38}$$

In other words, if we set

$$E_c(v) = U_c'(U_c^{-1}(v)), \quad \text{for } 0 < v < 1,$$

then we know that

$$E_\infty(v) \le E_c(v),$$

for any U_c that satisfies (4.38) with some $c \ge c_*$. Therefore, the limit $\tilde{u}_\infty(x)$ is the traveling wave that is the steepest among all traveling wave solutions. Exercise 4.4 implies that

$$\tilde{u}_\infty(x) = U_*(x)$$

is the minimal speed traveling wave. This finishes the proof of Theorem 4.8. □

4.3 Generalizations to other initial conditions

We will later need the corresponding results for other initial conditions, that are not necessarily steeper than the minimal speed traveling wave. Let $H(t,x)$ be the solution to the initial value problem that starts with a step function:

$$\begin{aligned} &H_t = \frac{\sigma^2}{2} H_{xx} + f(H),\ t > 0,\ x \in \mathbb{R}, \\ &H(0,x) = \mathbb{1}(x \le 0). \end{aligned} \tag{4.39}$$

We assume that $f(u) = 1 - u - g(1-u)$ is of the McKean type. Theorem 4.8 applies to $H(t,x)$ and says that there is a function $m(t)$ and $x_0 \in \mathbb{R}$ so that

$$H(t, x + m(t)) \to U_*(x + x_0), \quad \text{as } t \to +\infty, \text{ uniformly in } x \in \mathbb{R}, \tag{4.40}$$

and

$$\dot{m}(t) \to c_*, \quad \text{as } t \to +\infty. \tag{4.41}$$

A fundamental result of Bramson says that $m(t)$ can be taken to be

$$m(t) = c_* t - \frac{3}{2\lambda_*} \log(t+1). \tag{4.42}$$

It will be convenient for us to accept (4.42) for now without proof. We will explain how this asymptotics comes about in Section 6.2 below.

The first generalization concerns other initial conditions compactly supported on the right and uniformly positive on the left. It shows that they converge to a traveling wave in shape and stay a fixed shift away from $H(t,x)$ as $t \to +\infty$.

Theorem 4.9 *Let f be a McKean type nonlinearity, and $u(t,x)$ be the solution to the initial value problem*

$$\begin{aligned} &u_t = \frac{\sigma^2}{2} u_{xx} + f(u),\ t > 0,\ x \in \mathbb{R}, \\ &u(0,x) = \eta(x). \end{aligned} \tag{4.43}$$

Assume that the initial condition $\eta(x)$ is such that (i) $0 \le \eta(x) \le 1$, and $\eta(x) \not\equiv 0$, (ii) $\liminf_{x\to-\infty} \eta(x) > 0$, and (iii) there is $L_0 \in \mathbb{R}$ such that $\eta(x) = 0$ for all $x \ge L_0$. Then, there is a constant $s[\eta] \in \mathbb{R}$, known as the Bramson shift of η such that

$$u(t, x + m(t)) \to U_*(x + s[\eta]) \quad \text{as } t \to +\infty, \text{ uniformly on } \mathbb{R}. \tag{4.44}$$

Here, $m(t)$ is defined by (4.42).

A key assumption above is (iii): the initial condition vanishes on the right. The result can be generalized to other initial conditions as long as the decay is faster than $\exp(-\lambda_* x)$.

The next theorem shows that if the initial condition $\eta(x)$ is compactly supported, then the solution develops two fronts, one moving to the right and another to the left, one located a fixed distance away from $m(t)$, and another a fixed distance away from the position $(-m(t))$.

Theorem 4.10 *Let f be a McKean type nonlinearity, and $u(t,x)$ be the solution to*

$$\begin{aligned} &u_t = \frac{\sigma^2}{2} u_{xx} + f(u),\ t>0,\ x\in\mathbb{R},\\ &u(0,x) = \eta(x), \end{aligned} \tag{4.45}$$

with a compactly supported initial condition $\eta(x)$ such that $0 \le \eta(x) \le 1$ and $\eta(x) \not\equiv 0$. Then, there are two constant $s_\ell[\eta] \in \mathbb{R}$ and $s_r[\eta] \in \mathbb{R}$, such that for every $K \in \mathbb{R}$ we have

$$u(t, x+m(t)) \to U_*(x+s_r[\eta]) \quad \text{as } t\to+\infty, \text{ uniformly in } x \ge K, \tag{4.46}$$

and

$$u(t, x-m(t)) \to U_*(-(x+s_\ell[\eta])) \quad \text{as } t\to+\infty, \text{ uniformly in } x \le K. \tag{4.47}$$

Here, again, $m(t)$ is defined by (4.42).

Exercise 4.11 (**) Prove Theorems 4.9 and 4.10, without assuming Bramson's asymptotics (4.42). Instead, define $m(t)$ via $H(t, m(t)) = 1/2$. The proofs that we are aware of are quite long and simultaneously prove (4.42). There should be a way around using this fact: the proof of (4.42) should be harder than the claims of these two theorems.

5 The randomized Gumbel distribution of the maximum of BBM

A lot of intuition about BBM can be deduced from the martingale property of several of its functionals. This is what we introduce in this section.

5.1 The traveling wave martingale

Let us go back to branching Brownian motion and connect various BBM objects to the traveling wave we have just constructed, and the long time convergence results. It will be convenient for us not to work with the Fisher-KPP equation (4.1) directly but to use the solution to

$$\begin{aligned} &v_t = \frac{\sigma^2}{2} v_{xx} - v + g(v),\ t>0,\ x\in\mathbb{R},\\ &v(0,x) = r(x). \end{aligned} \tag{5.1}$$

We will again assume that $g(v)$ is of the McKean form

$$g(v) = \sum_{k=1}^{\infty} p_k v^k, \quad \sum_{k=1}^{\infty} p_k = 1, \quad p_k \ge 0. \tag{5.2}$$

Recall that if v is a solution to (5.1), then $u = 1 - v$ satisfies the Fisher-KPP equation (4.1):

$$\begin{aligned} &u_t = \frac{\sigma^2}{2} u_{xx} + f(u),\ t>0,\ x\in\mathbb{R},\\ &u(0,x) = 1 - r(x), \end{aligned} \tag{5.3}$$

with

$$f(u) = 1 - u - g(1-u).$$

McKean's representation tells us that

$$v(t,x) = \mathbb{E}\prod_{k=1}^{N_t} r(X_k^x(t)) = \mathbb{E}\prod_{k=1}^{N_t} r(x + X_k(t)). \tag{5.4}$$

Here, $X_k(t)$ are the BBM particles starting at $x = 0$ at $t = 0$, and $X_k^x(t)$ are the BBM particles starting at $t = 0$ at the position $x \in \mathbb{R}$.

Let us go into the frame moving with the speed c_* and note that the function

$$\hat{v}(t,x) = v(t, x + c_* t) \tag{5.5}$$

satisfies the initial value problem

$$\begin{aligned} &\hat{v}_t - c_*\hat{v}_x = \frac{\sigma^2}{2}\hat{v}_{xx} - \hat{v} + g(\hat{v}),\ t > 0,\ x \in \mathbb{R}, \\ &\hat{v}(0,x) = r(x). \end{aligned} \tag{5.6}$$

Using (5.4) we obtain a representation

$$\hat{v}(t,x) = \mathbb{E}\prod_{k=1}^{N_t} r(x + c_* t + X_k(t)). \tag{5.7}$$

Let us introduce the minimal speed traveling wave $\omega(x)$ for (5.1), the solution to

$$\begin{aligned} &-c_*\omega' = \frac{\sigma^2}{2}\omega'' - \omega + g(\omega),\ x \in \mathbb{R}, \\ &\omega(-\infty) = 0, \quad \omega(+\infty) = 1. \end{aligned} \tag{5.8}$$

It is related to $U_*(x)$, the minimal speed traveling wave for (4.1) by $\omega(x) = 1 - U_*(x)$. We fix the translation of $\omega(x)$ by the normalization $\omega(0) = 1/2$. Note that $\omega(x)$ is a time-independent solution to (5.6). Hence, (5.7) implies an identity

$$\omega(x) = \mathbb{E}\prod_{k=1}^{N_t} \omega(x + c_* t + X_k(t)), \tag{5.9}$$

that holds for all $t \ge 0$. This is a manifestation of the following important fact.

Proposition 5.1 *For a fixed $x \in \mathbb{R}$, the function*

$$W(t,x) = \prod_{k=1}^{N_t} \omega(x + c_* t + X_k(t)). \tag{5.10}$$

is a martingale. Moreover, $W(t,x)$ converges almost surely and in L^1 to a limit $W_\infty(x)$ such that

$$\mathbb{E}W_\infty(x) = \omega(x). \tag{5.11}$$

Proof. Integrability of W is immediate from the fact that $0 < \omega(x) < 1$. To show the martingale property of $W(t,x)$, let us take $t > 0$ and $s > 0$. The Markov nature of the exponential clocks and BBM allows us to write

$$\begin{aligned}\mathbb{E}(W(t+s)|\mathcal{F}_s) &= \mathbb{E}\Big(\prod_{m=1}^{N_{t+s}} \omega(x + c_*(t+s) + X_m(t+s))\Big|\mathcal{F}_s\Big) \\ &= \prod_{k=1}^{N_s} \mathbb{E}\Big(\prod_{j=1}^{N_t} \omega(x + X_k(s) + c_*(t+s) + X_{kj}(t))\Big|\mathcal{F}_s\Big) \\ &= \prod_{k=1}^{N_s} \omega(x + X_k(s) + c_* s) = W(s).\end{aligned} \tag{5.12}$$

We used (5.9) in the last step above. Since $W(t)$ is a non-negative martingale, it converges as $t \to +\infty$, almost surely and in L^1 to a limit W_∞. Taking the expectation in (5.10) and passing to the limit $t \to +\infty$ gives (5.11). □

Exercise 5.2 Use Proposition 5.1, the boundary conditions for $\omega(x)$ in (5.8), as well as the fact that $0 < \omega(x) < 1$ to understand informally where we should expect the minimum of $X_k(t)$ to be, relative to the position $(-c_* t)$.

The next exercise gives a martingale characterization of the traveling wave profile.

Exercise 5.3 Let $\phi(x)$ be a smooth function such that $\phi(-\infty) = 0$ and $\phi(+\infty) = 1$, and also $0 < \phi(x) < 1$ for all $x \in \mathbb{R}$. Show that if

$$W_\phi(t,x) = \prod_{k=1}^{N_t} \phi(x + c_* t + X_k(t)) \tag{5.13}$$

is a martingale for all $x \in \mathbb{R}$ fixed, then there is $x_0 \in \mathbb{R}$ so that

$$\phi(x) = \omega(x - x_0), \quad \text{for all } x \in \mathbb{R}.$$

5.2 The additive and derivative martingales

The next two martingales are constructed out of solutions not to the Fisher-KPP equation but to the linearized Fisher-KPP equation written in the frame moving with the speed c_*:

$$\begin{aligned}& z_t - c_* z_x = \frac{\sigma^2}{2} z_{xx} + (\bar{N} - 1) z, \\ & z(0,x) = r(x).\end{aligned} \tag{5.14}$$

Here, $\bar{N}$ is the expected number of the off-spring at each branching:

$$\bar{N} = \sum_{k=1}^{\infty} k p_k, \tag{5.15}$$

As we have seen, solutions to (5.14) have the probabilistic representation similar to McKean's for the Fisher-KPP equation but with the product replaced by the sum:

$$z(t,x) = \mathbb{E}\sum_{k=1}^{N_t} r(x + c_* t + X_k(t)). \tag{5.16}$$

In particular, if $\bar{z}(x)$ is a steady solution to (5.14):

$$-c_*\bar{z}_x = \frac{\sigma^2}{2}\bar{z}_{xx} + (\bar{N}-1)\bar{z}, \tag{5.17}$$

then we have an identity, similar to (5.9):

$$\bar{z}(x) = \mathbb{E}\sum_{k=1}^{N_t} \bar{z}(x + c_* t + X_k(t)), \tag{5.18}$$

that holds for all $t \ge 0$. There are two explicit steady solutions to (5.14). The first one is

$$\eta(x) = e^{-\lambda_* x}, \tag{5.19}$$

with λ_* given by (4.12):

$$\lambda_* = \frac{c_*}{\sigma^2}. \tag{5.20}$$

The second is

$$\bar{z}(x) = xe^{-\lambda_* x}. \tag{5.21}$$

They give us the following two objects:

$$Y(t) = \sum_{k=1}^{N_t} \exp(-\lambda_*(c_* t + X_k(t))), \tag{5.22}$$

and

$$Z(t) = \sum_{k=1}^{N_t} (c_* t + X_k(t)) \exp(-\lambda_*(c_* t + X_k(t))), \tag{5.23}$$

that we may expect to be martingales, for the same reason as $W(t,x)$ is a martingale.

Exercise 5.4 Using the proof of Proposition 5.1, show that $Y(t)$ is a martingale.

The martingale $Y(t)$ is non-negative and

$$\mathbb{E}(Y(t)) = 1, \tag{5.24}$$

for all $t \ge 0$. Hence, it converges as $t \to +\infty$, almost surely to a finite limit $Y_\infty \ge 0$. This has the following important consequence.

Exercise 5.5 Use the fact that $Y(t)$ has a finite limit to show that

$$\lim_{t\to+\infty}\Big(c_* t + \min_{1\le k\le N_t} X_k(t)\Big) = +\infty, \tag{5.25}$$

almost surely. Hint: all exponentials in the definition of $Y(t)$ must tend to negative infinity as $t \to +\infty$ in order for the limit to exist.

Similarly to (5.25), we also have

$$\lim_{t\to+\infty}\Big(c_*t-\max_{1\le k\le N_t}X_k(t)\Big)=+\infty,\ \text{a.s.} \tag{5.26}$$

This, roughly speaking, says that there are no BBM particles to the right of the position c_*t, or to the left of $(-c_*t)$, with a very high probability. The obvious question is how far behind the position c_*t the maximal particle lags, we will look at this a bit later.

Exercise 5.6 Use the proof of Proposition 5.1, and (5.25) and (5.26) to show that $Z(t)$ is a martingale.

We will refer to $Y(t)$ and $Z(t)$ as, respectively, the additive martingale and the derivative martingale.

5.3 The limit of the derivative martingale

The martingale $Z(t,x)$ is not necessarily positive, hence we do not a priori know that it has a limit as $t\to+\infty$. On the other hand, (5.25) hints that it should become positive as $t\to+\infty$, at least with a very high probability. To show that, indeed, it does have a limit, let us look again at the traveling wave martingale $W(t,x)$ and represent it as

$$\begin{aligned} W(t,x)&=\prod_{k=1}^{N_t}\omega(x+c_*t+X_k(t))=\prod_{k=1}^{N_t}\exp(\log\omega(x+c_*t+X_k(t)))\\ &=\exp\Big(\sum_{k=1}^{N_t}\log\omega(x+c_*t+X_k(t))\Big). \end{aligned} \tag{5.27}$$

The arguments in the right side of (5.27) are all very large as $t\to+\infty$, because of (5.25). Let us recall the corresponding asymptotics (4.14):

$$U_*(x)\sim xe^{-\lambda_*x},\quad \text{as } x\to+\infty, \tag{5.28}$$

which translates into

$$1-\omega(x)\sim xe^{-\lambda_*x},\quad \text{as } x\to+\infty. \tag{5.29}$$

With (5.25) in mind, we see that

$$\begin{aligned} \log\omega(x+c_*t+X_k(t))&\approx\log\big[1-(x+c_*t+X_k(t))\exp(-\lambda_*(x+c_*t+X_k(t)))\big]\\ &\approx-(x+c_*t+X_k(t))\exp(-\lambda_*(x+c_*t+X_k(t))). \end{aligned} \tag{5.30}$$

This gives the asymptotics

$$\begin{aligned} &\sum_{k=1}^{N_t}\log\omega(x+c_*t+X_k(t)))\approx-\sum_{k=1}^{N_t}(x+c_*t+X_k(t))\exp(-\lambda_*(x+c_*t+X_k(t)))\\ &=-(xY(t)+Z(t))e^{-\lambda_*x}. \end{aligned} \tag{5.31}$$

Using this in (5.27) gives

$$W(t,x)\approx\exp\big(-e^{-\lambda_*x}(xY(t)+Z(t))\big). \tag{5.32}$$

Exercise 5.7 Make the above argument rigorous and show that almost surely we have

$$W_\infty(x) = \lim_{t\to+\infty} W(t,x) = \lim_{t\to+\infty} \exp\big(-e^{-\lambda_* x}(xY(t)+Z(t))\big). \tag{5.33}$$

Since we know that the limits $W_\infty(x)$ and Y_∞ of $W(t,x)$ and $Y(t)$ as $t\to+\infty$ exist almost surely, it follows from (5.32) that the limit

$$Z_\infty = \lim_{t\to+\infty} Z(t) \tag{5.34}$$

exists and is finite. On the other hand, as we also have (5.25), existence of the limit of $Z(t)$ means that we must have $Y_\infty = 0$ almost surely. Thus, passing to the limit $t\to+\infty$ in (5.32) we obtain

$$W_\infty(x) = \exp\big(-e^{-\lambda_* x} Z_\infty\big). \tag{5.35}$$

Recalling (5.11), we deduce a representation for the traveling wave as the Laplace transform of the limit of the derivative martingale

$$\omega(x) = \mathbb{E}\exp\big(-e^{-\lambda_* x} Z_\infty\big). \tag{5.36}$$

Viewed differently, this identifies the law of the derivative martingale via its Laplace transform in terms of the traveling wave: for every $\kappa > 0$ we have

$$\mathbb{E}\exp\big(-\kappa Z_\infty\big) = \omega\Big(-\frac{1}{\lambda_*}\log\kappa\Big). \tag{5.37}$$

As we will see, the derivative martingale plays an important qualitative role in the description of the limiting law of BBM.

5.4 The law of the maximum and the derivative martingale

We now derive the law of the maximum of BBM and connect it to the derivative martingale. Let us recall that in Section 2.1 we have considered the maximum of 2^N independent Gaussian random variables $G_1,\dots,G_{2^N}$, with each G_k having variance N. We have shown that if we set

$$m_N = c_* N - \frac{1}{2\lambda_*}\log N, \tag{5.38}$$

with $c_* = \lambda_* = 2\sqrt{2}$, and define

$$u_N(y) = \mathbb{P}\Big(\max_{1\le k\le 2^N} G_k < m_N + y\Big), \tag{5.39}$$

then in the limit $N\to+\infty$ we get the Gumbel distribution

$$u_N(y) \to \exp\Big(-\frac{1}{c_*\sqrt{2\pi}}e^{-\lambda_* y}\Big), \quad \text{as } N\to+\infty. \tag{5.40}$$

It is helpful to think of (5.40) as

$$u_N(y) \to \exp\Big(-e^{-\lambda_*(y-\ell_*)}\Big), \quad \text{as } N\to+\infty, \tag{5.41}$$

with the shift

$$\ell_* = -\log(c_* \sqrt{2\pi}). \tag{5.42}$$

We now describe a similar result for the branching Brownian motion – the obvious difference with the above situation is that the locations of the BBM are not uncorrelated. Let $X_1(t), \dots, X_{N_t}(t)$ be the BBM particles present at time $t > 0$. Recall that the function

$$v(t,x) = \mathbb{P}\Big(\max_{1\le k\le N_t} X_k(t) \le x\Big) \tag{5.43}$$

satisfies (5.1)

$$\begin{aligned} &v_t = \frac{\sigma^2}{2} v_{xx} - v + g(v),\ \ t > 0,\ \ x \in \mathbb{R},\\ &v(0,x) = \mathbb{1}(x \ge 0). \end{aligned} \tag{5.44}$$

We know from Theorem 4.8 that if we fix $m(t)$ by (4.42):

$$m(t) = c_* t - \frac{3}{2}\log(t+1) \tag{5.45}$$

then there exists x_0 so that

$$v(t, x + m(t)) \to \omega(x + x_0) \ \text{ as } t \to +\infty, \text{ uniformly on } \mathbb{R}. \tag{5.46}$$

Let us now set

$$M(t) = \max_{1\le k\le N_t} X_k(t). \tag{5.47}$$

Then (5.46) says that

$$\mathbb{P}\Big(M(t) - m(t) \le x\Big) \to \omega(x + x_0) \ \text{ as } t \to +\infty, \text{ uniformly on } \mathbb{R}. \tag{5.48}$$

In order to compare to (5.40) and see the Gumbel distribution, we go back to the traveling wave representation (5.36), which gives

$$\mathbb{P}\Big(M(t) - m(t) \le x\Big) \to \mathbb{E}\exp\big(-e^{-\lambda_*(x+x_0)} Z_\infty\big), \quad \text{as } t \to +\infty, \text{ uniformly on } \mathbb{R}. \tag{5.49}$$

Comparing it to (5.40), we see that the probability distribution of the maximum of BBM in the reference frame $m(t)$ is a "randomized Gumbel distribution" that is re-centered by a random shift

$$x \to x + x_0 - \frac{1}{\lambda_*}\log Z_\infty. \tag{5.50}$$

In other words, if we set

$$L_\infty = \frac{1}{\lambda_*}\log(Z_\infty) - x_0, \tag{5.51}$$

then we can write (5.49) as a random shift of (5.41):

$$\mathbb{P}\Big(M(t) - m(t) \le x\Big) \to \mathbb{E}\exp\big(-e^{-\lambda_*(x - L_\infty)}\big), \quad \text{as } t \to +\infty, \text{ uniformly on } \mathbb{R}. \tag{5.52}$$

The law of the random shift L_∞ is determined by (5.51) and the Laplace transform of Z_∞ in (5.37). We should also mention that the Bramson centering (4.42)

$$m(t) = c_* t - \frac{3}{2\lambda_*} \log(t+1), \tag{5.53}$$

for the location $m(t)$ is different from (5.38) for the independent Gaussians – the pre-factor in front of $\log t$ changes from $1/2$ to $3/2$. The reason is the slowdown of BBM relative to independent Gaussians because the BBM particles are correlated. The intuitive reason for the random shift L_∞ is different: it is accumulated from the randomness at the initial stage of the branching process.

6 The limiting extremal process

In this section, we look at the limit of the point process of BBM centered at the Bramson centering location

$$m(t) = c_* t - \frac{3}{2\lambda_*} \log(t+1), \tag{6.1}$$

and defined via

$$\mathcal{E}_t(x) = \sum_{k=1}^{N_t} \delta(x - (X_k(t) - m(t))). \tag{6.2}$$

Recall that we have considered in Section 2.4 the point process for 2^N independent Gaussian random variables $G_1, \dots, G_{2^N}$, with each G_k having variance N. We proved in Theorem 2.5 that the corresponding limit is a Poisson point process. Here, we will derive an analog of this result for the branching Brownian motion. The limit will end up being more complicated than just a Poisson process: it will consist of Poisson points, known as cluster centers, with a cluster of points attached to each of the Poisson points.

6.1 Existence of the limit process and its Laplace transform

The first step is to prove existence of the limit of the point process $\mathcal{E}_t$.

Theorem 6.1 *The process $\mathcal{E}_t$ converges as $\to +\infty$ to a measure-valued process $\mathcal{E}$.*

Proof. Let $\phi(x)$ be a non-negative, bounded, smooth, and compactly supported test function. We will look at the limit of the Laplace transform of $\mathcal{E}_t$

$$\Psi_t(\phi) = \mathbb{E}\Big(\exp\Big(-\int \phi(x) d\mathcal{E}_t(x)\Big)\Big). \tag{6.3}$$

We need to show that $\Psi_t(\phi)$ converges for all such ϕ, as this will prove existence of the limit of $\mathcal{E}_t(x)$. We will also identify, albeit at this stage very implicitly, the Laplace transform $\Psi(\phi)$ of the limit $\mathcal{E}$.

Let us re-write $\Psi_t(\phi)$ as

$$\begin{aligned}\Psi_t(\phi) &= \mathbb{E}\Big(\exp\Big(-\int \phi(x)d\mathcal{E}_t(x)\Big)\Big) = \mathbb{E}\Big(\exp\Big(-\sum_{k=1}^{N_t}\phi(X_k(t)-m(t))\Big)\Big)\\ &= \mathbb{E}\Big(\prod_{k=1}^{N_t}\exp(-\phi(X_k(t)-m(t))\ \Big) = \mathbb{E}\Big(\prod_{k=1}^{N_t} r(-X_k(t)+m(t))\Big)\\ &= \mathbb{E}\Big(\prod_{k=1}^{N_t} r(X_k(t)+m(t))\Big).\end{aligned} \tag{6.4}$$

Here, we have set

$$r(x) = e^{-\phi(-x)}. \tag{6.5}$$

Note that in the last step in (6.4) we used the reflection symmetry of the law of BBM starting at $x=0$: the laws of $X_k(t)$ and $-X_k(t)$ are identical. Let us recall that the function

$$v(t,x) = \mathbb{E}\Big(\prod_{k=1}^{N_t} r(X_k(t)+x)\Big) \tag{6.6}$$

is the solution to the initial value problem (5.1)

$$\begin{aligned}&v_t = \frac{\sigma^2}{2}v_{xx} - v + g(v),\ t>0,\ x\in\mathbb{R},\\ &v(0,x) = r(x).\end{aligned} \tag{6.7}$$

Thus, (6.4) says that the Laplace transform can be written as

$$\Psi_t(\phi) = v(t,m(t)). \tag{6.8}$$

Note that the function

$$u(t,x) = 1 - v(t,x) \tag{6.9}$$

satisfies the Fisher-KPP equation

$$u_t = \frac{\sigma^2}{2}u_{xx} + f(u),\ t>0,\ x\in\mathbb{R}, \tag{6.10}$$

with the usual relation between the functions f and g:

$$f(u) = 1 - u - g(1-u), \tag{6.11}$$

and the initial condition

$$u(0,x) = 1 - v(0,x) = 1 - r(-x) = 1 - e^{-\phi(-x)}. \tag{6.12}$$

As the function $\phi(x)$ is compactly supported, the function $u(0,x)$ is also compactly supported. Thus, its long time behavior is governed by Theorem 4.10. In particular, the claim (4.46) of that theorem implies that there exists a constant

$$\gamma[\phi] = s_r\big[\zeta\big],\quad \zeta(x) = 1 - e^{-\phi(-x)}, \tag{6.13}$$

so that for each $K \in \mathbb{R}$ we have

$$u(t, x + m(t)) \to U_*(x + \gamma[\phi]), \quad \text{as } t \to +\infty, \text{ uniformly in } x \geq K. \tag{6.14}$$

Taking $x = 0$ in (6.14) and recalling (6.8)-(6.9) gives

$$\Psi_t[\phi] = v(t, m(t)) = 1 - u(t, m(t)) \to 1 - U_*(\gamma[\phi]), \quad \text{as } t \to +\infty. \tag{6.15}$$

This shows that the Laplace transforms $\Psi_t[\phi]$ converge as $t \to +\infty$, proving existence of a the limit of the point processes $\mathcal{E}_t$. □

Let us note again that we have shown in the course of the proof of Theorem 6.1 that the point process $\mathcal{E}_t$ of BBM converges as $t \to +\infty$ to a measure-valued process $\mathcal{E}$ with the Laplace transform given by (6.15), with $\gamma[\phi]$ defined by (6.13). This description of the limit is extremely implicit but, surprisingly, one can untangle it to obtain a nice description of the limit process. In particular, one can show that the limit is a point process. This is what we will do below.

In order to understand what we can hope for, let us go back to the corresponding result for a family of independent Gaussian random variables $G_1, \ldots, G_{2^N}$ of variance N. Theorem 2.5 says that the point processes

$$\mathcal{E}_N = \sum_{k=1}^{2^N} \delta(x - (G_k - m_N)), \tag{6.16}$$

converges in law as $N \to +\infty$ to a Poisson point process with intensity

$$\mu(x) = \frac{1}{\sqrt{2\pi}} e^{-\lambda_* x}. \tag{6.17}$$

Here, the median location is

$$m_N = c_* N - \frac{1}{2\lambda_*} \log N. \tag{6.18}$$

Recall that in Section 5.4 we recovered the probability distribution of the maximal particle of BBM as the law of the maximum of the independent collection $G_1, \ldots, G_{2^N}$:

$$\mathbb{P}\Big(M(t) - m(t) \leq x\Big) \to \mathbb{E} \exp\big(-e^{-\lambda_*(x + x_0)} Z_\infty\big), \quad \text{as } t \to +\infty, \text{ uniformly on } \mathbb{R}. \tag{6.19}$$

but with the random shift (5.50)-(5.51):

$$x \to x + L_\infty, \quad L_\infty = \frac{1}{\lambda_*} \log Z_\infty - x_0, \tag{6.20}$$

expressed in the terms of the limit Z_∞ of the derivative martingale Z_t.

Our goal is to find a similar interpretation for the law of the extremal process of BBM, as a Poisson point process, hopefully with the familiar random shift by L_∞. As we have mentioned, the true limit is not quite that, but it is a good naive picture to have in mind for now. We go back to (6.15) and write it as

$$\Psi[\phi] = \omega(s_r[\zeta]), \quad \zeta(x) = 1 - e^{-\phi(-x)}, \tag{6.21}$$

with $s_r[\zeta]$ being the right Bramson shift of the compactly supported function $\zeta(x)$. Let us recall (5.36) and re-write (6.21) as

$$\Psi[\phi] = \omega(s_r[\zeta]) = \mathbb{E}\exp\big(-Z_\infty e^{-\lambda_* s_r[\zeta]}\big). \tag{6.22}$$

We will need to understand why the right side can be interpreted in terms of some relative of a Poisson point process. For this, we will need an expression for the Bramson shift $s_r[\zeta]$ and we will next take a detour to this end.

6.2 How the asymptotics for the Bramson shift comes about

Let us now make a rather long digression to obtain informally an expression for the Bramson $s_r[\zeta]$ that enters (6.22) and also explain the Bramson centering

$$m(t) = c_* t - \frac{3}{2\lambda_*}\log(t+1). \tag{6.23}$$

For simplicity of notation, from now on we will set $\sigma = 1$. We start with the initial value problem Fisher-KPP equation

$$\begin{aligned} &u_t = \frac{1}{2}u_{xx} + f(u),\ t>0,\ x\in\mathbb{R},\\ &u(0,x) = \zeta(x), \end{aligned} \tag{6.24}$$

with a compactly supported initial condition $\zeta(x)$. Theorem 4.10 shows that there exists $s_r[\zeta]$ so that

$$u(t, x+m(t)) \to U_*(x+s_r[\zeta]), \quad \text{as } x\to+\infty, \tag{6.25}$$

uniformly on sets of the form $x > K$.

Our goal will be to establish the following formula for $s_r[\zeta]$

$$e^{-\lambda_* s_r[\zeta]} = \lim_{t\to+\infty}\sqrt{\frac{2}{\pi}}\int_{-\infty}^{\infty} x e^{\lambda_* x} u(t, x+c_* t)dx. \tag{6.26}$$

The proof of (6.26) is surprisingly technical, so we will not give the details but only convey the gist of the argument. Along the way, we will also explain how the Bramson centering (6.23) comes about.

The fact that the right side of (6.26) is finite already tells us informally what the Bramson centering should be. To see this, let us recall that we have shown, at least, for the step function initial condition $u(0,x) = \mathbb{1}(x\le 0)$ that there exists a reference frame $\tilde{m}(t)$ such that

$$u(t, x+\tilde{m}(t)) \to U_*(x), \tag{6.27}$$

and that $\tilde{m}(t) = 2t + o(t)$ as $t\to+\infty$. One can think of $\tilde{m}(t)$ as being a finite distance away from the median location of the BBM.

The traveling wave asymptotics

$$U_*(x) \sim x e^{-\lambda_* x}, \quad \text{as } x\to+\infty, \tag{6.28}$$

shows that if we naively insert the limit in (6.27) into the integral in the right side of (6.26), the result would be a diverging integral. This would be circumvented if we would have a more precise asymptotics than (6.27):

$$u(t, x + \tilde{m}(t)) \sim U_*(x)e^{-x^2/(2t)}, \tag{6.29}$$

with the additional Gaussian decay coming from the heat equation. The reason for this correction is that it is natural to expect that, on the scales $x \sim O(\sqrt{t})$ away from the front, the Gaussian decay of the heat equation dominates the exponential decay of the traveling wave. This would give

$$u(t, x + c_*t) \sim U_*(x + c_*t - \tilde{m}(t))e^{-(x+c_*t-\tilde{m}(t))^2/(2t)}. \tag{6.30}$$

Recalling (5.26)

$$\lim_{t\to+\infty}\Big(c_*t - \max_{1\le k\le N_t} X_k(t)\Big) = +\infty, \text{ a.s.}, \tag{6.31}$$

and that $\tilde{m}(t)$ is close to the location of the BBM maximum, we expect that

$$d(t) := c_*t - \tilde{m}(t) \to +\infty, \quad \text{as } t \to +\infty. \tag{6.32}$$

Hence, the integral in (6.26) is, to the leading order,

$$\begin{aligned}
&\int_{-\infty}^{\infty} xe^{\lambda_* x}u(t, x + c_*t)dx \approx \int_{-\infty}^{\infty} xe^{\lambda_* x}U_*(x + d(t))e^{-(x+d(t))^2/(2t)}dx\\
&\approx \int_{-\infty}^{\infty} xe^{\lambda_* x}(x + d(t))e^{-\lambda_*(x+d(t))}e^{-(x+d(t))^2/(2t)}dx\\
&= \int_{-\infty}^{\infty} x(x + d(t))e^{-\lambda_* d(t)}e^{-(x+d(t))^2/(2t)}dx.
\end{aligned} \tag{6.33}$$

Exercise 6.2 Use the change of variable $x = y\sqrt{t}$ to show that the last integral in the right side of (6.33) has a finite limit only if

$$d(t) \sim \frac{3}{2\lambda_*}\log t, \quad \text{as } t \to +\infty. \tag{6.34}$$

Thus, the existence of the limit in the right side of (6.25) is intricately related to the Bramson centering (6.23). In the rest of this subsection we explain how (6.26) comes about, and also in more detail how the Bramson centering arises.

6.2.1 The approximate Dirichlet problem

Let us consider the solution to (6.24) in a reference frame $x \to x-(c_*t-\mu\log(t+1))$, with $\mu \in \mathbb{R}$ to be determined. The goal is to find μ such that the function $u(t, x)$ stays strictly between 0 and 1 in the new reference frame, and to see that this necessitates taking $\mu = 3/(2\lambda_*)$. The function

$$\tilde{u}(t, x) = u(t, x + c_*t - \mu\log(t + 1)) \tag{6.35}$$

satisfies the initial value problem

$$
\begin{aligned}
&\tilde{u}_t - \Big(c_* - \frac{\mu}{t+1}\Big)\tilde{u}_x = \frac{\sigma^2}{2}\tilde{u}_{xx} + f(\tilde{u}),\\
&\tilde{u}(0,x) = \zeta(x).
\end{aligned}
\tag{6.36}
$$

Next, we take out the exponential factor: set

$$
\tilde{u}(t,x) = w(t,x)e^{-\lambda_* x}. \tag{6.37}
$$

The function $w(t,x)$ satisfies

$$
\begin{aligned}
&w_t - (c_* - \frac{\mu}{t+1}\Big)(w_x - \lambda_* w) = \frac{1}{2}w_{xx} - \lambda_* w_x + \frac{1}{2}\lambda_*^2 w + e^{\lambda_* x} f\Big(we^{-\lambda_* x}\Big),\\
&w(0,x) = \zeta(x)e^{\lambda_* x}.
\end{aligned}
\tag{6.38}
$$

Let us recall that c_* and λ_* satisfy (4.11)-(4.12), with $\sigma = 1$:

$$
\frac{1}{2}\lambda_*^2 - c_*\lambda_* + f'(0) = 0, \quad \lambda_* = c_*. \tag{6.39}
$$

Hence, (6.38) is

$$
\begin{aligned}
&w_t + \frac{\mu}{t+1}(w_x - \lambda_* w) = \frac{1}{2}w_{xx} + \Big[e^{\lambda_* x} f\Big(we^{-\lambda_* x}\Big) - f'(0)w\Big],\\
&w(0,x) = \zeta(x)e^{\lambda_* x}.
\end{aligned}
\tag{6.40}
$$

The Fisher-KPP property of the nonlinearity $f(u)$ says that

$$
f(u) \le f'(0)u, \quad \text{for all } u \in (0,1). \tag{6.41}
$$

Therefore, the function

$$
q(u) = f'(0)u - f(u) \tag{6.42}
$$

is non-negative:

$$
q(u) \ge 0, \quad \text{for all } u \in (0,1). \tag{6.43}
$$

In addition, we have $q'(0) = 0$, and $q(u)$ is smooth. Hence, there exists a constant $C_q > 0$ so that

$$
C_q u^2 \le q(u), \quad \text{for all } u \in (0,1). \tag{6.44}
$$

Let us now consider the last term in the right side of (6.40):

$$
\begin{aligned}
&e^{\lambda_* x} f\Big(we^{-\lambda_* x}\Big) - f'(0)w = e^{\lambda_* x}\Big[f'(0)we^{-\lambda_* x} - q\Big(we^{-\lambda_* x}\Big)\Big] - f'(0)w\\
&= -e^{\lambda_* x} q\Big(we^{-\lambda_* x}\Big) \le -e^{\lambda_* x} w^2 e^{-2\lambda_* x} = -w^2 e^{-\lambda_* x}.
\end{aligned}
\tag{6.45}
$$

This term is negative and exponentially large as $x \to -\infty$, while it is exponentially small as $x \to +\infty$. Thus, it is natural to expect that, on one hand, it plays the role of a huge absorption as $x \to -\infty$, forcing $w(t,x)$ to be very small for very negative x, and on the other, (6.40) behaves as the linear heat equation for large positive x and t. Thus, a toy model for (6.40) for $t \gg 1$ is the heat equation on the half-line $x > 0$, with the Dirichlet boundary condition at $x = 0$:

$$\begin{aligned} &w_t = \frac{1}{2} w_{xx}, \quad x > 0, \\ &w(t, 0) = 0. \end{aligned} \tag{6.46}$$

This toy model is a bit too simplistic – we expect the solution to (6.40) to converge a traveling wave, modulo the exponential factor $\exp(\lambda_* x)$, while solutions to (6.46) decay in time as $t^{-3/2}$. Hence, we need to be more careful even if we stick to informal arguments.

6.2.2 The asymptotics in the self-similar variables and the Bramson shift

In order to be slightly less careless, and also to account for the term involving $\mu/(t+1)$ in (6.40), let us consider (6.40) in the self-similar variables, writing

$$w(t, x) = z\Big(\log(t+1), \frac{x}{\sqrt{t+1}}\Big). \tag{6.47}$$

The function $z(\tau, \eta)$ satisfies

$$\begin{aligned} &z_\tau - \frac{\eta}{2} z_\eta + \mu(e^{-\tau/2} z_\eta - \lambda_* z) = \frac{1}{2} z_{\eta\eta} - e^{\tau} e^{\lambda_* \eta e^{\tau/2}} q(z e^{-\lambda_* \eta e^{\tau/2}}), \\ &z(0, \eta) = \zeta(\eta) e^{\lambda_* \eta}. \end{aligned} \tag{6.48}$$

Note that term involving μ has been "promoted" to order $O(1)$ as $\tau \to +\infty$, and it will have a non-trivial effect.

Let us see what happens as $\tau \to +\infty$ in (6.48). The last term in the right side of (6.48) plays the role of the absorption that is (i) exponentially large for $\eta < 0$ and $|\eta| \gg e^{-\lambda_* \tau/2}$ and (ii) exponentially small for $\eta > 0$ and $\eta \gg e^{-\lambda_* \tau/2}$. The first property says that $z(\tau, \eta)$ is very small for $\eta < 0$ – hence, it is reasonable to use the approximate Dirichlet boundary condition $z(\tau, 0) = 0$ for $\tau \gg 1$. The second means that the nonlinear term can be dropped for $\eta > 0$. In addition, the term with z_η in the left side of (6.48) is exponentially small for $\tau \gg 1$. Thus, for $\tau \gg 1$ a good approximation to (6.48) is the Dirichlet boundary value problem

$$\begin{aligned} &z_\tau - \frac{\eta}{2} z_\eta - \mu \lambda_* z = \frac{1}{2} z_{\eta\eta}, \quad \eta > 0, \\ &z(\tau, 0) = 0. \end{aligned} \tag{6.49}$$

The difference with the "more reckless" approximation (6.46) in the original variables is that we have retained the last term in the left side of (6.49). This will play a crucial role, leading to the correct choice $\mu = 3/(2\lambda_*)$.

To understand the long time behavior of the solutions to (6.49), note that the function

$$\bar{z}(\eta) = \eta e^{-\eta^2/2} \tag{6.50}$$

satisfies

$$\begin{aligned} \frac{1}{2} \bar{z}_{\eta\eta} + \frac{\eta}{2} \bar{z}_\eta &= \frac{\eta}{2} (1 - \eta^2) e^{-\eta^2/2} + \frac{1}{2} \big(-2\eta - (1 - \eta^2)\eta \big) e^{-\eta^2/2} \\ &= \Big(\frac{\eta}{2} - \frac{\eta^3}{2} - \frac{3\eta}{2} + \frac{\eta^3}{2} \Big) e^{-\eta^2/2} = -\bar{z}(\eta). \end{aligned} \tag{6.51}$$

Thus, the function

$$z_0(\tau,\eta) = e^{(\mu\lambda_*-1)\tau}\bar{z}(\eta) \tag{6.52}$$

is a particular solution to (6.49). In particular, it satisfies the boundary condition $\bar{z}(0) = 0$.

Exercise 6.3 Show that if $z(\tau,\eta)$ is a positive solution to (6.49) then there exists a constant $K[z] > 0$ so that

$$z(\tau,\eta)e^{(1-\mu\lambda_*)\tau} \to K[z]\bar{z}(\eta), \quad \text{as } t \to +\infty. \tag{6.53}$$

Hint: find $\beta \in \mathbb{R}$ so that the operator

$$\mathcal{M}z = \frac{1}{2}z_{\eta\eta} + \frac{\eta}{2}z_\eta \tag{6.54}$$

is compact and self-adjoint on the weighted space $L^2(e^{\beta x^2}dx)$ and has a discrete spectrum. All its eigenfunctions are actually explicit. Moreover, $\bar{z}(\eta)$ is its principal eigenfunction. This leads to (6.53).

The conclusion of Exercise 6.3 means that $z(\tau,\eta)$ has the asymptotics

$$z(\tau,\eta) \approx K[z]\bar{z}(\eta)e^{-(1-\mu\lambda_*)\tau} = K[z]\eta e^{-\eta^2/2}e^{-(1-\mu\lambda_*)\tau}, \quad \text{as } \tau \to +\infty. \tag{6.55}$$

The corresponding asymptotics for the function $w(t,x)$ is

$$w(t,x) = z\Big(\log(t+1), \frac{x}{\sqrt{t+1}}\Big) \approx K[z]\frac{x}{\sqrt{t+1}}(t+1)^{-(1-\mu\lambda_*)}e^{-x^2/(2(t+1))}. \tag{6.56}$$

Recalling the changes of variables (6.35) and (6.37), we obtain the following asymptotics for $u(t,x)$:

$$\begin{aligned} u(t,x+c_*t-\mu\log(t+1)) &= w(t,x)e^{-\lambda_* x} \\ &\approx K[z]xe^{-\lambda_* x}(t+1)^{-(1-\mu\lambda_*)-1/2}e^{-x^2/(2(t+1))}, \end{aligned} \tag{6.57}$$

as $t \to +\infty$. This is why we expected a Gaussian correction in (6.29).

Here comes another delicate technical point. The asymptotics (6.55) holds as $\tau \to +\infty$ and $\eta \sim O(1)$. This translates into $t \to +\infty$ and $x \sim O(\sqrt{t+1})$. This spatial scale is too large for our purposes. With a bit more work we can extend (6.55) to hold for $\eta \sim O(\exp(-\delta\tau))$ as $\tau \to +\infty$. This means that (6.57) holds for $x \sim O(t^\delta)$ with $\delta > 0$ but arbitrarily small.

On the other hand, the limit (4.46) in Theorem 4.10 says that $u(t,x)$ converges to a shift of the traveling wave in the frame where it stays away from being close to 0 or 1. This is exactly what we want the frame

$$m(t) = c_*t - \mu\log(t+1) \tag{6.58}$$

to be. Thus, our goal is to choose μ so that

$$u(t,x+m(t)) \to U_*(x+s_r[\zeta]) \ \text{ as } t \to +\infty, \text{ uniformly in } x \geq K, \tag{6.59}$$

The final ingredient is the asymptotics (5.28) for $U_*(x)$ for $x \gg 1$:

$$U_*(x) \sim xe^{-\lambda_* x}, \quad \text{as } x \to +\infty, \tag{6.60}$$

In order for (6.57) and (6.59) to be consistent for $x \sim O(t^\delta)$, we must have

$$-(1-\mu\lambda_*) - \frac{1}{2} = 0, \tag{6.61}$$

which means that

$$\mu = \frac{3}{2\lambda_*}. \tag{6.62}$$

This leads to the Bramson centering (6.23)

$$m(t) = c_* t - \frac{3}{2\lambda_*}\log(t+1). \tag{6.63}$$

Comparing (6.56), with μ chosen as in (6.62), and (6.59), and keeping in mind the asymptotics (6.60) gives

$$K[z] = e^{-\lambda_* s_r[\zeta]}, \tag{6.64}$$

so that the Bramson shift $s_r[\zeta]$ is related to the coefficient $K[z]$ by

$$e^{-\lambda_* s_r[\zeta]} = K[z]. \tag{6.65}$$

Thus, to derive (6.26), we need to show that

$$K[z] = \lim_{t\to+\infty}\sqrt{\frac{2}{\pi}}\int_{-\infty}^{\infty} x e^{\lambda_* x} u(t, x + c_* t)dx. \tag{6.66}$$

This is what we will do next.

6.2.3 Using an integral of motion

To obtain an expression for the coefficient $K[z]$, we will use an integral of motion. Let us go back to the Dirichlet boundary value problem (6.49), with $\mu = 3/(2\lambda_*)$

$$\begin{aligned} &z_\tau - \frac{\eta}{2}z_\eta - \frac{3}{2}z = \frac{1}{2}z_{\eta\eta}, \quad \eta > 0,\\ &z(\tau, 0) = 0. \end{aligned} \tag{6.67}$$

Let us multiply (6.67) by η and integrate:

$$\begin{aligned} \frac{d}{dt}\int_0^\infty \eta z(\tau,\eta)d\eta &= \frac{1}{2}\int_0^\infty \eta z_{\eta\eta}(\tau,\eta)d\eta + \frac{1}{2}\int_0^\infty \eta^2 z_\eta(\tau,\eta)d\eta + \frac{3}{2}\int_0^\infty \eta z(\tau,\eta)d\eta\\ &= -\frac{1}{2}\int_0^\infty z_\eta(\tau,\eta)d\eta - \int_0^\infty \eta z(\tau,\eta)d\eta + \frac{3}{2}\int_0^\infty \eta z(\tau,\eta)d\eta\\ &= \frac{1}{2}\int_0^\infty \eta z(\tau,\eta)d\eta. \end{aligned} \tag{6.68}$$

It follows that for any $\tau > \tau' > 0$ we have

$$\int_0^\infty \eta z(\tau,\eta)d\eta = e^{(\tau-\tau')/2}\int_0^\infty \eta z(\tau',\eta)d\eta, \tag{6.69}$$

or, equivalently

$$I(\tau) := e^{-\tau/2}\int_0^\infty \eta z(\tau,\eta)d\eta = I(\tau'), \tag{6.70}$$

is an integral of motion.

Let us now recall the asymptotics (6.55), written with $\mu = 3/(2\lambda_*)$:

$$z(\tau,\eta) \approx K[z]\eta e^{-\eta^2/2}e^{\tau/2}, \quad \text{as } \tau \to +\infty. \tag{6.71}$$

It follows from (6.70) that

$$K[z]\int_0^\infty \eta^2 e^{-\eta^2/2}d\eta = \lim_{\tau\to+\infty} e^{-\tau/2}\int_0^\infty \eta z(\tau,\eta)d\eta. \tag{6.72}$$

A simple computation shows that

$$\int_0^\infty \eta^2 e^{-\eta^2/2}d\eta = \int_0^\infty \eta^2 e^{-\eta^2/2}d\eta = \frac{1}{2}\int_\infty^\infty e^{-\eta^2/2}d\eta = \frac{\sqrt{2\pi}}{2}. \tag{6.73}$$

Thus, (6.72) is

$$K[z] = \frac{2}{\sqrt{2\pi}}\lim_{\tau\to+\infty}\int_0^\infty e^{-\tau/2}\eta z(\tau,\eta)d\eta. \tag{6.74}$$

Of course, if $z(\tau,\eta)$ is a solution to (6.67), then (6.72) holds for all $\tau > 0$, without the limit in the right side. The point is that (6.67) gives a better and better approximation to the original Fisher-KPP problem in the self-similar variables as $\tau \to +\infty$. With some more work, one can show that (6.74), with the limit in the right side, is correct for the true solution. Using (6.64) and expressing $z(\tau,\eta)$ in terms of the changes of variables (6.47) gives

$$\begin{aligned} e^{-\lambda_* s_r[\zeta]} &= \frac{2}{\sqrt{2\pi}}\lim_{\tau\to+\infty}\int_0^\infty \eta e^{-\tau/2}z(\tau,\eta)d\eta = \frac{2}{\sqrt{2\pi}}\lim_{\tau\to+\infty}\int_0^\infty \eta e^{-\tau/2}w(e^\tau-1,\eta e^{\tau/2})d\eta \\ &= \frac{2}{\sqrt{2\pi}}\lim_{t\to+\infty}\int_0^\infty \eta w(t,\eta\sqrt{t+1})\frac{d\eta}{\sqrt{t+1}} = \frac{2}{\sqrt{2\pi}}\lim_{t\to+\infty}\int_{-\infty}^\infty xw(t,x)\frac{dx}{(t+1)^{3/2}}. \end{aligned} \tag{6.75}$$

Note that we extended the integration in x to the whole line in the last step because the contribution of the integral over $x < 0$ vanishes as $t \to +\infty$ due to the factor $1/(t+1)^{3/2}$ since the function $w(t,x)$ is rapidly decaying as $x \to -\infty$. Next, we undo the change of variables (6.35)-(6.37):

$$w(t,x) = e^{\lambda_* x}u(t, x + c_* t - \frac{3}{2\lambda_*}\log(t+1)), \tag{6.76}$$

and (6.75) becomes

$$\begin{aligned} e^{-\lambda_* s_r[\zeta]} &= \frac{2}{\sqrt{2\pi}}\lim_{t\to+\infty}\int_{-\infty}^\infty xe^{\lambda_* x}u(t, x + c_* t - \frac{3}{2\lambda_*}\log(t+1))\frac{dx}{(t+1)^{3/2}} \\ &= \frac{2}{\sqrt{2\pi}}\lim_{t\to+\infty}\int_{-\infty}^\infty \Big(x + \frac{3}{2\lambda_*}\log(t+1)\Big)e^{\lambda_*(x+3/(2\lambda_*)\log(t+1))}u(t,x+c_* t)\frac{dx}{(t+1)^{3/2}} \\ &= \frac{2}{\sqrt{2\pi}}\lim_{t\to+\infty}\int_{-\infty}^\infty \Big(x + \frac{3}{2\lambda_*}\log(t+1)\Big)e^{\lambda_* x}u(t,x+c_* t)dx. \end{aligned} \tag{6.77}$$

Exercise 6.4 Use the previous asymptotics we have obtained in this section to argue that

$$u(t, x + c_* t) \sim \frac{x}{t^{3/2}} e^{-\lambda_* x} e^{-x^2/(2t)}, \quad \text{as } t \to +\infty. \tag{6.78}$$

Conclude that the integral in the right side of (6.77) is dominated by $x \sim \sqrt{t}$, and the contribution of the term involving $\log(t+1)$ is small.

Exercise 6.4 shows that (6.77) can be re-written as

$$e^{-\lambda_* s_r[\zeta]} = \frac{2}{\sqrt{2\pi}} \lim_{t\to+\infty} \int_{-\infty}^{\infty} x e^{\lambda_* x} u(t, x + c_* t) dx, \tag{6.79}$$

which is (6.26)!

6.3 An auxiliary point process

Let us go back to the limit $\mathcal{E}$ of the point process $\mathcal{E}_t$. So far ,we have shown that the Laplace transform of $\mathcal{E}$ is given by (6.22)

$$\Psi[\phi] = \mathbb{E} \exp\big(- Z_\infty e^{-\lambda_* s_r[\zeta]}\big), \tag{6.80}$$

and the Bramson shift $s_r[\zeta]$ is given by (6.79)

$$e^{-\lambda_* s_r[\zeta]} = \lim_{t\to+\infty} \sqrt{\frac{2}{\pi}} \int_{-\infty}^{\infty} x e^{\lambda_* x} u(t, x + c_* t) dx, \tag{6.81}$$

Here, $u(t, x)$ is the solution to the initial value problem (6.24):

$$\begin{aligned} &u_t = \frac{1}{2} u_{xx} + f(u), \ t > 0, \ x \in \mathbb{R}, \\ &u(0, x) = \zeta(x) := 1 - e^{-\phi(-x)}. \end{aligned} \tag{6.82}$$

The first step in identifying $\mathcal{E}$ will be to construct an auxiliary point process Π_t such that the limits of Π_t and $\mathcal{E}_t$ are the same. Then, we will identify the limit of Π_t. To construct Π_t we first take a collection of Poisson points η_k, $k = 1, 2, \dots$ that lie in $(-\infty, 0)$, distributed with the intensity

$$\Gamma(x) = \sqrt{\frac{2}{\pi}} (-x) e^{-\lambda_* x}. \tag{6.83}$$

For each k we construct independent BBM processes $X_i^{(k)}(t)$, $i = 1, \dots N_k(t)$. The process Π_t combines all these points shifted by $L_\infty + \eta_k$, with

$$L_\infty = \frac{1}{\lambda_*} \log Z_\infty. \tag{6.84}$$

That is, Π_t is given by

$$\Pi_t(x) = \sum_{k=1}^{\infty} \sum_{i=1}^{N_k(t)} \delta\big(x - [X_i^{(k)}(t) - c_* t + L_\infty + \eta_k]\big). \tag{6.85}$$

Note that for each k fixed we have

$$c_* t - L_\infty - \eta_k - \max_{1 \le i \le N^{(k)}(t)} X_i^{(k)}(t) \to +\infty, \quad \text{as } t \to +\infty. \tag{6.86}$$

Hence, each individual term in the sum goes to zero weakly as $t \to +\infty$. However, there are infinitely many terms, and that is what saves the day, allowing Π_t to have a non-trivial limit.

The Laplace transform of a Poisson point process

In order to identify the limit, we will need to compute the Laplace transform of Π_t. To do this, we will need the formula for the Laplace transform of a Poisson point process

$$\mathcal{P}(x) = \sum_k \delta(x - Y_k). \tag{6.87}$$

Here, Y_k are Poisson points distributed with an intensity $\Lambda(dx)$. We claim that the Laplace transform of $\mathcal{P}(x)$ is

$$\mathbb{E}\Big(\exp\Big(-\int \phi(x) d\mathcal{P}(x)\Big)\Big) = \mathbb{E}\Big(\exp\Big(-\sum_k \phi(Y_k)\Big)\Big) = \exp\Big(-\int (1-e^{-\phi(x)})\Lambda(dx)\Big). \tag{6.88}$$

To see why (6.88) holds, consider a non-negative simple function

$$\phi(x) = \sum_k c_k \mathbb{1}(x \in A_k). \tag{6.89}$$

Here, A_k are pairwise disjoint measurable sets and $c_k > 0$. Since A_k are pairwise disjoint, the random variables $N_k = \#\big[Y_j \in A_k\big]$ are independent. With this in mind, an explicit computation gives

$$\begin{aligned} &\mathbb{E}\Big(\exp\Big(-\int \phi(x) d\mathcal{P}(x)\Big)\Big) = \mathbb{E}\Big(\exp\Big(-\sum_j \phi(Y_j)\Big)\Big) = \mathbb{E}\Big(\exp\Big(-\sum_k c_k \#\big[Y_j \in A_k\big]\Big)\Big) \\ &= \mathbb{E}\Big(\prod_k \exp\big(-c_k N_k\big)\Big) = \prod_k \mathbb{E}\Big(\exp\big(-c_k N_k\big)\Big). \end{aligned} \tag{6.90}$$

Note that for each k we have

$$\begin{aligned} &\mathbb{E}\Big(\exp\big(-c_k N_k\big)\Big) = \sum_{m=0}^{\infty} e^{-c_k m} e^{-\Lambda(A_k)} \frac{(\Lambda(A_k))^m}{m!} = e^{-\Lambda(A_k)} \exp\Big(\Lambda(A_k) e^{-c_k}\Big) \\ &= \exp\Big(-(1-e^{-c_k})\Lambda(A_k)\Big). \end{aligned} \tag{6.91}$$

Inserting this into (6.90) gives

$$\begin{aligned} &\mathbb{E}\Big(\exp\Big(-\int \phi(x) d\mathcal{P}(x)\Big)\Big) = \prod_k \exp\Big(-(1-e^{-c_k})\Lambda(A_k)\Big) \\ &= \exp\Big(-\sum_k \Big((1-e^{-c_k})\Lambda(A_k)\Big) = \exp\Big(-\int (1-e^{-\phi(x)})\Lambda(dx)\Big). \end{aligned} \tag{6.92}$$

This proves (6.88) for simple functions, and the conclusion for continuous functions follows by a standard approximation argument.

The limits of Π_t and $\mathcal{E}_t$ are the same

We are now ready to prove the following theorem. It will allow us to understand the limit process $\mathcal{E}$ in terms of the limit of Π_t.

Theorem 6.5 *The process Π_t converges to a limit measure-valued process Π, and the law of Π is the same as that of $\mathcal{E}$.*

Proof. The Laplace transform of Π_t is

$$\begin{aligned}\mathbb{E}\Big(\exp\Big(-\int\phi(x)\Pi_t(dx)\Big)\Big) &= \mathbb{E}\Big(\exp\Big(-\sum_k\sum_{i=1}^{N_k(t)}\phi(X_i^{(k)}(t)-c_*t+L_\infty+\eta_k)\Big)\Big)\\ &= \mathbb{E}\Big(\prod_k v(t,\eta_k-c_*t+L_\infty)\Big)=\mathbb{E}\Big(\exp\Big(\sum_k\log v(t,\eta_k-c_*t+L_\infty)\Big).\end{aligned} \tag{6.93}$$

Here, we have taken the expectation with respect to the independent BBM $X^{(k)}$, and have denoted $v(t,x)=1-\tilde{u}(t,x)$, where $\tilde{u}(t,x)$ is the solution to (6.82)

$$\begin{aligned}&\tilde{u}_t=\frac{1}{2}\tilde{u}_{xx}+f(\tilde{u}),\ t>0,\ x\in\mathbb{R},\\ &\tilde{u}(0,x)=1-e^{-\phi(x)}.\end{aligned} \tag{6.94}$$

Recall that

$$v(t,x)=\mathbb{E}\Big(\exp\Big(-\sum_{i=1}^{N_t}\phi(X_i(t)+x)\Big)\Big), \tag{6.95}$$

where $X_i(t)$ is the standard BBM.

Since η_k are Poisson points with the intensity (6.83):

$$\Gamma(x)=\sqrt{\frac{2}{\pi}}(-x)e^{-\lambda_*x}, \tag{6.96}$$

we may use (6.88) to re-write (6.93) as

$$\begin{aligned}\mathbb{E}\Big(\exp\Big(-\int\phi(x)\Pi_t(dx)\Big)\Big) &= \mathbb{E}\Big(\exp\Big(-\sum_k\log v(t,\eta_k-c_*t+L_\infty)\Big)\\ &= \mathbb{E}\exp\Big(-\int_{-\infty}^0\Big[1-e^{\log(v(t,x-c_*t+L_\infty))}\Big]\sqrt{\frac{2}{\pi}}(-x)e^{-\lambda_*x}dx\Big)\\ &= \mathbb{E}\exp\Big(-\int_{-\infty}^0\tilde{u}(t,x-c_*t+L_\infty))\sqrt{\frac{2}{\pi}}(-x)e^{-\lambda_*x}dx\Big)\\ &= \mathbb{E}\exp\Big(-\sqrt{\frac{2}{\pi}}\int_0^\infty\tilde{u}(t,-x-c_*t+L_\infty)xe^{\lambda_*x}dx\Big)\\ &= \mathbb{E}\exp\Big(-\sqrt{\frac{2}{\pi}}\int_0^\infty u(t,x+c_*t-L_\infty)xe^{\lambda_*x}dx\Big).\end{aligned} \tag{6.97}$$

Here, $u(t,x) = \tilde{u}(t,-x)$ is the solution to

$$\begin{aligned} u_t &= \frac{1}{2}u_{xx} + f(u),\ t>0,\ x\in\mathbb{R}, \\ u(0,x) &= 1-e^{-\phi(-x)}. \end{aligned} \tag{6.98}$$

This, of course, is nothing but (6.82). Next, recalling (6.84), we write (6.97) as

$$\begin{aligned} &\mathbb{E}\Big(\exp\Big(-\int\phi(x)\Pi_t(dx)\Big)\Big) = \mathbb{E}\exp\Big(-\sqrt{\frac{2}{\pi}}\int_{-L_\infty}^{\infty}u(t,x+c_*t)(x+L_\infty)e^{\lambda_*(x+L_\infty)}dx\Big) \\ &= \mathbb{E}\exp\Big(-\sqrt{\frac{2}{\pi}}Z_\infty\int_{-L_\infty}^{\infty}u(t,x+c_*t)(x+\frac{1}{\lambda_*}\log Z_\infty)e^{\lambda_* x}dx\Big). \end{aligned} \tag{6.99}$$

Exercise 6.6 Show that (6.99) can be simplified in the limit $t\to+\infty$ to

$$\lim_{t\to\infty}\mathbb{E}\Big(\exp\Big(-\int\phi(x)\Pi_t(dx)\Big)\Big) = \lim_{t\to\infty}\mathbb{E}\exp\Big(-\sqrt{\frac{2}{\pi}}Z_\infty\int_{-\infty}^{\infty}u(t,x+c_*t)xe^{\lambda_* x}dx\Big). \tag{6.100}$$

Hint: Exercise 6.4 may be useful here.

We now recall (6.81)

$$e^{-\lambda_* s_r[\zeta]} = \lim_{t\to+\infty}\sqrt{\frac{2}{\pi}}\int_{-\infty}^{\infty}xe^{\lambda_* x}u(t,x+c_*t)dx, \tag{6.101}$$

which, combined with (6.100) gives

$$\lim_{t\to\infty}\mathbb{E}\Big(\exp\Big(-\int\phi(x)\Pi_t(dx)\Big)\Big) = \mathbb{E}\exp\Big(-Z_\infty e^{-\lambda_* s_r[\zeta]}\Big). \tag{6.102}$$

Recall also that the Laplace transform of the limit $\mathcal{E}$ of $\mathcal{E}_t$ is given by (6.80):

$$\Psi[\phi] = \mathbb{E}\exp\big(-Z_\infty e^{-\lambda_* s_r[\zeta]}\big). \tag{6.103}$$

Comparing (6.102) and (6.103), we see that the limits of the point processes Π_t and $\mathcal{E}_t$ are the same, finishing the proof of Theorem 6.5. □

6.4 The maxima of the auxiliary BBMs

So far, we have shown that the limit $\mathcal{E}$ of the BBM point process

$$\mathcal{E}_t = \sum_{k=1}^{N_t}\delta(x-(X_k(t)-m(t))), \tag{6.104}$$

as seen from the location

$$m(t) = c_*t - \frac{3}{2\lambda_*}\log(t+1), \tag{6.105}$$

has the same law as the limit Π of the process defined in (6.85)

$$\Pi_t(x) = \sum_{k=1}^{\infty} \sum_{i=1}^{N_k(t)} \delta\big(x - [X_i^{(k)}(t) - c_* t + L_\infty + \eta_k]\big). \tag{6.106}$$

Here, $X^{(k)}$ are independent BBM, the points η_k are Poisson, distributed on $(-\infty, 0]$ with the intensity (6.83)

$$\Gamma(x) = \sqrt{\frac{2}{\pi}}(-x)e^{-\lambda_* x}, \tag{6.107}$$

and $L_\infty = (1/\lambda_*)Z_\infty$.

Our next goal is to better understand Π, and hence $\mathcal{E}$. Let us recall that for each k fixed, the maximum of $X^{(k)}(t)$ satisfies

$$c_* t - \max_{1 \le i \le N_t^{(k)}} X_i^{(k)}(t) \to +\infty, \quad \text{as } t \to +\infty. \tag{6.108}$$

Thus, the only k that contribute to the weak limit of the sum in (6.106) are those with "anomalously large" maxima.

To get a handle on the anomalosuly large maxima, let us first consider the process that looks only at the maxima: set

$$M_k(t) = \max_{1 \le k \le N_k(t)} X_i^{(k)}(t), \tag{6.109}$$

and define the point process of the maxima

$$\Pi_t^{ext}(x) = \sum_{k=1}^{\infty} \delta\big(x - [M_k(t) - c_* t + L_\infty + \eta_k]\big). \tag{6.110}$$

Proposition 6.7 *The process Π_t^{ext} converges in law as $t \to +\infty$, to a Poisson point process with the intensity*

$$K_0 Z_\infty e^{-\lambda_* x}, \tag{6.111}$$

with a deterministic constant $K_0 > 0$.

Proposition 6.7 says that the anomalously large maxima are Poisson points with the intensity $\exp(-\lambda_* x)$, that is much smaller than the intensity $(-x)\exp\{-\lambda_* x\}$ of the Poisson starting points η_k as $x \to -\infty$. This thinning out comes from the fact that most of maxima are much smaller than $c_* t$ and do not contribute to the limit.

Proof. As usual, we will look at the Laplace transform of Π_t^{ext}:

$$\mathbb{E}\Big(\exp\Big(-\int \phi(x)\Pi_t^{ext}(dx)\Big)\Big) = \mathbb{E}\Big(\exp - \Big(\sum_k \phi(M_k(t) - c_* t + L_\infty + \eta_k)\Big)\Big). \tag{6.112}$$

Let us set

$$q(t,x) = \mathbb{E}e^{-\phi(-x+M(t))}, \tag{6.113}$$

where $M(t)$ is the maximum of the standard BBM. Taking the expectation with respect to $M_k(t)$, we re-write (6.112) as

$$\begin{aligned}&\mathbb{E}\Big(\exp\Big(-\int\phi(x)\Pi_t^{ext}(dx)\Big)\Big)=\mathbb{E}\Big(\exp\Big(-\sum_k\phi(M_k(t)-c_*t+L_\infty+\eta_k)\Big)\Big)\\&=\mathbb{E}\Big(\prod_k\exp\Big(-\phi(M_k(t)-c_*t+L_\infty+\eta_k)\Big)\Big)\\&=\mathbb{E}\Big(\prod_k q(c_*t-L_\infty-\eta_k)\Big)=\mathbb{E}\Big(\exp\Big(-\sum_k\big[-\log q(t,c_*t-L_\infty-\eta_k)\big]\Big)\Big).\end{aligned}\tag{6.114}$$

Now, we can use the formula (6.88)

$$\mathbb{E}\Big(\exp\Big(-\int\phi(x)d\mathcal{P}(x)\Big)\Big)=\mathbb{E}\Big(\exp\Big(-\sum_k\phi(Y_k)\Big)\Big)=\exp\Big(-\int(1-e^{-\phi(x)})\Lambda(dx)\Big),\tag{6.115}$$

for the Laplace transform of a Poisson point process Y_k with intensity $\Lambda(x)$, to obtain from (6.114) that

$$\begin{aligned}&\mathbb{E}\Big(\exp\Big(-\int\phi(x)\Pi_t^{ext}(dx)\Big)\Big)=\mathbb{E}\Big(\exp\Big(-\sum_k\big[-\log q(t,c_*t-L_\infty-\eta_k)\big]\Big)\Big)\\&=\mathbb{E}\exp\Big(-\sqrt{\frac{2}{\pi}}\int_0^\infty(1-q(t,x+c_*t-L_\infty))xe^{\lambda_*x}dx\Big)\\&=\mathbb{E}\exp\Big(-\sqrt{\frac{2}{\pi}}\int_0^\infty w(t,x+c_*t-L_\infty)xe^{\lambda_*x}dx\Big).\end{aligned}\tag{6.116}$$

Here, we have introduced

$$w(t,x)=1-q(t,x)=\mathbb{E}\Big(1-e^{-\phi(-x+M(t))}\Big).\tag{6.117}$$

In order to compute the right side of (6.116), recall that the cumulative distirbution function of the maximum

$$u(t,x)=\mathbb{P}(M(t)>x),\tag{6.118}$$

is the solution to the initial value problem

$$\begin{aligned}&u_t=\frac{1}{2}u_{xx}+f(u),\quad t>0,\\&u(0,x)=\mathbb{1}(x\le 0).\end{aligned}\tag{6.119}$$

Therefore, the integral in the right side of (6.116) can be re-written as

$$\begin{aligned}&\int_0^\infty w(t,x+c_*t-L_\infty)xe^{\lambda_*x}dx=\int_0^\infty\mathbb{E}\Big(1-e^{-\phi(-x-c_*t+L_\infty+M(t))}\Big)xe^{\lambda_*x}dx\\&=\int_0^\infty\int_{-\infty}^\infty\Big(1-e^{-\phi(-x-c_*t+L_\infty+y)}\Big)(-u_y(t,y))xe^{\lambda_*x}dydx\\&=\int_0^\infty\int_{-\infty}^\infty\Big(1-e^{-\phi(-x+L_\infty+y)}\Big)(-u_y(t,y+c_*t))xe^{\lambda_*x}dydx\\&=\int_{-\infty}^\infty\int_{-\infty}^y\Big(1-e^{-\phi(\xi+L_\infty)}\Big)(-u_y(t,y+c_*t))(y-\xi)e^{\lambda_*(y-\xi)}d\xi dy.\end{aligned}\tag{6.120}$$

Exercise 6.8 Go back to Exercise 6.4 to show that there exists x_0 so that we have the asymptotics

$$u(t, y + c_*t) \approx e^{-\lambda_* x_0} \frac{ye^{-\lambda_* y}}{t^{3/2}} e^{-y^2/(2t)}, \tag{6.121}$$

and

$$(-u_y)(t, y + c_*t) \approx C_0 \frac{ye^{-\lambda_* y}}{t^{3/2}} e^{-y^2/(2t)}. \tag{6.122}$$

with

$$C_0 = \lambda_* e^{-\lambda_* x_0}.$$

Using (6.122) in (6.120) we obtain

$$\begin{aligned} &\int_0^\infty w(t, x + c_*t - L_\infty) x e^{\lambda_* x} dx \\ &\approx C_0 \int_{-\infty}^\infty \int_{-\infty}^y \Big(1 - e^{-\phi(\xi + L_\infty)}\Big) \frac{ye^{-\lambda_* y}}{t^{3/2}} e^{-y^2/(2t)} (y - \xi) e^{\lambda_*(y-\xi)} d\xi dy \\ &= C_0 \int_{-\infty}^\infty \int_{-\infty}^y \Big(1 - e^{-\phi(\xi + L_\infty)}\Big) \frac{y}{t^{3/2}} e^{-y^2/(2t)} (y - \xi) e^{-\lambda_* \xi} d\xi dy. \end{aligned} \tag{6.123}$$

Making a change of variables $y = z\sqrt{t}$ gives

$$\begin{aligned} &\int_0^\infty w(t, x + c_*t - L_\infty) x e^{\lambda_* x} dx \\ &\approx C_0 \int_{-\infty}^\infty \int_{-\infty}^{z\sqrt{t}} \Big(1 - e^{-\phi(\xi + L_\infty)}\Big) z e^{-z^2/2} \Big(z - \frac{\xi}{\sqrt{t}}\Big) e^{-\lambda_* \xi} d\xi dz \\ &\approx C_0 \int_{-\infty}^\infty \int_{-\infty}^{\infty} \Big(1 - e^{-\phi(\xi + L_\infty)}\Big) z^2 e^{-z^2/2} e^{-\lambda_* \xi} d\xi dz. \end{aligned} \tag{6.124}$$

Integrating out the z variable we get

$$\int_0^\infty w(t, x + c_*t - L_\infty) x e^{\lambda_* x} dx \approx C_1 \int_{-\infty}^\infty \Big(1 - e^{-\phi(\xi + L_\infty)}\Big) e^{-\lambda_* \xi} d\xi. \tag{6.125}$$

Going back to (6.116), we have shown that

$$\begin{aligned} &\lim_{t \to +\infty} \mathbb{E}\Big(\exp\Big(-\int \phi(x) \Pi_t^{ext}(dx)\Big)\Big) = C_1 \int_{-\infty}^\infty \Big(1 - e^{-\phi(\xi + L_\infty)}\Big) e^{-\lambda_* \xi} d\xi \\ &= C_1 \int_{-\infty}^\infty \Big(1 - e^{-\phi(\xi)}\Big) e^{-\lambda_*(\xi - L_\infty)} d\xi = C_1 Z \int_{-\infty}^\infty \Big(1 - e^{-\phi(\xi)}\Big) e^{-\lambda_* \xi} d\xi. \end{aligned} \tag{6.126}$$

Recalling the formula (6.88)

$$\mathbb{E}\Big(\exp\Big(-\int \phi(x) d\mathcal{P}(x)\Big)\Big) = \exp\Big(-\int (1 - e^{-\phi(x)}) \Lambda(dx)\Big). \tag{6.127}$$

for the Laplace transform of a Poisson point process, we have shown that the limit of the point process Π_t^{ext} as $t \to +\infty$ is a Poisson point process on $(-\infty, \infty)$ with the intensity

$$C_1 Z e^{-\lambda_* x}. \tag{6.128}$$

This finishes the proof of Proposition 6.7. □

6.5 The clusters

Now, we know that the limit process Π comes from the limit of the anomalously large maxima of BBM and their families. The points correspoding to the anomalous maxima are Poisson distributed points, as in Proposition 6.7, and the limits of the families form what is known as clusters around those maxima. The law of the cluster is simply the limit of the law of BBM conditioned to have a maximum that is larger than $c_* t$. We omit the details.

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