Moderate dispersion in scalar conservation laws

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Abstract

We consider the weakly dissipative and weakly dispersive Burgers-Hopf-Korteweg-de-Vries equation with the diffusion coefficient ε and the dispersion rate δ in the range $\delta/\varepsilon \to 0$. We study the travelling wave connecting $u(-\infty) = 1$ to $u(+\infty) = 0$ and show that it converges strongly to the entropic shock profile as $\varepsilon, \delta \to 0$.

Key-words Travelling waves, moderate dispersion, Korteweg de Vries equation, entropy solutions, dissipative-dispersive scalar conservation laws.
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1 Introduction

This paper is concerned with the hyperbolic conservation laws approximated by a weakly dissipative and weakly dispersive equation, that is, we are interested in the limit as $\varepsilon \to 0$, $\delta \to 0$ of the solutions of the following problem

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\left(A(u)\right) = \varepsilon \frac{\partial^2 u}{\partial x^2} - \delta \frac{\partial^3 u}{\partial x^3}, \qquad (1)$$
$$u(t,0) = u^0(x).$$

It is well known that when the parameter ε vanishes too fast compared to δ, the dispersive effects dominate and produce oscillations. In that case the (weak) limit is not a weak solution to the conservation law with $\varepsilon = \delta = 0$ ([7]). Therefore several authors have considered the *weak dispersion* case $\delta = \alpha\varepsilon^2$, with $0 < \alpha < \infty$ fixed, showing the solutions of (1)converge strongly ([10, 6]) to weak solution to the equation with $\varepsilon = \delta = 0$. Such limits may lead to non-entropic solutions ([10, 6, 5, 1, 8]) for non-convex fluxes $A(u)$. However, for strictly convex fluxes and in this weak dispersion regime $\delta = \alpha\varepsilon^2$ with $\alpha > 0$ fixed, one expects that the (strong) limits always satisfy the family of Kruzkov entropies; this is proved in [8] for instance for travelling waves. For general solutions to the initial value problem in the weak dispersion regime, it is easy to prove that the square entropy satisfies the entropy inequality but there is no direct derivation of the full family of entropy inequalities. However an indirect argument due to R. DiPerna indicates that for convex fluxes a single entropy inequality implies all the others; the original argument uses the BV regularity of solutions (such a bound is not

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available here) and this regularity assumption has been removed recently in [3, 9]. But we are not aware of any related result in the moderate dispersion regime, i.e. a priori strong limits.

The purpose of this paper is to further investigate the case of convex fluxes, on the simple example of the Burgers-Hopf equation

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\left(\frac{u^2}{2}\right) = \varepsilon \frac{\partial^2 u}{\partial x^2} - \delta \frac{\partial^3 u}{\partial x^3}. \tag{2}$$

Then we sustain, by means of the study of travelling waves, the idea that the entropy inequalities are reached in the more general limit

$$\frac{\delta}{\varepsilon} \to 0 \qquad \text{(moderate dispersion)}, \tag{3}$$

and that the condition $\delta = O(\varepsilon^2)$ is perhaps too stringent. More precisely we prove the

Theorem 1.1 *There exists a (unique up to translation) travelling wave solution $u(t,x) = S_{\varepsilon,\delta}(x-t/2)$ of (2) connecting the states $u(-\infty) = 1$ to $u(+\infty) = 0$ which converges strongly, as $\varepsilon, \delta \to 0$, together with (3), to the entropic shock profile $\bar{u}(t,x) = \bar{S}(x - t/2)$ with*

$$\bar{S}(x) = \begin{cases} 1, & \textit{for } x < 0, \\ 0, & \textit{for } x > 0. \end{cases}$$

The proof of this theorem is given in the next section. We first recall several facts on travelling waves and rescale the problem to settle a clearer asymptotic problem. Then, we study the limiting case $c = 0$ which serves as a basis for the expansions performed in the study of the general travelling wave. Our analysis indicates that when $\delta = \alpha\varepsilon$ with $\alpha > 0$ fixed then the travelling wave $S_{\varepsilon,\delta}$ converges only weakly and convergence to the entropic shock breaks down at this level. In that sense, the result of Theorem 1.1 is sharp.

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2 Convergence of Burgers-KdV travelling wave solutions

2.1 Rescaled travelling waves

We now look for travelling wave solutions of the Burgers-KdV equation

$$u_t + uu_x = \varepsilon u_{xx} - \delta u_{xxx} \tag{4}$$

that connect $u = 1$ as $x \to -\infty$ and $u = 0$ as $x \to +\infty$, and move with the correct speed $c_0 = 1/2$ (the fact that this is the only possible speed for a travelling wave connecting these two states follows immediately from the Rankin-Hugoniot condition), as it stands for an entropic shock of the Burgers equation. Such solutions have the form $u(t,x) = S_{\varepsilon,\delta}(x - t/2)$ with the function $S_{\varepsilon,\delta}(x)$ which solves

$$-\frac{1}{2}S'_{\varepsilon,\delta} + S_{\varepsilon,\delta}S'_{\varepsilon,\delta} = \varepsilon S''_{\varepsilon,\delta} - \delta S'''_{\varepsilon,\delta}, \quad S_{\varepsilon,\delta}(-\infty) = 1, \quad S_{\varepsilon,\delta}(+\infty) = 0. \tag{5}$$

Integrating between $-\infty$ and x we obtain

$$-\frac{1}{2}S_{\varepsilon,\delta} + \frac{S^2_{\varepsilon,\delta}}{2} = \varepsilon S'_{\varepsilon,\delta} - \delta S''_{\varepsilon,\delta}, \quad S_{\varepsilon,\delta}(-\infty) = 1, \quad S_{\varepsilon,\delta}(+\infty) = 0. \tag{6}$$

We also rescale and reverse the direction: $t = -x/\sqrt{\delta}$ and arrive at

$$S_{\varepsilon,\delta}(x) = S_c(-x/\sqrt{\delta}) \quad \text{with } \; c = \varepsilon/\sqrt{\delta}, \tag{7}$$

and

$$-\frac{1}{2}S_c + \frac{S_c^2}{2} = -cS_c' - S_c'', \quad S_c(-\infty) = 0, \quad S_c(+\infty) = 1. \tag{8}$$

Note that the function $\phi(t) = 1 - S_c(t)$ satisfies the familiar Fisher-KPP equation

$$-c\phi' = \phi'' + \frac{1}{2}\phi(1-\phi), \qquad \phi(-\infty) = 1, \quad \phi(+\infty) = 0. \tag{9}$$

It follows that there are *three regimes* for the travelling front solutions of (2).
• The first one arises when $c \geq c_* = \sqrt{2}$, the minimal KPP speed. Then equation (9) admits monotonic travelling wave solutions. In terms of (4) this means that such solutions exist for $\varepsilon^2/\delta \geq 2$. After rescaling as in (7) and normalizing $S_c(0) = 1/2$ we observe that $S_{\varepsilon,\delta}(x)$ converges pointwise to the entropic shock profile

$$\bar{S}(x) = \begin{cases} 1, & \text{for } \; x < 0 \\ 0, & \text{for } \; x > 0. \end{cases} \tag{10}$$

This regime has been widely studied, see [4] and the references therein.
• The second regime, when $0 < c < c_*$, is different – a travelling wave still exists for all $\varepsilon > 0$, $\delta > 0$, that is, for all $c > 0$ but it is no longer monotonic in x. Nevertheless, as long as we keep $\varepsilon/\sqrt{\delta} \geq c_o$ for any fixed $c_o > 0$ and let $\varepsilon, \delta \to 0$, the wave $S_{\varepsilon,\delta}$ converges to the shock (10), and here again an additional normalization is needed to fix the location of the travelling wave:

$$S_c'(x) > 0 \text{ for } x < 0 \qquad \text{and } S_c'(0) = 0, \tag{11}$$

a normalization we will used throughout the paper. This has been shown in [2], see also [6, 5, 8] for the same conclusions on the initial value problem. In terms of (9) this means that the speed c is bounded away from zero: $c \geq c_o > 0$.
• Here we are interested in what happens in the third regime, when c becomes small but so that δ/ε vanishes. We would like to show that the wave $S_{\varepsilon,\delta}(x)$ picture is as follows (arguing backward from $+\infty$ to $-\infty$: for $x > 0$ solution decays to zero on the length scale $\lambda = \sqrt{\delta}/c$, this region is followed by an interval of a comparable length where solution oscillates between $u = 0$ and $u = 3/2$, which is finally followed by a region where solution oscillates on the length scale $\lambda_1 = \sqrt{\delta}$ and approaches the value $u = 1$ at the exponential rate $k = 1/\lambda$. This means that the travelling wave converges pointwise to a shock profile as $\varepsilon, \delta \to 0$, as long as $\lambda \to 0$. In terms of the parameters ε and δ this translates into $\delta/\varepsilon \to 0$ as opposed to the case $\delta \leq c_o^2\varepsilon^2$ studied previously in [2]. The main difference compared to the latter regime is that there is a region inside the "viscous shock" where oscillations are strong.

In terms of the wave $S_c(t)$ we have to show the following: (i) In the first zone (the monotonicity region), $t < 0$, we have $S_c'(t) > 0$ and there exists constant $B > 0$, independent of $c > 0$ so that $0 < S_c(t) \leq Be^{t/B}$; (ii) In the second zone (the transient oscillations region), between $x = 0$ and $x = O(1/c)$, the wave $S_c(t)$ oscillates between the values 0 and 3/2 with the period $O(c^{-1/4})$; (iii) In the third zone (the exponential damping region), $x > B/c$, we have $|1 - S_c(x)| \leq Be^{-c(x-B/c)}$ but S_c is still an oscillatory function with the period equal to $O(1)$ – and these bounds should hold with a constant B independent of $c > 0$. A profile of $S_c(x)$ with $c = .05$ is depicted in Figure 2 with translated abscissae.

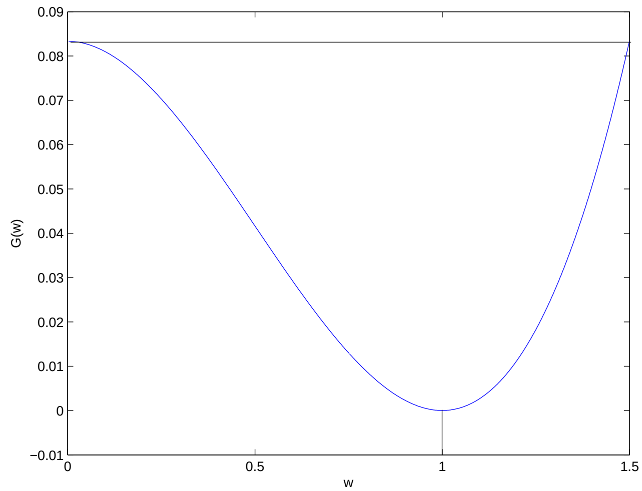

Figure 1: The potential energy, the function $G(w)$.

For this purpose, it is convenient to introduce the phase space variables $w = S_c$ and $v = -S_c'$ to re-write (8)–(11) as

$$\begin{cases} w' = -v \\ v' = -cv - \frac{1}{2}w(1-w) \\ w(-\infty) = v(-\infty) = 0, \quad v(0) = 0, \quad w(+\infty) = 1, \ v(+\infty) = 0. \end{cases} \tag{12}$$

This system has an energy

$$H(v,w) = \frac{v^2}{2} + G(w), \qquad \frac{d}{dt}H(v(t),w(t)) = -cv(t)^2, \tag{13}$$

with $G'(w) = -w(1-w)/2$. We choose the normalization so that $\min G = G(1) = 0$ which means that

$$G(w) = \frac{w^3}{6} - \frac{w^2}{4} + \frac{1}{12} = \frac{(1-w)^2(1+2w)}{12}.$$

This function is plotted in Figure 1.

2.2 The periodic solutions for $c = 0$

In order to study the problem with $c > 0$ we first recall the basic properties of the periodic solutions that exist when $c = 0$. Then, the system (12) becomes

$$\begin{cases} w' = -v, \\ v' = -\frac{1}{2}w(1-w), \end{cases} \tag{14}$$

and the energy is conserved. The energy level, $0 \le H = H(0, w(0)) = G(w(0)) < 1/12 = G(0)$, characterizes the solution.

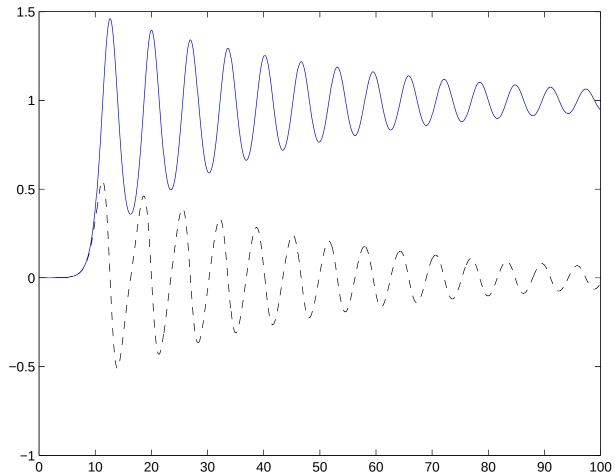

Figure 2: Solid line: a profile of $S_c(x)$ with $c = .05$, $w(0) = 5 \cdot 10^{-5}$. Dashed line: the derivative $S_c'(x)$.

The period. There exists a global solution (corresponding to $T = \infty$) with the maximal energy $H = 1/12$ and $w > 0$, which corresponds to a homoclinic orbit of (14) that connects $(0,0)$ to itself. Other solutions with energies $0 \le H < 1/12$ are periodic, with the period T given in terms of the energy H by

$$T = 2 \int_{w_1(H)}^{w_2(H)} \frac{dw}{\sqrt{2\big(H - G(w)\big)}}. \tag{15}$$

Here $w_1(H) < w_2(H)$ are the two solutions of $G(w_{1,2}(H)) = H$, that is, the minimum and maximum values of the function $w(t)$ on the trajectory. We have two limiting cases: (maximal energy) $w_1(1/12) = 0$ and $w_2(1/12) = 3/2$, (minimal energy) $w_1(0) = w_2(0) = 1$.

Large energies, $H \lesssim 1/12$. Such a periodic solution, with $w(0) = 5 \cdot 10^{-5}$, is depicted in Figure 3. The periodic solutions with H close to $H = 1/12$ spend most of the time close to the minimal value $w_1(H) \approx 0$ as can be seen from Figure 3 – the reason is that they "follow" the bound state for a long time. However, the time they spend between $w = 1/2$ and $w = w_2(H) \approx 3/2$ is uniformly (in H) bounded from above. Indeed, assume that $H > H(1/2, 0) = 1/24$, $w(t_1) = 1/2$ with $v(t_1) > 0$ and t_2 is the first time larger than t_1 such that $w(t_2) = w_2(H)$. Then we have

$$t_2 - t_1 = 2 \int_{1/2}^{w_2(H)} \frac{dw}{\sqrt{2(H - G(w))}} \le C,$$

since $G'(w_2(H))$ is bounded away from zero for $H \ge 1/24$. On the other hand, as $G'(0) = 0$, the period $T(H)$ for $H \ge 1/24$ is bounded from below:

$$T(H) = 2 \int_{w_1(H)}^{w_2(H)} \frac{dw}{\sqrt{2(H - G(w))}} \ge \frac{1}{\sqrt{2}} \int_{w_1(H)}^{1/2} \frac{dw}{\sqrt{2\big(H - G(w)\big)}} \ge T_0 > 0.$$

Moreover, as $H \to 1/12$ we have

$$T(H) = 2 \int_{w_1(H)}^{w_2(H)} \frac{dw}{\sqrt{2(H - G(w))}} \to +\infty.$$

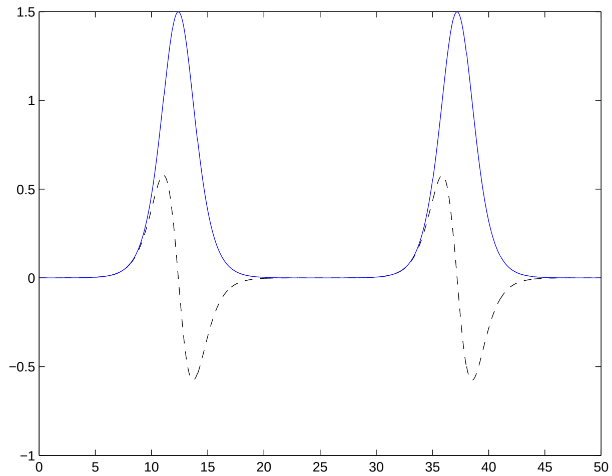

Figure 3: Large energy periodic solution. Solid line: a profile of $S_c(x)$ with $c = 0$, $w(0) = 5 \cdot 10^{-5}$. Dashed line: the derivative $S_c'(x)$.

For $H \lesssim 1/12$, one can also prove that

$$w_1(H) = 2\sqrt{\frac{1}{12} - H} + \frac{4}{3}\left(\frac{1}{12} - H\right) + O((\frac{1}{12} - H)^{3/2}), \quad w_2(H) = \frac{8}{3}\left(\frac{1}{12} - H\right) + O((\frac{1}{12} - H)^2),$$
$$T(H) = -2\sqrt{2}\ln(w_1(H)) + O(1). \tag{16}$$

Indeed, we have, by a simple Taylor expansion, $w_1(H) = O(\sqrt{\frac{1}{12} - H})$. Then we obtain from the expression (15) for the period that

$$\begin{aligned} T(H) &= 2\int_{w_1}^{\cdot} \frac{dw}{\sqrt{2(H - \frac{1}{12} + \frac{w^2}{4} - \frac{w^3}{6})}} = 2\int_0^{\cdot} \frac{ds}{\sqrt{s(w_1 - w_1^2) + s^2(\frac{1}{2} - w_1) - \frac{s^3}{3}}} \\ &= 2\int_0^{\cdot/w_1} \frac{du}{\sqrt{u(1 - w_1) + u^2(\frac{1}{2} - w_1) - \frac{u^3}{3}w_1}} \\ &\approx 2\int_0^{\cdot/w_1} \frac{du}{\sqrt{u + \frac{u^2}{2}}} \approx -2\sqrt{2}\ln w_1. \end{aligned}$$

Small energies, $H \gtrsim 0$. The situation is different for small energies: the period is bounded both from below and from above. Indeed, note that $G(w)$ is convex for $w \geq 1/2$ so that

$$\begin{aligned} H - G(w) &\leq -G'(w_1(H))(w - w_1) \quad \text{for } w_1(H) \leq w \leq 1, \\ H - G(w) &\leq G'(w_2(H))(w_2 - w) \quad \text{for } 1 \leq w \leq w_2(H). \end{aligned}$$

Using these bounds we obtain by an elementary calculation

$$T = 2\int_{w_1(H)}^{w_2(H)} \frac{dw}{\sqrt{2(H - G(w))}} \geq C\left(\frac{1}{\sqrt{w_1(H)}} + \frac{1}{\sqrt{w_2(H)}}\right) \geq T_1 > 0.$$

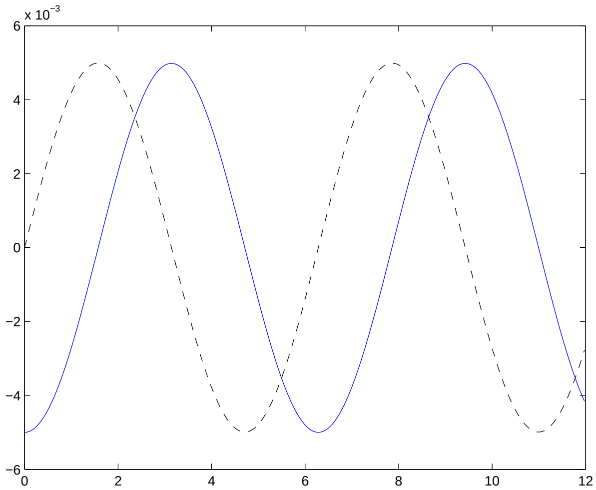

Figure 4: Small energy periodic solution. Solid line: a profile of $S_c(x)-1$ with $c=0$, $w(0)=1-0.005$. Dashed line: the derivative $S_c'(x)$.

On the other hand, we have an upper bound for $G(w)$:

$$G(w) \leq H - \frac{H}{1-w_1(H)}(w - w_1(H)) \text{ for } w_1(H) \leq w \leq 1,$$

and

$$G(w) \leq H + \frac{H}{w_2(H)-1}(w - w_2(H)) \text{ for } 1 \leq w \leq w_2(H).$$

It follows with the help of another elementary computation that

$$T = 2\int_{w_1(H)}^{w_2(H)} \frac{dw}{\sqrt{2(H-G(w))}} \leq C\left(\frac{1-w_1}{\sqrt{H}} + \frac{w_2-1}{\sqrt{H}}\right) \leq T_2 < +\infty.$$

In fact, the asymptotic behavior for small energy is clear. The solution approaches the harmonic oscillator $1-w(x) \approx (1-w(0))cos(2\pi t)$.

2.3 Travelling waves for a small $c > 0$

As we have mentioned, existence of travelling wave solutions for all $0 < c < c_* = \sqrt{2}$ has been established in [2]. It has been shown that they look as follows: In the first zone $S_c(t)$ is increasing for $t < 0$ and has its first maximum at $x_0 = 0$. Then, $S_c(t)$ oscillates with a decreasing amplitude in the following sense: there exists infinite sequences of maxima $0 = x_0 < x_1 < \cdots < x_n < \dots$ and minima $0 < y_1 < y_2 < \cdots < y_n < \dots$ with $x_{n-1} < y_n < x_n$ so that $1 < S_c(x_{n+1}) < S_c(x_n) < 3/2$ and $0 < S_c(y_n) < S_c(y_{n+1}) < 1$ – see Figure 2 for a typical profile (with translated abscissae). Our task is to estimate the differences of S_c at successive maxima and minima as well as the distances $L_n = x_{n+1} - x_n$ and $l_n = y_{n+1} - y_n$ as well as the energy drop $H_c(x_{n+1}) - H_c(x_n)$ where $H_c(t)$ denotes the energy of S_c as introduced in (13). We do that in distinguishing two additional zones (transitory oscillations and exponential damping).

Transitory oscillations. Here, we wish to prove the

Lemma 2.1 *For $c > 0$ small enough, and as long as $\frac{1}{24} \leq H(y_n) < \frac{1}{12}$, for some constant K, we have*

$$y_{n+1} - y_n \leq a_n := \frac{K}{(cn)^{1/4}}, \qquad H(y_{n+1}) - H(y_n) \leq -Kc.$$

Proof. We denote by K a universal constant independent of c throughout the proof. We need a preliminary and straightforward estimate on solutions $(w_c(t), v_c(t))$ with $c > 0$ by those $(w_0(t_0; t), v_0(t_0; t))$ obtained with $c = 0$ and the same initial data at $t = t_0$, $(w_c(t_0), v_c(t_0)) = (w_0(t_0; t_0), v_0(t_0; t_0))$. This estimate is

$$|v_c(t) - v_0(t_0; t)| + |w_c(t) - w_0(t_0; t)| \leq c\, C_T \sup_{|s-t_0| \leq T} |v_c(s)|, \text{ for all } |t - t_0| \leq T. \tag{17}$$

It is a consequence of Gronwall lemma applied to the system (12) where the term $-cv$ is considered as a source.

Next, we can consider the energy drop between $-\infty$ and $x_0 = 0$. It is estimated as

$$\frac{1}{12} - H_c(0) = c \int_{-\infty}^{0} v_c^2(t)dt.$$

We claim that $H_c(0) \leq 1/12 - cM$ with a constant M independent of $0 < c < c_0$, for some c_0 small enough. Indeed, we have for $-1 \leq t \leq 0$:

$$|v_c(t) - v_0(0; t)| + |w_c(t) - w_0(0; t)| \leq Kc \sup_{-1 \leq s \leq 0} |v_c(s)| \leq K'c, \text{ for all } -1 \leq t \leq 0.$$

This means that

$$\int_{-\infty}^{0} v_c^2(t)dt \geq \int_{-1}^{0} v_c^2(t)dt \geq \int_{-1}^{0} v_0^2(0; t)dt - Kc \geq K,$$

and thus

$$H_c(0) \leq \frac{1}{12} - Kc. \tag{18}$$

We also have the following fact: if $H_c(x_n) \geq 1/24$ then $x_n - x_{n-1} \geq T_0$ with T_0 independent of n. On the other hand, we have

$$H_c(x_{n-1}) - H_c(x_n) = c \int_{x_{n-1}}^{x_n} v_c^2(t)dt. \tag{19}$$

By the same argument as above we deduce that

$$H_c(x_n) \leq H_c(x_{n-1}) - Kc \quad \text{ if } H_c(x_n) \geq 1/24.$$

This means that

$$H_c(x_n) \leq \frac{1}{12} - Kcn \quad \text{ if } H_c(x_n) \geq 1/24.$$

As a consequence, we have $w_c \geq K\sqrt{cn}$ for $x_{n-1} \leq t \leq x_n$ and, in particular, $w_c(y_n) \geq K\sqrt{cn}$.

Let $z_n \in (y_n, x_n)$ be the first point to the right of y_n where $w_c = 3/4$. In order to estimate $|x_n - y_n|$ we will now first estimate the distance between y_n and z_n. On the interval between y_n and z_n the function w_c satisfies

$$-w_c'' = cw_c' - \frac{1}{2} w_c(1 - w_c),$$

and $w_c \geq K\sqrt{cn}$. Therefore, on this interval we have

$$w_c'' + cw_c' \geq K\sqrt{cn}, \quad w_c(y_n) \geq K\sqrt{nc}, \quad w_c'(y_n) = 0, \quad w_c' \geq 0 \text{ for } y_n \leq t \leq z_n.$$

It follows that

$$w_c'(t) \geq \frac{K\sqrt{n}}{\sqrt{c}} \left(1 - e^{-c(t-y_n)}\right) \text{ for } y_n \leq t \leq z_n. \tag{20}$$

Therefore, as long as (20) holds and $t \leq y_n + \frac{1}{100c}$ we get

$$\begin{aligned} w_c &\geq K\sqrt{cn} + \frac{K\sqrt{n}}{\sqrt{c}}(t - y_n) - \frac{K\sqrt{n}}{c\sqrt{c}} \left(1 - e^{-c(t-y_n)}\right) \\ &\geq K\sqrt{cn} + \frac{K\sqrt{n}}{\sqrt{c}}(t - y_n) - \frac{K\sqrt{n}}{c\sqrt{c}} \left(1 - 1 + c(t - y_n) - \frac{c^2}{4}(t - y_n)^2\right) \\ &= K\sqrt{cn}(1 + (t - y_n)^2). \end{aligned}$$

It follows that

$$z_n - y_n \leq \frac{K}{(cn)^{1/4}}.$$

On the other hand, it is straightforward to compute that $|z_n - x_{n+1}| \leq T_0$ with T_0 uniform in the energies. We see that

$$x_n - y_{n-1} \leq \frac{K}{(cn)^{1/4}}.$$

A similar computation shows that

$$y_n - x_n \leq \frac{K}{(cn)^{1/4}},$$

so that

$$y_n - y_{n-1} \leq a_n = \frac{K}{(cn)^{1/4}},$$

provided that $H(y_n) \geq 1/24$. $\square$

We can now conclude the analysis of the transitory zone. Because, we have seen that energy drops by Kc between y_n and y_{n+1}, we conclude that the number N of oscillations before the energy $H = 1/24$ is reached is bounded by $N_c = K/c$. Then the total time it takes to reach this energy level is bounded by

$$L = \sum_{n=1}^{N=K/c} a_n = \sum_{n=1}^{N=K/c} \frac{K}{(cn)^{1/4}} \leq \frac{K}{c^{1/4}} \left(\frac{K}{c}\right)^{3/4} \leq \frac{K}{c}.$$

Exponential damping. The third zone is when the energy $H_c(t)$ is smaller than $1/24$. There, using the inequality

$$H_c'(t) = -cv^2 \geq -cH(t),$$

we deduce the lower bound

$$H(t) \geq H(t_0)e^{-c(t-t_0)}.$$

Our purpose is to prove the reverse inequality.

We begin by arguing as before for (17)

$$|v_c(t) - v_0(t_0;t)| + |w_c(t) - w_0(t_0;t)| \leq c\sqrt{H_c(t_0)}e^{M(t-t_0)}. \tag{21}$$

This proves, again by continuity as $c \to 0$ on the rescaled quantities $v_c(t)/\sqrt{H_c(x_n)}$, $w_c(t)/\sqrt{H_c(x_n)}$, that

$$x_{n+1} - x_n = O(1), \quad \text{and } \; y_{n+1} - y_n = O(1). \tag{22}$$

Finally, with the similar argument and using again (19), we have a bound for energy drop over each oscillation:

$$H_c(x_n) \le (1 - Kc) H_c(x_{n-1}).$$

This means that, in this third zone, energy decays exponentially at the rate cK:

$$H_c(x_n) \le K e^{-Kc(x_n - K/c)},$$

in other words $w_c(t) \to 1$ exponentially as we have claimed. This finishes the proof of Theorem 1.1.

2.4 The general nonlinearities

Our results may be generalized to any strictly convex flux $f(u)$ and boundary conditions $u(-\infty) = u_l$ and $u(+\infty) = u_r$:

$$u_t + (f(u))_x = \varepsilon u_{xx} - \delta u_{xxx}, \quad u(-\infty) = u_l, \quad u(+\infty) = u_r. \tag{23}$$

We look for a travelling wave that moves with the speed $s = (f(u_r) - f(u_l))/(u_r - u_l)$:

$$-sS'_{\varepsilon,\delta} + [f(S_{\varepsilon,\delta})]' = \varepsilon S''_{\varepsilon,\delta} - \delta S'''_{\varepsilon,\delta}.$$

Integrating between $-\infty$ and x we get

$$-sS_{\varepsilon,\delta} + su_l + f(S_{\varepsilon,\delta}) - f(u_l) = \varepsilon S'_{\varepsilon,\delta} - \delta S''_{\varepsilon,\delta}.$$

Again, after rescaling x by $\sqrt{\delta}$ and setting $c = \varepsilon/\sqrt{\delta}$ we arrive at

$$cS'_c - S''_c = F(S_c), \quad S_c(-\infty) = u_l, \quad S_c(+\infty) = u_r.$$

The nonlinearity F has the form

$$F(\phi) = -s\phi + su_l + f(\phi) - f(u_l). \tag{24}$$

Note that $F(u_l) = F(u_r) = 0$ – this follows from the Rankin-Hugoniot condition on the speed s. Moreover, for ϕ between u_l and u_r we have

$$F(\phi) = f(\phi) - f(u_l) - \frac{f(u_r) - f(u_l)}{u_r - u_l}(\phi - u_l) \le 0,$$

since f is convex. Thus, the situation is again reduced to the KPP since the nonlinearity $F(\phi)$ is convex in ϕ.

The situation is, of course, completely different in the non-convex case as travelling waves may not exist even if $\delta = 0$ – that is, in the absence of dispersion. Indeed, if we look for a travelling wave solution of (23) with $\delta = 0$ we arrive simply at

$$cS'_c = F(S_c), \quad S_c(-\infty) = u_l, \quad S_c(+\infty) = u_r,$$

with F given by (24). This equation may not have a solution if there exists a point ϕ between u_l and u_r such that $F(\phi) = 0$, that is, if the line joining $(u_l, f(u_l))$ and $(u_r, f(u_r))$ intersects the graph of $f(u)$ between u_l and u_r. For that to happen the function f has to be non-convex.

Conclusion

We have studied the oscillatory travelling waves for the dissipative-dispersive Burgers-Hopf equation and estimated precisely the periods and damping rate. From this study, we deduce that when δ/ε vanishes, the travelling wave converges to an entropic shock. We conjecture that the entropic solutions should be reached in this regime even for the initial value problem.

It is therefore natural to ask the question of the regime $\delta = \varepsilon$ (or may be larger). It is easy to get convinced that the corresponding travelling wave converges only weakly and thus does not reach the entropic shock in the limit. Indeed, in the third region of our analysis, the oscillation length of S_c is of length 0(1) with a damping rate to 1 is e^{-ct}. Once rescaled to the actual travelling wave $S_{\varepsilon,\delta}$ is oscillates period of order $\sqrt{\delta}$ and the damping rate to 1 is now e^{-x}. In other words in the third region, $S_{\varepsilon,\delta}$ converges weakly to 1. It is plausible that the second region remains of size $O(1)$ and a smooth transition is generated in the weak limit.

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