Finite time singularity for the modified SQG patch equation

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September 4, 2015

Abstract

It is well known that the incompressible Euler equations in two dimensions have globally regular solutions. The inviscid surface quasi-geostrophic (SQG) equation has a Biot-Savart law which is one derivative less regular than in the Euler case, and the question of global regularity for its solutions is still open. We study here the patch dynamics in the half-plane for a family of active scalars which interpolates between these two equations, via a parameter $\alpha \in [0, \frac{1}{2}]$ appearing in the kernels of their Biot-Savart laws. The values $\alpha = 0$ and $\alpha = \frac{1}{2}$ correspond to the 2D Euler and SQG cases, respectively. We prove global in time regularity for the 2D Euler patch model, even if the patches initially touch the boundary of the half-plane. On the other hand, for any sufficiently small $\alpha > 0$, we exhibit initial data which lead to a singularity in finite time. Thus, these results show a phase transition in the behavior of solutions to these equations, and provide a rigorous foundation for classifying the 2D Euler equations as critical.

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1 Introduction

The question of global regularity of solutions is still open for many fundamental equations of fluid dynamics. In the case of the three dimensional Navier-Stokes and Euler equations, it remains one of the central open problems of classical mathematical physics and partial differential equations. Much more is known in two dimensions, though the picture is far from complete even in that case. Global regularity of solutions to the 2D incompressible Euler equations in smooth domains has been known since the works of Wolibner [51] and Hölder [29]. However, even in 2D the estimates necessary for the Euler global regularity barely close, and the best upper bound on the growth of derivatives is double exponential in time. Recently, Kiselev and Šverák showed that this upper bound is sharp by constructing an example of a solution to the 2D Euler equations on a disk whose gradient indeed grows double exponentially in time [35]. Exponential growth on a domain without a boundary (the torus $\mathbb{T}^2$) was recently shown to be possible by Zlatoš [56]. Some earlier examples of unbounded growth are due to Yudovich [31, 54], Nadirashvili [43], and Denisov [18,19]. In a certain sense that will be made precise below, the 2D Euler equations may be regarded as critical, even though we are not aware of a simple scaling argument for such a classification.

The SQG and modified SQG equations

As opposed to the 2D Euler equations, the global regularity vs finite time singularity question has not been resolved for the two-dimensional surface quasi-geostrophic (SQG) equation, which appears in atmospheric science models (and shares many of its features with the 3D Euler equation — see, e.g., [10, 40, 48]). The SQG equation is given by

$$\partial_t \omega + (u \cdot \nabla)\omega = 0, \tag{1.1}$$

with $\omega(\cdot, 0) = \omega_0$ and the Biot-Savart law for the velocity

$$u := \nabla^{\perp}(-\Delta)^{-1/2}\omega,$$

where $\nabla^{\perp} := (\partial_{x_2}, -\partial_{x_1})$. Equation (1.1) has the same form as the 2D Euler equations in the vorticity formulation, but the latter has the more regular (by one derivative) Biot-Savart law

$$u := \nabla^{\perp}(-\Delta)^{-1}\omega.$$

The SQG equation is usually considered on either $\mathbb{R}^2$ or $\mathbb{T}^2$, and the fractional Laplacian can be defined via the Fourier transform. The equation appears, for instance, in the book [44] by Pedlosky and was first rigorously studied in the work of Constantin, Majda and Tabak [10] where, in particular, a closing saddle scenario for a finite time singularity has been suggested. This scenario and some other related ones have been ruled out in the later works of Córdoba [14] and Córdoba and Fefferman [15]. Also, existence of global weak solutions was proved by Resnick [47].

We should mention that a lot of work has focused on the SQG equation and related active scalars with a fractional dissipation term of the form $-(-\Delta)^{\beta}\omega$ on the right-hand side of (1.1). Global regularity for the critical viscous SQG equation, with $\beta = \frac{1}{2}$, was proved independently by Caffarelli and Vasseur [3], and by Kiselev, Nazarov, and Volberg [34] (see also the subsequent works [11,12,33] for alternative proofs). The global regularity proof is standard for $\beta \in (\frac{1}{2}, 1]$ (see e.g. [30]), while in the super-critical case $\beta < \frac{1}{2}$ the question of global regularity vs finite time blow-up remains open. The best available result in this direction is global regularity for the logarithmically super-critical SQG equation by Dabkowski, Kiselev, Silvestre, and Vicol [17].

A natural family of active scalars which interpolates between the 2D Euler and SQG equations is given by (1.1) with the Biot-Savart law

$$u := \nabla^{\perp}(-\Delta)^{-1+\alpha}\omega.$$

This family has been called modified or generalized SQG equations in the literature (see, e.g., Constantin et al [9], or Pierrehumbert et al [45] and Smith et al [49] for geophysical literature references). The cases $\alpha = 0$ and $\alpha = \frac{1}{2}$ correspond to the 2D Euler and SQG equations, respectively. The question of global regularity of the solutions with smooth initial data has been open for all $\alpha > 0$, that is, for any of these models which are more singular than the 2D Euler equations. Ironically, even though the SQG and the modified SQG equations are more singular than the 2D Euler equations, no examples of solutions with unbounded growth of derivatives in time are known. The best result in this direction is arbitrary bounded growth of high Sobolev norms on finite time intervals by Kiselev and Nazarov [32]. The reason is that due to nonlinearity and nonlocality of active scalars, it is difficult to control the solutions at large times, and this task gets harder as the Biot-Savart law becomes more singular. This issue will be evident in the present paper as well.

Vortex patches

While the above discussion concerns active scalars with sufficiently smooth initial data, an important class of solutions to these equations are vortex patches

$$\omega(x,t) = \sum_k \theta_k \chi_{\Omega_k(t)}(x).$$

Here θ_j are some constants, $\Omega_j(t)$ are (evolving in time) open sets with non-zero mutual distances and smooth boundaries, and χ_D denotes the characteristic function of a domain D. Vortex patches model flows with abrupt variations in vorticity, which are common in nature. Existence and uniqueness of appropriately defined vortex patch solutions to the 2D Euler equations in the whole plane goes back to the work of Yudovich [55], and regularity in this setting refers to sufficient smoothness of the patch boundaries as well as to the lack of both self-intersections of each patch boundary and touches of different patches.

Singularity formation for 2D Euler patches had initially been conjectured based on the numerical simulations by Buttke [2], see Majda [39] for a discussion. Later, simulations by Dritschel, McIntyre, and Zabusky [21,22] questioning the singularity formation prediction appeared; we refer to [46] for a review of these and related works. This controversy was settled in 1993, when Chemin [7] proved that the boundary of a 2D Euler patch remains regular for all times, with a double exponential upper bound on the temporal growth of its curvature (see also the work by Bertozzi and Constantin [1] for a different proof).

The patch problem for the SQG equation is more involved. Local existence and uniqueness in the class of weak solutions of the special type

$$\omega(\cdot, t) = \chi_{\{x_2 < \varphi(x_1,t)\}},$$

with $\varphi \in C^\infty$ and periodic in x_1, corresponding to a (single patch) initial condition of the same form, was proved by Rodrigo [48]. For the SQG and modified SQG patches with boundaries which are simple closed H^3 curves, local existence was established by Gancedo [25] via a study of a contour equation whose solutions parametrize the patch boundary (uniqueness of solutions was also proved for the contour equation for $\alpha \in (0, \frac{1}{2})$, although not for the original modified SQG equation). Local existence of such contour solutions in the more singular case $\alpha \in (\frac{1}{2}, 1]$ was obtained by Chae, Constantin, Córdoba, Gancedo, and Wu [6]. Existence of splash singularities (touching of exactly two segments of a patch boundary, which remains uniformly H^3) for the SQG equation was ruled out by Gancedo and Strain [26].

A computational study of the SQG and modified SQG patches by Córdoba, Fontelos, Mancho, and Rodrigo [16] (where the patch problem for the modified SQG equation first appeared) suggested a finite time singularity, with two patches touching each other and simultaneously developing corners at the touching point. A more careful numerical study by Mancho [41] suggests involvement of self-similar elements in this singularity formation process, but its rigorous confirmation and understanding is still lacking. We note that even local well-posedness is far from trivial for many interface evolution models of fluid dynamics, see e.g. [13] where the Muskat problem is discussed. We refer to [4, 5, 52, 53] for other recent advances in some of the interface problems of fluid dynamics.

Vortex patches in domains with boundaries

In this paper, we consider the patch evolution for the 2D Euler equations and for the modified SQG equations in the presence of boundaries. The latter are important in many applications, in particular, in the onset of turbulence and in the creation of small scales in the motion of fluids. The global existence of a single $C^{1,\gamma}$ patch for the 2D Euler equations on the half-plane $D := \mathbb{R} \times \mathbb{R}^+$ was proved by Depauw [20] when the patch does not touch the boundary ∂D initially. If it is, then [20] only proved that the patch will remain $C^{1,\gamma}$ for a finite time, while Dutrifoy [23] proved a result which can be used to obtain global existence in the weaker space $C^{1,s}$ for some $s \in (0, \gamma)$. Uniqueness of solutions in the 2D Euler case follows from the work of Yudovich [55].

Since we are not aware of a global existence result without a loss of regularity for (either one or multiple) 2D Euler patches on the half-plane which may touch its boundary, we will provide a proof of the global existence for such $C^{1,\gamma}$ patches here. This contrasts with our main goal, proving finite time singularity formation for the modified SQG patch evolution with $\alpha > 0$ in domains with a boundary. These two results together will then also establish existence of a phase transition in the behavior of solutions at $\alpha = 0$. For the sake of minimizing the technicalities, we do not strive for the greatest generality, and will consider H^3 patches (as in [6, 25, 26]) on the half-plane, with small enough $\alpha > 0$ (that is, slightly more singular than the 2D Euler case $\alpha = 0$). Our initial condition ω_0 will be the difference of characteristic functions of two patches with smooth boundaries. The patches will initially touch the boundary of the half-plane and, as was explained above, the loss of H^3 regularity or self-intersections of their boundaries, as well as touches of the two patches, will all constitute a singularity.

The possible importance of boundaries in the formation of singularities in fluids has been illustrated by recent numerical simulations of Luo and Hou [37,38], which suggested a new scenario for singularity formation in the 3D Euler equations. The flow in this scenario is axi-symmetric on a cylinder and so, in a way, can be viewed as two-dimensional (see [8] for a more detailed discussion). The rapid growth of the vorticity in these simulations happens on the boundary of the cylinder. The geometry of the construction we carry out in this work bears some similarity to this scenario, as well as to the geometry of the Kiselev-Šverák example of a solution to the 2D Euler equations with a double exponential growth of its vorticity gradient. In particular, in all three instances, a hyperbolic fixed point of the flow located on the boundary is involved. However the construction itself and the methods we use are quite different from earlier works.

The main results

Let us now turn to the specifics. As we said above, we will only consider modified SQG evolution for small enough $\alpha > 0$, specifically $\alpha \in (0, \frac{1}{24})$. The constraint $\alpha < \frac{1}{24}$ comes from the currently available local well-posedness results, while the singularity formation argument by itself allows a somewhat larger value. The Bio-Savart law for the patch evolution on the half-plane $D := \mathbb{R} \times \mathbb{R}^+$ is

$$u = \nabla^\perp (-\Delta)^{-1+\alpha} \omega,$$

with the Dirichlet Laplacian on D, which can also be written as

$$u(x,t) := \int_D \left(\frac{(x-y)^\perp}{|x-y|^{2+2\alpha}} - \frac{(x-\bar{y})^\perp}{|x-\bar{y}|^{2+2\alpha}} \right) \omega(y,t) dy \tag{1.2}$$

for $x \in \bar{D}$ (up to a positive pre-factor, which can be dropped without loss due to scaling). We use here the notation

$$v^\perp := (v_2, -v_1) \qquad \text{and} \qquad \bar{v} := (v_1, -v_2)$$

for $v = (v_1, v_2)$. The vector field u given by (1.2) is divergence free and tangential to the boundary ∂D, that is,

$$u_2(x,t) = 0 \text{ when } x_2 = 0.$$

A traditional approach to the 2D Euler ($\alpha = 0$) vortex patch evolution, going back to Yudovich (see [42] for an exposition) is via the corresponding flow map. The active scalar ω is advected by u from (1.2) via

$$\omega(x,t) = \omega\left(\Phi_t^{-1}(x), 0\right), \tag{1.3}$$

where

$$\frac{d}{dt}\Phi_t(x) = u\left(\Phi_t(x), t\right) \qquad \text{and} \qquad \Phi_0(x) = x. \tag{1.4}$$

The initial condition ω_0 for (1.2)-(1.4) is patch-like,

$$\omega_0 = \sum_{k=1}^N \theta_k \chi_{\Omega_{0k}}, \tag{1.5}$$

with $\theta_1, \dots, \theta_N \neq 0$ and $\Omega_{01}, \dots, \Omega_{0N} \subseteq D$ bounded open sets, whose closures $\overline{\Omega_{0,k}}$ are pairwise disjoint and whose boundaries $\partial\Omega_{0k}$ are simple closed curves.

One reason the Yudovich theory works for the 2D Euler equations is that for ω which is (uniformly in time) in $L^1 \cap L^\infty$, the velocity field u given by (1.2) with $\alpha = 0$ is log-Lipschitz in space, and the flow map Φ_t is everywhere well-defined. In our situation, when ω is a patch solution and $\alpha > 0$, the flow u from (1.2) is smooth away from the patch boundaries $\partial\Omega_k(t)$ but is only Hölder at $\partial\Omega_k(t)$ which is exactly where one needs to use the flow map (see Lemma 4.1 for the corresponding Hölder estimate). Thus, the Yudovich definition of the evolution may not be applied directly, as the flow trajectories need not be unique when u is only Hölder continuous. We will instead use a natural alternative definition of patch solutions to (1.1)-(1.2), which will be equivalent to the usual definition in the 2D Euler case, and closely related to the definitions used in earlier works on modified SQG patches. In order to not interrupt this introduction, we postpone the precise discussion of these points to Section 2 — see Definition 2.2 and the rest of that section.

The following local well-posedness result is proved in the companion paper [36].

Theorem 1.1. *([36]) If $\alpha \in (0, \frac{1}{24})$, then for each H^3 patch-like initial data ω_0, there exists a unique local H^3 patch solution ω to (1.1)-(1.2) with $\omega(\cdot, 0) = \omega_0$. Moreover, if the maximal time T_ω of existence of ω is finite, then at T_ω a singularity forms: either two patches touch, or a patch boundary touches itself or loses H^3 regularity.*

The hypothesis $\alpha < \frac{1}{24}$ in Theorem 1.1 may well be an artifact of the local existence proof, but we still will need a "small α" assumption, even though less restrictive, in the finite time singularity proof below. The last claim in this theorem means that either

$$\partial\Omega_k(T_\omega) \cap \partial\Omega_i(T_\omega) \neq \emptyset$$

for some $k \neq i$, or $\partial\Omega_k(T_\omega)$ is not a simple closed curve for some k, or

$$\lim_{t \nearrow T_\omega} \|\Omega_k(t)\|_{H^3} = \infty$$

for some k, where the above norm is the H^3 norm of any constant-speed parametrization of $\partial\Omega_k(t)$ (see Definition 2.1 below). Note that the sets

$$\partial\Omega_k(T_\omega) := \lim_{t \nearrow T_\omega} \partial\Omega_k(t),$$

with the limit taken with respect to the Hausdorff distance d_H, are well defined if $T_\omega < \infty$ because u is uniformly bounded — see Lemma 4.1 below. In fact, [36, Lemma 4.10] yields

$$d_H(\partial\Omega(t), \partial\Omega(s)) \leq \|u\|_{L^\infty}|t - s|$$

for $t, s \in [0, T_\omega)$.

We can now state the main results of the present paper — global regularity of $C^{1,\gamma}$ patch solutions in the 2D Euler case $\alpha = 0$, and existence of H^3 patch solutions which develop a singularity in finite time for small $\alpha > 0$.

Theorem 1.2. *Let $\alpha = 0$ and $\gamma \in (0,1]$. Then for each $C^{1,\gamma}$ patch-like initial data ω_0, there exists a unique global $C^{1,\gamma}$ patch solution ω to* (1.1)-(1.2) *with $\omega(\cdot,0) = \omega_0$.*

Theorem 1.3. *Let $\alpha \in (0, \frac{1}{24})$. Then there are H^3 patch-like initial data ω_0 for which the unique local H^3 patch solution ω to* (1.1)-(1.2) *with $\omega(\cdot,0) = \omega_0$ becomes singular in finite time (i.e., its maximal time of existence T_ω is finite).*

To the best of our knowledge, Theorem 1.3 is the first rigorous proof of finite time singularity formation in this class of fluid dynamics models. Moreover, Theorems 1.2 and 1.3 show that the α-patch model undergoes a phase transition at $\alpha = 0$, which provides a reason for calling the 2D Euler equations "critical".

Let us now describe the type initial conditions, depicted in Figure 1, which will lead to a singularity for $\alpha > 0$. As we have mentioned above, our choice of initial data is motivated

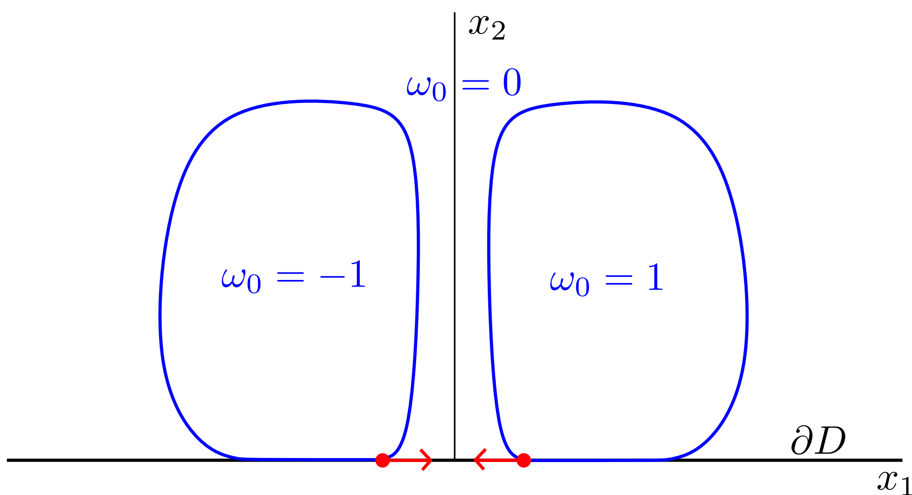

Figure 1: Initial data ω_0 which leads to a finite time singularity.

by the numerical simulations of the three-dimensional Euler equations in [37, 38], as well as by the example of smooth solutions to the 2D Euler equations with a double exponential temporal growth of their vorticity gradients in [35]. The initial condition consist of two patches with opposite signs, symmetric with respect to the x_2-axis and touching the x_1-axis. The patches are sufficiently close to the origin and have a sufficiently large area. It can then be seen from (1.2) that the rightmost point of the left patch on the x_1-axis and the leftmost point of the right patch on the x_1-axis will move toward each other (see Figure 1). In the case of the 2D Euler equations $\alpha = 0$, Theorem 1.2 shows that the two points never reach the origin. When $\alpha > 0$ is small, however, we are able to control the evolution sufficiently well to show that — unless the solution develops another singularity earlier — both points will reach the origin in a finite time. The argument yielding such control is fairly subtle, and the estimates do not extend to all $\alpha < \frac{1}{2}$, even though one would expect the singularity formation to persist for more singular equations.

We note that we will actually run the singularity formation argument for the less regular $C^{1\gamma}$ patch solutions. However, we do not currently have local well-posedness theorem in this class for $\alpha > 0$, even though existence of such solutions follows from existence of the more regular H^3 patch solutions. Since our argument requires odd symmetry, which would follow from uniqueness due to the symmetries of the equation, it effectively shows that there exist $C^{1,\gamma}$ patch solutions which either have a finite maximal time of existence (i.e., exhibit singularity formation) or lose uniqueness (and odd symmetry).

Throughout the paper we denote by C, C_γ, etc. various universal constants, which may change from line to line.

Acknowledgment. We thank Peter Guba, Bob Hardt, and Giovanni Russo for useful discussions. We acknowledge partial support by NSF-DMS grants 1056327, 1159133, 1311903, 1411857, 1412023, and 1535653.

2 Vortex patches and low regularity velocity fields

In this section, we make precise the notion of the patch evolution for $\alpha > 0$ and recall additional existence results from [36] which we will need in the proof of Theorem 1.3.

The definition of the patch evolution

As we mentioned above, Hölder regularity of the fluid velocity u at the patch boundaries is not sufficient for a unique definition of the trajectories from (1.4) when $\alpha > 0$. We start with a definition of the Hölder and Sobolev norms of the boundaries of domains in $\mathbb{R}^2$ which will make the notions of $C^{1,\gamma}$ and H^3 patches precise.

Definition 2.1. *Let $\Omega \subseteq \mathbb{R}^2$ be a bounded open set whose boundary $\partial\Omega$ is a simple closed C^1 curve with arc-length $|\partial\Omega|$. A constant speed parametrization of $\partial\Omega$ is any counter-clockwise parametrization $z : \mathbb{T} \to \mathbb{R}^2$ of $\partial\Omega$ with $|z'| \equiv \frac{1}{2\pi}|\partial\Omega|$ on the circle $\mathbb{T} := [-\pi, \pi]$ (with $\pm\pi$ identified), and we define $\|\Omega\|_{C^{m,\gamma}} := \|z\|_{C^{m,\gamma}}$ and $\|\Omega\|_{H^m} := \|z\|_{H^m}$.*

It is not difficult to see (using [36, Lemma 3.4]), that a domain Ω as above satisfies $\|\Omega\|_{C^{m,\gamma}} < \infty$ (resp. $\|\Omega\|_{H^m} < \infty$) precisely when for some $r > 0$, $M < \infty$, and each $x \in \partial\Omega$, the set $\partial\Omega \cap B(x,r)$ is (in the coordinate system centered at x and with the axes given by the tangent and normal vectors to $\partial\Omega$ at x) the graph of a function with $C^{m,\gamma}$ (resp. H^m) norm less than M.

We denote by $d_H(\Gamma, \tilde{\Gamma})$ the Hausdorff distance between two sets $\Gamma, \tilde{\Gamma}$. For a set $\Gamma \subseteq \mathbb{R}^2$, a vector field $v : \Gamma \to \mathbb{R}^2$, and $h \in \mathbb{R}$, we let

$$X_v^h[\Gamma] := \{x + hv(x) \, : \, x \in \Gamma\}.$$

Our definition of a patch solution to (1.1)-(1.2) in the half-plane is as follows.

Definition 2.2. *Let $D := \mathbb{R} \times \mathbb{R}^+$, let $\theta_1, \dots, \theta_N \in \mathbb{R} \setminus \{0\}$, and for each $t \in [0,T)$, let $\Omega_1(t), \dots, \Omega_N(t) \subseteq D$ be bounded open sets with pairwise disjoint closures whose boundaries $\partial\Omega_k(t)$ are simple closed curves, such that each $\partial\Omega_k(t)$ is also continuous in $t \in [0,T)$ with respect to d_H. Denote $\Omega(t) := \bigcup_{k=1}^N \Omega_k(t)$ and let*

$$\omega(x,t) := \sum_{k=1}^{N} \theta_k \chi_{\Omega_k(t)}(x). \tag{2.1}$$

If for each $t \in (0,T)$ and u from (1.2), we have

$$\lim_{h \to 0} \frac{d_H\Big(\partial\Omega(t+h), X_{u(\cdot,t)}^h[\partial\Omega(t)]\Big)}{h} = 0, \tag{2.2}$$

then ω is a patch solution to (1.1)-(1.2) on the time interval $[0,T)$. If we also have

$$\sup_{t \in [0,T']} \|\Omega_k(t)\|_{C^{m,\gamma}} < \infty \qquad \left(\text{resp. } \sup_{t \in [0,T']} \|\Omega_k(t)\|_{H^m} < \infty \right)$$

for each k and $T' \in (0,T)$, then ω is a $C^{m,\gamma}$ (resp. H^m) patch solution to (1.1)-(1.2) on $[0,T)$.

Lemma 4.1 below shows that u is Hölder continuous for patch solutions, thus (2.2) says that $\partial\Omega$ is moving with velocity $u(x,t)$ at any $t \in [0,T)$ and $x \in \partial\Omega(t)$.

This definition generalizes the well-known definitions for the 2D Euler equations in terms of (1.3)-(1.4) or in terms of the normal velocity at $\partial\Omega$. Indeed, if ω satisfies $\partial\Omega_k(t) = \Phi_t(\partial\Omega_k(0))$ for each k and $t \in [0,T)$, the patches have pairwise disjoint closures, and their boundaries remain simple closed curves, then continuity of u, compactness of $\partial\Omega(t)$, and (1.4) show that ω is a patch solution to (1.1)-(1.2) on $[0,T)$. Moreover, if $\partial\Omega(t)$ is C^1 and $n_{x,t}$ is the outer unit normal vector at $x \in \partial\Omega(t)$, then (2.2) is equivalent to the motion of $\partial\Omega(t)$ with the outer normal velocity $u(x,t) \cdot n_{x,t}$ at each $x \in \partial\Omega(t)$ (which can be defined in a natural way by (2.2) with $u(\cdot,t)$ replaced by $(u(\cdot,t) \cdot n_{\cdot,t})n_{\cdot,t}$). However, Definition 2.2 makes sense even if $\Phi_t(x)$ cannot be uniquely defined for some x, or when $\partial\Omega(t)$ is not C^1.

It is not difficult to show (see [36], Remark 3 after Definition 1.2) that C^1 patch solutions to (1.1)-(1.2) are also weak solutions to (1.1) in the sense that for each $f \in C^1(\bar{D})$ we have

$$\frac{d}{dt}\int_D \omega(x,t)f(x)dx = \int_D \omega(x,t)[u(x,t) \cdot \nabla f(x)]dx \tag{2.3}$$

for all $t \in (0,T)$, with both sides continuous in t. Also, weak solutions to (1.1)-(1.2) which are of the form (2.1) and have C^1 boundaries $\partial\Omega_k(t)$ which move with some continuous velocity $v : \mathbb{R}^2 \times (0,T) \to \mathbb{R}^2$ (in the sense of (2.2) with v in place of u), do satisfy (2.2) with u (hence they are patch solutions if those boundaries are simple closed curves and the domains have pairwise disjoint closures). Moreover, (2.3) also leads to $|\Omega_k(t)| = |\Omega_k(0)|$ for each k and $t \in [0,T)$ — see an elementary argument at the end of the introduction of [36].

We also note that in the 2D Euler case $\alpha = 0$, it is not difficult to show via the standard approach of Yudovich theory that there is a unique global weak solution ω to (1.1)-(1.2) on D with a given $\omega(\cdot,0)$ as in Definition 2.2, and it is of the form (2.1) with $\partial\Omega_k(t) = \Phi_t(\partial\Omega_k(0))$. (We spell out this argument in Section 3.) Thus, the above shows that as long as the patch boundaries remain pairwise disjoint simple closed curves, ω is also the unique patch solution to (1.1)-(1.2).

Relation of patch solutions to the flow map Φ_t in the modified SQG case $\alpha > 0$

The companion paper [36], which proves Theorem 1.1 as well as the same result on $\mathbb{R}^2$ for all $\alpha \in (0, \frac{1}{2})$ (thus extending the results of [48] for infinitely smooth SQG patches of a special type on $\mathbb{R}^2$ to all H^3 modified SQG patches), also provides a link between patch solutions and the flow map Φ_t from (1.4) which will be important in our finite time singularity proof. Note that since u is smooth away from $\partial\Omega$, the trajectories $\Phi_t(x)$ remain unique at least until they hit $\partial\Omega$ (in the Euler case, $\Phi_t(x)$ is always unique because u is log-Lipschitz). However, after hitting a patch boundary, the trajectory still exists but need not be unique. Part (a) of the following result from [36] shows that for $\alpha < \frac{1}{4}$ and patch solutions with H^3 boundaries, the flow lines which start away from $\partial\Omega(0)$ will stay away from $\partial\Omega(t)$ as long as the solution remains regular (note that we have $H^3(\mathbb{T}) \subseteq C^{1,1}(\mathbb{T})$).

Theorem 2.3. *([36]) For ω as in the first paragraph of Definition 2.2 and $x \in \bar{D} \setminus \partial\Omega(0)$, let $t_{\omega,x} \in [0,T]$ be the maximal time such that the solution of* (1.4) *with u from* (1.2) *satisfies $\Phi_t(x) \in \bar{D} \setminus \partial\Omega(t)$ for each $t \in [0, t_{\omega,x})$.*

(a) If $\alpha \in (0, \frac{1}{4})$, $\gamma \in (\frac{2\alpha}{1-2\alpha}, 1]$, and ω is a $C^{1,\gamma}$ patch solution to (1.1)*-*(1.2) *on $[0,T)$, then $t_{\omega,x} = T$ for each $x \in \bar{D} \setminus \partial\Omega(0)$ and*

$$\Phi_t : [\bar{D} \setminus \partial\Omega(0)] \to [\bar{D} \setminus \partial\Omega(t)]$$

is a bijection for each $t \in [0,T)$.

(b) If $\alpha \in (0, \frac{1}{2})$, $t_{\omega,x} = T$ for each $x \in \bar{D} \setminus \partial\Omega(0)$, and $\Phi_t : [\bar{D} \setminus \partial\Omega(0)] \to [\bar{D} \setminus \partial\Omega(t)]$ is a bijection for each $t \in [0,T)$, then ω is a patch solution to (1.1)*-*(1.2) *on $[0,T)$. Moreover, Φ_t is measure preserving on $\bar{D} \setminus \partial\Omega(0)$ and it also maps each $\Omega_k(0)$ onto $\Omega_k(t)$ as well as $\bar{D} \setminus \overline{\Omega(0)}$ onto $\bar{D} \setminus \overline{\Omega(t)}$. Finally, we have*

$$\Phi_t(\partial\Omega_k(0)) = \partial\Omega_k(t)$$

for each k and $t \in [0,T)$, in the sense that any solution of (1.4) *with $x \in \partial\Omega_k(0)$ satisfies $\Phi_t(x) \in \partial\Omega_k(t)$, and for each $y \in \partial\Omega_k(t)$, there is $x \in \partial\Omega_k(0)$ and a solution of* (1.4) *such that $\Phi_t(x) = y$.*

3 Global well-posedness for the Euler case $\alpha = 0$

In this section we prove Theorem 1.2.

3.1 Proof of Theorem 1.2 in the single patch case

For the sake of simplicity of presentation, we first consider a single patch $\Omega(t) \subseteq D$, with

$$\omega(x,t) = \theta\chi_{\Omega(t)}(x).$$

Later, we will show how to generalize this to finitely many patches. We may assume without loss of generality that both $\theta = 1$ and $|\Omega(t)| = |\Omega(0)| = 1$, as the general single patch case then follows by a simple scaling. The local-in-time existence and uniqueness of $C^{1,\gamma}$ patch solutions for this initial value problem was proved in [20]. We will therefore focus on estimates which will allow the solution to be continued indefinitely.

Our approach is a combination of the techniques form [1] and a refinement of the estimates in [20]. Following [1], we reformulate the vortex-patch evolution in terms of the evolution of a function $\varphi(x,t)$, which defines the patch via

$$\Omega(t) = \{\varphi(x,t) > 0\}.$$

First, if $\partial\Omega(0)$ is a simple closed $C^{1,\gamma}$ curve, there exists a function $\varphi_0 \in C^{1,\gamma}(\overline{\Omega(0)})$ such that $\varphi_0 > 0$ on $\Omega(0)$, $\varphi_0 = 0$ on $\partial\Omega(0)$, and

$$\inf_{\partial\Omega(0)} |\nabla\varphi_0| > 0. \tag{3.1}$$

One can obtain such φ_0, for instance, by solving the Dirichlet problem

$$\begin{aligned} -\Delta\varphi_0 &= f \text{ on } \Omega(0), \\ \varphi_0 &= 0 \text{ on } \partial\Omega(0), \end{aligned}$$

with $0 \le f \in C_0^\infty(\Omega(0))$. It follows from the standard elliptic estimates (see, e.g., [27, Theorem 8.34]) that $\varphi_0 \in C^{1,\gamma}(\overline{\Omega(0)})$, while (3.1) is a consequence of Hopf's lemma, which holds for $C^{1,\gamma}$ domains by a result of Finn and Gilbarg [24] (see also [28, Section 10]).

Consider now the flow map (1.4) corresponding to the Biot-Savart law for the Euler equation on the half plane,

$$u(x,t) = \int_{\Omega(t)} \frac{(x-y)^\perp}{|x-y|^2}dy - \int_{\tilde\Omega(t)} \frac{(x-y)^\perp}{|x-y|^2}dy =: v(x,t) - \tilde v(x,t),$$

with $\tilde\Omega(t)$ the reflection of $\Omega(t)$ across the x_1-axis. For $x \in \Omega(t)$, we set

$$\varphi(x,t) = \varphi_0(\Phi_t^{-1}(x)),$$

with Φ_t^{-1} the inverse map, so that φ solves

$$\partial_t\varphi + u\cdot\nabla\varphi = 0 \tag{3.2}$$

on $\{(t,x) \,:\, t > 0 \text{ and } x \in \Omega(t)\}$. Thus, for each $t \ge 0$, $\varphi(\cdot,t) > 0$ on $\Omega(t)$, it vanishes on $\partial\Omega(t)$, and it is not defined on $\mathbb{R}^2 \setminus \overline{\Omega(t)}$. We now let

$$w = (w_1, w_2) := \nabla^\perp\varphi = (\partial_{x_2}\varphi, -\partial_{x_1}\varphi), \tag{3.3}$$

and define

$$\begin{aligned} A_\gamma(t) &:= \|w(\cdot,t)\|_{\dot C^\gamma(\Omega(t))} := \sup_{x,y\in\Omega(t)} \frac{|w(x,t)-w(y,t)|}{|x-y|^\gamma}, \\ A_\infty(t) &:= \|w(\cdot,t)\|_{L^\infty(\Omega(t))}, \\ A_{\inf}(t) &:= \inf_{x\in\partial\Omega(t)} |w(x,t)|. \end{aligned}$$

By our choice of φ_0, we have

$$A_\gamma(0), A_\infty(0), A_{\inf}(0)^{-1} < \infty.$$

Moreover, w is divergence free and satisfies

$$w_t + u \cdot \nabla w = (\nabla u) w. \tag{3.4}$$

Proposition 1 in [1] and $|\Omega(t)| = 1$ yield

$$\|\nabla v(\cdot,t)\|_{L^\infty(\mathbb{R}^2)} + \|\nabla \tilde{v}(\cdot,t)\|_{L^\infty(\mathbb{R}^2)} \leq C_\gamma \left(1 + \log_+ \frac{A_\gamma(t)}{A_{\inf}(t)}\right), \tag{3.5}$$

with $\log_+ x := \max\{\log x, 0\}$ and some universal $C_\gamma < \infty$. Hence, we obtain from (3.4) (after doubling C_γ):

$$A'_\infty(t) \leq C_\gamma A_\infty(t) \left(1 + \log_+ \frac{A_\gamma(t)}{A_{\inf}(t)}\right), \tag{3.6}$$

$$A'_{\inf}(t) \geq -C_\gamma A_{\inf}(t) \left(1 + \log_+ \frac{A_\gamma(t)}{A_{\inf}(t)}\right). \tag{3.7}$$

The main step in the proof will be to get an appropriate bound on $A_\gamma(t)$. A simple calculation and (3.4) yield

$$A'_\gamma(t) \leq \gamma \|\nabla u(\cdot,t)\|_{L^\infty(\mathbb{R}^2)} A_\gamma(t) + \|\nabla u(\cdot,t) w(\cdot,t)\|_{\dot{C}^\gamma(\Omega(t))}. \tag{3.8}$$

Our goal will now be to show

$$\|\nabla u(\cdot,t) w(\cdot,t)\|_{\dot{C}^\gamma(\Omega(t))} \leq C_\gamma A_\gamma(t) \left(1 + \log_+ \frac{A_\gamma(t)}{A_{\inf}(t)}\right) \tag{3.9}$$

with some universal $C_\gamma < \infty$. This and (3.5) turn (3.8) into

$$A'_\gamma(t) \leq C_\gamma A_\gamma(t) \left(1 + \log_+ \frac{A_\gamma(t)}{A_{\inf}(t)}\right). \tag{3.10}$$

It follows from (3.10) and (3.7) that the ratio

$$A(t) := \frac{A_\gamma(t)}{A_{\inf}(t)}$$

satisfies

$$A'(t) \leq C_\gamma A(t)(1 + \log_+ A(t)).$$

Therefore, $A(t)$ grows at most double-exponentially in time, and the same estimate for $A_\infty(t)$, $A_{\inf}(t)^{-1}$, and $A_\gamma(t)$ follows from (3.6), (3.7), and (3.10), respectively. This then proves Theorem 1.2 for a single patch because the above bounds on $A_\gamma(t)$, $A_{\inf}(t)$ and $A_\infty(t)$ imply that the patch boundary cannot touch itself and must be $C^{1,\gamma}$ at time t (hence the local-in-time solution can be extended indefinitely).

Thus, the proof for a single patch is reduced to (3.9). The time variable will not play a role here, so we will drop the argument t in what follows. We split $(\nabla u)w$ as

$$(\nabla u)w = (\nabla v)w + (\nabla \tilde{v})w.$$

Since v is generated by the patch Ω, and w is tangential to $\partial\Omega$, [1, Corollary 1] gives

$$\|(\nabla v)w\|_{\dot{C}^\gamma(\Omega)} \le C_\gamma \|\nabla v\|_{L^\infty(\mathbb{R}^2)} \|w\|_{\dot{C}^\gamma(\Omega)} \tag{3.11}$$

with a universal C_γ. Note that in [1], w is defined in $\mathbb{R}^2$ and all the norms are over $\mathbb{R}^2$. We can use Whitney-type extension theorems [50, Sec 6.2, Theorem 4] to extend our φ to all of $\mathbb{R}^2$ so that its $C^{1,\gamma}$ norm increases at most by a universal factor $\tilde{C}_\gamma < \infty$. This and [1] now yield (3.11). Notice that this extended φ does not necessarily solve (3.2).

By (3.11) and (3.5), $\|(\nabla v)w\|_{\dot{C}^\gamma(\Omega)}$ is indeed bounded by the right-hand side of (3.9). Thus, it suffices to show that $\|(\nabla \tilde{v})w\|_{\dot{C}^\gamma(\Omega)}$ satisfies the same estimate. As w is not tangential to the boundary of $\tilde{\Omega}$, which generates $\tilde{v}$, we cannot directly apply the methods from [1]. Let us take the above extension of φ to $\mathbb{R}^2$ and define

$$\tilde{\varphi}(x) := \varphi(\bar{x}) \qquad \text{and} \qquad \tilde{w} := -\nabla^\perp \tilde{\varphi},$$

with $\bar{x} = (x_1, -x_2)$. Then $\tilde{w}$ is tangential to $\partial\tilde{\Omega}$ and

$$\|\tilde{w}\|_{\dot{C}^\gamma(\mathbb{R}^2)} \le \tilde{C}_\gamma A_\gamma,$$

Thus, Corollary 1 of [1] again yields

$$\|(\nabla \tilde{v})\tilde{w}\|_{\dot{C}^\gamma(\Omega)} \le C_\gamma \|\nabla \tilde{v}\|_{L^\infty(\mathbb{R}^2)} \|\tilde{w}\|_{\dot{C}^\gamma(\mathbb{R}^2)} \le C_\gamma A_\gamma \left(1 + \log_+ \frac{A_\gamma}{A_{\inf}}\right).$$

Hence, it suffices to prove the following bound.

Proposition 3.1. *If $\varphi : \Omega \to [0, \infty)$ is positive on $\Omega \subseteq D$ and vanishes on $\partial\Omega$, then, with $\tilde{v}, w, \tilde{w}, A_\gamma, A_{\inf}$ as above (and some universal $C_\gamma < \infty$) we have*

$$\|\nabla \tilde{v}(w - \tilde{w})\|_{\dot{C}^\gamma(\Omega)} \le C_\gamma A_\gamma \left(1 + \log_+ \frac{A_\gamma}{A_{\inf}}\right). \tag{3.12}$$

Let us introduce some notation: for any $x \in \mathbb{R}^2 \setminus \tilde{\Omega}$, define

$$d(x) := \operatorname{dist}(x, \tilde{\Omega}),$$

let $P_x \in \partial\tilde{\Omega}$ be such that $\operatorname{dist}(x, P_x) = d(x)$ (if there are multiple such points, we pick any of them), and let $\bar{P}_x$ be the reflection of P_x across the x_1-axis. For an illustration of $w, \tilde{w}, d(x), P_x, \bar{P}_x$, see Figure 2.

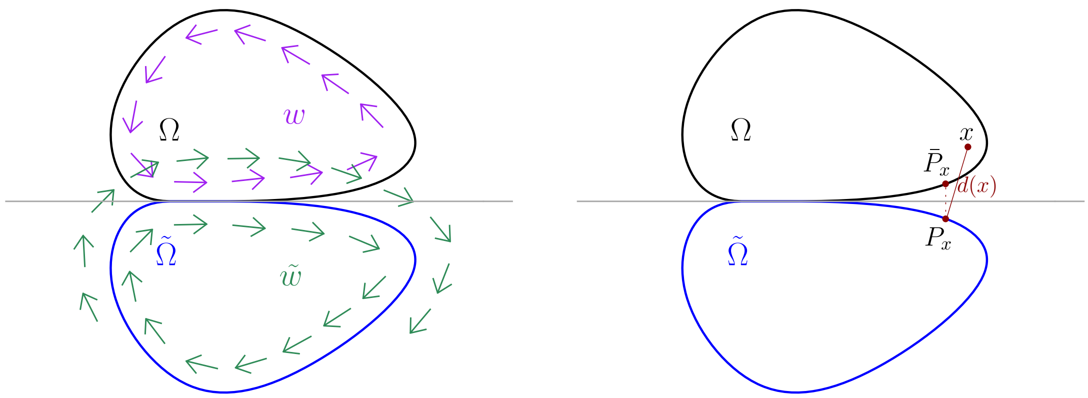

Figure 2: Vector fields w and $\tilde{w}$, and $d(x)$, P_x, $\bar{P}_x$ for $x \in \Omega$.

For arbitrary $x, y \in \Omega$ we can assume, without loss of generality, that $d(x) \le d(y)$. Then, with $g := (\nabla\tilde{v})(w - \tilde{w})$, we have

$$\frac{|g(x) - g(y)|}{|x-y|^\gamma} \le |\nabla\tilde{v}(y)| \|w - \tilde{w}\|_{\dot{C}^\gamma(\Omega)} + \underbrace{\frac{|\nabla\tilde{v}(x) - \nabla\tilde{v}(y)|}{|x-y|^\gamma}}_{T_1(x,y)} \underbrace{\big|w(x) - \tilde{w}(x)\big|}_{T_2(x)}.$$

Since the first term on the right-hand side is bounded by the right-hand side of (3.12) due to (3.5) and the definition of $\tilde{w}$, we only need to obtain the same bound for the second term. We will estimate T_1 and T_2 separately, in terms of A_γ, $A_{\inf}$, $d(x)$, and $|\tilde{w}(P_x)| = |w(\bar{P}_x)|$.

Let us start with T_2. We estimate

$$T_2(x) \le \big|w(\bar{P}_x) - \tilde{w}(P_x)\big| + \big|w(\bar{P}_x) - w(x)\big| + \big|\tilde{w}(P_x) - \tilde{w}(x)\big| \le 2\tilde{C}_\gamma A_\gamma d(x)^\gamma + 2|w_2(\bar{P}_x)|,$$

where we used the inequality

$$\operatorname{dist}(x, \bar{P}_x) \le \operatorname{dist}(x, P_x) = d(x)$$

to bound the last two terms in the middle expression by $\tilde{C}_\gamma A_\gamma d(x)^\gamma$, while the first term equals $2|w_2(\bar{P}_x)|$ because

$$\tilde{w}(P_x) = (w_1(\bar{P}_x), -w_2(\bar{P}_x)).$$

The following lemma will allow us to control $|w_2(\bar{P}_x)|$.

Lemma 3.2. *For any $P = (p_1, p_2) \in \partial\Omega$, we have $|w_2(P)| \le 2\left(A_\gamma p_2^\gamma |w(P)|^\gamma\right)^{\frac{1}{1+\gamma}}$.*

Proof. Denote by $\theta \in [0, \frac{\pi}{2}]$ the angle between $\nabla\varphi(P)$ and the x_2-axis (see Figure 3), so that

$$|w_2(P)| = |\nabla\varphi(P)| \sin\theta \le 2|\nabla\varphi(P)| \sin\frac{\theta}{2}. \tag{3.13}$$

If $\theta = 0$, then we are done. Otherwise, let ν denote the unit vector such that the angle between ν and $\nabla\varphi(P)$ is $\frac{\pi}{2} - \frac{\theta}{2}$ (so ν points inside Ω at P) and $\nu_2 < 0$. Draw a ray in the direction ν and originating at P, and denote by Q its intersection with the x_1-axis. Note that $Q \neq P$ since $p_2 > 0$ due to $\theta \neq 0$.

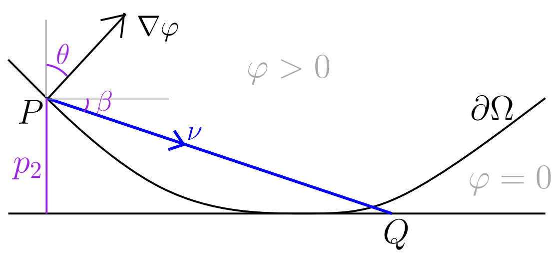

Figure 3: The definitions of θ, β, ν, Q.

The length of the segment PQ is

$$|PQ| = \frac{p_2}{\sin\beta},$$

where either $\beta = \frac{\theta}{2}$ or $\beta = \frac{3\theta}{2}$, the latter if $(\nabla\varphi(P))_2 < 0$. In either case we have

$$|PQ| \le \frac{p_2}{\sin(\theta/2)}.$$

We also have

$$\nabla\varphi(P) \cdot \nu = |\nabla\varphi(P)| \sin\frac{\theta}{2} > 0,$$

and $\nabla\varphi \cdot \nu$ must change sign on the segment PQ because $Q \notin \Omega$ and $\varphi = 0$ on $\partial\Omega$. As

$$\|\nabla\varphi\|_{\dot{C}^\gamma(\Omega)} \le A_\gamma,$$

we obtain

$$|\nabla\varphi(P)|\sin\frac{\theta}{2} \le A_\gamma \left(\frac{p_2}{\sin(\theta/2)}\right)^\gamma .$$

Raising this to power $\frac{1}{1+\gamma}$ and using (3.13) yields

$$|w_2(P)| \le 2|\nabla\varphi(P)|\sin\frac{\theta}{2} \le 2\left(A_\gamma p_2^\gamma |\nabla\varphi(P)|^\gamma\right)^{\frac{1}{1+\gamma}} .$$

Since $|\nabla\varphi(P)| = |w(P)|$, the proof is complete. □

The above lemma applied at $P := \bar{P}_x$, along with $|w(\bar{P}_x)| = |\tilde{w}(P_x)|$, now yields

$$T_2(x) \le 2\tilde{C}_\gamma A_\gamma d(x)^\gamma + 4\left(A_\gamma\, d(x)^\gamma |\tilde{w}(P_x)|^\gamma\right)^{\frac{1}{1+\gamma}} . \tag{3.14}$$

Next we bound T_1.

Proposition 3.3. *With the hypotheses of Proposition 3.1, for $x, y \in \Omega$ with $d(x) \le d(y)$ we have*

$$T_1(x,y) := \frac{|\nabla\tilde{v}(x) - \nabla\tilde{v}(y)|}{|x-y|^\gamma} \le C_\gamma \left(1 + \log_+ \frac{A_\gamma}{A_{\inf}}\right) \min\left\{\frac{A_\gamma}{|\tilde{w}(P_x)|}, d(x)^{-\gamma}\right\}. \tag{3.15}$$

A related but weaker bound (which does not suffice here) was proved in [20]. Before proving Proposition 3.3, let us first complete the proof of Proposition 3.1.

Proof of Proposition 3.1. The bound (3.15) implies

$$T_1(x,y) \le C_\gamma \left(1 + \log_+ \frac{A_\gamma}{A_{\inf}}\right) \min\left\{\left(\frac{A_\gamma}{|\tilde{w}(P_x)|}\right)^{\frac{\gamma}{1+\gamma}} d(x)^{-\frac{\gamma}{1+\gamma}}, d(x)^{-\gamma}\right\}.$$

Multiplying this by (3.14) gives

$$T_1(x,y)T_2(x) \le C_\gamma A_\gamma \left(1 + \log_+ \frac{A_\gamma}{A_{\inf}}\right).$$

As we have explained above, this yields (3.12) and concludes the proof. □

We are left with proving Proposition 3.3. We start with the following simple lemma.

Lemma 3.4. *When $d(x) \le d(y)$ for $x, y \in \Omega$, we have (with a universal $C < \infty$)*

$$\frac{|\nabla\tilde{v}(x) - \nabla\tilde{v}(y)|}{|x-y|^\gamma} \le \frac{C}{\gamma} d(x)^{-\gamma}. \tag{3.16}$$

Proof. The mean value theorem yields

$$\frac{|\nabla\tilde{v}(x)-\nabla\tilde{v}(y)|}{|x-y|^\gamma} \le |\nabla^2\tilde{v}(Z_{xy})||x-y|^{1-\gamma}$$

for some point Z_{xy} on the segment connecting x and y. Since $\tilde{\Omega}$ is the reflection of $\Omega \subseteq D$ with respect to the x_1-axis, we have $d(x) \in [x_2, 2x_2]$ and $d(y) \in [y_2, 2y_2]$. As $d(x) \le d(y)$, we then obtain

$$d(Z_{xy}) \ge \min\{x_2, y_2\} \ge \frac{d(x)}{2}.$$

Moreover, for any $Z \in \mathbb{R}^2 \setminus \tilde{\Omega}$ we have (with a universal $C < \infty$)

$$|\nabla^2\tilde{v}(Z)| \le \int_{\mathbb{R}^2\setminus B(Z,d(Z))} \frac{C}{|Z-z|^3}\,dz \le Cd(Z)^{-1}. \tag{3.17}$$

Combining these estimates, we obtain

$$\frac{|\nabla\tilde{v}(x)-\nabla\tilde{v}(y)|}{|x-y|^\gamma} \le Cd(Z_{xy})^{-1}|x-y|^{1-\gamma} \le 2Cd(x)^{-1}|x-y|^{1-\gamma}.$$

If $|x-y| \le d(x)$, then (3.16) follows because $\gamma \le 1$.

If $|x-y| \ge d(x)$, let

$$Q_{xy} = (x_1, x_2 + 2|x-y|),$$

and connect x and y by a path consisting of the segments $[xQ_{xy}]$ and $[Q_{xy}y]$. Then

$$|Q_{xy}-y| \le 3|x-y| \tag{3.18}$$

yields

$$\frac{|\nabla\tilde{v}(x)-\nabla\tilde{v}(y)|}{3|x-y|} \le \int_0^1 |\nabla^2\tilde{v}(x+s(Q_{xy}-x))|\,ds + \int_0^1 |\nabla^2\tilde{v}(y+s(Q_{xy}-y))|\,ds. \tag{3.19}$$

Note that

$$d(x+s(Q_{xy}-x)) \ge \max\{d(x), 2s|x-y|\}, \tag{3.20}$$

and we also have

$$d(y+s(Q_{xy}-y)) \ge s|x-y| \ge \frac{s}{3}|Q_{xy}-y|$$

due to

$$(Q_{xy}-y)_2 \ge |x-y|$$

and (3.18). It then follows that

$$d(y) \le d(y+s(Q_{xy}-y)) + s|Q_{xy}-y| \le 4d(y+s(Q_{xy}-y)),$$

so by $d(x) \le d(y)$ and the above we have

$$d(y + s(Q_{xy} - y)) \ge \max\left\{\frac{d(x)}{4}, s|x-y|\right\},$$

in addition to (3.20). Combining these estimates with (3.17), we obtain

$$\begin{aligned}\frac{|\nabla\tilde{v}(x) - \nabla\tilde{v}(y)|}{|x-y|^\gamma} &\le C|x-y|^{1-\gamma}\left(\int_0^{\frac{d(x)}{|x-y|}} d(x)^{-1} ds + \int_{\frac{d(x)}{|x-y|}}^1 (s|x-y|)^{-1} ds\right) \\ &\le C|x-y|^{-\gamma}\left(1 + \log\frac{|x-y|}{d(x)}\right) \le \frac{C}{\gamma} d(x)^{-\gamma}\end{aligned}$$

because $|x - y| \ge d(x)$. $\square$

We continue the proof of Proposition 3.3. Due to Lemma 3.4, to prove (3.15) we only need to consider the case

$$d(x) \le \tilde{C}^{-1}\left(\frac{|\tilde{w}(P_x)|}{A_\gamma}\right)^{1/\gamma}$$

for any fixed $\tilde{C} < \infty$. Let us pick $\tilde{C} := 16(4\tilde{C}_\gamma)^{1/\gamma}$, with the universal constant $\tilde{C}_\gamma$ from the remark about Whitney extensions after (3.11), so that if we let $\tilde{A}_\gamma := \|\tilde{w}\|_{\dot{C}^\gamma(\mathbb{R}^2)}$ and

$$r_x := \left(\frac{|\tilde{w}(P_x)|}{2\tilde{A}_\gamma}\right)^{1/\gamma},$$

it suffices to consider $d(x) \le 2^{-4-1/\gamma} r_x$ (because $\tilde{A}_\gamma \le \tilde{C}_\gamma A_\gamma$).

Hence, the next lemma finishes the proof of Proposition 3.3.

Lemma 3.5. *When $d(x) \le \min\{d(y), 2^{-4-1/\gamma} r_x\}$ for $x, y \in \Omega$, we have (with a universal constant $C_\gamma < \infty$)*

$$\frac{|\nabla\tilde{v}(x) - \nabla\tilde{v}(y)|}{|x-y|^\gamma} \le C_\gamma\left(1 + \log_+\frac{A_\gamma}{A_{\inf}}\right)\frac{A_\gamma}{|\tilde{w}(P_x)|}. \tag{3.21}$$

In the proof of this lemma, the following improvement of (3.17) will be used to control $|\nabla\tilde{v}(x) - \nabla\tilde{v}(y)|$. Its proof is postponed until the end of this section.

Lemma 3.6. *For any $x \in \mathbb{R}^2 \backslash \tilde{\Omega}$ with $d(x) \in (0, \frac{1}{4} r_x]$, we have (with a universal $C_\gamma < \infty$)*

$$|\nabla^2\tilde{v}(x)| \le C_\gamma d(x)^{-1+\gamma} r_x^{-\gamma}.$$

Proof of Lemma 3.5. Let us first assume

$$|x-y| \geq 2^{-4-1/\gamma} r_x,$$

so that

$$|x-y|^{-\gamma} \leq 64\tilde{C}_\gamma \frac{A_\gamma}{|\tilde{w}(P_x)|}.$$

Then (3.21) follows from the estimate

$$|\nabla \tilde{v}(x) - \nabla \tilde{v}(y)| \leq 2\|\nabla \tilde{v}\|_{L^\infty(\mathbb{R}^2)}$$

and (3.5) (the latter holds for any Ω, φ as in Proposition 3.1 — see [1, Proposition 1]).

Assume now that $|x-y| < 2^{-4-1/\gamma} r_x$. As in Lemma 3.4, let

$$Q_{xy} = (x_1, x_2 + 2|x-y|),$$

and connect the points x and y by a path consisting of the two segments $[xQ_{xy}]$, $[Q_{xy}y]$, again parametrized by

$$z_1(s) = x + s(Q_{xy} - x) \qquad \text{and} \qquad z_2(s) = y + s(Q_{xy} - y),$$

for $s \in [0,1]$ (see Figure 4). Then we again have

$$d(z_i(s)) \geq s|x-y|$$

for $i = 1, 2$ and $s \in [0,1]$.

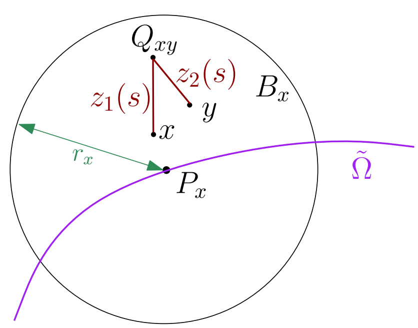

Figure 4: The point Q_{xy} and the paths $z_1(s)$ and $z_2(s)$.

We also have

$$|z_i(s) - P_x| \leq |z_i(s) - x| + d(x) \leq 2|x-y| + d(x),$$

so

$$d(z_i(s)) \leq 2^{-2-1/\gamma} r_x. \tag{3.22}$$

These imply

$$P_{z_i(s)} \in B(P_x, 2^{-1-1/\gamma} r_x) \subseteq B_x := B(P_x, r_x). \tag{3.23}$$

Note that for all $z \in B_x$ we have

$$|\tilde{w}(z) - \tilde{w}(P_x)| \leq \frac{|\tilde{w}(P_x)|}{2}. \tag{3.24}$$

Thus, (3.23) gives

$$|\tilde{w}(P_{z_i(s)})| \geq \frac{1}{2} |\tilde{w}(P_x)|,$$

implying

$$r_{z_i(s)} \geq 2^{-1/\gamma} r_x.$$

From (3.22) it now follows that

$$d(z_i(s)) \leq \tfrac{1}{4} r_{z_i(s)}.$$

Thus, Lemma 3.6 applies to $z_i(s)$ and yields (together with the above estimates)

$$|\nabla^2 \tilde{v}(z_i(s))| \leq C_\gamma d(z_i(s))^{-1+\gamma} r_{z_i(s)}^{-\gamma} \leq 2C_\gamma (s|x-y|)^{-1+\gamma} r_x^{-\gamma}.$$

Then (3.19) implies

$$\frac{|\nabla \tilde{v}(x) - \nabla \tilde{v}(y)|}{|x-y|^\gamma} \leq 12C_\gamma |x-y|^{1-\gamma} \int_0^1 (s|x-y|)^{-1+\gamma} \, r_x^{-\gamma} ds \leq \frac{12C_\gamma}{\gamma} r_x^{-\gamma} \leq \frac{24C_\gamma \tilde{C}_\gamma}{\gamma} \frac{A_\gamma}{|\tilde{w}(P_x)|},$$

which gives (3.21). □

3.2 Proof of Theorem 1.2 in the general case

We now consider an initial condition ω_0 with an arbitrary number of patches and arbitrary values of θ_k as in the statement of Theorem 1.2, and extend it as an odd function to $x_2 < 0$. By [40, Theorems 8.1 and 8.2], there is a unique global weak solution ω to (1.1) with the whole plane flow

$$u(x,t) = \int_{\mathbb{R}^2} \frac{(x-y)^\perp}{|x-y|^2} \omega(y,t) dy, \tag{3.25}$$

and the initial data $\omega(\cdot, 0) = \omega_0$, in the sense that

$$\int_D \omega(x,T) g(x,T) dx - \int_D \omega_0(x) g(x,0) dx = \int_{D \times (0,T)} \omega(x,t) [\partial_t g(x,t) + u(x,t) \cdot \nabla g(x,t)] dx dt$$

for all $T < \infty$ and $g \in C^1(\bar{D} \times [0,T])$. This solution is also a collection of vortex patches

$$\omega(\cdot,t) = \sum_{k=1}^{N} \theta_k \chi_{\Omega_k(t)},$$

with $\Omega_k(t) = \Phi_t(\Omega_k(0))$ for each k [42, Chapter 2, Theorem 3.1]. Note that $\Phi_t(x)$ is uniquely defined for any $x \in \mathbb{R}^2$, due to the time-uniform log-Lipschitz apriori bound

$$|u(x,t) - u(y,t)| \le C_{\omega_0}|x-y| \log\left(1 + |x-y|^{-1}\right) \tag{3.26}$$

for u (see, e.g., [40, Lemma 8.1]), with the constant depending only on $\|\omega_0\|_{L^1}$ and $\|\omega_0\|_{L^\infty}$. Uniqueness shows that ω remains odd in x_2, thus its restriction to $D \times [0,\infty)$ is also the unique weak solution to (1.1), (3.25) (and it is unique such with $\omega(\cdot,0) = \omega_0$ because an odd-in-x_2 extension of a weak solution on $D \times [0,\infty)$ is a weak solution on $\mathbb{R}^2 \times [0,\infty)$). It follows from (1.4), continuity of u (which is obtained as the last claim in Lemma 4.1 below but using (3.26) instead of (4.7)), and compactness of $\partial\Omega(t) \times \{t\}$ that (2.2) holds for each $t > 0$. Hence, if we show that $\{\partial\Omega_k(t)\}_{k=1}^N$ is a family of disjoint simple closed curves for each $t \ge 0$, and

$$\sup_{t\in[0,T]} \max_k \|\Omega_k(t)\|_{C^{1,\gamma}} < \infty$$

for each $T < \infty$, then ω will also be a $C^{1,\gamma}$ patch solution to (1.1)-(1.2) on $[0,\infty)$. Moreover, since $C^{1,\gamma}$ patch solutions are weak solutions in the above sense as well (it is easy to see that (2.3) implies this), ω must then also be the unique patch solution.

Note that (3.26) yields

$$\min_{i\neq k} \operatorname{dist}(\Omega_i(t), \Omega_k(t)) \ge \delta(t) > 0$$

for all $t \ge 0$, where $\delta(t)$ decreases double exponentially in time. This will ensure that the effects of the patches on each other will be controlled. Therefore, it remains to prove that each $\partial\Omega_k(t)$ is a simple closed curve with $\|\partial\Omega_k(t)\|_{C^{1,\gamma}}$ uniformly bounded on bounded intervals.

Let us decompose

$$u = \sum_{i=1}^{N} u_i,$$

with each u_i coming from the contribution of the patch Ω_i to u. If $i \neq k$, then obviously

$$\|\nabla^n u_i(\cdot,t)\|_{L^\infty(\Omega_k(t))} \le C(\omega_0,n)\delta(t)^{-n-1},$$

for all $n \ge 0$. This yields

$$\|\nabla u_i(\cdot,t)\|_{\dot{C}^\gamma(\Omega_k(t))} \le C(\omega_0)\delta(t)^{-3}$$

for $i \neq k$. Also, simple scaling shows that (3.5) now becomes (for each i and with $v_i, \tilde{v}_i$ defined analogously to $v, \tilde{v}$)

$$\|\nabla v_i(\cdot,t)\|_{L^\infty(\mathbb{R}^2)} + \|\nabla \tilde{v}_i(\cdot,t)\|_{L^\infty(\mathbb{R}^2)} \le C_\gamma |\theta_i| \left(1 + \log_+ \frac{A_\gamma(t)|\Omega_i(t)|^{\gamma/2}}{A_{\inf}(t)}\right).$$

We now consider a separate φ_k and $w_k := \nabla^\perp \varphi_k$ for each Ω_k, all φ_k evolving with velocity u. We also add $\sup_k$ in the definitions of A_γ and A_∞ and $\inf_k$ in the definition of $A_{\inf}$. We can repeat the proof above, with (3.9) replaced by (for each k and $t > 0$)

$$\begin{aligned}
\|(\nabla u) w_k\|_{\dot{C}^\gamma(\Omega_k)} &\le C_\gamma \Theta A_\gamma \left(|\Omega| + \log_+ \frac{A_\gamma}{A_{\inf}}\right) \\
&\quad + \sum_{i \neq k} \left(\|\nabla u_i\|_{L^\infty(\Omega_k)} \|w_k\|_{\dot{C}^\gamma(\Omega_k)} + \|\nabla u_i\|_{\dot{C}^\gamma(\Omega_k)} \|w_k\|_{L^\infty(\Omega_k)} \right) \\
&\le C_\gamma N \Theta A_\gamma \left(|\Omega| + \log_+ \frac{A_\gamma}{A_{\inf}}\right) + C(\omega_0) N \delta^{-3} A_\infty,
\end{aligned}$$

where

$$\Theta := \max_{1 \le k \le N} |\theta_k| \qquad \text{and} \qquad |\Omega| := 1 + \max_{1 \le k \le N} |\Omega_k(t)| = 1 + \max_{1 \le k \le N} |\Omega_k(0)|.$$

Then (3.10) is replaced by

$$A'_\gamma(t) \le C_\gamma N \Theta A_\gamma(t) \left(|\Omega| + \log_+ \frac{A_\gamma(t)}{A_{\inf}(t)}\right) + C(\omega_0) N \delta(t)^{-3} A_\infty(t).$$

From this and (3.6), (3.7) a simple computation shows that

$$\tilde{A}(t) := A_\gamma(t) A_{\inf}(t)^{-1} + A_\infty(t)$$

satisfies

$$\tilde{A}'(t) \le C(\gamma, N, \omega_0) \tilde{A}(t) \left(\delta(t)^{-3} + \log_+ \tilde{A}(t)\right).$$

Since $\delta(t)^{-3}$ increases at most double exponentially in time, it follows that $\tilde{A}(t)$ increases at most triple exponentially. As before, this implies that each $\partial\Omega_k(t)$ is a simple closed curve with $\|\partial\Omega_k(t)\|_{C^{1,\gamma}}$ uniformly bounded on bounded intervals. Hence ω is a global $C^{1,\gamma}$ patch solution to (1.1)-(1.2), thus finishing the proof.

3.3 Proof of Lemma 3.6

Let us start with a simple geometric result concerning the behavior of $\partial\tilde{\Omega}$ near P_x, which is similar to the Geometric Lemma in [1]. It says that $\partial\tilde{\Omega} \cap B_x$ is sufficiently "flat".

Lemma 3.7. *Given $x \in \mathbb{R}^2 \setminus \tilde{\Omega}$, let $n_x := \nabla\tilde{\varphi}(P_x)/|\nabla\tilde{\varphi}(P_x)|$, and*

$$S_x := \left\{ P_x + \rho\nu \,:\, \rho \in [0, r_x),\ |\nu| = 1,\ \left(\frac{\rho}{r_x}\right)^\gamma \geq 2|\nu \cdot n_x| \right\}. \tag{3.27}$$

If ν is a unit vector and $\rho \in [0, r_x)$, then the following hold. If $\nu \cdot n_x \geq 0$ and $P_x + \rho\nu \notin S_x$, then $P_x + \rho\nu \in \tilde{\Omega}$. If $\nu \cdot n_x \leq 0$ and $P_x + \rho\nu \notin S_x$, then $P_x + \rho\nu \in \mathbb{R}^2 \setminus \tilde{\Omega}$.

In particular, $\partial\tilde{\Omega} \cap B_x \subseteq S_x$ (see Figure 5).

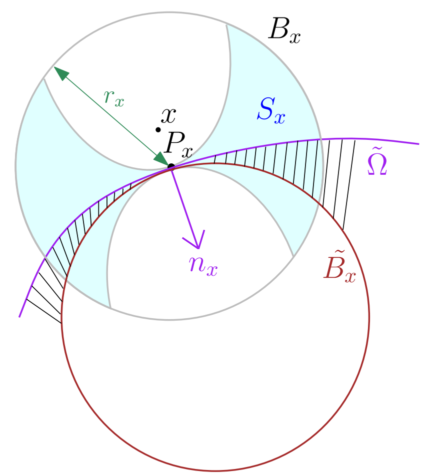

Figure 5: The sets S_x (shaded), $\tilde{B}_x$, and $\tilde{\Omega}\triangle\tilde{B}_x$ (lined).

Proof. We only prove the first statement, as the proof of the second is analogous. Let us assume $\nu \cdot n_x \geq 0$ and $P_x + \rho\nu \notin \tilde{\Omega}$, with $|\nu| = 1$ and $\rho \geq 0$. Then

$$\nabla\tilde{\varphi}(P_x) \cdot \nu \geq 0 \qquad \text{and} \qquad \tilde{\varphi}(P_x + \rho\nu) \leq 0,$$

so we must have $\nabla\tilde{\varphi}(P_x) \cdot \nu \leq \tilde{A}_\gamma \rho^\gamma$ because $\tilde{\varphi}(P_x) = 0$. Thus

$$2\nu \cdot n_x \leq \frac{2\tilde{A}_\gamma \rho^\gamma}{|\nabla\tilde{\varphi}(P_x)|} = \left(\frac{\rho}{r_x}\right)^\gamma,$$

so either $\rho \geq r_x$ or $P_x + \rho\nu \in S_x$. $\square$

Proof of Lemma 3.6. Let n_x, S_x be from Lemma 3.7 and let $\tilde{B}_x := B(O_x, r_x)$, where

$$O_x := P_x + r_x n_x.$$

Then $P_x \in \partial\tilde{B}_x$ and the unit inner normal to $\partial\tilde{B}_x$ at P_x is n_x. We have $\partial\tilde{B}_x \cap B_x \subseteq S_x$ because if $P_x + \rho\nu \in \partial\tilde{B}_x \cap B_x$ with $|\nu| = 1$ and $\rho > 0$, then

$$r_x > \rho = 2r_x|\nu \cdot n_x|.$$

Combining this with Lemma 3.7 directly yields

$$(\tilde{\Omega}\triangle\tilde{B}_x) \cap B_x \subseteq S_x, \tag{3.28}$$

with

$$\tilde{\Omega}\triangle\tilde{B}_x := (\tilde{\Omega}\backslash\tilde{B}_x) \cup (\tilde{B}_x\backslash\tilde{\Omega})$$

the symmetric difference of $\tilde{\Omega}$ and $\tilde{B}_x$ (the lined region in Figure 5). Let

$$u_{\tilde{B}_x}(z) := \int_{\tilde{B}_x} \frac{(z-y)^\perp}{|z-y|^2} dy = 2\pi(\nabla^\perp\Delta^{-1}\chi_{\tilde{B}_x})(z)$$

be the velocity field corresponding to the disc $\tilde{B}_x$. When $|z - O_x| > r_x$, we have by the rotational invariance of $u_{\tilde{B}_x}$ (and with n the outer unit normal vector to $\partial B(O_x, |z-O_x|)$)

$$\begin{aligned}
u_{\tilde{B}_x}(z) &= \frac{(z-O_x)^\perp}{|z-O_x|}\left|u_{\tilde{B}_x}(z)\right| \\
&= \frac{(z-O_x)^\perp}{|z-O_x|} \fint_{\partial B(O_x,|z-O_x|)} n \cdot 2\pi\nabla\Delta^{-1}\chi_{\tilde{B}_x} d\sigma \\
&= \frac{(z-O_x)^\perp}{|z-O_x|^2} \int_{B(O_x,|z-O_x|)} \chi_{\tilde{B}_x}(y) dy \\
&= \pi r_x^2 \frac{(z-O_x)^\perp}{|z-O_x|^2}.
\end{aligned}$$

Differentiating this and noting that

$$|x - O_x| = r_x + d(x) > r_x,$$

yields

$$|\nabla^2 u_{\tilde{B}_x}(x)| \le \frac{C}{r_x}. \tag{3.29}$$

From the definitions of $\tilde{v}$ and $u_{\tilde{B}_x}$ we also have (with some $\tilde{C} < \infty$ and a new $C < \infty$)

$$|\nabla^2\tilde{v}(x) - \nabla^2 u_{\tilde{B}_x}(x)| \le \underbrace{\int_{\mathbb{R}^2\backslash B_x} \frac{\tilde{C}}{|x-y|^3} dy}_{\le Cr_x^{-1}} + \underbrace{\int_{(\tilde{\Omega}\triangle\tilde{B}_x)\cap B_x} \frac{\tilde{C}}{|x-y|^3} dy}_{=:I}. \tag{3.30}$$

Finally, note that

$$\operatorname{dist}(x, S_x) \geq \frac{d(x)}{2}.$$

This holds because if $P_x + \rho\nu \in B(x, \frac{1}{2}d(x))$ with $|\nu| = 1$ and $\rho \geq 0$, then

$$\nu \cdot n_x \geq \cos\frac{\pi}{6} > \frac{1}{2} \qquad \text{and} \qquad \rho \leq \frac{3}{2}d(x) < r_x$$

(due to $d(x) \leq \frac{1}{4}r_x$), hence $P_x + \rho\nu \notin S_x$. Also, if $|P_x - y| \geq 2d(x)$, then

$$|P_x - y| \leq |x - y| + d(x) \leq 2|x - y|.$$

From these, (3.28), and $|\theta| \leq 2|\sin\theta|$ for $|\theta| \leq \frac{\pi}{2}$ we now have

$$\begin{aligned}
I &\leq \int_{S_x} \frac{\tilde{C}}{|x-y|^3} dy \\
&\leq \int_{S_x \setminus B(P_x, 2d(x))} \frac{\tilde{C}}{|x-y|^3} dy + \tilde{C}\left(\frac{d(x)}{2}\right)^{-3} \big|S_x \cap B(P_x, 2d(x))\big| \\
&\leq \int_{S_x \setminus B(P_x, 2d(x))} \frac{8\tilde{C}}{|P_x-y|^3} dy + \frac{8\tilde{C}}{d(x)^3} \big|S_x \cap B(P_x, 2d(x))\big| \\
&\leq \int_{2d(x)}^{r_x} \frac{8\tilde{C}}{\rho^3} 4\left(\frac{\rho}{r_x}\right)^{\gamma} \rho\, d\rho + \frac{8\tilde{C}}{d(x)^3} \int_0^{2d(x)} 4\left(\frac{\rho}{r_x}\right)^{\gamma} \rho\, d\rho \\
&\leq C_\gamma d(x)^{-1+\gamma} r_x^{-\gamma}.
\end{aligned}$$

This, (3.30), and (3.29) now yield

$$|\nabla^2 \tilde{v}(x)| \leq C_\gamma d(x)^{-1+\gamma} r_x^{-\gamma} + C r_x^{-1},$$

so the result follows from $d(x) \leq \frac{1}{4}r_x$. □

4 Finite time blow-up for small $\alpha > 0$

In this section we prove Theorem 1.3, which is an immediate corollary of Theorem 4.6 below.

Let $\alpha \in (0, \frac{1}{24})$ and $\epsilon > 0$ be a small α-dependent number, to be determined later. Let $D^+ := \mathbb{R}^+ \times \mathbb{R}^+$, $\Omega_1 := (\epsilon, 4) \times (0, 4)$, $\Omega_2 := (2\epsilon, 3) \times (0, 3)$, and let $\Omega_0 \subseteq D^+$ be an open set whose boundary is a smooth simple closed curve and which satisfies $\Omega_2 \subseteq \Omega_0 \subseteq \Omega_1$. Let ω be the unique H^3 patch solution to (1.1)-(1.2) with the initial data

$$\omega(\cdot, 0) := \chi_{\Omega_0} - \chi_{\tilde{\Omega}_0} \tag{4.1}$$

and the maximal time of existence $T_\omega > 0$. Here, $\tilde{\Omega}_0$ is the reflection of Ω_0 with respect to the x_2-axis. Then oddness of ω_0 in x_1 and the local uniqueness of the solution imply that

$$\omega(\cdot, t) = \chi_{\Omega(t)} - \chi_{\tilde{\Omega}(t)} \tag{4.2}$$

for $t \in [0, T_\omega)$, with $\Omega(t) := \Phi_t(\Omega_0)$ and $\tilde{\Omega}(t)$ the reflection of $\Omega(t)$ with respect to the x_2-axis. Note that $\Omega(t)$ is well-defined due to Theorem 2.3(a) and $H^3(\mathbb{T}) \subseteq C^{1,1}(\mathbb{T})$). We will show that $T_\omega < \infty$, that is, ω becomes singular in finite time.

More specifically, let

$$T := 50(3\epsilon)^{2\alpha} \qquad \text{and} \qquad X(t) := \left[(3\epsilon)^{2\alpha} - \frac{t}{50}\right]^{1/2\alpha} \quad \text{for } t \in [0, T], \tag{4.3}$$

so that

$$X'(t) = -\frac{1}{100\alpha} X(t)^{1-2\alpha}, \tag{4.4}$$

on $[0, T]$, with $X(0) = 3\epsilon$ and $X(T) = 0$, and let

$$K(t) := \{x \in D^+ : x_1 \in (X(t), 2) \text{ and } x_2 \in (0, x_1)\} \tag{4.5}$$

for $t \in [0, T]$. We will show that if $T_\omega > T$, then $K(t) \subseteq \Omega(t)$ for all $t \in [0, T]$. This yields a contradiction because then $\Omega(T)$ and $\tilde{\Omega}(T)$ touch at the origin (and thus they also cannot remain H^3).

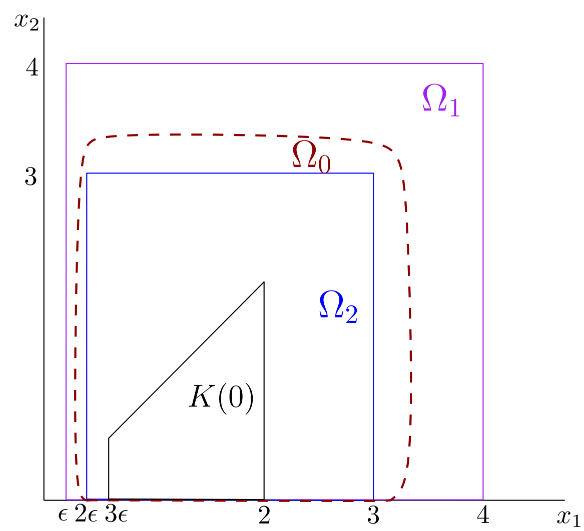

Figure 6: The domains $\Omega_1, \Omega_2, \Omega_0$, and $K(0)$ (with $\omega_0 = \chi_{\Omega_0} - \chi_{\tilde{\Omega}_0}$).

This result will, in fact, hold for the less regular $C^{1,\gamma}$ patches, but in this case we need to assume oddness of ω in x_1 (this is not immediate from the same property of ω_0 without knowing local uniqueness in this class). Before we can prove the result, however, we need to obtain some estimates on the velocity u, the most crucial of which is Proposition 4.5.

Remark. The fact that the fraction on the right-hand side of (4.4) blows-up as $\alpha \to 0$ may seem worrying but ϵ will go to zero quickly as $\alpha \to 0$ (and $X(t) \in [0, 3\epsilon]$), so this growth will be compensated by the term $X(t)^{1-2\alpha}$ which decays as $\alpha \to 0$.

4.1 Some estimates on the velocity fields

Let us start with some basic estimates on the fluid velocities for a general ω.

Lemma 4.1. *For $\alpha \in (0, \frac{1}{2})$ and $u(\cdot, t)$ as in (1.2) with $\omega(\cdot, t) \in L^1(D) \cap L^\infty(D)$, we have*

$$\|u(\cdot, t)\|_{L^\infty} \leq \frac{2\pi}{1-2\alpha}\|\omega(\cdot, t)\|_{L^\infty} + 2\|\omega(\cdot, t)\|_{L^1} \tag{4.6}$$

and

$$\|u(\cdot, t)\|_{C^{1-2\alpha}} \leq \frac{8\pi}{\alpha(1-2\alpha)}\|\omega(\cdot, t)\|_{L^\infty} + 2\|\omega(\cdot, t)\|_{L^1}. \tag{4.7}$$

Furthermore, if ω is weak-$*$ continuous as an $L^\infty(D)$-valued function on a time interval $[a, b]$, and is supported inside a fixed compact subset of $\bar{D}$ for every $t \in [a, b]$, then u is continuous on $\bar{D} \times [a, b]$.

Proof. Let $\eta : \mathbb{R}^2 \to \mathbb{R}$ be the odd extension of $\omega(\cdot, t)$ to the whole plane. The Bio-Savart law (1.2) for $x \in D$ then becomes

$$u(x, t) = \int_{\mathbb{R}^2} \frac{(x-y)^\perp}{|x-y|^{2+2\alpha}}\eta(y)dy, \tag{4.8}$$

and (4.6) follows from

$$\begin{aligned} |u(x, t)| &\leq \int_{|x-y|\leq 1} \frac{|\eta(y)|}{|x-y|^{1+2\alpha}}dy + \int_{|x-y|>1} \frac{|\eta(y)|}{|x-y|^{1+2\alpha}}dy \\ &\leq \|\eta\|_{L^\infty} \int_{|x-y|\leq 1} \frac{1}{|x-y|^{1+2\alpha}}dy + \|\eta\|_{L^1} \\ &\leq \frac{2\pi}{1-2\alpha}\|\omega(\cdot, t)\|_{L^\infty} + 2\|\omega(\cdot, t)\|_{L^1}. \end{aligned}$$

To prove (4.7), consider any $x, z \in \bar{D}$ with $r := |x - z|$. Then

$$\begin{aligned}
|u(x,t) - u(z,t)| \le & \int_{B(x,2r)} \frac{1}{|x-y|^{1+2\alpha}} \eta(y)\, dy + \int_{B(x,2r)} \frac{1}{|z-y|^{1+2\alpha}} \eta(y)\, dy \\
& + \int_{\mathbb{R}^2 \setminus B(x,2r)} \left| \frac{(x-y)^\perp}{|x-y|^{2+2\alpha}} - \frac{(z-y)^\perp}{|z-y|^{2+2\alpha}} \right| \eta(y)\, dy \\
\le & 4\pi \|\eta\|_{L^\infty} \int_0^{3r} s^{-2\alpha}\, ds + 32 \|\eta\|_{L^\infty} \int_{2r}^\infty r s^{-1-2\alpha}\, ds \\
\le & \left(\frac{12\pi}{1-2\alpha} + \frac{32}{2\alpha} \right) \|\eta\|_{L^\infty} |x-z|^{1-2\alpha}.
\end{aligned}$$

Combining this with (4.6) yields (4.7).

It remains to prove the last claim. Since the kernel in (4.8) is L^1 on any compact subset of $\bar{D}$, the assumptions show that u is continuous in $t \in [a,b]$ for any fixed $x \in \bar{D}$. The claim now follows from uniform continuity of u in $x \in \bar{D}$, see (4.7). $\square$

For $y = (y_1, y_2) \in \bar{D}^+ = \mathbb{R}^+ \times \mathbb{R}^+$, we denote $\bar{y} := (y_1, -y_2)$ and $\tilde{y} := (-y_1, y_2)$. If $\omega(\cdot, t) \in L^\infty(D)$ is odd in x_1, then (1.2) becomes (we drop t from the notation in this sub-section)

$$\begin{aligned}
u_1(x) &= -\int_{D^+} K_1(x,y) \omega(y) dy, \\
u_2(x) &= \int_{D^+} K_2(x,y) \omega(y) dy,
\end{aligned} \tag{4.9}$$

where

$$K_1(x,y) = \underbrace{\frac{y_2 - x_2}{|x-y|^{2+2\alpha}}}_{K_{11}(x,y)} - \underbrace{\frac{y_2 - x_2}{|x-\tilde{y}|^{2+2\alpha}}}_{K_{12}(x,y)} - \underbrace{\frac{y_2 + x_2}{|x+y|^{2+2\alpha}}}_{K_{13}(x,y)} + \underbrace{\frac{y_2 + x_2}{|x-\bar{y}|^{2+2\alpha}}}_{K_{14}(x,y)}, \tag{4.10}$$

$$K_2(x,y) = \underbrace{\frac{y_1 - x_1}{|x-y|^{2+2\alpha}}}_{K_{21}(x,y)} + \underbrace{\frac{y_1 + x_1}{|x-\tilde{y}|^{2+2\alpha}}}_{K_{22}(x,y)} - \underbrace{\frac{y_1 + x_1}{|x+y|^{2+2\alpha}}}_{K_{23}(x,y)} - \underbrace{\frac{y_1 - x_1}{|x-\bar{y}|^{2+2\alpha}}}_{K_{24}(x,y)}. \tag{4.11}$$

Let us start with some simple observations about K_1 and K_2.

Lemma 4.2. *For $\alpha \in (0, \frac{1}{2})$ and $x, y \in D^+$ we have the following:*

(a) $K_1(x,y) \ge K_{11}(x,y) - K_{12}(x,y)$.

(b) $\operatorname{sgn}(y_2 - x_2)(K_{11}(x,y) - K_{12}(x,y)) \ge 0$.

(c) $K_2(x,y) \ge K_{21}(x,y) - K_{24}(x,y)$.

(d) $\operatorname{sgn}(y_1 - x_1)(K_{21}(x,y) - K_{24}(x,y)) \ge 0$.

Proof. Part (a) is immediate from $|x-\bar{y}| \le |x+y|$ and (b) from $|x-y| \le |x-\tilde{y}|$. Exchanging $\bar{y}$ and $\tilde{y}$ yields the proofs of (c) and (d). $\square$

Our goal will be to show that if the solution with the initial data from (4.1) exists globally , in which case $0 \le \omega \le 1$ on D^+ by symmetry, then the patch $\Omega(t)$ and its reflection across the x_2 axis must touch at the origin in finite time, which is a contradiction. In particular, we will need to show that u_1 is sufficiently negative in an appropriate subset of D^+ (at least for some time). We will do this by separately estimating the "bad" part

$$u_1^{bad}(x) := -\int_{\mathbb{R}^+\times(0,x_2)} K_1(x,y)\omega(y)dy$$

of the integral in (4.9), (where $K_{11}-K_{12}<0$) and the "good" part

$$u_1^{good}(x) := -\int_{\mathbb{R}^+\times(x_2,\infty)} K_1(x,y)\omega(y)dy$$

(where $K_{11}-K_{12}\ge 0$). We will also obtain similar estimates for the u_2 analogs

$$u_2^{bad}(x) := \int_{(0,x_1)\times\mathbb{R}^+} K_2(x,y)\omega(y)dy,$$

$$u_2^{good}(x) := \int_{(x_1,\infty)\times\mathbb{R}^+} K_2(x,y)\omega(y)dy.$$

Lemma 4.3. *Let $\alpha \in (0,\frac{1}{2})$ and assume that ω is odd in x_1 and $0\le\omega\le 1$ on D^+.*

(a) If $x \in \overline{D^+}$ and $x_2 \le x_1$, then

$$u_1^{bad}(x) \le \frac{1}{\alpha}\left(\frac{1}{1-2\alpha} - 2^{-\alpha}\right) x_1^{1-2\alpha}.$$

(b) If $x \in \overline{D^+}$ and $x_2 \ge x_1$, then

$$u_2^{bad}(x) \ge -\frac{1}{\alpha}\left(\frac{1}{1-2\alpha} - 2^{-\alpha}\right) x_2^{1-2\alpha}.$$

Proof. (a) As $0\le\omega\le 1$ on D^+, it follows from Lemma 4.2(a,b) that

$$\begin{aligned}
u_1^{bad}(x) &\le -\int_{\mathbb{R}^+\times(0,x_2)} \left(\frac{y_2-x_2}{|x-y|^{2+2\alpha}} - \frac{y_2-x_2}{|x-\tilde{y}|^{2+2\alpha}}\right)\omega(y)dy \\
&\le -\int_{\mathbb{R}^+\times(0,x_2)} \left(\frac{y_2-x_2}{|x-y|^{2+2\alpha}} - \frac{y_2-x_2}{|x-\tilde{y}|^{2+2\alpha}}\right)dy \\
&= -\int_{(0,2x_1)\times(0,x_2)} \frac{y_2-x_2}{|x-y|^{2+2\alpha}}dy.
\end{aligned}$$

The equality holds due to identity

$$\int_{\mathbb{R}^+\times(0,x_2)} \frac{y_2-x_2}{|x-\tilde{y}|^{2+2\alpha}}dy = \int_{(2x_1,\infty)\times(0,x_2)} \frac{y_2-x_2}{|x-y|^{2+2\alpha}}dy,$$

that can be verified by a change of variables $y_1 \mapsto y_1 + 2x_1$. Now, the change of variables $z := x - y$, symmetry, together with the assumption $x_2 \leq x_1$, yield

$$\begin{aligned}
u_1^{bad}(x) &\leq 2\int_{(0,x_1)\times(0,x_2)} \frac{z_2}{(z_1^2+z_2^2)^{1+\alpha}}dz \\
&= \frac{1}{\alpha}\int_0^{x_1}\left(\frac{1}{z_1^{2\alpha}} - \frac{1}{(z_1^2+x_2^2)^\alpha}\right)dz_1 \\
&\leq \frac{1}{\alpha(1-2\alpha)}x_1^{1-2\alpha} - \frac{1}{\alpha}\int_0^{x_1}\frac{1}{(2x_1^2)^\alpha}dz_1 \\
&= \frac{1}{\alpha}\left(\frac{1}{1-2\alpha} - 2^{-\alpha}\right)x_1^{1-2\alpha}.
\end{aligned} \tag{4.12}$$

The proof of part (b) is analogous to (a). $\square$

In the estimate of the "good" parts of u_1, u_2 we will in addition assume that for some $x \in D^+$ we have $\omega = 1$ on the triangle

$$A(x) := \{y \,:\, y_1 \in (x_1, x_1+1) \text{ and } y_2 \in (x_2, x_2+y_1-x_1)\}, \tag{4.13}$$

which is depicted in Figure 7. This assumption will feature in the proof of the comparison-principle-type result $K(t) \subseteq \Omega(t)$ (mentioned above) in the next sub-section.

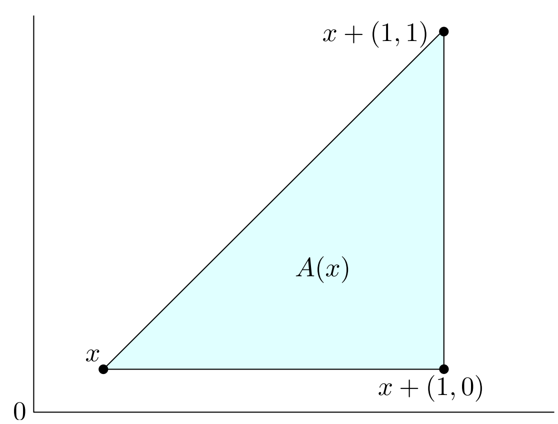

Figure 7: The domain $A(x)$.

Lemma 4.4. *Let $\alpha \in (0, \frac{1}{2})$ and assume that ω is odd in x_1 and for some $x \in \overline{D^+}$ we have $\omega \geq \chi_{A(x)}$ on D^+, with $A(x)$ from (4.13). There exists $\delta_\alpha \in (0,1)$, depending only on α, such that the following hold.*

(a) If $x_1 \leq \delta_\alpha$, then

$$u_1^{good}(x) \leq -\frac{1}{6 \cdot 20^\alpha \alpha} x_1^{1-2\alpha}.$$

(b) If $x_2 \leq \delta_\alpha$, then

$$u_2^{good}(x) \geq \frac{1}{5 \cdot 8^\alpha \alpha} x_2^{1-2\alpha}.$$

Proof. (a) Using Lemma 4.2(a) and then changing variables $y_1 \mapsto y_1 + 2x_1$, we obtain

$$\begin{aligned} u_1^{good}(x) &\leq -\int_{A(x)} \left(\frac{y_2 - x_2}{|x-y|^{2+2\alpha}} - \frac{y_2 - x_2}{|x - \tilde{y}|^{2+2\alpha}} \right) dy \\ &= -\int_{A(x)} \frac{y_2 - x_2}{|x-y|^{2+2\alpha}} dy + \int_{A(x)+2x_1 e_1} \frac{y_2 - x_2}{|x-y|^{2+2\alpha}} dy, \end{aligned} \tag{4.14}$$

with $e_1 := (1,0)$. Since the last two integrands are the same, after a cancellation due to the opposite signs we obtain

$$u_1^{good}(x) \leq -\underbrace{\int_{A_1} \frac{y_2 - x_2}{|x-y|^{2+2\alpha}} dy}_{T_1} + \underbrace{\int_{A_2} \frac{y_2 - x_2}{|x-y|^{2+2\alpha}} dy}_{T_2},$$

with the domains

$$\begin{aligned} A_1 &:= \{ y \, : \, y_2 \in (x_2, x_2 + 1) \ \text{ and } y_1 \in (x_1 + y_2 - x_2, 3x_1 + y_2 - x_2) \}, \\ A_2 &:= (x_1 + 1, 3x_1 + 1) \times (x_2, x_2 + 1) \end{aligned}$$

illustrated in Figure 8. Since for $y \in A_2$ we have $y_2 - x_2 \leq 1 \leq |x - y|$, we obtain

$$T_2 \leq |A_2| = 2x_1.$$

To control T_1, we first note that its integrand is positive, so we can get a lower bound on T_1 by only integrating over $A_1' := A_1 \cap [\mathbb{R} \times (x_2 + 2x_1, \infty)]$. For $y \in A_1'$ we have

$$y_2 - x_2 \geq \frac{1}{2}(y_1 - x_1),$$

which yields

$$\sqrt{5}(y_2 - x_2) \geq |x - y|.$$

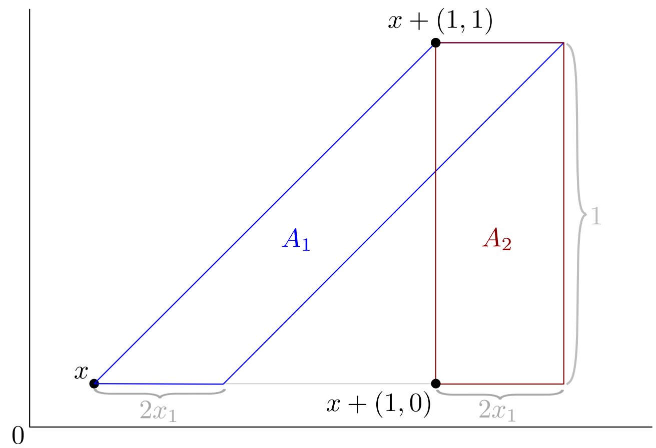

Figure 8: The domains A_1 and A_2.

This gives

$$\begin{aligned} T_1 &\geq 5^{-1-\alpha} \int_{A_1'} (y_2 - x_2)^{-(1+2\alpha)} dy \\ &= 5^{-1-\alpha} 2x_1 \int_{x_2+2x_1}^{x_2+1} (y_2 - x_2)^{-(1+2\alpha)} dy_2 \\ &= \frac{1}{5^{1+\alpha}\alpha} x_1 [(2x_1)^{-2\alpha} - 1]. \end{aligned} \tag{4.15}$$

Putting the estimates for T_1 and T_2 together yields

$$u_1^{good}(x) \leq - \left[\frac{1}{5 \cdot 20^\alpha \alpha} - \left(\frac{1}{5^{1+\alpha}\alpha} + 2 \right) x_1^{2\alpha} \right] x_1^{1-2\alpha}. \tag{4.16}$$

The result now follows for some small enough $\delta_\alpha > 0$.

(b) Using Lemma 4.2(c) and then the change of variables $y_2 \mapsto y_2 + 2x_2$, we obtain

$$\begin{aligned} u_2^{good}(x) &\geq \int_{A(x)} \left(\frac{y_1 - x_1}{|x-y|^{2+2\alpha}} - \frac{y_1 - x_1}{|x - \bar{y}|^{2+2\alpha}} \right) dy \\ &= \int_{A(x)} \frac{y_1 - x_1}{|x-y|^{2+2\alpha}} dy - \int_{A(x)+2x_2e_2} \frac{y_1 - x_1}{|x-y|^{2+2\alpha}} dy, \end{aligned} \tag{4.17}$$

with $e_2 := (0,1)$. Since the last two integrands are the same, after a cancellation due to the opposite signs we obtain

$$u_2^{good}(x) \geq \int_{B_1} \frac{y_1 - x_1}{|x-y|^{2+2\alpha}} dy - \int_{B_2} \frac{y_1 - x_1}{|x-y|^{2+2\alpha}} dy,$$

with the domains

$$B_1 := (x_1, x_1+1) \times (x_2, 3x_2)\,,$$
$$B_2 := \{y \,:\, y_1 \in (x_1, x_1+1) \text{ and } y_2 \in (x_2 + y_1 - x_1, 3x_2 + y_1 - x_1)\}$$

illustrated in Figure 9. The change of variables $y_2 \mapsto y_2 - (y_1 - x_1)$ in the second integral

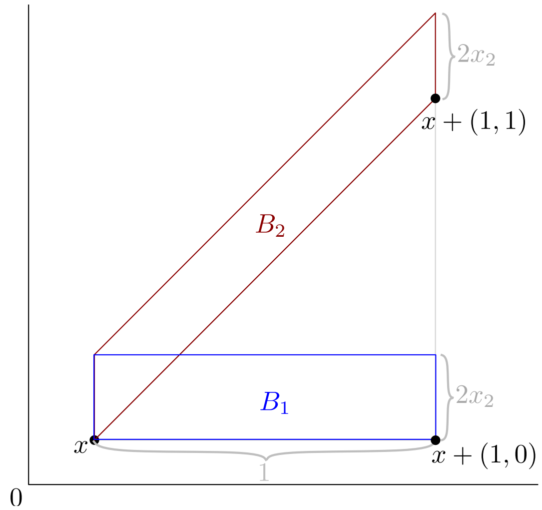

Figure 9: The domains B_1 and B_2.

then yields

$$u_2^{good}(x) \geq \int_{B_1} \left(\frac{y_1 - x_1}{|x-y|^{2+2\alpha}} - \frac{y_1 - x_1}{|x - (y_1, y_2 + y_1 - x_1)|^{2+2\alpha}} \right) dy.$$

Since the integrand is positive, and for $y \in (x_1 + 2x_2, x_1 + 1) \times (x_2, 3x_2)$ we have

$$|x - (y_1, y_2 + y_1 - x_1)|^2 = 2|x-y|^2 + (y_2 - x_2)[2(y_1 - x_1) - (y_2 - x_2)] > 2|x-y|^2$$

due to $y_1 - x_1 > 2x_2 > y_2 - x_2 > 0$, it follows that

$$u_2^{good}(x) \geq \left(1 - 2^{-1-\alpha}\right) \int_{(x_1+2x_2, x_1+1)\times(x_2,3x_2)} \frac{y_1 - x_1}{|x-y|^{2+2\alpha}} dy.$$

On this domain of integration we have $y_1 - x_1 \geq \frac{1}{\sqrt{2}}|x-y|$, so

$$\begin{aligned} u_2^{good}(x) &\geq 2^{-2-2\alpha} \int_{(x_1+2x_2, x_1+1)\times(x_2,3x_2)} (y_1 - x_1)^{-1-2\alpha} dy \\ &= 2^{-1-\alpha} x_2 \int_{x_1+2x_2}^{x_1+1} (y_1 - x_1)^{-1-2\alpha} dy_1 = \frac{1}{2^{2+\alpha}\alpha} x_2 [(2x_2)^{-2\alpha} - 1] \qquad (4.18) \\ &= \left[\frac{1}{4 \cdot 8^\alpha \alpha} - \frac{1}{2^{2+\alpha}\alpha} x_2^{2\alpha} \right] x_2^{1-2\alpha}. \end{aligned}$$

The result now follows for some small enough $\delta_\alpha > 0$. $\square$

The last two lemmas combine to the following result for small α.

Proposition 4.5. *Let $\alpha \in (0, \frac{1}{24})$ and assume that ω is odd in x_1 and for some $x \in \overline{D^+}$ we have $\chi_{A(x)} \le \omega \le 1$ on D^+, with $A(x)$ from* (4.13)*. Then there exists $\delta_\alpha \in (0,1)$, depending only on α, such that the following hold.*

(a) If $x_2 \le x_1 \le \delta_\alpha$, then

$$u_1(x) \le -\frac{1}{50\alpha} x_1^{1-2\alpha}.$$

(b) If $x_1 \le x_2 \le \delta_\alpha$, then

$$u_2(x) \ge \frac{1}{50\alpha} x_2^{1-2\alpha}.$$

Proof. (a) This is immediate from the last two lemmas and $u_1 = u_1^{bad} + u_1^{good}$ because

$$-\frac{1}{6 \cdot 20^\alpha} + \left(\frac{1}{1-2\alpha} - 2^{-\alpha} \right)$$

is increasing in α and its value for $\alpha = \frac{1}{24}$ is less than $-1/50$.

(b) Since $5 \cdot 8^\alpha < 6 \cdot 20^\alpha$, this is analogous to (a). $\square$

4.2 The finite time singularity analysis

Let us now return to the setting from the beginning of this section. The initial condition we consider is odd in x_1, and the resulting unique H^3 patch solution is also odd. We will run the blow-up argument in the class of the less regular $C^{1,\gamma}$ patch solutions to (1.1)-(1.2), and show that any such solution either has a finite maximal time of existence (i.e., loss of existence) or stops being odd (i.e., loss of uniqueness). Of course, the latter cannot happen for the H^3 patch solution.

Theorem 4.6. *Let $\alpha \in (0, \frac{1}{24})$ and $\epsilon > 0$ be small enough. Let $\omega(\cdot, 0)$ be given by* (4.1)*, with a bounded open $\Omega_0 \subseteq D^+$ such that $(2\epsilon, 3) \times (0,3) \subseteq \Omega_0 \subseteq (\epsilon, 4) \times (0,4)$ and $\partial\Omega_0$ is a smooth simple closed curve. Then for any $\gamma > \frac{2\alpha}{1-2\alpha}$, there is no odd-in-$x_1$ $C^{1,\gamma}$ patch solution ω to* (1.1)-(1.2) *on any interval $[0, T')$ with $T' > 50(3\epsilon)^{2\alpha}$.*

This immediately yields Theorem 1.3 because the (local) H^3 solution for this initial condition is odd in x_1 (due to of its uniqueness), and it is $C^{1,\gamma}$ for each $\gamma \in (0,1]$.

Proof. Let us assume that such a solution exists and let $T, X(t), K(t)$ be from (4.3)-(4.5). The solution then has the form (4.2), and we will show that $K(t) \subseteq \Omega(t)$ for each $t \in [0,T]$. This is a contradiction because then the patches $\Omega(T)$ and $\tilde{\Omega}(T)$ touch at 0.

As $|\Omega(t)| = |\Omega_0| \leq 16$, Lemma 4.1 implies

$$\|u(\cdot,t)\|_{L^\infty} \leq 100 \tag{4.19}$$

for all $t \in [0,T]$. Since $\partial\Omega(t)$ is continuous in $t \in [0,T]$ with respect to the Hausdorff distance of sets, the lemma also shows that u is continuous on $\bar{D} \times [0,T]$.

Consider $\delta_\alpha \in (0,1)$ from Proposition 4.5 and let the constant ϵ in (4.3) satisfy

$$\epsilon \leq \frac{\delta_\alpha^{1/2\alpha}}{3 \cdot 100^{1/\alpha}}.$$

We know from (4.19) that the function $f(t) := \operatorname{dist}(D^+ \setminus \overline{\Omega(t)}, K(t))$ is continuous on $[0,T]$. Hence, if $K(t)$ is not contained in $\Omega(t)$ at some $t \in [0,T]$, then there is the first time $t_0 \in [0,T]$ such that $f(t_0) = 0$. As $f(0) \geq \epsilon > 0$, we have $t_0 > 0$ and $K(t_0) \subseteq \Omega(t_0)$.

Let us assume that such t_0 exists and let

$$\Omega_3 := (\delta_\alpha, \frac{5}{2}) \times (0, \frac{5}{2}).$$

Then $T = 200^{-1}\delta_\alpha$, the estimate (4.19), and $2\epsilon < \frac{1}{2}\delta_\alpha < \frac{1}{2}$ imply

$$[D^+ \setminus \overline{\Omega(t_0)}] \cap \Omega_3 = \emptyset,$$

where we also used that symmetry and Theorem 2.3 yield

$$D^+ \setminus \overline{\Omega(t)} = \Phi_t(D^+ \setminus \overline{\Omega(0)}) \tag{4.20}$$

for any $t \in [0,T]$. As t_0 is the first time with $f(t_0) = 0$, it follows that there exists some

$$x \in \partial[D^+ \setminus \overline{\Omega(t_0)}] \cap [I_1 \cup I_2], \tag{4.21}$$

where $I_1 = \{X(t_0)\} \times [0, X(t_0))$ and I_2 is the closed straight segment connecting the points $(X(t_0), X(t_0))$ and $(\delta_\alpha, \delta_\alpha)$ (see Figure 10).

If $x \in I_1$, then the triangle $A(x)$ defined in (4.13) and depicted in Figure 7 satisfies

$$A(x) \subseteq K(t_0) \subseteq \Omega(t_0)$$

because

$$X(t_0) \leq 3\epsilon < \delta_\alpha < 1.$$

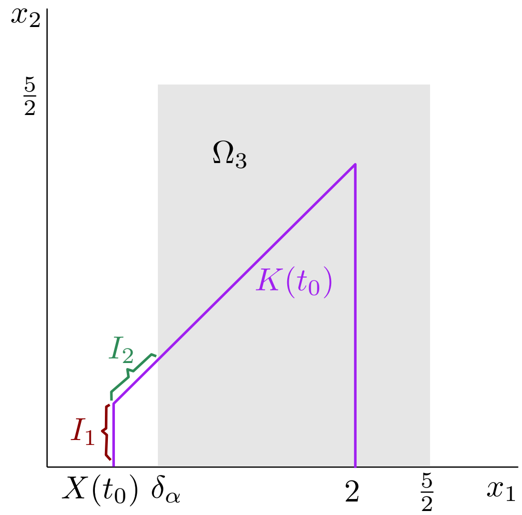

Figure 10: The segments I_1 and I_2 and the sets Ω_3 and $K(t_0)$.

Hence Proposition 4.5(a) and $x_1 = X(t_0)$ yield

$$u_1(x, t_0) \le -\frac{1}{50\alpha} x_1^{1-2\alpha} < -\frac{1}{100\alpha} x_1^{1-2\alpha} = X'(t_0).$$

Since

$$\Phi_{t_0}(D^+ \setminus \overline{\Omega(0)}) \cap B(x, r) \neq \emptyset$$

for any $r > 0$ and u is continuous, it follows from this and (4.19) that for any sufficiently small $s \in (0, \frac{1}{100}[X(t_0) - x_2])$ we have

$$\Phi_{t_0 - s}(D^+ \setminus \overline{\Omega(0)}) \cap [(X(t_0 - s), 2) \times (0, X(t_0))] \neq \emptyset.$$

From (4.20) and $(X(t_0 - s), 2) \times (0, X(t_0)) \subseteq K(t_0 - s)$ we now obtain $f(t_0 - s) = 0$ for these s, a contradiction with the choice of t_0.

If now $x \in I_2$, so that $x_1 = x_2 \le \delta_\alpha$, a similar argument and Proposition 4.5(a,b) yield

$$(-1)^{j-1} u_j(x, t_0) \le -\frac{1}{50\alpha} x_1^{1-2\alpha} < -\frac{1}{100\alpha} x_1^{1-2\alpha} \le X'(t_0)$$

for $j = 1, 2$, and thus

$$\Phi_{t_0 - s}(D^+ \setminus \overline{\Omega(0)}) \cap [(x_1 + X(t_0 - s) - X(t_0), 2) \times (0, x_1 - X(t_0 - s) + X(t_0))] \neq \emptyset$$

for all small enough $s > 0$. We again obtain a contradiction because $X(t_0) \le x_1 = x_2$ implies $(x_1 + X(t_0 - s) - X(t_0), 2) \times (0, x_1) \subseteq K(t_0 - s)$. $\square$

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