The Schrödinger equation with spatial white noise: the average wave function

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Abstract

We prove a representation for the average wave function of the Schrödinger equation with a white noise potential in $d = 1, 2$, in terms of the renormalized self-intersection local time of a Brownian motion.

1 Introduction

We consider the Schrödinger equation with a large, highly oscillatory random potential

$$i\partial_t \psi_\varepsilon + \frac{1}{2}\Delta\psi_\varepsilon - V_\varepsilon(x)\psi_\varepsilon = 0, \quad \psi_\varepsilon(0, x) = \phi_0(x), \quad x \in \mathbb{R}^d, \tag{1.1}$$

and the initial condition $\phi_0(x)$ that is a compactly supported C^∞ function. The random potential is a microscopically smoothed version of a spatial white noise:

$$V_\varepsilon(x) = \frac{1}{\varepsilon^{d/2}} V(\frac{x}{\varepsilon}).$$

Here, V is a stationary, zero-mean and isotropic Gaussian random field over a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ with the expectation denoted by $\mathbb{E}$. If the two-point correlation function

$$R(x) := \mathbb{E}[V(x + y)V(y)] = \rho(|x|), \quad x, y \in \mathbb{R}^d, \tag{1.2}$$

decays sufficiently fast, then $V_\varepsilon(x)$ converges to a spatial white noise $\dot{W}(x)$.

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In $d=2$, this problem was analyzed in [5] on a torus $\mathbb{T}^2$. The solution of (1.1) acquires a large phase by $t \sim O(1)$, and the main result of [5] is that the adjusted solution

$$\phi_\varepsilon(t,x) = \psi_\varepsilon(t,x) e^{-iC_\varepsilon t},$$

that satisfies

$$i\partial_t \phi_\varepsilon + \frac{1}{2}\Delta\phi_\varepsilon - (V_\varepsilon(x) + C_\varepsilon)\phi_\varepsilon = 0, \quad \phi_\varepsilon(0,x) = \phi_0(x), \quad x \in \mathbb{R}^d, \tag{1.3}$$

with $C_\varepsilon \sim \log \varepsilon^{-1}$, converges to the solution of the stochastic PDE that can be formally written as

$$i\partial_t \phi_{\mathrm{spde}} + \frac{1}{2}\Delta\phi_{\mathrm{spde}} - \dot{W}(x) \cdot \phi_{\mathrm{spde}} = 0. \tag{1.4}$$

The approach is based on a change of variable used in [8], together with the mass and energy conservations, and also applies to the nonlinear equations. By analyzing the Anderson Hamiltonian

$$-\frac{1}{2}\Delta + V_\varepsilon(x) + C_\varepsilon$$

with the paracontrolled calculus, a spectral theory has been established in [1], which also gives a meaning to the solution to (1.4) on $\mathbb{T}^2$.

When $d=1$, no renormalization is needed and $C_\varepsilon = 0$. It has been proved in [17] that the solution ϕ_ε of (1.3) converges to a solution to (1.4), defined as an infinite series of iterated Stratonovich integrals.

Unfortunately, the information on the limit from the above considerations is rather implicit. Our goal here is to understand some of the properties of the solution to (1.3), in a more direct way. In particular, we establish a representation of $\lim_{\varepsilon\to 0} \mathbb{E}[\widehat{\phi}_\varepsilon]$ in $d=1,2$, see Theorem 1.1 below. Here, and in what follows $\widehat{f}$ denotes the Fourier transform of a function f:

$$\widehat{f}(\xi) = \int_{\mathbb{R}^d} f(x) e^{-i\xi\cdot x} dx.$$

The self-intersection local time of Brownian motion

The representation for $\mathbb{E}[\widehat{\phi}_\varepsilon]$ we are pursuing relies on the self-intersection local time of Brownian motion. For the convenience of the reader, we provide a brief introduction here. Let $\{B_t, t \geq 0\}$ be a standard $d-$dimensional Brownian motion starting from the origin, defined on a probability space Σ, and $\mathbb{E}_{\mathrm{B}}$ denotes the respective expectation.

In $d=1$, one can show [4] that for any $f \in L^1(\mathbb{R})$ with $\int_{\mathbb{R}} f(x)dx = 1$ and $t > 0$, the following limit exists and represents the intersection time of the Brownian motion:

$$\beta([0,t]^2_<) := \lim_{\varepsilon\to 0} \frac{1}{\varepsilon}\int_0^t ds \int_0^s f\left(\frac{B_s - B_u}{\varepsilon}\right) du = \frac{1}{2}\int_{\mathbb{R}} l^2(t,x)dx, \quad \text{a.s. and in } L^1(\Sigma), \tag{1.5}$$

where $l(t,x)$ is the local time of $\{B_t, t \geq 0\}$. Here, given a subset $A \subset [0,+\infty)$, we denote by $A^2_< := \{(s,t) \in A^2 : \, s < t\}$. On the formal level, we can think of $\beta([0,t]^2_<)$ as

$$\beta([0,t]^2_<) = \int_0^t \int_0^s \delta(B_s - B_u) du ds.$$

The direct analogue of the self-intersection time in dimensions $d \geq 2$ becomes infinite, and a suitable renormalization is needed to recover a non-trivial object. The renormalized self-intersection local time of a planar Brownian motion $\gamma([0,t]^2_<)$ formally corresponds to

$$\gamma([0,t]^2_<) = \int_0^t \int_0^s (\delta(B_s - B_u) - \mathbb{E}_{\mathrm{B}}[\delta(B_s - B_u)]) du ds. \tag{1.6}$$

To make sense of (1.6) in $d = 2$, one defines the renormalized self-intersection local time as

$$\gamma([0,t]^2_<) := \lim_{\varepsilon \to 0} \int_0^t \int_0^s \Big\{ q_\varepsilon(B_s - B_u) - \mathbb{E}_{\mathrm{B}}[q_\varepsilon(B_s - B_u)] \Big\} du ds. \tag{1.7}$$

The limit exists for any $t > 0$ [10, 14, 16]. Here, we denote

$$q_\varepsilon(x) := \frac{1}{(2\pi\varepsilon)^{d/2}} e^{-|x|^2/(2\varepsilon)}, \quad \varepsilon > 0. \tag{1.8}$$

We refer to [10, Section VIII.4] for a detailed construction.

In $d = 1$, we simply let

$$\gamma([0,t]^2_<) := \beta([0,t]^2_<) - \mathbb{E}_{\mathrm{B}}[\beta([0,t]^2_<)] = \frac{1}{2} \left\{ \int_{\mathbb{R}} l^2(t,x) dx - \frac{8t^{3/2}}{3\sqrt{2\pi}} \right\}. \tag{1.9}$$

The main result

We will assume that the covariance function of the Gaussian random field $V(x)$ has the form (1.2), with a function $\rho(y)$ of the Schoenberg class [13]:

$$\rho(y) = \int_0^{+\infty} \exp\left\{-(\lambda y)^2/2\right\} \mu(d\lambda), \quad y \in \mathbb{R}, \tag{1.10}$$

for some finite Borel measure μ on $[0,+\infty)$. To ensure that $V_\varepsilon(x)$ scales to a spatial white noise with a finite variance, we assume that

$$\bar{R}_d := \int_{\mathbb{R}^d} R(x) dx = (2\pi)^{d/2} \int_0^{+\infty} \frac{\mu(d\lambda)}{\lambda^d} < +\infty. \tag{1.11}$$

With appropriate choices of $\mu(d\lambda)$, the covariance function $R(x)$ can be merely integrable. To simplify some considerations, we further require that

$$\int_0^{+\infty} \frac{|\log \lambda|}{\lambda^d} \mu(d\lambda) < +\infty, \tag{1.12}$$

and define

$$\bar{R}_2' = 2\pi \int_0^{+\infty} \frac{\log \lambda}{\lambda^d} \mu(d\lambda). \tag{1.13}$$

The constraint (1.12) on $\mu(d\lambda)$ near the origin can be relaxed, as discussed at the end of the proof of Lemma 4.1 below but we are not striving for the sharpest assumptions here.

Define the deterministic function

$$\rho_d(t) := \begin{cases} -\bar{R}_1 \dfrac{(2t)^{\frac{3}{2}}}{3\sqrt{i\pi}}, & d=1, \\[2ex] \bar{R}_2 \left[\dfrac{it}{2\pi} \log\left(\dfrac{t}{e}\right) - \dfrac{t}{4} \right] + \bar{R}_2' \dfrac{it}{\pi}, & d=2, \end{cases} \tag{1.14}$$

and the renormalization constant

$$C_\varepsilon := \begin{cases} 0, & d=1, \\[2ex] \dfrac{\bar{R}_2}{\pi} \log \varepsilon^{-1}, & d=2. \end{cases} \tag{1.15}$$

The following result is the main objective of this paper.

Theorem 1.1. *Suppose that $d = 1, 2$ and ϕ_ε is the solution to (1.3) with C_ε given by (1.15). Then, there exists $t_0 \in (0, +\infty)$ such that for $t \in [0, t_0], \xi \in \mathbb{R}^d$, we have*

$$\lim_{\varepsilon \to 0} \mathbb{E}[\widehat{\phi}_\varepsilon(t,\xi)] = \widehat{\phi}_0(\xi) e^{\rho_d(t)} \mathbb{E}_{\mathrm{B}} \left[\exp\left\{ i^{3/2} \xi \cdot B_t - i^{3d/2} \bar{R}_d \gamma([0,t]_<^2) \right\} \right]. \tag{1.16}$$

Without the random potential, the solution to the free Schrödinger equation can be written in the Fourier domain as

$$\widehat{\phi}_0(\xi) \exp\Big\{ -\frac{i}{2}|\xi|^2 t \Big\} = \widehat{\phi}_0(\xi) \mathbb{E}_{\mathrm{B}}[\exp(i^{3/2}\xi \cdot B_t)],$$

so Theorem 1.1 shows that the effect of the white noise potential is manifested by the term $\bar{R}_d \gamma([0,t]_<^2)$ in (1.16).

The stochastic and homogenization regimes

Equation (1.1) is written in terms of the macroscopic variables. If we start from the microscopic dynamics – the Schrödinger equation with a potential of a size $\delta > 0$ and a low frequency initial condition, varying on a spatial scale $l_{\mathrm{in}} \sim \varepsilon^{-1} \gg 1$,

$$i\partial_t \phi + \frac{1}{2}\Delta\phi - \delta V(x)\phi = 0, \quad \phi(0,x) = \phi_0(\varepsilon x) \tag{1.17}$$

then $\psi_\varepsilon(t,x) := \phi(t/\varepsilon^2, x/\varepsilon)$ solves (1.1) provided $\varepsilon = \delta^{1/(2-d/2)}$. In particular, in $d=2$, we need to choose $\varepsilon = \delta$ to be in the "white-noise" scaling of (1.1). In other words, the white-noise scaling in $d=2$ is equivalent to the weak coupling scaling with a low frequency initial condition.

It has been shown in [3, 18] that in $d \geq 3$, for the low frequency initial data $\phi(0,x) = \phi_0(\varepsilon x)$, the diffusively rescaled wave function $\phi_\varepsilon(t,x) = \phi(\varepsilon^{-2}t, \varepsilon^{-1}x)$ converges to a homogenized limit: the solution has a deterministic limit, and we only observe a phase shift of the wave function in the limit, by a factor proportional to

$$V_{\text{eff}} = \int_{\mathbb{R}^d} \frac{\widehat{R}(p)dp}{|p|^2}. \tag{1.18}$$

The integral in (1.18) blows up in $d=2$ due to the singularity at the origin, and the role of the large constant C_ε appearing in (1.3) is to compensate for this divergence, so that we can obtain a non-trivial limit, which is now random, unlike in $d \geq 3$. One may ask if there is a shorter time scale T_ε, on which the solution of (1.17) is affected in a non-trivial way but is still deterministic in $d=2$. The answer is given by the following theorem: $T_\varepsilon = \varepsilon^{-2}$, with $\delta = \varepsilon|\log\varepsilon|^{-1/2}$.

Theorem 1.2. *Consider*

$$i\partial_t\phi_\varepsilon + \frac{1}{2}\Delta\phi_\varepsilon - \frac{1}{\varepsilon|\log\varepsilon|^{\frac{1}{2}}}V(\frac{x}{\varepsilon})\phi_\varepsilon = 0, \quad \phi_\varepsilon(0,x) = \phi_0(x), \quad x \in \mathbb{R}^2, \tag{1.19}$$

and

$$i\partial_t\phi_{\text{hom}} + \frac{1}{2}\Delta\phi_{\text{hom}} + \frac{\bar{R}_2}{\pi}\phi_{\text{hom}} = 0, \quad \phi_{\text{hom}}(0,x) = \phi_0(x), \quad x \in \mathbb{R}^2, \tag{1.20}$$

with $\phi_0 \in L^2(\mathbb{R}^2)$. Then, there exists $t_0 \in (0,\infty)$ such that for all $t \in [0,t_0]$, we have

$$\lim_{\varepsilon\to 0}\int_{\mathbb{R}^2} \mathbb{E}[|\widehat{\phi}_\varepsilon(t,\xi) - \widehat{\phi}_{\text{hom}}(t,\xi)|^2]d\xi = 0. \tag{1.21}$$

The non-diagrammatic approach

The standard approach to the random Schrödinger equation in the weak coupling regime is through a diagram expansion: the solution to (1.17) is written in the mild formulation

$$\widehat{\phi}(t,\xi) = \widehat{\phi}(0,\xi)e^{-\frac{i}{2}|\xi|^2 t} + \delta\int_0^t e^{-\frac{i}{2}|\xi|^2(t-s)}\left(\int_{\mathbb{R}^d} \frac{\widehat{V}(dp)}{i(2\pi)^d}\widehat{\phi}(s,\xi-p)\right)ds. \tag{1.22}$$

Then (1.22) is iterated to produce an infinite series expansion of $\widehat{\phi}(t,\xi)$. Evaluating the average wave function $\mathbb{E}[\widehat{\phi}(t,\xi)]$, or the energy $\mathbb{E}[|\widehat{\phi}(t,\xi)|^2]$ leads to the Feynman diagrams arising from computing the high order moments of the form $\mathbb{E}[\widehat{V}(dp_1)\ldots\widehat{V}(dp_N)]$ for arbitrarily large N. To pass to the limit requires either delicate oscillatory phase estimates or some specific structure of the power spectrum so that explicit calculations can be carried out. It is unclear whether the diagram expansion can be applied in $d = 2$ when we need the renormalization.

We use a different approach in this paper, similar to the one applied to the parabolic setting in [6]. For the heat equation with a random potential

$$\partial_t u_\varepsilon = \frac{1}{2}\Delta u_\varepsilon + (V_\varepsilon - C_\varepsilon)u_\varepsilon, \quad u_\varepsilon(0,x) = u_0(x), \tag{1.23}$$

the Feynman-Kac formula implies

$$\begin{aligned}\mathbb{E}[u_\varepsilon(t,x)] &= \mathbb{E}\mathbb{E}_{\mathrm{B}}\Big[u_0(x+B_t)\exp\Big\{\int_0^t V_\varepsilon(x+B_s)ds - C_\varepsilon t\Big\}\Big] \\ &= \mathbb{E}_{\mathrm{B}}\Big[u_0(x+B_t)\exp\Big\{\int_0^t\int_0^s R_\varepsilon(B_s - B_u)duds - C_\varepsilon t\Big\}\Big].\end{aligned} \tag{1.24}$$

with R_ε the covariance function of $V_\varepsilon(x)$. Using (1.7) one can easily show – see (3.3) below, that, for $d = 2$ and the Schoenberg class covariance function $R(\cdot)$ satisfying condition (1.11), we have

$$\lim_{\varepsilon\to 0}\int_0^t\int_0^s (R_\varepsilon(B_s - B_u) - \mathbb{E}_{\mathrm{B}}[R_\varepsilon(B_s - B_u)])duds = \bar{R}_d\gamma([0,t]^2_<), \quad \text{in } L^2. \tag{1.25}$$

In this case, the average intersection time in $d = 2$ is

$$\int_0^t\int_0^s \mathbb{E}_{\mathrm{B}}[R_\varepsilon(B_s - B_u)]duds \sim C_\varepsilon t = \frac{\bar{R}_2 t}{\pi}\log\varepsilon^{-1}. \tag{1.26}$$

In $d = 1$, the mean on the left side converges and no renormalization is needed, so $C_\varepsilon = 0$. It was proved in [12] for $d = 1$ and in [8] for $d = 2$ that u_ε converges to the solution to a limiting SPDE. By passing to the limit on both sides of (1.24), a representation for the moments of u_ε can be obtained, see [6].

The idea of the proof of Theorem 1.1 is similar: (1.17) is rewritten as

$$\partial_t\phi = \frac{i}{2}\Delta\phi - i\delta V(x)\phi,$$

and the Feynman-Kac formula can be used to formally express ϕ as an average with respect to the Brownian motion with an "imaginary diffusivity", written as $\sqrt{i}B_t$. Thus, we need to design a Feynman-Kac type formula for $\mathbb{E}[\widehat{\phi}_\varepsilon(t,\xi)]$ similar to (1.24), and prove a parallel version of (1.25) with R_ε replaced by a corresponding complex function in the case of the Schrödinger equation.

It is natural to ask what happens in dimensions $d \geq 3$. The approach used here breaks down – in $d \geq 3$, the renormalized self-intersection local time of Brownian motion does not exist [2, 15] since the variance also blows up. For the parabolic setting in $d = 3$, the mean of

$$\int_0^t\int_0^s R_\varepsilon(B_s - B_u)duds$$

diverges as ε^{-1} and its variance diverges as $\log \varepsilon^{-1}$, so two renormalization constants are needed – it has been proved in [7] that with

$$C_\varepsilon = c_1 \varepsilon^{-1} + c_2 \log \varepsilon^{-1},$$

and appropriate c_1, c_2, the solution u_ε converges to a non-trivial random limit. However, $\mathbb{E}[u_\varepsilon]$ blows up in the limit [6].

The rest of the paper is organized as follows. In Section 2, we present a Feynman-Kac representation for the average wave function which corresponds to (1.24) in the parabolic setting. In Section 3, we prove the convergence to the renormalized self-intersection local time in (1.25), where the Schoenberg class R_ε is replaced by the respective "mixture" of free Schrödinger kernels. The proof relies on an application of the Clark-Ocone formula which is recalled in the appendix. In Section 4, we pass to the limit in the Feynman-Kac representation. The homogenization result is shown in Section 5.

Throughout the paper, we define $\sqrt{i} = (1+i)/\sqrt{2}$, and we use $a \lesssim b$ to denote $a \le Cb$ for some constant $C > 0$ independent of ε, and the constants denoted by C may differ from line to line.

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2 A Feynman-Kac formula for the average wave function

In this section, we prove the Feynmann-Kac representation for the average wave function. We understand the solution of the Schrödinger equation

$$i\partial_t \phi + \frac{1}{2}\Delta \phi - V(x)\phi = 0, \quad \phi(0,x) = \phi_0(x), \tag{2.1}$$

in terms of the corresponding Duhamel series expansion [3]. A standard argument, as, for instance, in [3, Proposition 2.2 part (iii)], shows that, even though the potential $V(x)$ is unbounded, (2.1) preserves the $L^2(\mathbb{R}^d)$ norm of the solution:

$$\mathbb{E}\|\widehat{\phi}(t,\cdot)\|^2_{L^2(\mathbb{R}^d)} = \|\widehat{\phi}_0\|^2_{L^2(\mathbb{R}^d)},$$

and the function $\bar{\phi}(t,\xi) := \mathbb{E}[\widehat{\phi}(t,\xi)]$ belongs to $L^2(\mathbb{R}^d)$ for each $t \ge 0$.

Proposition 2.1. *The solution of (2.1) satisfies, point-wise in (t,ξ):*

$$\mathbb{E}[\widehat{\phi}(t,\xi)] = \widehat{\phi}_0(\xi)\mathbb{E}_{\mathrm{B}}\left[\exp\left\{i\sqrt{i}\xi \cdot B_t - \frac{1}{2}\int_{[0,t]^2} R(\sqrt{i}(B_s - B_u))duds\right\}\right]. \tag{2.2}$$

To make sense of (2.2), we may extend the function $R(x)$ to the domain $\bar{D} \subset \mathbb{C}^d$, where

$$D := \{zx :\, x \in \mathbb{R}^d,\, z \in \mathbb{D}_0\}, \qquad \mathbb{D}_0 := \{z \in \mathbb{C} : \mathrm{Re}\; z^2 > 0\},$$

by setting $R(zx) = \rho(z|x|)$, with $\rho(r)$ given by (1.10). Then, $R(\sqrt{i}(B_s - B_u))$ is uniformly bounded for all $s, u \geq 0$ and the r.h.s. of (2.2) is well-defined.

We note that another expression for $\mathbb{E}[\widehat{\phi}(t,\xi)e^{\frac{i}{2}|\xi|^2 t}]$ was obtained in [3, Proposition 2.1] but it is less suitable for our analysis.

Proof of Proposition 2.1

We fix (t, ξ) and define the function

$$F_1(z) := \mathbb{E}_{\mathrm{B}}\Big[\exp\Big\{iz\xi \cdot B_t - \frac{1}{2}\int_{[0,t]^2} R(z(B_s - B_u))duds\Big\}\Big],$$

as well as the corresponding Taylor expansion

$$F_2(z) = \sum_{n=0}^{\infty} F_{2,n}(z), \quad z \in \bar{\mathbb{D}}_0,$$

with

$$F_{2,n}(z) := \frac{(-1)^n}{2^n(2\pi)^{nd}n!}\int_{[0,t]^{2n}}\int_{\mathbb{R}^{nd}}\prod_{j=1}^{n}\widehat{R}(p_j)\mathbb{E}_{\mathrm{B}}\left[e^{iz\xi\cdot B_t}e^{izp_j\cdot(B_{s_j}-B_{u_j})}\right]dsdudp.$$

It is straightforward to check that both F_1 and F_2 are analytic on $\mathbb{D}_0$ and continuous on $\bar{\mathbb{D}}_0$. Note that $\sqrt{i} \in \partial\mathbb{D}_0$. The goal is to show that

$$\mathbb{E}[\widehat{\phi}(t,\xi)] = \widehat{\phi}_0(\xi)F_1(\sqrt{i}). \tag{2.3}$$

Since $(z, s, u) \mapsto R(z(B_s - B_u))$ is bounded on $\bar{\mathbb{D}}_0 \times \mathbb{R}_+^2$, we have

$$\begin{aligned}
F_1(z) &= \sum_{n=0}^{\infty}\frac{(-1)^n}{2^n n!}\mathbb{E}_{\mathrm{B}}\left[e^{iz\xi\cdot B_t}\left(\int_{[0,t]^2} R(z(B_s - B_u))dsdu\right)^n\right]\\
&= \sum_{n=0}^{\infty}\frac{(-1)^n}{2^n n!}\mathbb{E}_{\mathrm{B}}\left[e^{iz\xi\cdot B_t}\int_{[0,t]^{2n}}\prod_{j=1}^{n} R(z(B_{s_j} - B_{u_j}))dsdu\right]\\
&= \sum_{n=0}^{\infty}\frac{(-1)^n}{2^n(2\pi)^{nd}n!}\mathbb{E}_{\mathrm{B}}\left[e^{iz\xi\cdot B_t}\int_{[0,t]^{2n}}\int_{\mathbb{R}^{nd}}\prod_{j=1}^{n}\widehat{R}(p_j)e^{izp_j\cdot(B_{s_j}-B_{u_j})}dpdsdu\right], \quad z \in \bar{\mathbb{D}}_0.
\end{aligned} \tag{2.4}$$

For $z = x \in \mathbb{R}$, we can apply the Fubini theorem to see that $F_1(x) = F_2(x)$. Due to the analyticity and continuity of F_1 and F_2, we therefore have $F_1(z) = F_2(z)$ for all $z \in \bar{\mathbb{D}}_0$. Hence, (2.3) is equivalent to

$$\mathbb{E}[\widehat{\phi}(t,\xi)] = \widehat{\phi}_0(\xi) \sum_{n=0}^{\infty} F_{2,n}(\sqrt{i}), \tag{2.5}$$

and this is what we will show. For a fixed n, we rewrite

$$F_{2,n}(\sqrt{i}) = \frac{(-1)^n}{2^n (2\pi)^{nd} n!} \int_{[0,t]^{2n}} \int_{\mathbb{R}^{2nd}} \prod_{j=1}^{n} \widehat{R}(p_{2j-1}) \delta(p_{2j-1} + p_{2j}) \\ \times \mathbb{E}_{\mathrm{B}} \left[e^{i\sqrt{i}\xi \cdot B_t} e^{-\sum_{j=1}^{2n} i\sqrt{i} p_j B_{s_j}} \right] ds dp.$$

Let σ denote a permutation of $\{1, \ldots, 2n\}$. After a suitable relabeling of the p-variables we can write

$$F_{2,n}(\sqrt{i}) = \frac{(-1)^n}{2^n (2\pi)^{nd} n!} \sum_{\sigma} \int_{[0,t]^{2n}_<} \int_{\mathbb{R}^{2nd}} \prod_{j=1}^{n} \widehat{R}(p_{\sigma(2j-1)}) \delta(p_{\sigma(2j-1)} + p_{\sigma(2j)}) \\ \times \mathbb{E}_{\mathrm{B}} \left[e^{i\sqrt{i}\xi \cdot B_t} e^{-\sum_{j=1}^{2n} i\sqrt{i} p_j B_{s_j}} \right] ds dp, \tag{2.6}$$

where $[0,t]^{2n}_< := \{(s_1, \ldots, s_{2n}) : 0 \le s_{2n} \le \ldots \le s_1 \le t\}$. Let $\mathcal{F}$ denote the pairings formed over $\{1, \ldots, 2n\}$. It is straightforward to check that

$$F_{2,n}(\sqrt{i}) = \frac{1}{i^{2n} (2\pi)^{nd}} \sum_{\mathcal{F}} \int_{[0,t]^{2n}_<} \int_{\mathbb{R}^{2nd}} \prod_{(k,l) \in \mathcal{F}} \widehat{R}(p_k) \delta(p_k + p_l) \\ \times \mathbb{E}_{\mathrm{B}} \left[e^{i\sqrt{i}\xi \cdot B_t} e^{-\sum_{j=1}^{2n} i\sqrt{i} p_j B_{s_j}} \right] ds dp. \tag{2.7}$$

See [3] for a detailed explanation of the mapping between the sets of permutations and pairings that accounts for the $2^n n!$ difference in the pre-factors in (2.6) and (2.7).

The phase factor inside the integral in (2.7) can be computed explicitly:

$$\mathbb{E}_{\mathrm{B}} \left[e^{i\sqrt{i}\xi \cdot B_t} e^{-\sum_{j=1}^{2n} i\sqrt{i} p_j B_{s_j}} \right] = e^{-\frac{i}{2}|\xi|^2(t-s_1) - \frac{i}{2}|\xi - p_1|^2(s_1 - s_2) - \ldots - \frac{i}{2}|\xi - \ldots - p_{2n}|^2 s_{2n}}. \tag{2.8}$$

On the other hand, using the Duhamel expansion, we can write the solution $\widehat{\phi}(t,\xi)$ as an infinite series

$$\widehat{\phi}(t,\xi) = \sum_{n=0}^{\infty} \int_{[0,t]^n_<} \int_{\mathbb{R}^{nd}} \prod_{j=1}^{n} \frac{\widehat{V}(dp_j)}{i(2\pi)^d} e^{-\frac{i}{2}|\xi|^2(t-s_1) - \frac{i}{2}|\xi - p_1|^2(s_1 - s_2) - \ldots - \frac{i}{2}|\xi - \ldots - p_n|^2 s_n} \\ \times \widehat{\phi}_0(\xi - p_1 - \ldots - p_n) ds. \tag{2.9}$$

Evaluating the expectation $\mathbb{E}[\widehat{\phi}(t,\xi)]$ in (2.9), using the pairing formula for computing the Gaussian moment

$$\mathbb{E}[\widehat{V}(dp_1)\dots\widehat{V}(dp_n)],$$

and the fact that

$$\mathbb{E}[\widehat{V}(dp_i)\widehat{V}(dp_j)] = (2\pi)^d\widehat{R}(p_i)\delta(p_i+p_j)dp_idp_j,$$

and comparing the result to (2.7)-(2.8), we conclude that (2.5) holds, completing the proof. □

3 Convergence to the renormalized self-intersection local time

By Proposition 2.1, the average of the solution to (1.3) is written as

$$\mathbb{E}[\widehat{\phi}_\varepsilon(t,\xi)] = \widehat{\phi}_0(\xi)\exp\{-iC_\varepsilon t\}\,\mathbb{E}_{\mathrm{B}}\left[\exp\left\{i\sqrt{i}\xi\cdot B_t - \int_{[0,t]^2_<} R_\varepsilon(\sqrt{i}(B_s-B_u))dsdu\right\}\right],$$

with

$$R_\varepsilon(x) := \frac{1}{\varepsilon^d}R\left(\frac{x}{\varepsilon}\right), \quad x\in\mathbb{R}^d. \tag{3.1}$$

We define

$$X_\varepsilon(t) := \int_{[0,t]^2_<} R_\varepsilon(\sqrt{i}(B_s-B_u))dsdu = \frac{1}{\varepsilon^d}\int_0^{+\infty}\mu(d\lambda)\int_{[0,t]^2_<} e^{-\frac{i\lambda^2}{2\varepsilon^2}|B_s-B_u|^2}dsdu.$$

The goal of this section is to prove the L^2 convergence of $X_\varepsilon(t)$, as $\varepsilon\to 0$. Let $q_t(x)$ be the Gaussian kernel given by (1.8). We denote by

$$s_t(x) := q_{it}(x) = \frac{1}{(2\pi it)^{d/2}}e^{-\frac{|x|^2}{2it}}, \quad t\in\mathbb{R},$$

the free Schrödinger kernel, the solution of

$$i\partial_t s_t + \frac{1}{2}\Delta s_t = 0,$$

and also set

$$\mathcal{X}_\tau(t) := \int_{[0,t]^2_<} s_\tau(B_s-B_u)dsdu. \tag{3.2}$$

It is straightforward to check that

$$X_\varepsilon(t) = (-2\pi i)^{d/2}\int_0^{+\infty}\mathcal{X}^*_{\varepsilon^2\lambda^{-2}}(t)\frac{\mu(d\lambda)}{\lambda^d}. \tag{3.3}$$

The expectation of the solution to (1.3) can be written as

$$\begin{aligned}\mathbb{E}[\widehat{\phi}_\varepsilon(t,\xi)] &= \widehat{\phi}_0(\xi)\exp\{-iC_\varepsilon t\}\,\mathbb{E}_{\mathrm{B}}\left[\exp\left\{i\sqrt{i}\xi\cdot B_t + X_\varepsilon(t)\right\}\right] \qquad (3.4)\\ &= \widehat{\phi}_0(\xi)\exp\{-iC_\varepsilon t\}\,\mathbb{E}_{\mathrm{B}}\left[\exp\left\{i\sqrt{i}\xi\cdot B_t - (-2\pi i)^{d/2}\int_0^{+\infty}\mathcal{X}^*_{\varepsilon^2\lambda^{-2}}(t)\lambda^{-d}\mu(d\lambda)\right\}\right],\end{aligned}$$

which, in turn, can be split as

$$\begin{aligned}\mathbb{E}[\widehat{\phi}_\varepsilon(t,\xi)] = \widehat{\phi}_0(\xi)\exp\{-iC_\varepsilon t\}\exp\Big\{-(-2\pi i)^{d/2}\int_0^{+\infty}\mathbb{E}_{\mathrm{B}}[\mathcal{X}^*_{\varepsilon^2\lambda^{-2}}(t)]\lambda^{-d}\mu(d\lambda)\Big\} \qquad (3.5)\\ \times\,\mathbb{E}_{\mathrm{B}}\Big[\exp\Big\{i\sqrt{i}\xi\cdot B_t - (-2\pi i)^{d/2}\int_0^{+\infty}[\mathcal{X}^*_{\varepsilon^2\lambda^{-2}}(t) - \mathbb{E}_{\mathrm{B}}[\mathcal{X}^*_{\varepsilon^2\lambda^{-2}}(t)]\lambda^{-d}\mu(d\lambda)\Big\}\Big].\end{aligned}$$

We will show that the terms in the first line in (3.5) compensate each other, and the term in the second line has a limit. We begin with the latter.

Proposition 3.1. *In $d=1,2$,*

$$\lim_{\tau\to 0}\{\mathcal{X}_\tau(t) - \mathbb{E}_{\mathrm{B}}[\mathcal{X}_\tau(t)]\} = \gamma([0,t]^2_<), \quad \textit{for any } t>0, \qquad (3.6)$$

in $L^2(\Sigma)$, with $\gamma([0,t]^2_<)$ defined in (1.7). In addition, we have

$$\sup_{\tau>0}\mathbb{E}_{\mathrm{B}}|\mathcal{X}_\tau(t) - \mathbb{E}_{\mathrm{B}}[\mathcal{X}_\tau(t)]|^2 < +\infty, \quad \textit{for any } t>0. \qquad (3.7)$$

If the free Schrödinger kernel in (3.2) is replaced by the heat kernel, Proposition 3.1 is classical and reduces to the convergence expressed in (1.7). Although, on the formal level, $q_\tau(x)$ and $s_\tau(x)$ both converge to the Dirac function as $\tau\to 0$, it is surprising that the oscillation in s_τ does not change the asymptotic behavior of $\mathcal{X}_\tau - \mathbb{E}_{\mathrm{B}}[\mathcal{X}_\tau]$.

For the analysis of the intersection local time of the Brownian motion (and more generally, the fractional Brownian motion), the Clark-Ocone formula turns out to be a convenient tool, see [9]. For a fixed $\tau>0$, and $t>0$, we let

$$\chi_\tau(t,r) := \int_r^t\left[\int_0^r \nabla q_{i\tau+s-r}(B_r - B_u)du\right]ds, \quad 0\le r\le t. \qquad (3.8)$$

The process $(\chi_\tau(t,r))$, $0\le r\le t$, is adapted with respect to the natural filtration $\mathcal{F}_r$ of the Brownian motion. As we show in the appendix, see (A.1), we have

$$\mathcal{X}_\tau(t) - \mathbb{E}_{\mathrm{B}}[\mathcal{X}_\tau(t)] = \int_0^t \chi_\tau(t,r)dB_r, \qquad (3.9)$$

with the stochastic integral understood in the Itô sense. The renormalized self-intersection local time has the stochastic integral representation:

$$\gamma([0,t]^2_<) = \int_0^t \chi_0(t,r)dB_r. \qquad (3.10)$$

Formally, the convergence of $\mathcal{X}_\tau(t) - \mathbb{E}_{\mathrm{B}}[\mathcal{X}_\tau(t)]$ towards $\gamma([0,t]^2_<)$, as $\tau\to 0$ follows from the fact that $\lim_{\tau\to 0}\chi_\tau(t,r) = \chi_0(t,r)$.

Proof of Proposition 3.1

Let $\mathcal{Y}_\tau(t) := \mathcal{X}_\tau(t) - \mathbb{E}_{\mathrm{B}}[\mathcal{X}_\tau(t)]$ and consider the covariance

$$\mathbb{E}_{\mathrm{B}}[\mathcal{Y}_{\tau_1}(t)\mathcal{Y}^*_{\tau_2}(t)] = \int_0^t \left(\int_{[r,t]^2} \int_{[0,r]^2} \mathbb{E}_{\mathrm{B}}[\nabla q_{i\tau_1+s_1-r}(B_r - B_{u_1}) \cdot \nabla q^*_{i\tau_2+s_2-r}(B_r - B_{u_2})] du ds \right) dr.$$

We write the expectation inside the integral in the Fourier domain

$$\begin{aligned}&\mathbb{E}_{\mathrm{B}}[\nabla q_{i\tau_1+s_1-r}(B_r - B_{u_1}) \cdot \nabla q^*_{i\tau_2+s_2-r}(B_r - B_{u_2})] \\ &= \frac{1}{(2\pi)^{2d}} \int_{\mathbb{R}^{2d}} \mathbb{E}_{\mathrm{B}}[e^{i\xi_1\cdot(B_r-B_{u_1})} e^{-i\xi_2\cdot(B_r-B_{u_2})}](\xi_1 \cdot \xi_2) e^{-\frac{1}{2}|\xi_1|^2(i\tau_1+s_1-r)} e^{-\frac{1}{2}|\xi_2|^2(-i\tau_2+s_2-r)} d\xi,\end{aligned}$$

and claim that the non-negative function

$$F(\xi, u, s, r) := \mathbb{E}_{\mathrm{B}}[e^{i\xi_1\cdot(B_r-B_{u_1})} e^{-i\xi_2\cdot(B_r-B_{u_2})}]|\xi_1||\xi_2| e^{-\frac{1}{2}|\xi_1|^2(s_1-r)} e^{-\frac{1}{2}|\xi_2|^2(s_2-r)}$$

satisfies

$$\mathcal{I}(t) := \int_0^t dr \int_{[r,t]^2} \int_{[0,r]^2} du ds \int_{\mathbb{R}^{2d}} F(\xi, u, s, r) d\xi < +\infty. \tag{3.11}$$

Then, by the dominated convergence theorem, we deduce that $\mathbb{E}_{\mathrm{B}}[\mathcal{Y}_{\tau_1}(t)\mathcal{Y}^*_{\tau_2}(t)]$ converges as $\tau_1, \tau_2 \to 0$, hence $\mathcal{Y}_\tau(t)$ converges in $L^2(\Sigma)$. The same argument also implies that

$$\lim_{\tau\to 0} \mathbb{E}_{\mathrm{B}}|\mathcal{Y}_\tau(t) - \gamma([0,t]^2_<)|^2 = 0,$$

because of (3.10).

We turn to the proof of (3.11). Fix $t > 0$ and note that

$$\int_0^t e^{-\lambda s} ds \le \frac{c(t)}{1+\lambda}$$

for any $\lambda > 0$ with

$$c(t) := \sup_{\lambda > 0} \frac{1+\lambda}{\lambda}(1 - e^{-\lambda t}).$$

Using this estimate, we first integrate in s, and then take the expectation, to obtain, with the constant in the "$\lesssim$" inequality dependent on t:

$$\begin{aligned}\mathcal{I}(t) &\lesssim \int_0^t \int_{[0,r]^2} \int_{\mathbb{R}^{2d}} \mathbb{E}_{\mathrm{B}}[e^{i\xi_1\cdot(B_r-B_{u_1})} e^{-i\xi_2\cdot(B_r-B_{u_2})}] \frac{|\xi_1||\xi_2|}{(1+|\xi_1|^2)(1+|\xi_2|^2)} dr du d\xi \\ &= 2\int_0^t \int_{[0,r]^2} \int_{\mathbb{R}^{2d}} \mathbb{E}_{\mathrm{B}}[e^{i\xi_1\cdot(B_r-B_{u_1})} e^{-i\xi_2\cdot(B_r-B_{u_2})}] \frac{1_{\{u_2<u_1\}}|\xi_1||\xi_2|}{(1+|\xi_1|^2)(1+|\xi_2|^2)} dr du d\xi \\ &= 2\int_0^t \int_{[0,r]^2} \int_{\mathbb{R}^{2d}} e^{-\frac{1}{2}|\xi_1-\xi_2|^2(r-u_1)} e^{-\frac{1}{2}|\xi_2|^2(u_1-u_2)} \frac{1_{\{u_2<u_1\}}|\xi_1||\xi_2|}{(1+|\xi_1|^2)(1+|\xi_2|^2)} dr du d\xi.\end{aligned}$$

We further integrate in u and r and see that

$$\mathcal{I}(t) \lesssim \int_{\mathbb{R}^{2d}} \frac{|\xi_1||\xi_2| d\xi_1 d\xi_2}{(1+|\xi_1-\xi_2|^2)(1+|\xi_1|^2)(1+|\xi_2|^2)^2} < +\infty, \tag{3.12}$$

as $d \le 2$, which is (3.11). To conclude that (3.7) holds, it suffices to observe that by virtue of (3.12) we have

$$\sup_{\tau>0} \mathbb{E}_{\mathrm{B}}[|\mathcal{Y}_\tau(t)|^2] \lesssim \mathcal{I}(t) < +\infty,$$

finishing the proof of Proposition 3.1.

Re-centering as the compensating constant

Going back to (3.5), we now show that the recentering of the intersection local time $\mathbb{E}_{\mathrm{B}}[\mathcal{X}_\tau(t)]$ coincides with the renormalization of the random PDE by the addition of the term C_ε, so that the two terms in the first line of (3.5) cancel up to a $O(1)$ constant.

Lemma 3.2. *We have, for each $t > 0$ fixed,*

$$\mathbb{E}_{\mathrm{B}}[\mathcal{X}_\tau(t)] = \left| \begin{array}{ll} \dfrac{(2t)^{\frac{3}{2}}}{3\sqrt{\pi}} + o(1), & \text{when } d = 1, \\ \\ \dfrac{t}{2\pi} \log\left(\dfrac{t}{e\tau}\right) - \dfrac{it}{4} + o(1), & \text{when } d = 2, \end{array} \right.$$

as $\tau \to 0$. In addition, we have $\sup_{\tau>1} |\mathbb{E}_{\mathrm{B}}[\mathcal{X}_\tau(t)]| \lesssim 1$.

Proof. By a direct calculation, we have

$$\mathbb{E}_{\mathrm{B}}[\mathcal{X}_\tau(t)] = \int_{[0,t]^2_<} \mathbb{E}_{\mathrm{B}}[s_\tau(B_s - B_u)] du ds = \frac{1}{(2\pi i)^{d/2}} \int_0^t ds \int_0^s \frac{du}{[\tau - i(s-u)]^{d/2}}, \tag{3.13}$$

so it is clear that $\sup_{\tau>1} |\mathbb{E}_{\mathrm{B}}[\mathcal{X}_\tau(t)]| \lesssim 1$.

Next, when $d = 1$, we have

$$\mathbb{E}_{\mathrm{B}}[\mathcal{X}_\tau(t)] = \frac{1}{\sqrt{2\pi i}} \int_0^t ds \int_0^s \frac{1}{\sqrt{-iu}} du + o(1) = \frac{4t^{\frac{3}{2}}}{3\sqrt{2\pi}} + o(1).$$

When $d = 2$, we have

$$\begin{aligned} \mathbb{E}_{\mathrm{B}}[\mathcal{X}_\tau(t)] =& \frac{1}{2\pi i} \int_0^t ds \int_0^s \frac{\tau du}{\tau^2 + u^2} + \frac{1}{2\pi} \int_0^t ds \int_0^s \frac{u du}{\tau^2 + u^2} \\ =& \frac{1}{2\pi i} \int_0^t ds \int_0^{s/\tau} \frac{du}{1+u^2} + \frac{1}{2\pi} \int_0^t ds \int_0^{s/\tau} \frac{u du}{1+u^2}. \end{aligned} \tag{3.14}$$

The first integral is uniformly bounded in $\tau > 0$ and converges as $\tau \to 0$. For the second integral, we have

$$\frac{1}{2\pi}\int_0^t ds \int_0^{s/\tau} \frac{udu}{1+u^2} = \frac{1}{4\pi}\int_0^t \log \frac{\tau^2+s^2}{\tau^2} ds.$$

Passing to the limit $\tau \to 0$ in the integral on the right side completes the proof. □

4 Uniform integrability and passing to the limit

We now pass to the limit in (3.5) that we write as

$$\mathbb{E}[\widehat{\phi}_\varepsilon(t,\xi)] = \widehat{\phi}_0(\xi) \exp\Big\{ - (-2\pi i)^{d/2} \int_0^{+\infty} \mathbb{E}_{\mathrm{B}}[\mathcal{X}^*_{\varepsilon^2\lambda^{-2}}(t)]\lambda^{-d}\mu(d\lambda) - iC_\varepsilon t\Big\}\mathbb{E}_{\mathrm{B}}[Z_\varepsilon(t,\xi)], \quad (4.1)$$

where

$$Z_\varepsilon(t,\xi) := \exp\left\{ i\sqrt{i}\xi \cdot B_t - (-2\pi i)^{d/2} \int_0^{+\infty} \{\mathcal{X}^*_{\varepsilon^2\lambda^{-2}}(t) - \mathbb{E}_{\mathrm{B}}[\mathcal{X}^*_{\varepsilon^2\lambda^{-2}}(t)]\}\lambda^{-d}\mu(d\lambda)\right\}. \quad (4.2)$$

We first prove the convergence of the constant factor.

Lemma 4.1. *With the C_ε given in (1.15) and $\rho_d(t)$ given in (1.14), we have*

$$- (-2\pi i)^{d/2} \int_0^{+\infty} \mathbb{E}_{\mathrm{B}}[\mathcal{X}^*_{\varepsilon^2\lambda^{-2}}(t)]\lambda^{-d}\mu(d\lambda) - iC_\varepsilon t \to \rho_d(t). \quad (4.3)$$

Proof. We fix $t > 0$ and apply Lemma 3.2. In $d = 1$, using the fact that

$$\lim_{\tau\to 0} \mathbb{E}_{\mathrm{B}}[\mathcal{X}_\tau] = \frac{(2t)^{\frac{3}{2}}}{3\sqrt{\pi}} \text{ and } \sup_{\tau>0} |\mathbb{E}_{\mathrm{B}}[\mathcal{X}_\tau]| \lesssim 1,$$

we send $\varepsilon \to 0$ in (4.3) to obtain the result.

In $d = 2$, we write

$$\int_0^{+\infty} \mathbb{E}_{\mathrm{B}}[\mathcal{X}^*_{\varepsilon^2\lambda^{-2}}(t)]\lambda^{-d}\mu(d\lambda) = \left(\int_0^{\varepsilon} + \int_{\varepsilon}^{+\infty}\right) \mathbb{E}_{\mathrm{B}}[\mathcal{X}^*_{\varepsilon^2\lambda^{-2}}(t)]\lambda^{-d}\mu(d\lambda).$$

For the integral over the interval $(0,\varepsilon)$, we have $\varepsilon^2\lambda^{-2} > 1$. As

$$\sup_{\tau>1} |\mathbb{E}_{\mathrm{B}}[\mathcal{X}_\tau]| \lesssim 1,$$

we conclude the integral goes to zero in the limit. For the integral over $[\varepsilon, +\infty)$, we have the estimate

$$\left| \mathbb{E}_{\mathrm{B}}[\mathcal{X}^*_{\varepsilon^2\lambda^{-2}}(t)] - \frac{t}{2\pi}\log\left(\frac{t\lambda^2}{e\varepsilon^2}\right) - \frac{it}{4}\right| \lesssim 1$$

uniformly in $\lambda \geq \varepsilon$, and the left side above goes to zero as $\varepsilon \to 0$ for each such fixed λ. Now we only need to note that

$$2\pi i \int_{\varepsilon}^{+\infty} \left(\frac{t}{2\pi} \log\left(\frac{t\lambda^2}{e\varepsilon^2}\right) + \frac{it}{4} \right) \lambda^{-d} \mu(d\lambda) - iC_\varepsilon t \to \rho_2(t) \tag{4.4}$$

to complete the proof. $\square$

Assumption (1.12) is used in (4.4) to pass to the limit. For the above integral in λ to be finite, we only need

$$\int_1^{+\infty} \lambda^{-d} (\log \lambda) \mu(d\lambda) < +\infty.$$

If

$$\int_0^1 \lambda^{-d} |\log \lambda| \mu(d\lambda) = +\infty,$$

we only need to change C_ε to remove also the divergent integral

$$\int_{\varepsilon}^{+\infty} \lambda^{-d} (\log \lambda) \mu(d\lambda).$$

The uniform integrability of $Z_\varepsilon(t, \xi)$

By Proposition 3.1, we have

$$Z_\varepsilon(t, \xi) \to Z_0(t, \xi) := \exp\left\{ i\sqrt{i}\xi \cdot B_t - i^{\frac{3d}{2}} \bar{R}_d \gamma([0, t]^2_<) \right\}, \quad \text{as } \varepsilon \to 0, \tag{4.5}$$

in probability. To pass to the limit of $\mathbb{E}_{\mathrm{B}}[Z_\varepsilon(t, \xi)]$ in (4.1), it suffices to show the uniform integrability of the random variables $Z_\varepsilon(t, \xi)$. For a fixed $t > 0$, define the processes

$$\begin{aligned} M^\tau(s; t) &:= \int_0^s \chi_\tau(t, r) dB_r, \quad \tau > 0, \\ N^\varepsilon(s; t) &:= \int_0^{+\infty} M^{\varepsilon^2 \lambda^{-2}}(s; t) \frac{\mu(d\lambda)}{\lambda^d}, \quad \varepsilon > 0, \quad 0 \leq s \leq t, \end{aligned}$$

where $\chi_\tau(t, r)$ is given by (3.8). Then, $Z_\varepsilon(t, \xi)$ can be rewritten as

$$Z_\varepsilon(t, \xi) = \exp\left\{ i\sqrt{i}\xi \cdot B_t - (-2\pi i)^{d/2} (N^\varepsilon)^*(t; t) \right\}. \tag{4.6}$$

Note that for fixed $t, \tau, \varepsilon > 0$, the processes $(M^\tau(s; t))_{s \in [0,t]}$ and $(N^\varepsilon(s; t))_{s \in [0,t]}$ are continuous trajectory, square integrable martingales. Their respective quadratic variations are

$$\begin{aligned} \langle M^\tau(\cdot; t) \rangle_s &= \int_0^s |\chi_\tau(t, r)|^2 dr, \quad \tau > 0, \\ \langle N^\varepsilon(\cdot; t) \rangle_s &= \int_0^s \left| \int_0^{+\infty} \chi_{\varepsilon^2 \lambda^{-2}}(t, r) \frac{\mu(d\lambda)}{\lambda^d} \right|^2 dr, \quad \varepsilon > 0, \quad 0 \leq s \leq t. \end{aligned} \tag{4.7}$$

Using the Cauchy-Scwartz inequality and (1.11), followed by Jensen's inequality, we conclude that

$$\begin{aligned}\mathbb{E}_{\rm B}\left[\exp\left\{\theta\langle N^{\varepsilon}(\cdot;t)\rangle_s\right\}\right] &\le \mathbb{E}_{\rm B}\left[\exp\left\{\theta\bar{R}_d(2\pi)^{-d/2}\int_0^{+\infty}\langle M^{\varepsilon^2\lambda^{-2}}(\cdot;t)\rangle_s\frac{\mu(d\lambda)}{\lambda^d}\right\}\right]\\ &\le \frac{(2\pi)^{d/2}}{\bar{R}_d}\int_0^{+\infty}\frac{\mu(d\lambda)}{\lambda^d}\mathbb{E}_{\rm B}\left[\exp\left\{\theta\bar{R}_d^2(2\pi)^{-d}\langle M^{\varepsilon^2\lambda^{-2}}(\cdot;t)\rangle_s\right\}\right],\end{aligned}\tag{4.8}$$

for any $\theta>0$. We have the following result:

Proposition 4.2. *For any $\theta>0$, there exists $t_0>0$ such that*

$$\sup_{t\in[0,t_0]}\sup_{\varepsilon>0}\mathbb{E}_{\rm B}\left[\exp\left\{\theta\langle N^{\varepsilon}(\cdot;t)\rangle_t\right\}\right]<+\infty.\tag{4.9}$$

Proof. Thanks to (4.8), the estimate (4.9) is a result of the following claim: for any $\theta>0$ there exists $t_0>0$ such that

$$\sup_{t\in[0,t_0]}\sup_{\tau>0}\mathbb{E}_{\rm B}\left[\exp\left\{\theta\langle M^{\tau}(\cdot;t)\rangle_t\right\}\right]<+\infty.\tag{4.10}$$

Let us recall (3.8):

$$\chi_\tau(t,r)=\frac{1}{(2\pi)^{\frac{d}{2}}}\int_0^r du\left\{\int_0^{t-r}\frac{B_u-B_r}{(i\tau+s)^{\frac{d}{2}+1}}e^{-\frac{|B_r-B_u|^2}{2(i\tau+s)}}ds\right\}.\tag{4.11}$$

The case $d=1$

We shall need the following.

Lemma 4.3. *There exists a constant $C>0$ such that for all $t,\lambda,\tau>0$, we have*

$$\left|\int_0^t\frac{1}{(i\tau+s)^{\frac{3}{2}}}e^{-\frac{\lambda}{i\tau+s}}ds\right|\le\frac{C}{\sqrt{\lambda}}.$$

Proof. We have

$$\begin{aligned}\left|\int_0^t\frac{1}{(i\tau+s)^{\frac{3}{2}}}e^{-\frac{\lambda}{i\tau+s}}ds\right| &\le\int_0^t\frac{1}{(\tau^2+s^2)^{\frac{3}{4}}}e^{-\frac{\lambda s}{\tau^2+s^2}}ds\\ &\lesssim\left(\int_0^\tau+\int_\tau^t\right)\frac{1}{(\tau+s)^{\frac{3}{2}}}\exp\left\{-\frac{\lambda}{s+\tau^2/s}\right\}ds:=I_1+I_2.\end{aligned}$$

When $s\in(0,\tau)$, we have $s+\tau^2/s\le 2\tau^2/s$, so

$$I_1\le\int_0^\tau\frac{1}{(\tau+s)^{\frac{3}{2}}}e^{-\frac{\lambda s}{2\tau^2}}ds=\frac{1}{\sqrt{\tau}}\int_0^1\frac{1}{(1+s)^{\frac{3}{2}}}e^{-\frac{\lambda s}{2\tau}}ds\le\frac{2\sqrt{\tau}}{\lambda}(1-e^{-\frac{\lambda}{2\tau}})\lesssim\frac{1}{\sqrt{\lambda}}.$$

When $s \in (\tau, t)$, we have $s + \tau^2/s \leq 2s$, so

$$I_2 \leq \int_\tau^t \frac{1}{(\tau+s)^{\frac{3}{2}}} e^{-\frac{\lambda}{2s}} ds \leq \int_0^\infty \frac{1}{s^{\frac{3}{2}}} e^{-\frac{\lambda}{2s}} ds \lesssim \frac{1}{\sqrt{\lambda}}.$$

The proof is complete. $\square$

Using the above lemma we conclude that there exists a constant $C > 0$ such that

$$\left| \int_0^{t-r} \frac{B_r - B_u}{(i\tau + s)^{\frac{3}{2}}} e^{-\frac{|B_r - B_u|^2}{2(i\tau+s)}} ds \right| \leq C$$

for all $r \in (0, t)$, which, in light of (4.7) and (4.11), implies

$$\langle M^\tau(\cdot; t)\rangle_t \lesssim \int_0^t r^2 dr = \frac{t^3}{3}.$$

The case $d = 2$

Integrating out the s variable gives

$$\chi_\tau(t, r) = -\int_0^r \frac{(B_r - B_u)}{\pi |B_r - B_u|^2} \Big(e^{-\frac{|B_r - B_u|^2}{2(i\tau + t - r)}} - e^{-\frac{|B_r - B_u|^2}{2i\tau}} \Big) du,$$

which, together with (4.7), implies that there exists $C > 0$ such that

$$\langle M^\tau(\cdot; t)\rangle_t \leq C \int_0^t \left(\int_0^r |B_r - B_u|^{-1} du \right)^2 dr.$$

Therefore, by the above and Jensen's inequality, we conclude that

$$\begin{aligned} \mathbb{E}_{\mathrm{B}}[\exp\{\theta \langle M^\tau(\cdot; t)\rangle_t\}] \leq & \mathbb{E}_{\mathrm{B}}\Big[\exp\Big\{ \theta C \int_0^t \Big(\int_0^r |B_r - B_u|^{-1} du \Big)^2 dr \Big\} \Big] \\ \leq & \frac{1}{t} \int_0^t \mathbb{E}_{\mathrm{B}}\Big[\exp\Big\{ \theta C t \Big(\int_0^r |B_r - B_u|^{-1} du \Big)^2 \Big\} \Big] dr. \end{aligned} \tag{4.12}$$

Note that, for a fixed $r > 0$, we have

$$\int_0^r |B_r - B_u|^{-1} du \overset{\text{law}}{=} \int_0^r \mathscr{R}_u^{-1} du,$$

where $\mathscr{R}_u := |B_u|$, $u \geq 0$ is a Bessel process of dimension 2. An application of the Itô formula shows that $(\mathscr{R}_r)_{r \geq 0}$ satisfies

$$\int_0^r \mathscr{R}_u^{-1} du = 2(\mathscr{R}_r - b_r), \quad r \geq 0.$$

Here, $(b_r)_{r\geq 0}$ is a standard one dimensional Brownian motion. Having this in mind, we estimate the utmost right hand side of (4.12) using the Cauchy-Schwarz inequality and obtain

$$\mathbb{E}_{\mathrm{B}}[\exp\{\theta\langle M^\tau(\cdot;t)\rangle_t\}] \leq \frac{1}{t}\int_0^t \Big\{\mathbb{E}_{\mathrm{B}}\Big[\exp\Big\{\theta Ct\mathscr{R}_r^2\Big\}\Big]\Big\}^{1/2}\Big\{\mathbb{E}_{\mathrm{B}}\Big[\exp\Big\{\theta Ctb_r^2\Big\}\Big]\Big\}^{1/2} dr.$$

It is clear that when t is sufficiently small, the last expression is bounded independent of τ, which completes the proof of (4.10), and thus that of Proposition 4.2. $\square$

Proof of Theorem 1.1

Now we can finish the proof of the main result. By (4.1), (4.3) and (4.5), it remains to prove the uniform integrability of random variables $Z_\varepsilon(t,\xi)$ given in (4.6). To do so, we bound their second moments. Using the Cauchy-Schwarz inequality we get

$$\begin{aligned}\mathbb{E}_{\mathrm{B}}[|Z_\varepsilon(t,\xi)|^2] =&\mathbb{E}_{\mathrm{B}}\Big[e^{-\sqrt{2}\xi\cdot B_t}\exp\Big\{-2(2\pi)^{d/2}\mathrm{Re}[(-i)^{\frac{d}{2}}(N^\varepsilon)^*(t;t)]\Big\}\Big]\\ &\leq \Big\{\mathbb{E}_{\mathrm{B}}\big[e^{-2\sqrt{2}\xi\cdot B_t}\big]\Big\}^{1/2}\Big\{\mathbb{E}_{\mathrm{B}}\Big[\exp\Big\{-4(2\pi)^{d/2}\mathrm{Re}[(-i)^{\frac{d}{2}}(N^\varepsilon)^*(t;t)]\Big\}\Big]\Big\}^{1/2}.\end{aligned}$$

We wish to show that there exists $t_0>0$ such that the right side of the above estimate is uniformly bounded in $\varepsilon>0$ for $t\in(0,t_0)$. This will obviously imply the uniform integrability of $Z_\varepsilon(t,\xi)$ and complete the proof of Theorem 1.1 after passing to the limit in (4.1). To obtain the desired bound we consider the following martingale for a fixed $t>0$:

$$\mathbf{N}^\varepsilon(s;t) := -4(2\pi)^{d/2}\mathrm{Re}[(-i)^{\frac{d}{2}}(N^\varepsilon)^*(s;t)],\quad 0\leq s\leq t.$$

We have

$$\langle \mathbf{N}^\varepsilon(\cdot;t)\rangle_t \leq 16(2\pi)^d\langle N^\varepsilon(\cdot;t)\rangle_t,$$

for any $\theta>0$. By Proposition 4.2, there exists $t_0>0$ depending on θ such that

$$\sup_{t\in[0,t_0]}\sup_{\varepsilon>0}\mathbb{E}_{\mathrm{B}}[e^{\theta\langle\mathbf{N}^\varepsilon(\cdot;t)\rangle_t}] \leq \sup_{t\in[0,t_0]}\sup_{\tau>0}\mathbb{E}_{\mathrm{B}}[e^{16(2\pi)^d\theta\langle N^\varepsilon(\cdot;t)\rangle_t}] < +\infty.$$

For $\theta=2$, we adjust the respective t_0 as in the statement of Proposition 4.2. We have then

$$\begin{aligned}&\sup_{t\in[0,t_0]}\sup_{\varepsilon>0}\mathbb{E}_{\mathrm{B}}[e^{\mathbf{N}^\varepsilon(t;t)}] = \sup_{t\in[0,t_0]}\sup_{\varepsilon>0}\mathbb{E}_{\mathrm{B}}\Big[\exp\Big\{\mathbf{N}^\varepsilon(t;t)-\langle\mathbf{N}^\varepsilon(\cdot;t)\rangle_t\Big\}e^{\langle\mathbf{N}^\varepsilon(\cdot;t)\rangle_t}\Big]\\ &\leq \sup_{t\in[0,t_0]}\sup_{\varepsilon>0}\Big\{\mathbb{E}_{\mathrm{B}}\Big[\exp\Big\{2\mathbf{N}^\varepsilon(t;t)-2\langle\mathbf{N}^\varepsilon(\cdot;t)\rangle_t\Big\}\Big]\Big\}^{1/2}\Big\{\mathbb{E}_{\mathrm{B}}\Big[e^{2\langle\mathbf{N}^\varepsilon(\cdot;t)\rangle_t}\Big]\Big\}^{1/2}\\ &= \sup_{t\in[0,t_0]}\sup_{\varepsilon>0}\Big\{\mathbb{E}_{\mathrm{B}}\Big[e^{2\langle\mathbf{N}^\varepsilon(\cdot;t)\rangle_t}\Big]\Big\}^{1/2} < +\infty.\end{aligned}$$

The proof of Theorem 1.1 is complete.

5 Proof of the homogenization result

We now prove Theorem 1.2. Assume without loss of generality that the initial condition $\widehat{\phi}_0(\xi)$ for (1.19) is compactly supported. For an arbitrary $\widehat{\phi}_0 \in L^2(\mathbb{R}^2)$, we can argue by an approximation, since both (1.19) and (1.20) preserve the $L^2(\mathbb{R}^2)$ norm.

By Proposition 2.1 and (3.4), we have for any (t,ξ),

$$\begin{aligned}\mathbb{E}[\widehat{\phi}_\varepsilon(t,\xi)] =&\widehat{\phi}_0(\xi)\mathbb{E}_{\mathrm{B}}\left[\exp\left\{i\sqrt{i}\xi\cdot B_t - \frac{1}{\log\varepsilon^{-1}}\int_{[0,t]^2_<} R_\varepsilon(\sqrt{i}(B_s-B_u))dsdu\right\}\right]\\ =&\widehat{\phi}_0(\xi)\mathbb{E}_{\mathrm{B}}\left[\exp\left\{i\sqrt{i}\xi\cdot B_t + \frac{2\pi i}{\log\varepsilon^{-1}}\int_0^{+\infty} \mathcal{X}^*_{\varepsilon^2\lambda^{-2}}(t)\lambda^{-2}\mu(d\lambda)\right\}\right].\end{aligned} \tag{5.1}$$

By Lemma 4.1 and (1.15), we have

$$\lim_{\varepsilon\to0}\frac{2\pi i}{\log\varepsilon^{-1}}\int_0^{+\infty}\mathbb{E}_{\mathrm{B}}[\mathcal{X}^*_{\varepsilon^2\lambda^{-2}}(t)]\lambda^{-2}\mu(d\lambda) = \frac{it\bar{R}_2}{\pi}.$$

Combining with Proposition 3.1, we further derive

$$\lim_{\varepsilon\to0}\frac{2\pi i}{\log\varepsilon^{-1}}\int_0^{+\infty}\mathcal{X}^*_{\varepsilon^2\lambda^{-2}}(t)\lambda^{-2}\mu(d\lambda) = \frac{it\bar{R}_2}{\pi} \quad \text{in } L^2(\Sigma).$$

Applying Proposition 4.2 and following the proof of Theorem 1.1, one sees that there exists $t_0>0$ such that for all $t\in[0,t_0], \xi\in\mathbb{R}^2$,

$$\begin{aligned}\lim_{\varepsilon\to0}\mathbb{E}[\widehat{\phi}_\varepsilon(t,\xi)] &= \widehat{\phi}_0(\xi)\mathbb{E}_{\mathrm{B}}\Big[\exp\Big\{i\sqrt{i}\xi\cdot B_t + it\frac{\bar{R}_2}{\pi}\Big\}\Big] = \widehat{\phi}_0(\xi)\exp\Big\{-it\Big(\frac{|\xi|^2}{2}-\frac{\bar{R}_2}{\pi}\Big)\Big\}\\ &= \widehat{\phi}_{\mathrm{hom}}(t,\xi).\end{aligned}$$

In addition, we have the simple estimate

$$|\mathbb{E}[\widehat{\phi}_\varepsilon(t,\xi)]| \lesssim |\widehat{\phi}_0(\xi)|e^{C|\xi|^2t}.$$

As $\widehat{\phi}_0$ has compact support, we have

$$|\mathbb{E}[\widehat{\phi}_\varepsilon(t,\xi)]\widehat{\phi}^*_{\mathrm{hom}}(t,\xi)| \lesssim |\widehat{\phi}_0(\xi)|^2e^{C|\xi|^2t} \in L^1(\mathbb{R}^2),$$

thus, by the dominated convergence theorem we have

$$\int_{\mathbb{R}^2}\mathbb{E}[|\widehat{\phi}_\varepsilon(t,\xi)-\widehat{\phi}_{\mathrm{hom}}(t,\xi)|^2]d\xi = 2\int_{\mathbb{R}^2}|\widehat{\phi}_0(\xi)|^2d\xi - 2\mathrm{Re}\left[\int_{\mathbb{R}^2}\mathbb{E}[\widehat{\phi}_\varepsilon(t,\xi)]\widehat{\phi}^*_{\mathrm{hom}}(t,\xi)d\xi\right] \to 0,$$

as $\varepsilon\to0$. The proof of Theorem 1.2 is complete.

A The Clark-Ocone formula

We recall some facts from the Malliavin calculus for a standard d-dimensional Brownian motion $B_r = (B_r^1, \ldots, B_r^d), r \geq 0$ on $(\Sigma, \mathcal{F}, \mathbb{P}_{\mathrm{B}})$ that are used in our argument. We refer to [11] for a more detailed presentation. Let $H = L^2([0, \infty), \mathbb{R}^d)$ be the Hilbert space corresponding to the standard inner product $\langle \cdot, \cdot \rangle_H$. We define a mapping $B : H \to L^2(\Sigma, \mathcal{F}, \mathbb{P}_{\mathrm{B}})$ by letting

$$B(h) = \int_0^\infty h(r) dB_r = \sum_{j=1}^d \int_0^\infty h^j(r) dB_r^j, \quad h = (h^1, \ldots, h^d) \in H,$$

so that

$$\mathbb{E}_{\mathrm{B}}[B(h_1)B(h_2)] = \langle h_1, h_2 \rangle_H \quad \text{for } h_1, h_2 \in H.$$

For a random variable of the form $F = f(B(h_1), \ldots, B(h_n))$, where $f : \mathbb{R}^n \to \mathbb{R}$ is a smooth function of polynomial growth and $h_k \in H, k = 1, \ldots, n$, the derivative operator is defined as

$$D_r^j F = \sum_{k=1}^n \partial_{x_k} f(B(h_1), \ldots, B(h_n)) h_k^j(r), \quad j = 1, \ldots, d$$

and we write $D_r = (D_r^1, \ldots, D_r^d)$. The derivative is a closeable operator on $L^2(\Sigma)$ with values in $L^2(\Sigma; H)$. Denote by $\mathbb{D}^{1,2}$ the Hilbert space defined as the completion of the random variables F with respect to the product

$$\langle F, G \rangle_{1,2} := \mathbb{E}_{\mathrm{B}}[FG] + \mathbb{E}_{\mathrm{B}} \left[\sum_{j=1}^d \int_0^\infty (D_r^j F)(D_r^j G) dr \right].$$

The Clark-Ocone formula, see [11, Proposition 1.3.14 p. 46], says that if $F \in \mathbb{D}^{1,2}$, then

$$F = \mathbb{E}_{\mathrm{B}}[F] + \int_0^\infty \mathbb{E}_{\mathrm{B}}[D_r F | \mathcal{F}_r] dB_r = \mathbb{E}_{\mathrm{B}}[F] + \sum_{j=1}^d \int_0^\infty \mathbb{E}_{\mathrm{B}}[D_r^j F | \mathcal{F}_r] dB_r^j,$$

with $(\mathcal{F}_r)$ the natural filtration corresponding the Brownian. In our case, with $F = \mathcal{X}_\tau(t)$ for fixed $t, \tau > 0$, we have

$$\mathcal{X}_\tau(t) = \mathbb{E}_{\mathrm{B}}[\mathcal{X}_\tau(t)] + \int_0^\infty \mathbb{E}_{\mathrm{B}}[D_r \mathcal{X}_\tau(t) | \mathcal{F}_r] dB_r.$$

Recall that

$$\mathcal{X}_\tau(t) = \int_0^t \int_0^s s_\tau(B_s - B_u) du ds,$$

therefore,

$$D_r \mathcal{X}_\tau(t) = \int_0^t \int_0^s \nabla s_\tau(B_s - B_u) 1_{[u,s]}(r) du ds = 1_{[0,t]}(r) \int_r^t \int_0^r \nabla s_\tau(B_s - B_u) du ds.$$

This implies

$$\mathbb{E}_{\mathrm{B}}[D_r\mathcal{X}_\tau(t)|\mathcal{F}_r] = 1_{[0,t]}(r)\int_r^t\int_0^r \mathbb{E}_{\mathrm{B}}[\nabla s_\tau(B_s-B_u)|\mathcal{F}_r]duds$$
$$= 1_{[0,t]}(r)\int_r^t\int_0^r \nabla s_\tau \star q_{s-r}(B_r-B_u)duds = 1_{[0,t]}(r)\int_r^t\int_0^r \nabla q_{i\tau+s-r}(B_r-B_u)duds.$$

Here, we have used the fact that $s_\tau = q_{i\tau}$ and $q_{i\tau}\star q_{s-r} = q_{i\tau+s-r}$. Thus, we have

$$\mathcal{X}_\tau(t) - \mathbb{E}_{\mathrm{B}}[\mathcal{X}_\tau(t)] = \int_0^t\left(\int_r^t\int_0^r \nabla q_{i\tau+s-r}(B_r-B_u)duds\right)dB_r, \tag{A.1}$$

which is (3.9).

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