The random Schrödinger equation: homogenization in time-dependent potentials

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Abstract

We analyze the solutions of the Schrödinger equation with the low frequency initial data and a time-dependent weakly random potential. We prove a homogenization result for the low frequency component of the wave field. We also show that the dynamics generates a non-trivial energy in the high frequencies, which do not homogenize – the high frequency component of the wave field remains random and the evolution of its energy is described by a kinetic equation. The transition from the homogenization of the low frequencies to the random limit of the high frequencies is illustrated by understanding the size of the small random fluctuations of the low frequency component.

1 Introduction

We consider the Schrödinger equation

$$i\partial_t\phi(t,x)+\frac{1}{2}\Delta\phi(t,x)-\varepsilon V(t,x)\phi(t,x)=0 \tag{1.1}$$

with a low frequency initial condition of the form

$$\phi(0,x)=\phi_0(\varepsilon^\alpha x), \tag{1.2}$$

with some $\alpha>0$. Our goal is to analyze the long time behavior of $\phi(t,x)$, and understand the energy transfer from the low to high frequencies that comes about from the inhomogeneities in the random media.

We assume that $V(t,x)$ is a stationary mean-zero Gaussian random field with a spectral representation

$$V(t,x)=\int_{\mathbb{R}^d}e^{ip\cdot x}\frac{\tilde{V}(t,dp)}{(2\pi)^d}. \tag{1.3}$$

Its covariance function and power spectrum are

$$R(t,x)=\mathbb{E}\{V(s,y)V(s+t,y+x)\},\quad \hat{R}(\omega,\xi)=\int_{\mathbb{R}^{d+1}}R(t,x)e^{-i\omega t-i\xi\cdot x}dtdx.$$

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The spatial power spectrum (the Fourier transform of $R(t,x)$ in x only) has the form

$$\tilde{R}(t,\xi) = \int_{\mathbb{R}^d} R(t,x) e^{-i\xi\cdot x} dx = e^{-\mathfrak{g}(\xi)|t|} \hat{R}(\xi), \tag{1.4}$$

where $\hat{R}(\xi) \in L^1(\mathbb{R}^d)$, and the spectral gap $\mathfrak{g}(\xi) \geq 0$, so that

$$\hat{R}(\omega,\xi) = \frac{2\mathfrak{g}(\xi)\hat{R}(\xi)}{\omega^2 + \mathfrak{g}^2(\xi)}. \tag{1.5}$$

Throughout the paper, we assume that

$$\frac{\hat{R}(p)}{\mathfrak{g}(p)} \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d). \tag{1.6}$$

The compensated wave function

The standard approach to an understanding of the behavior of the solutions of the weakly random Schrödinger equation is in the context of the kinetic limit [6, 7, 3, 12, 8, 11, 5], through the study of the Wigner transform of the solution (the phase space resolved energy density) [9]. Our work here is closer in spirit to [4, 10] that focused not on the weak limit of the energy density of the solution but on the strong limit of the wave field itself. In order to motivate the "correct" way to this end, let us mention that after a long time the phase of the wave field acquires a large factor: for instance, setting $V = 0$ in (1.1) leads to an explicit expression

$$\hat{\phi}(t,\xi) = e^{-i|\xi|^2 t/2} \hat{\phi}(0,\xi)$$

for the Fourier transform of the solution. The Fourier transform is defined as

$$\hat{f}(\xi) = \int_{\mathbb{R}^d} e^{-i\xi\cdot x} f(x) dx.$$

Thus, a convenient object in the context of long time behaviors is the compensated wave function

$$\hat{\psi}(t,\xi) = e^{i|\xi|^2 t/2} \hat{\phi}(t,\xi), \tag{1.7}$$

which eliminates the deterministic component of the phase. This procedure is also known as phase conjugation in the engineering and physical literature. The surprising miracle is that after this simple-minded phase compensation, the wave field has a non-trivial limit.

Loose end #1: the high frequency initial data

We first describe the results of [4] obtained when the initial data for (1.1) is not slowly varying:

$$\phi(0,x) = \phi_0(x),$$

that is, $\alpha = 0$ in (1.2). Let us set

$$D(p,\xi) = \frac{2\hat{R}(p)}{(2\pi)^d[\mathfrak{g}(p) - i(|\xi|^2 - |\xi - p|^2)/2]}, \qquad D(\xi) = \int_{\mathbb{R}^d} D(p,\xi) dp. \tag{1.8}$$

It is straightforward to check that

$$\mathrm{Re}D(p,\xi)=\frac{2\hat{R}(p)\mathfrak{g}(p)}{(2\pi)^d[\mathfrak{g}^2(p)+(|\xi|^2-|\xi-p|^2)^2/4]}=\frac{1}{(2\pi)^d}\hat{R}(\frac{|\xi|^2-|\xi-p|^2}{2},p). \tag{1.9}$$

One of the results of [4] is that if

$$\frac{\hat{R}(p)}{\mathfrak{g}(p)}\in L^1(\mathbb{R}^d),$$

then on the time scale $t\sim\varepsilon^{-2}$, the compensated wave function corresponding to the initial data with $\alpha=0$ converges pointwise in distribution to a Gaussian random variable:

$$\hat{\phi}(\frac{t}{\varepsilon^2},\xi)e^{\frac{i|\xi|^2t}{2\varepsilon^2}}\Rightarrow\hat{\phi}_0(\xi)e^{-\frac{1}{2}D(\xi)t}+Z(t,\xi). \tag{1.10}$$

Here, $Z(t,\xi)$ is a centered, complex valued Gaussian with the variance

$$\mathbb{E}\{|Z(t,\xi)|^2\}=\widehat{W}(t,\xi)-|\hat{\phi}_0(\xi)|^2e^{-\mathrm{Re}D(\xi)t}. \tag{1.11}$$

The function $\widehat{W}$ solves a (space-homogeneous) kinetic equation

$$\partial_t\widehat{W}=\int_{\mathbb{R}^d}\hat{R}(\frac{|p|^2-|\xi|^2}{2},p-\xi)(\widehat{W}(t,p)-\widehat{W}(t,\xi))\frac{dp}{(2\pi)^d}, \tag{1.12}$$

with the initial condition

$$\widehat{W}(0,\xi)=|\hat{\phi}_0(\xi)|^2.$$

This result is consistent with the aforementioned "traditional" kinetic equation approaches.

Loose end #2: homogenization of the very low frequencies

The results in the high frequency regime ($\alpha=0$) should be contrasted with the analysis of Bal and Zhang in [13, 14] for the case $\alpha=1$ in (1.2), performed for time-independent potentials. For the initial value problem

$$\begin{aligned}&i\phi_t+\frac{1}{2}\Delta\phi-\varepsilon V(x)\phi=0,\\&\phi(0,x)=\phi_0(\varepsilon x),\end{aligned} \tag{1.13}$$

with a mean-zero Gaussian random potential $V(x)$, they have established a homogenization result:

$$\phi^\varepsilon(t,x)=\phi\Big(\frac{t}{\varepsilon^2},\frac{x}{\varepsilon}\Big)$$

converges in probability, as $\varepsilon\to0$ to a deterministic limit $\bar{\phi}(t,x)$, which satisfies the Schrödinger equation

$$\begin{aligned}&i\bar{\phi}_t+\frac{1}{2}\Delta\bar{\phi}-\bar{V}\bar{\phi}=0,\\&\bar{\phi}(0,x)=\phi_0(x).\end{aligned} \tag{1.14}$$

The effective potential is constant and is given by

$$\bar{V} = \int_{\mathbb{R}^d} \frac{\hat{R}(p)dp}{|p|^2}.$$

Let us mention that the choice $\alpha = 1$ is special, as then the overall phase of the solution at the times $t \sim \varepsilon^{-2}$ is

$$\frac{t}{\varepsilon^2}\varepsilon^2|\xi|^2 = O(1),$$

so that no phase compensation is needed.

Homogenization of the low frequencies

Summarizing the above results, while solutions of (1.1) with the high frequency initial data have a random limit on the time scale $t \sim \varepsilon^{-2}$, as in (1.10), solutions with the "very slowly varying" initial data as in (1.13) are homogenized on this time scale – their limit is deterministic. The first goal of this paper is to understand where the transition between the two regimes occurs – this is the motivation for introducing a general $\alpha > 0$ in (1.2). It will turn out that the homogenization result (formulated for the compensated wave function) holds for all $\alpha > 0$ – that is, no matter how "relatively high" the low frequency of the initial condition is, solution has a deterministic limit at times $t \sim \varepsilon^{-2}$. However, we will see that, unlike in the setting of [13, 14], the temporal fluctuations of the random potential lead to an effective potential with a non-trivial imaginary part. This means that the homogenized field loses mass in the limit. This loss of mass is attributed to the energy transfer to the high frequencies, which, as we show, account for the mass missing in the low frequencies, do not homogenize, and satisfy a kinetic type limit. We also analyze the random fluctuations of the low frequency component of the wave field and characterize the corrector to the homogenized limit.

More precisely, we consider the Schrödinger equation

$$i\partial_t\phi(t,x) + \frac{1}{2}\Delta\phi(t,x) - \varepsilon V(t,x)\phi(t,x) = 0 \tag{1.15}$$

with a low frequency initial condition

$$\phi(0,x) = \phi_0(\kappa x), \tag{1.16}$$

with $\kappa \ll 1$. The Fourier transform of the initial condition is

$$\hat{\phi}(0,\xi) = \kappa^{-d}\hat{\phi}_0\big(\frac{\xi}{\kappa}\big).$$

Thus, if the function $\hat{\phi}_0(\xi)$ is of the Schwartz class, $\hat{\phi}(0,\xi)$ is concentrated on the wave vectors ξ of the size $O(\kappa)$. While the Schrödinger equation with a time-dependent potential conserves the total mass:

$$M(t) = \int_{\mathbb{R}^d} |\phi(t,x)|^2 dx = \int_{\mathbb{R}^d} |\phi(0,x)|^2 dx, \tag{1.17}$$

the total energy

$$E(t) = \int_{\mathbb{R}^d} [|\nabla\phi|^2 + \varepsilon V|\phi|^2]dx \tag{1.18}$$

is not conserved, unlike for time-independent potentials. Thus, even if the mass is initially concentrated in the low wave numbers, after a long time evolution it may spread to $O(1)$ frequencies as well. As the potential is weak, the time it takes for the mass to spread over a range of frequencies will be long.

We consider the long time behavior of the solution, on the time scale of the order $t \sim \varepsilon^{-2}$, when the effect of the weak random potential will be non-trivial. We will first consider the "low frequency" rescaled compensated wave function:

$$\psi_\varepsilon(t,\xi) = \kappa^d \hat{\phi}(\frac{t}{\varepsilon^2}, \kappa\xi) e^{\frac{i\kappa^2|\xi|^2 t}{2\varepsilon^2}} \tag{1.19}$$

with the initial data $\psi_\varepsilon(0,\xi) = \hat{\phi}_0(\xi)$. This allows us to study the low frequency component of the solution – wave numbers of the order $O(\kappa)$. A straightforward computation shows that this function is a solution of the following integral equation

$$\psi_\varepsilon(t,\xi) = \hat{\phi}_0(\xi) + \frac{1}{i\varepsilon}\int_0^t \int_{\mathbb{R}^d} \frac{\tilde{V}(\frac{s}{\varepsilon^2}, dp)}{(2\pi)^d} e^{i\kappa^2(|\xi|^2 - |\xi - \frac{p}{\kappa}|^2)\frac{s}{2\varepsilon^2}} \psi_\varepsilon(s, \xi - \frac{p}{\kappa}) ds. \tag{1.20}$$

We have the following result for the low frequencies.

Theorem 1.1. *Assume that $\kappa = \varepsilon^\alpha$ with $\alpha > 0$. Then, for fixed $t > 0$ and $\xi \in \mathbb{R}^d$,*

$$\psi_\varepsilon(t,\xi) \to \bar{\psi}(t,\xi) = \hat{\phi}_0(\xi) e^{-\frac{1}{2}D(0)t} \text{ in probability as } \varepsilon \to 0. \tag{1.21}$$

Let us stress that $\xi = O(1)$ in the argument of the function $\psi_\varepsilon(t,\xi)$ corresponds to $\xi = O(\kappa)$ in the argument of the function ϕ – Theorem 1.1 addresses the evolution of the low frequencies of the solution of the Schrödinger equation with a slowly varying initial condition. Recall that

$$D(0) = \int_{\mathbb{R}^d} \frac{2\hat{R}(p)}{(2\pi)^d(\mathfrak{g}(p) + i|p|^2/2)} dp, \tag{1.22}$$

and, as $\mathfrak{g}(p) \geq 0$, we have $\mathrm{Re}D(0) > 0$. Therefore, the passage to limit $\varepsilon \to 0$ in (1.21) induces a loss of the $L^2(\mathbb{R}^d)$ norm: while

$$\|\psi_\varepsilon(t,\cdot)\|_{L^2} = \|\phi_0\|_{L^2},$$

as can be seen simply from the definition of $\psi_\varepsilon(t,\xi)$, we have

$$\|\bar{\psi}(t,\cdot)\|_{L^2} = \|\phi_0\|_{L^2} e^{-\frac{1}{2}D(0)t} < \|\phi_0\|_{L^2}.$$

The natural question is how does the loss of mass happen, and where does the mass go? Mathematically, there is no contradiction, as we will show the convergence in Theorem 1.1 is not uniform with respect to $\xi \in \mathbb{R}^d$. From a physical point of view, as we have mentioned, the time dependence of the random potential breaks the conservation of the energy (1.18), which allows the mass to escape to the high frequencies. Let us mention that in the *time-independent* case [2], where the conservation of the energy prevents the escape of mass from the low frequencies, it is shown that the mass is conserved as well.

Generation of the high frequencies

We now investigate the generation of the high frequencies in the above setting. Once again, we consider the solution $\phi(t,x)$ of (1.15) with the initial data (1.16). We stress that in all our results the initial condition (1.16) is the same – various rescalings in Theorem 1.1 above and Theorems 1.2, 1.3 and 1.4 below correspond to zooming into various frequency ranges in the same solution. Our next goal is to understand how the mass escapes from the low frequencies (those of the initial condition) to the high frequencies, generated by the interaction with the random potential. As we are now interested in the high and not the low frequencies, we define the compensated wave function not quite as in (1.19), but as

$$\Psi_\varepsilon(t,\xi) = \kappa^{\frac{d}{2}}\hat{\phi}(\frac{t}{\varepsilon^2},\xi)e^{\frac{i|\xi|^2 t}{2\varepsilon^2}}, \tag{1.23}$$

so that the frequency is not rescaled. The initial condition for Ψ_ε is

$$\Psi_\varepsilon(0,\xi) = \kappa^{-d/2}\hat{\phi}_0(\xi/\kappa).$$

The pre-factor $\kappa^{d/2}$ in (1.23) is chosen so that we get a non-trivial limit. This function solves the integral equation

$$\Psi_\varepsilon(t,\xi) = \frac{1}{\kappa^{d/2}}\hat{\phi}_0(\frac{\xi}{\kappa}) + \frac{1}{i\varepsilon}\int_0^t\int_{\mathbb{R}^d}\frac{\tilde{V}(\frac{s}{\varepsilon^2},dp)}{(2\pi)^d}e^{i(|\xi|^2-|\xi-p|^2)\frac{s}{2\varepsilon^2}}\Psi_\varepsilon(s,\xi-p)ds. \tag{1.24}$$

The following result explains the loss of mass observed in Theorem 1.1, and tracks the generation of the high frequencies.

Theorem 1.2. *Assume that $\kappa=\varepsilon^\alpha$ with $\alpha>0$, then for fixed $t>0$ and $\xi\neq 0$, we have*

$$\Psi_\varepsilon(t,\xi)\Rightarrow \bar{Z}(t,\xi) \text{ in law as } \varepsilon\to 0,$$

where $\bar{Z}(t,\xi)$ is a centered, complex valued Gaussian random variable. Its variance $\widehat{W}_\delta(t,\xi)$ is the solution of (1.12) with the initial condition $\widehat{W}_\delta(0,\xi)=\|\hat{\phi}_0\|_2^2\delta(\xi)$.

The variance $\widehat{W}_\delta(t,\xi)$ can be explicitly written as a series expansion

$$\widehat{W}_\delta(t,\xi) = \widehat{W}_{\delta,b}(t,\xi) + \widehat{W}_{\delta,s}(t,\xi), \tag{1.25}$$

with the ballistic part

$$\widehat{W}_{\delta,b}(t,\xi) = \|\hat{\phi}_0\|^2 e^{-\mathrm{Re}D(0)t}\delta(\xi),$$

and the scattering part

$$\widehat{W}_{\delta,s}(t,\xi) = \sum_{k=1}^{\infty}\|\hat{\phi}_0\|_2^2\int_{0=v_{k+1}\le v_k\le\ldots\le v_1\le v_0=t}dv\int_{\mathbb{R}^{kd}}dP\prod_{j=0}^{k}e^{-(v_j-v_{j+1})\mathrm{Re}D(\xi-\ldots-P_j)}$$
$$\times\prod_{j=1}^{k}\mathrm{Re}D(P_j,\xi-\ldots-P_{j-1})\delta(\xi-P_1-\ldots-P_k).$$

Let us mention that $\widehat{W}_\delta(t,\xi)=\widehat{W}_{\delta,s}(t,\xi)$ when $\xi\neq 0$, that is, only the scattering part contributes to the variance in Theorem 1.2. We also observe

$$\int_{\mathbb{R}^d}\widehat{W}_{\delta,b}(t,\xi)d\xi=\|\hat{\phi}_0\|^2e^{-\mathrm{Re}D(0)t},$$

which equals to the mass lost in the low frequencies.

Theorems 1.1 and 1.2 describe the dynamics of (1.1) on different scales of the frequency domain. In the former case, the low frequencies are zoomed in, and we find a deterministic evolution (homogenzation). In the latter, we track the high frequency component of the solution, so that the low frequency initial condition shrinks to a point source at the origin, which generates the high frequency waves.

The fluctuation analysis in homogenization regime

We now return to the analysis of the behavior of the low frequencies. According to Theorem 1.1, the compensated wave function homogenizes for the low frequencies, hence the next interesting object is the fluctuation, which we define as

$$\mathcal{U}_\varepsilon(t,\xi)=\frac{1}{\kappa^{d/2}}(\psi_\varepsilon(t,\xi)-\mathbb{E}\{\psi_\varepsilon(t,\xi)\}).$$

Here, $\psi_\varepsilon(t,\xi)$ is defined as in (1.19). Heuristically, since the homogenization limit in Theorem 1.1 captures the ballistic component of the wave field, we expect small random fluctuations consisting of the remaining scattering components. Indeed, we will see that the fluctuation exhibits a kinetic-like behavior. Let us set

$$\mathcal{W}_\alpha(t,\xi)=\begin{cases}0 & \text{if } \alpha\in(0,1),\\ -D(0,0)e^{-D(0)t}\displaystyle\int_0^t\int_{\mathbb{R}^d}\hat{\phi}_0(\xi-p)\hat{\phi}_0(\xi+p)e^{-i|p|^2v}dpdv & \text{if } \alpha=1,\\ -D(0,0)te^{-D(0)t}\displaystyle\int_{\mathbb{R}^d}\hat{\phi}_0(\xi-p)\hat{\phi}_0(\xi+p)dp & \text{if } \alpha>1.\end{cases}\tag{1.26}$$

Theorem 1.3. *Assume that $\kappa=\varepsilon^\alpha$, then for fixed $t>0$ and $\xi\in\mathbb{R}^d$, we have*

$$\mathcal{U}_\varepsilon(t,\xi)\Rightarrow Z_\delta(t,\xi)=X_\delta(t,\xi)+iY_\delta(t,\xi)\ \text{as } \varepsilon\to 0,$$

where X_δ,Y_δ are centered, jointly Gaussian random variables such that

$$\mathbb{E}\{|Z_\delta(t,\xi)|^2\}=\widehat{W}_{\delta,s}(t,0),$$

and

$$\mathbb{E}\{Z_\delta(t,\xi)^2\}=\mathcal{W}_\alpha(t,\xi).$$

Therefore, we can write

$$\psi_\varepsilon(t,\xi)=\mathbb{E}\{\psi_\varepsilon(t,\xi)\}+\kappa^{d/2}\mathcal{U}_\varepsilon(t,\xi),$$

and Theorem 1.3 shows that when $\kappa = \varepsilon^\alpha$, with $\alpha < 1$, the fluctuation $\mathcal{U}_\varepsilon(t,\xi)$ is approximately distributed as $Z_\delta(t,0)$, a centered complex Gaussian random variable with variance $\widehat{W}_{\delta,s}(t,0)$. This is similar to the result of Theorem 1.2 for the high frequency, albeit the variance is now given by the transport solution evaluated at the origin $\xi = 0$, since we are now in the low frequency regime. If we let $\alpha \to 0$ (which is the same as $\kappa \to 1$, so that the initial condition is less and less slowly varying), then, formally, $\psi_\varepsilon(t,\xi)$ is distributed as

$$\hat{\phi}_0(\xi)e^{-\frac{1}{2}D(0)t} + Z_\delta(t,0),$$

which is consistent with (1.10). That is, Theorem 1.3 also interpolates between the deterministic limit for the low frequencies and the random behavior of the high frequency component of the solution.

The limit of the Wigner transform

Besides the pointwise fluctuation for a fixed $\xi \in \mathbb{R}^d$, we also consider the fluctuation of $\psi_\varepsilon(t,\xi)$ as a wave field. The tool we use is the Wigner transform for some $\beta \geq 0$:

$$W_\varepsilon(t,x,\xi) = \int_{\mathbb{R}^d} \mathcal{U}_\varepsilon(t,\xi + \frac{\varepsilon^\beta \eta}{2})\mathcal{U}_\varepsilon^*(t,\xi - \frac{\varepsilon^\beta \eta}{2})e^{i\eta\cdot x}\frac{d\eta}{(2\pi)^d}. \tag{1.27}$$

Let $\bar{W}_\delta$ be the solution to the kinetic equation

$$\partial_t \bar{W} + \xi\cdot\nabla_x \bar{W} = \int_{\mathbb{R}^d} \hat{R}(\frac{|p|^2-|\xi|^2}{2}, p-\xi)(\bar{W}(t,x,p) - \bar{W}(t,x,\xi))\frac{dp}{(2\pi)^d}, \tag{1.28}$$

with the initial condition

$$\bar{W}_\delta(0,x,\xi) = \|\hat{\phi}_0\|_2^2\delta(\xi)\delta(x),$$

and $\bar{W}_{\delta,b}, \bar{W}_{\delta,s}$ be the ballistic and scattering component of $\bar{W}_\delta$, respectively:

$$\bar{W}_{\delta,b}(t,x,\xi) = \|\hat{\phi}_0\|_2^2\delta(\xi)\delta(x)e^{-\mathrm{Re}D(0)t},$$

and

$$\begin{aligned}\bar{W}_{\delta,s}(t,x,\xi) = \sum_{k=1}^\infty \|\hat{\phi}_0\|_2^2 \int_{0=v_{k+1}\leq v_k\leq\ldots\leq v_1\leq v_0=t} dv \int_{\mathbb{R}^{kd}} dP \prod_{j=0}^k e^{-(v_j-v_{j+1})\mathrm{Re}D(\xi-\ldots-P_j)} \\ \times \prod_{j=1}^k \mathrm{Re}D(P_j,\xi-\ldots-P_{j-1})\delta(\xi-P_1-\ldots-P_k)\delta(x-\xi t+\sum_{j=1}^k P_j v_j).\end{aligned}$$

Theorem 1.4. *Assume that $\kappa = \varepsilon^\alpha$, $\alpha \in (0,1)$ and $\alpha+\beta = 2$, then for any test function $\varphi \in \mathcal{S}(\mathbb{R}^{2d})$ and $t > 0$,*

$$\int_{\mathbb{R}^{2d}} W_\varepsilon(t,x,\xi)\varphi^*(x,\xi)dxd\xi \to \int_{\mathbb{R}^{2d}} \bar{W}_{\delta,s}(t,x,0)\varphi^*(x,\xi)dxd\xi$$

in probability as $\varepsilon \to 0$.

As Theorem 1.1 indicates that the ballistic component of transport solution gives the low frequency behavior, we conclude from Theorems 1.3 and 1.4 that the small random fluctuations are described by the scattering component of the solution of the kinetic equation.

This paper is organized as follows. First, in Section 2 we present the Duhamel expansion and the corresponding diagrammatic expansions and the moment estimates that are needed for the proofs of all theorems. Section 3 contains the proof of Theorem 1.1. Theorem 1.2 is proved in Section 4. Finally, Theorems 1.3 and 1.4 are proved in Section 5.

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2 The Duhamel expansion and the moment estimates

Theorems 1.1, 1.2, 1.3 and 1.4 are all proved using the moment method. For the convergence

$$\psi_\varepsilon(t,\xi) \to \hat{\phi}_0(\xi) e^{-\frac{1}{2}D(0)t},$$

in probability (Theorem 1.1), it suffices to show the convergence of $\mathbb{E}\{\psi_\varepsilon(t,\xi)\}$ and $\mathbb{E}\{|\psi_\varepsilon(t,\xi)|^2\}$ to their respective limits. For the convergence in law of $\Psi_\varepsilon(t,\xi)$ and $\mathcal{U}_\varepsilon(t,\xi)$ to a Gaussian in Theorems 1.2 and 1.3, respectively, we need to show the convergence of the corresponding moments $\mathbb{E}\{\Psi_\varepsilon(t,\xi)^M(\Psi_\varepsilon^*(t,\xi))^N\}$ and $\mathbb{E}\{\mathcal{U}_\varepsilon(t,\xi)^M(\mathcal{U}_\varepsilon^*(t,\xi))^N\}$ for any $M,N\in\mathbb{N}$ to their respective limits, which makes the analysis slightly more computationally heavy. In this section, we perform the preliminary moment estimates that are needed in the proofs of the theorems.

The Duhamel expansions

All moment estimates rely on the Duhamel expansions that we now recall. From now on, we will set $\kappa = \varepsilon^\alpha$. For the low frequencies, we can iterate the integral equation (1.20) for the function $\psi_\varepsilon(t,\xi)$, and write the solution as a series

$$\psi_\varepsilon(t,\xi) = \sum_{n=0}^{\infty} f_{n,\varepsilon}(t,\xi), \tag{2.1}$$

with the individual terms

$$f_{n,\varepsilon}(t,\xi) = \frac{1}{(i\varepsilon)^n}\int_{\Delta_n(t)}\int_{\mathbb{R}^{nd}}\prod_{j=1}^n \frac{\tilde{V}(\frac{s_j}{\varepsilon^2},dp_j)}{(2\pi)^d} e^{iG_n(\varepsilon^\alpha\xi,s^{(n)},p^{(n)})/\varepsilon^2}\hat{\phi}_0(\xi-\frac{p_1+\ldots+p_n}{\varepsilon^\alpha}), \tag{2.2}$$

and the phase factor

$$G_n(\xi,s^{(n)},p^{(n)}) = \sum_{k=1}^n (|\xi-p_1-\ldots-p_{k-1}|^2 - |\xi-p_1-\ldots-p_k|^2)\frac{s_k}{2}. \tag{2.3}$$

Here, we used the convention $f_{0,\varepsilon}(t,\xi)=\hat{\phi}_0(\xi)$, and have set $p_0=0$, $p^{(n)}=(p_1,\ldots,p_n)$, as well as $s^{(n)}=(s_1,\ldots,s_n)$. We have also defined the time simplex

$$\Delta_n(t) = \{0\le s_n\le\ldots\le s_1\le t\}.$$

For the high frequencies, the solution $\Psi_\varepsilon(t,\xi)$ to (1.24) is similarly written as

$$\Psi_\varepsilon(t,\xi) = \sum_{n=0}^{\infty} F_{n,\varepsilon}(t,\xi) \tag{2.4}$$

with

$$F_{n,\varepsilon}(t,\xi) = \frac{1}{(i\varepsilon)^n}\int_{\Delta_n(t)}\int_{\mathbb{R}^{nd}}\prod_{j=1}^{n}\frac{\tilde V(\frac{s_j}{\varepsilon^2},dp_j)}{(2\pi)^d}e^{iG_n(\xi,s^{(n)},p^{(n)})/\varepsilon^2}\frac{1}{\varepsilon^{\alpha d/2}}\hat\phi_0(\frac{\xi-p_1-\ldots-p_n}{\varepsilon^\alpha}) \tag{2.5}$$

and

$$F_{0,\varepsilon} = \frac{1}{\varepsilon^{\alpha d/2}}\hat\phi_0(\frac{\xi}{\varepsilon^\alpha}).$$

The key “bureaucratic” difference between the Duhamel expansions (2.2) and (2.5) for the functions $\psi_\varepsilon(t,\xi)$ and $\Psi_\varepsilon(t,\xi)$ is that $\varepsilon^\alpha\xi\mapsto\xi$. This will make the limits very different.

The following lemma ensures that the solutions given by (2.1) and (2.4) are well-defined and we can interchange the summation and the expectation when computing the moments. Its proof is exactly as that of [4, Proposition 3.8].

Lemma 2.1. *Fix $\varepsilon>0, M,N\in\mathbb{N}$. Let $g_{n,\varepsilon}=f_{n,\varepsilon}$ or $F_{n,\varepsilon}$, then*

$$|\mathbb{E}\{g_{m_1,\varepsilon}\ldots g_{m_M,\varepsilon}g^*_{n_1,\varepsilon}\ldots g^*_{n_N,\varepsilon}\}| \le C_\varepsilon(m_1,\ldots,m_M,n_1,\ldots,n_N) \tag{2.6}$$

with

$$\sum_{m_1,\ldots,m_M=0}^{\infty}\sum_{n_1,\ldots,n_N=0}^{\infty} C_\varepsilon(m_1,\ldots,m_M,n_1,\ldots,n_N)<\infty.$$

The pairings

Now, we discuss in detail the calculation of the moments

$$\mathbb{E}\{g_{m_1,\varepsilon}\ldots g_{m_M,\varepsilon}g^*_{n_1,\varepsilon}\ldots g^*_{n_N,\varepsilon}\},$$

where $g_{n,\varepsilon}=f_{n,\varepsilon}$ or $F_{n,\varepsilon}$, and

$$\sum_{i=1}^{M} m_i+\sum_{j=1}^{N} n_j = 2k,$$

for some $k\in\mathbb{N}$ (if the sum is odd, then the moment is zero by the Gaussian property). We have

$$\begin{aligned}\mathbb{E}\{g_{m_1,\varepsilon}\ldots g_{m_M,\varepsilon}g^*_{n_1,\varepsilon}\ldots g^*_{n_N,\varepsilon}\} &= (i\varepsilon)^{-\sum_{i=1}^M m_i}(-i\varepsilon)^{-\sum_{j=1}^N n_j}\\ &\times\int_{\Delta_{m_1}(t)\times\ldots\times\Delta_{n_N}(t)} dsdu\int_{\mathbb{R}^{2kd}}\mathbb{E}\{I_{M,N}\}e^{i\mathcal{G}_M}e^{-i\mathcal{G}_N}\prod_{i=1}^{M}h_{M,i}\prod_{j=1}^{N}h^*_{N,j},\end{aligned} \tag{2.7}$$

with

$$I_{M,N} = \frac{1}{(2\pi)^{2kd}} \tilde{V}(\frac{s_{1,1}}{\varepsilon^2}, dp_{1,1}) \ldots \tilde{V}(\frac{s_{1,m_1}}{\varepsilon^2}, dp_{1,m_1}) \ldots \tilde{V}(\frac{s_{M,1}}{\varepsilon^2}, dp_{M,1}) \ldots \tilde{V}(\frac{s_{M,m_M}}{\varepsilon^2}, dp_{M,m_M})$$
$$\times \tilde{V}^*(\frac{u_{1,1}}{\varepsilon^2}, dq_{1,1}) \ldots \tilde{V}^*(\frac{u_{1,n_1}}{\varepsilon^2}, dq_{1,n_1}) \ldots \tilde{V}^*(\frac{u_{N,1}}{\varepsilon^2}, dq_{N,1}) \ldots \tilde{V}^*(\frac{u_{N,n_N}}{\varepsilon^2}, dq_{N,n_N}),$$

and the phases

$$\mathcal{G}_M = \sum_{i=1}^{M} G_{m_i}(\eta, s_i^{(m_i)}, p_i^{(m_i)})/\varepsilon^2, \quad \mathcal{G}_N = \sum_{i=1}^{N} G_{n_i}(\eta, u_i^{(n_i)}, q_i^{(n_i)})/\varepsilon^2,$$

with $\eta = \varepsilon^\alpha \xi$ or ξ, depending on whether $g_{n,\varepsilon} = f_{n,\varepsilon}$ or $F_{n,\varepsilon}$. The initial conditions appear as

$$h_{M,i} = \hat{\phi}_0(\xi - \frac{p_{i,1} + \ldots + p_{i,m_i}}{\varepsilon^\alpha}), \quad h^*_{N,j} = \hat{\phi}^*_0(\xi - \frac{q_{j,1} + \ldots + q_{j,n_j}}{\varepsilon^\alpha})$$

when $g_{n,\varepsilon} = f_{n,\varepsilon}$, and as

$$h_{M,i} = \varepsilon^{-\alpha d/2} \hat{\phi}_0(\frac{\xi - p_{i,1} - \ldots - p_{i,m_i}}{\varepsilon^\alpha}), \quad h^*_{N,j} = \varepsilon^{-\alpha d/2} \hat{\phi}^*_0(\frac{\xi - q_{j,1} - \ldots - q_{j,n_j}}{\varepsilon^\alpha}),$$

when $g_{n,\varepsilon} = F_{n,\varepsilon}$.

Using the rules of computing the $2k-$th joint moment of mean zero Gaussian random variables, we obtain

$$\mathbb{E}\{I_{M,N}\} = \sum_{\mathcal{F}} \prod_{(v_l, v_r) \in \mathcal{F}} (2\pi)^{-d} e^{-\mathfrak{g}(w_l)|v_l - v_r|/\varepsilon^2} \delta(w_l + w_r) \hat{R}(w_l) dw_l dw_r. \tag{2.8}$$

The summation $\sum_{\mathcal{F}}$ extends over all pairings $\mathcal{F}$ formed over the vertices

$$\{s_{1,1}, \ldots, s_{1,m_1}, \ldots, s_{M,1}, \ldots, s_{M,m_M} u_{1,1}, \ldots, u_{1,n_1}, \ldots, u_{N,1}, \ldots, u_{N,n_N}\}.$$

In (2.8), v_l, v_r are the two vertices of a given pair, and w_l, w_r are the respective p, q variables, that is, $w_l = p_{i,j}$ if $v_l = s_{i,j}$ and $w_l = -q_{i,j}$ if $v_l = u_{i,j}$. The same holds for w_r. We will also write a pair as an edge $e = (v_l, v_r)$. Note that the order of v_l, v_r does not matter here since $\mathfrak{g}, \hat{R}$ are both even.

A uniform bound on the pairings

We recall the following general bound.

Lemma 2.2. *Let $g_{n,\varepsilon} = f_{n,\varepsilon}$ or $F_{n,\varepsilon}$, then we have, for all $\varepsilon \in (0,1]$,*

$$|\mathbb{E}\{g_{m_1,\varepsilon} \ldots g_{m_M,\varepsilon} g^*_{n_1,\varepsilon} \ldots g^*_{n_N,\varepsilon}\}| \le \frac{(2k-1)!!}{\prod_{i=1}^M (m_i)! \prod_{j=1}^N (n_j)!} C^k \tag{2.9}$$

with some constant C depending on $t, \xi, \hat{R}, \mathfrak{g}$.

Proof. The proof is close to the case $g = f_{n,\varepsilon}$ and $\alpha = 0$ which is already contained in [4]. We present it, together with the required modifications, for the convenience of the reader. By symmetry, the RHS of (2.7) can be bounded by

$$\frac{1}{\prod_{i=1}^{M}(m_i)!\prod_{j=1}^{N}(n_j)!}\frac{1}{\varepsilon^{2k}}\int_{[0,t]^{2k}} dsdu \int_{\mathbb{R}^{2kd}} |\mathbb{E}\{I_{M,N}\}| \prod_{i=1}^{M}|h_{M,i}|\prod_{j=1}^{N}|h^*_{N,j}|. \tag{2.10}$$

In the case when $g_{n,\varepsilon} = f_{n,\varepsilon}$, we bound

$$\prod_{i=1}^{M}|h_{M,i}|\prod_{j=1}^{N}|h^*_{N,j}| \leq \|\hat{\phi}_0\|_\infty^{M+N},$$

then for a given pairing $\mathcal{F}$, we have

$$\frac{1}{\varepsilon^{2k}}\int_{[0,t]^{2k}} dsdu \int_{\mathbb{R}^{2kd}} \prod_{(v_l,v_r)\in\mathcal{F}} (2\pi)^{-d} e^{-\mathfrak{g}(w_l)|v_l-v_r|/\varepsilon^2}\delta(w_l+w_r)\hat{R}(w_l)dw_l dw_r \leq C^k,$$

where we used the integrability of $\hat{R}(p)/\mathfrak{g}(p)$. Thus, (2.10) can be bounded by

$$\frac{\#(\mathcal{F})}{\prod_{i=1}^{M}(m_i)!\prod_{j=1}^{N}(n_j)!}C^k\|\hat{\phi}_0\|_\infty^{M+N} = \frac{(2k-1)!!}{\prod_{i=1}^{M}(m_i)!\prod_{j=1}^{N}(n_j)!}C^k\|\hat{\phi}_0\|_\infty^{M+N}. \tag{2.11}$$

In the case when $g_{n,\varepsilon} = F_{n,\varepsilon}$, we integrate w_r and bound (2.10) by

$$\frac{1}{\prod_{i=1}^{M}(m_i)!\prod_{j=1}^{N}(n_j)!}\frac{1}{\varepsilon^{2k}}\int_{[0,t]^{2k}} dsdu \int_{\mathbb{R}^{kd}} \sum_{\mathcal{F}} \prod_{(v_l,v_r)\in\mathcal{F}} e^{-\mathfrak{g}(w_l)|v_l-v_r|/\varepsilon^2}\hat{R}(w_l)\prod_{i=1}^{M}|h_{M,i}|\prod_{j=1}^{N}|h^*_{N,j}|\prod_{l=1}^{k}\frac{dw_l}{(2\pi)^d}. \tag{2.12}$$

For a given pairing $\mathcal{F}$, we have

$$|h_{M,i}| = \varepsilon^{-\alpha d/2}|\hat{\phi}_0|\Big(\frac{P_i}{\varepsilon^\alpha}\Big), \quad |h^*_{N,j}| = \varepsilon^{-\alpha d/2}|\hat{\phi}_0|\Big(\frac{Q_j}{\varepsilon^\alpha}\Big), \tag{2.13}$$

where

$$P_i = \xi - p_{i,1} - \ldots - p_{i,m_i}, \quad Q_j = \xi - q_{j,1} - \ldots - q_{j,n_j},$$

subject to the conditions

$$w_l + w_r = 0 \text{ when } (v_l, v_r) \in \mathcal{F}. \tag{2.14}$$

The difference with the previous case are the factors $\varepsilon^{-\alpha d/2}$ in (2.13). Note that if $P_i = \xi$ or $Q_j = \xi$ (this may happen because of (2.14)), as $\xi \neq 0$ is fixed and $\hat{\phi}_0$ is rapidly decaying, we may simply use the bound

$$\varepsilon^{-\alpha d/2}|\hat{\phi}_0|\Big(\frac{\xi}{\varepsilon^\alpha}\Big) \leq C.$$

For i, j such that $P_i, Q_j \neq \xi$, to deal with the large factors in (2.13), we change variables as follows. Take some i with $P_i \neq \xi$, so that

$$p_{i,1} + \ldots + p_{i,m_i} \neq 0.$$

We pick any variable p from $\{p_{i,1}, \dots, p_{i,m_i}\}$ (note the number of elements here can be strictly smaller than m_i since we have already integrated out the variables w_r), and change p to $p' = P_i/\varepsilon^\alpha$. The variable $p = w_l$ was paired to some p_j or $q_j = w_r$ as in (2.14). Thus, after the integration of w_r, p' will also appear in a unique $\tilde{h}_{M,i}$ which equals to some $h_{M,j}$ or $h^*_{N,j}$. We use the bound

$$|\tilde{h}_{M,i}| \le \varepsilon^{-\alpha d/2} C.$$

Thus, after the change of variable and taking into account the Jacobian of the change of variables, we have, with a slight abuse of notation

$$|h_{M,i}\tilde{h}_{M,i}|dp \le \varepsilon^{-\alpha d/2}|\hat{\phi}_0|(p')\varepsilon^{-\alpha d/2}C\varepsilon^{\alpha d} = C|\hat{\phi}_0|(p')dp'. \tag{2.15}$$

Since the change of variable only relates to p_i, all other $h_{M,i}, h^*_{N,j}$ are not affected. We continue the procedure, integrating out the p-variables one by one. If we are left with a single

$$|h| = \varepsilon^{-\alpha d/2}|\hat{\phi}_0|(P_i/\varepsilon^\alpha) \text{ or } \varepsilon^{-\alpha d/2}|\hat{\phi}_0|(Q_i/\varepsilon^\alpha)$$

in the end, we change variable similarly, and estimate this term, together with the Jacobian as

$$\varepsilon^{\alpha d/2}|\hat{\phi}_0|(p')dp' \le |\hat{\phi}_0|(p')dp'. \tag{2.16}$$

Overall, this change of variables will involve $M + N$ momenta, and will eliminate all factors $h_{M,i}$ and $h^*_{N,j}$, and we will be left with an expression of the form

$$\begin{aligned}
&\frac{1}{\varepsilon^{2k}}\int_{[0,t]^{2k}} dsdu \int_{\mathbb{R}^{kd}} \prod_{(v_l,v_r)\in\mathcal{F}} (2\pi)^{-d} e^{-\mathfrak{g}(w_l)|v_l-v_r|/\varepsilon^2}\hat{R}(w_l)\prod_{i=1}^{M}|h_{M,i}|\prod_{j=1}^{N}|h^*_{N,j}|dw \\
\le &\frac{C^{M+N}}{\varepsilon^{2k}}\int_{[0,t]^{2k}} dsdu \int_{\mathbb{R}^{kd}} \prod_{(v_l,v_r)\in\mathcal{F}_1} (2\pi)^{-d} e^{-\mathfrak{g}(w_l)|v_l-v_r|/\varepsilon^2}\hat{R}(w_l) \prod_{(v_l,v_r)\in\mathcal{F}_2} (2\pi)^{-d} e^{-\mathfrak{g}(z_l)|v_l-v_r|/\varepsilon^2}\hat{R}(z_l) \\
&\times \prod_{(v_l,v_r)\in\mathcal{F}_2} |\hat{\phi}_0|(w_l)dw.
\end{aligned} \tag{2.17}$$

Here, $(v_l, v_r) \in \mathcal{F}_1$ denotes the pairings in which the momenta do not participate in the change of variables and $(v_l, v_r) \in \mathcal{F}_2$ denotes the affected pairings. The explicit form of z_l that appears above is not important, so we do not specify them. The bounds (2.15) and (2.16) mean that the "participating" w_l give us the factor

$$\prod_{(v_l,v_r)\in\mathcal{F}_2} |\hat{\phi}_0|(w_l)$$

that appears in the last line of (2.17).

Next, we integrate in time. This brings about the product

$$C^k \prod_{(v_l,v_r)\in\mathcal{F}_1} \frac{\hat{R}(w_l)}{\mathfrak{g}(w_l)} \prod_{(v_l,v_r)\in\mathcal{F}_2} \frac{\hat{R}(z_l)}{\mathfrak{g}(z_l)}.$$

Using the fact that $\hat{R}(w_l)/\mathfrak{g}(w_l)$ is integrable for the vertices in $\mathcal{F}_1$, and that $\hat{R}(z_l)/\mathfrak{g}(z_l)$ is uniformly bounded for the vertices in $\mathcal{F}_2$, we may integrate out all the momenta variables, showing that (2.12) is bounded by

$$\frac{\#(\mathcal{F})}{\prod_{i=1}^{M}(m_i)!\prod_{j=1}^{N}(n_j)!}C^k = \frac{(2k-1)!!}{\prod_{i=1}^{M}(m_i)!\prod_{j=1}^{N}(n_j)!}C^k \tag{2.18}$$

This finishes the proof. □

Lemma 2.2 ensures we can interchange the limit $\varepsilon \to 0$ and the summation.

An estimate on non-simple pairings

Now we need to consider more carefully the contribution from different types of pairings. First we can decompose the temporal domain $\Delta_{m_1}(t) \times \ldots \times \Delta_{n_N}(t)$ according to all possible permutations of $\{s_{1,1}, \ldots u_{N,n_N}\}$ and write

$$\begin{aligned}&\mathbb{E}\{g_{m_1,\varepsilon}\ldots g_{m_M,\varepsilon}g^*_{n_1,\varepsilon}\ldots g^*_{n_N,\varepsilon}\}\\ &= \sum_{\sigma}\frac{1}{(i\varepsilon)^{\sum_{i=1}^{M}m_i}}\frac{1}{(-i\varepsilon)^{\sum_{j=1}^{N}n_j}}\int_{\sigma_{2k}(t)}dsdu\int_{\mathbb{R}^{2kd}}\mathbb{E}\{I_{M,N}\}e^{i\mathcal{G}_M}e^{-i\mathcal{G}_N}\prod_{i=1}^{M}h_{M,i}\prod_{j=1}^{N}h^*_{N,j},\end{aligned} \tag{2.19}$$

where $\sigma_{2k}(t) = \{0 \le v_{2k} \le \ldots \le v_1 \le t\}$ and $\sigma = \{v_1, \ldots, v_{2k}\}$ denotes all possible permutations of $\{s_{1,1}, \ldots u_{N,n_N}\}$ such that $\sigma_{2k}(t) \neq \varnothing$. By (2.8),

$$\mathbb{E}\{I_{M,N}\} = \sum_{\mathcal{F}}\prod_{(v_l,v_r)\in\mathcal{F}}(2\pi)^{-d}e^{-\mathfrak{g}(w_l)|v_l-v_r|/\varepsilon^2}\delta(w_l+w_r)\hat{R}(w_l)dw_ldw_r, \tag{2.20}$$

where $\mathcal{F}$ are pairings obtained by computing joint moments of Gaussian. We can write

$$\mathbb{E}\{g_{m_1,\varepsilon}\ldots g_{m_M,\varepsilon}g^*_{n_1,\varepsilon}\ldots g^*_{n_N,\varepsilon}\} = \sum_{\sigma}\sum_{\mathcal{F}}J^{\varepsilon}_{m_1,\ldots,n_N^*}(\sigma,\mathcal{F},\xi,g) \tag{2.21}$$

with

$$\begin{aligned}J^{\varepsilon}_{m_1,\ldots,n_N^*}(\sigma,\mathcal{F},\xi,g) &= \frac{1}{(i\varepsilon)^{\sum_{i=1}^{M}m_i}}\frac{1}{(-i\varepsilon)^{\sum_{j=1}^{N}n_j}}\\ \times\int_{\sigma_{2k}(t)}dsdu\int_{\mathbb{R}^{2kd}}&\prod_{(v_l,v_r)\in\mathcal{F}}(2\pi)^{d}e^{-\mathfrak{g}(w_l)|v_l-v_r|/\varepsilon^2}\delta(w_l+w_r)\hat{R}(w_l)dw_ldw_re^{i\mathcal{G}_M}e^{-i\mathcal{G}_N}\prod_{i=1}^{M}h_{M,i}\prod_{j=1}^{N}h^*_{N,j}\end{aligned} \tag{2.22}$$

and the symbol $g = f$ or F indicates the dependence of $J^{\varepsilon}_{m_1,\ldots,n_N^*}$ on $g_{n,\varepsilon} = f_{n,\varepsilon}$ or $F_{n,\varepsilon}$.

Given a permutation σ, we say that $\mathcal{F}_\sigma$ is a *simple pairing* if v_{2i-1}, v_{2i} form a pair for every index $i = 1, \ldots, k$. The next lemma shows that the overall contribution of the non-simple pairings vanishes in the limit $\varepsilon \to 0$.

Lemma 2.3. *Let $g_{n,\varepsilon} = f_{n,\varepsilon}$ or $F_{n,\varepsilon}$, then we have*

$$\sum_{\sigma}\sum_{\mathcal{F}\neq\mathcal{F}_\sigma}J^{\varepsilon}_{m_1,\ldots,n_N^*}(\sigma,\mathcal{F},\xi,g) \to 0, \text{ as } \varepsilon \to 0.$$

Proof. When $g_{n,\varepsilon} = f_{n,\varepsilon}$, this is proved in [4, Lemma 3.6]. The proof for $g_{n,\varepsilon} = F_{n,\varepsilon}$ is similar, using the same change of variables as in the proof of Lemma 2.2. We do not provide all details – but just mention the main simple point: if non-consecutive times are paired, then the time integration of the exponentials brings out too large power of ε, as, essentially, you collapse the intervals of the time integration "too much". We write

$$
\begin{aligned}
&|J^{\varepsilon}_{m_1,\ldots,n^*_N}(\sigma,\mathcal{F},\xi,F)| \\
\leq&\frac{1}{\varepsilon^{2k}}\int_{\sigma_{2k}(t)} dsdu \int_{\mathbb{R}^{2kd}} \prod_{(v_l,v_r)\in\mathcal{F}} (2\pi)^{-d} e^{-\mathfrak{g}(w_l)|v_l-v_r|/\varepsilon^2}\delta(w_l+w_r)\hat{R}(w_l)dw_l dw_r \prod_{i=1}^{M}|h_{M,i}|\prod_{j=1}^{N}|h^*_{N,j}|,
\end{aligned}
$$

and by the proof of (2.17), we have

$$
\begin{aligned}
&|J^{\varepsilon}_{m_1,\ldots,n^*_N}(\sigma,\mathcal{F},\xi,F)| \\
\leq&\frac{C^{M+N}}{\varepsilon^{2k}}\int_{\sigma_{2k}(t)} dsdu \int_{\mathbb{R}^{kd}} \prod_{(v_l,v_r)\in\mathcal{F}_1} (2\pi)^{-d} e^{-\mathfrak{g}(w_l)|v_l-v_r|/\varepsilon^2}\hat{R}(w_l) \prod_{(v_l,v_r)\in\mathcal{F}_2} (2\pi)^{-d} e^{-\mathfrak{g}(\cdot)|v_l-v_r|/\varepsilon^2}\hat{R}(\cdot) \\
&\times \prod_{(v_l,v_r)\in\mathcal{F}_2} |\hat{\phi}_0|(w_l)dw.
\end{aligned}
$$

Then, using the fact that $\hat{R}(p)/\mathfrak{g}(p)$ is integrable and uniformly bounded, we only need to follow the proof of [4, Lemma 3.6] using the aforementioned observation that the time integration will bring about too high power of ε because of the exponential in time factors. □

The vanishing of the crossing pairings

By Lemma 2.3, we have

$$
\lim_{\varepsilon\to 0}\mathbb{E}\{g_{m_1,\varepsilon}\ldots g_{m_M,\varepsilon}g^*_{n_1,\varepsilon}\ldots g^*_{n_N,\varepsilon}\} = \sum_{\sigma}\lim_{\varepsilon\to 0} J^{\varepsilon}_{m_1,\ldots,n^*_N}(\sigma,\mathcal{F}_\sigma,\xi,g). \tag{2.23}
$$

Let us define sets

$$
A_i = \{s_{i,1},\ldots,s_{i,m_i}\},\ B_j = \{u_{j,1},\ldots,u_{j,n_j}\} \text{ with } i=1,\ldots,M,\ j=1,\ldots,N.
$$

Given a pairing $\mathcal{F}_\sigma$, we say

$$
S_1, S_2 \in \{A_i, B_j : i=1,\ldots,M, j=1,\ldots,N\}
$$

interact with each other if there is an edge $(v_l,v_r)\in\mathcal{F}_\sigma$ such that $v_l\in S_1, v_r\in S_2$, and we write $S_1\leftrightarrow S_2$. We say they are connected if there exist other sets such that $S_1\leftrightarrow\ldots\leftrightarrow S_2$. Thus, for a given permutation σ, we may decompose $\{A_i,B_j : i=1,\ldots,M, j=1,\ldots,N\}$ into connected components. For example, if all variables in A_1 pair inside A_1, then A_1 itself is a connected component. If all variables in A_1 and A_2 either pair inside the corresponding set or pair with variables in the other set, and we have at least one edge joining A_1 and A_2, then $\{A_1,A_2\}$ is a connected component, and so on. We let $N_c(\mathcal{F}_\sigma)$ be the size of largest connected component corresponding to $\mathcal{F}_\sigma$. The following lemma shows the permutations with more than triple interactions do not contribute in the limit. This leads to a Gaussian limit in Theorems 1.2 and 1.3.

Lemma 2.4. *We have*

$$\sum_{\sigma:N_c(\mathcal{F}_\sigma)\geq 2} \lim_{\varepsilon\to 0} J^{\varepsilon}_{m_1,\ldots,n^*_N}(\sigma,\mathcal{F}_\sigma,\xi,f)=0.$$

and

$$\sum_{\sigma:N_c(\mathcal{F}_\sigma)\geq 3} \lim_{\varepsilon\to 0} J^{\varepsilon}_{m_1,\ldots,n^*_N}(\sigma,\mathcal{F}_\sigma,\xi,F)=0.$$

Proof. We first consider the case $g_{n,\varepsilon}=f_{n,\varepsilon}$. For a given permutation σ, if $N_c(\sigma)\geq 2$, we can find the sets

$$S_1,S_2\in\{A_i,B_j: i=1,\ldots,M, j=1,\ldots,N\},$$

such that $S_1\leftrightarrow S_2$. Let e be an edge joining S_1 and S_2, and h_{S_1},h_{S_2} be the initial conditions corresponding to S_1,S_2, then we have

$$\begin{aligned}&|J^{\varepsilon}_{m_1,\ldots,n^*_N}(\sigma,\mathcal{F}_\sigma,\xi,f)|\\ \leq&\frac{C^{M+N-2}}{\varepsilon^{2k}}\int_{\sigma_{2k}(t)}dsdu\int_{\mathbb{R}^{2kd}}\prod_{(v_l,v_r)\in\mathcal{F}_\sigma}(2\pi)^d e^{-\mathfrak{g}(w_l)|v_l-v_r|/\varepsilon^2}\delta(w_l+w_r)\hat{R}(w_l)dw_ldw_r|h_{S_1}h_{S_2}|.\end{aligned} \tag{2.24}$$

Recall that when $g_{n,\varepsilon}=f_{n,\varepsilon}$, we have

$$h_{M,i}=\hat{\phi}_0(\xi-\frac{p_{i,1}+\ldots+p_{i,m_i}}{\varepsilon^\alpha}),\quad h^*_{N,j}=\hat{\phi}^*_0(\xi-\frac{q_{j,1}+\ldots+q_{j,n_j}}{\varepsilon^\alpha}).$$

We can assume

$$|h_{S_1}|=|\hat{\phi}_0|(\xi-\frac{P_1}{\varepsilon^\alpha}) \text{ and } |h_{S_2}|=|\hat{\phi}_0|(\xi-\frac{P_2}{\varepsilon^\alpha}),$$

for some P_1,P_2 after integrating out w_r in (2.24). It is clear that $P_1,P_2\neq 0$ since they both contain the w_l variable corresponding to the edge e. Now we have

$$|J^{\varepsilon}_{m_1,\ldots,n^*_N}(\sigma,\mathcal{F}_\sigma,\xi,f)|\leq C^{M+N}\int_{\mathbb{R}^{kd}}\prod_{(v_l,v_r)\in\mathcal{F}_\sigma}\frac{\hat{R}(w_l)}{\mathfrak{g}(w_l)}|\hat{\phi}_0|(\xi-\frac{P_1}{\varepsilon^\alpha})|\hat{\phi}_0|(\xi-\frac{P_2}{\varepsilon^\alpha})|dw_l\to 0, \tag{2.25}$$

as $\varepsilon\to 0$ by dominated convergence theorem.

Next we consider the case $g_{n,\varepsilon}=F_{n,\varepsilon}$. The following estimate holds

$$\begin{aligned}&|J^{\varepsilon}_{m_1,\ldots,n^*_N}(\sigma,\mathcal{F}_\sigma,\xi,F)|\\ \leq&\frac{1}{\varepsilon^{2k}}\int_{\sigma_{2k}(t)}dsdu\int_{\mathbb{R}^{2kd}}\prod_{(v_l,v_r)\in\mathcal{F}_\sigma}(2\pi)^d e^{-\mathfrak{g}(w_l)|v_l-v_r|/\varepsilon^2}\delta(w_l+w_r)\hat{R}(w_l)dw_ldw_r\prod_{i=1}^M|h_{M,i}|\prod_{j=1}^N|h^*_{N,j}|.\end{aligned}$$

Recall that now

$$h_{M,i}=\varepsilon^{-\alpha d/2}\hat{\phi}_0(\frac{\xi-p_{i,1}-\ldots-p_{i,m_i}}{\varepsilon^\alpha}),\quad h^*_{N,j}=\varepsilon^{-\alpha d/2}\hat{\phi}^*_0(\frac{\xi-q_{j,1}-\ldots-q_{j,n_j}}{\varepsilon^\alpha}).$$

If $N_c(\sigma)\geq 3$, we can find

$$S_1,S_2,S_3\in\{A_i,B_j: i=1,\ldots,M, j=1,\ldots,N\}$$

such that $S_1 \leftrightarrow S_2 \leftrightarrow S_3$. We pick two edges linking S_1 to S_2 and S_2 to S_3, and denote them by $e_{1,2}$ and $e_{2,3}$, respectively. We also denote the variables corresponding to $e_{1,2}, e_{2,3}$ by $w_{1,2}, w_{2,3}$. Let h_{S_i} be the initial condition corresponding to $S_i, i = 1, 2, 3$, then we have

$$|h_{S_i}| = \varepsilon^{-\alpha d/2} |\hat{\phi}_0| (\frac{\xi - P_i}{\varepsilon^\alpha}) \text{ for some } P_i, i = 1, 2, 3.$$

After integrating out the w_r variables, it is clear that P_1 contains the variable $w_{1,2}$, P_2 contains the variables $w_{1,2}, w_{2,3}$ and P_3 contains the variable $w_{2,3}$. We do a similar change of variable as in the proof of Lemma 2.2. First, we change $w_{1,2}$ so that $(\xi - P_1)/\varepsilon^\alpha \mapsto P_1$. Second, we change $w_{2,3}$ so that $(\xi - P_3)/\varepsilon^\alpha \mapsto P_3$. Then we have

$$\begin{aligned} |h_{S_1} h_{S_2} h_{S_3}| =& \varepsilon^{-3\alpha d/2} |\hat{\phi}_0 (\frac{\xi - P_1}{\varepsilon^\alpha}) \hat{\phi}_0 (\frac{\xi - P_2}{\varepsilon^\alpha}) \hat{\phi}_0 (\frac{\xi - P_3}{\varepsilon^\alpha})| dw_{1,2} dw_{2,3} \\ \mapsto& \varepsilon^{-3\alpha d/2} \varepsilon^{2\alpha d} |\hat{\phi}_0(P_1) \hat{\phi}_0(z) \hat{\phi}_0(P_3)| dP_1 dP_3 \le C \varepsilon^{\alpha d/2} |\hat{\phi}_0(P_1) \hat{\phi}_0(P_3)| dP_1 dP_3, \end{aligned} \tag{2.26}$$

with some z that does not matter to us, as we simply bound $\hat{\phi}_0(z)$ by C. Now, we only need to carry out the same change of variable as in the proof of Lemma 2.2 for the remaining h. In the end, we obtain

$$\begin{aligned} &|J^\varepsilon_{m_1,\ldots,n_N^*}(\sigma, \mathcal{F}_\sigma, \xi, F)| \le \frac{C^{M+N} \varepsilon^{\alpha d/2}}{\varepsilon^{2k}} \int_{[0,t]^{2k}} ds du \int_{\mathbb{R}^{kd}} \prod_{(v_l, v_r) \in \mathcal{F}_{\sigma,1}} (2\pi)^{-d} e^{-\mathfrak{g}(w_l)|v_l - v_r|/\varepsilon^2} \hat{R}(w_l) \\ &\times \prod_{(v_l, v_r) \in \mathcal{F}_{\sigma,2}} (2\pi)^{-d} e^{-\mathfrak{g}(z_l)|v_l - v_r|/\varepsilon^2} \hat{R}(z_l) \prod_{(v_l, v_r) \in \mathcal{F}_{\sigma,2}} |\hat{\phi}_0|(w_l) dw, \end{aligned} \tag{2.27}$$

where $(v_l, v_r) \in \mathcal{F}_{\sigma,1}$ denotes the pairings which are not affected by the change of variables, and $(v_l, v_r) \in \mathcal{F}_{\sigma,2}$ denotes the affected pairings, and, as in the analysis of (2.17), the precise expression for z_l is not important to us. Clearly, the RHS of (2.27) goes to zero as $\varepsilon \to 0$ because of the extra factor $\varepsilon^{\alpha d/2}$ compared to (2.17). $\square$

Pairings for the correctors

We now describe analogous estimates that are needed in the analysis of the corrector

$$\mathscr{U}_\varepsilon(t, \xi) = \varepsilon^{-\alpha d/2} \sum_{n=0}^\infty \mathscr{F}_{n,\varepsilon}(t, \xi),$$

with

$$\begin{aligned} \mathscr{F}_{n,\varepsilon}(t, \xi) &= f_{n,\varepsilon}(t, \xi) - \mathbb{E}\{f_{n,\varepsilon}(t, \xi)\} \\ &= \frac{1}{(i\varepsilon)^n} \int_{\Delta_n(t)} \int_{\mathbb{R}^{nd}} \mathcal{V}(\frac{s_1}{\varepsilon^2}, \ldots, \frac{s_n}{\varepsilon^2}, dp_1, \ldots, dp_n) e^{iG_n(\varepsilon^\alpha \xi, s^{(n)}, p^{(n)})/\varepsilon^2} \hat{\phi}_0(\xi - \frac{p_1}{\varepsilon^\alpha} - \ldots - \frac{p_n}{\varepsilon^\alpha}) ds, \end{aligned}$$

where

$$\mathcal{V}(s_1, \ldots, s_n, dp_1, \ldots, dp_n) = (2\pi)^{-nd} \left(\prod_{j=1}^n \tilde{V}(s_j, dp_j) - \mathbb{E}\{\prod_{j=1}^n \tilde{V}(s_j, dp_j)\} \right). \tag{2.28}$$

Let us discuss in detail the calculation of moments

$$\varepsilon^{-\alpha d(M+N)/2}\mathbb{E}\{\mathscr{F}_{m_1,\varepsilon}\ldots\mathscr{F}_{m_M,\varepsilon}\mathscr{F}^*_{n_1,\varepsilon}\ldots\mathscr{F}^*_{n_N,\varepsilon}\},\ \text{for } m_1,\ldots,n_N\in\mathbb{N},$$

with

$$\sum_{i=1}^M m_i+\sum_{j=1}^N n_j=2k.$$

Similar to (2.7), we can write

$$\begin{aligned}&\varepsilon^{-\alpha d(M+N)/2}\mathbb{E}\{\mathscr{F}_{m_1,\varepsilon}\ldots\mathscr{F}_{m_M,\varepsilon}\mathscr{F}^*_{n_1,\varepsilon}\ldots\mathscr{F}^*_{n_N,\varepsilon}\}\\&=\frac{1}{(i\varepsilon)^{\sum_{i=1}^M m_i}}\frac{1}{(-i\varepsilon)^{\sum_{j=1}^N n_j}}\int_{\Delta_{m_1}(t)\times\ldots\times\Delta_{n_N}(t)}dsdu\int_{\mathbb{R}^{2kd}}\mathbb{E}\{\mathcal{I}_{M,N}\}e^{i\mathcal{G}_M}e^{-i\mathcal{G}_N}\prod_{i=1}^M h_{M,i}\prod_{j=1}^N h^*_{N,j},\end{aligned}\tag{2.29}$$

where

$$\begin{aligned}\mathcal{I}_{M,N}=&\mathcal{V}(\frac{s_{1,1}}{\varepsilon^2},\ldots,\frac{s_{1,m_1}}{\varepsilon^2},dp_{1,1},\ldots,dp_{1,m_1})\ldots\mathcal{V}(\frac{s_{M,1}}{\varepsilon^2},\ldots,\frac{s_{M,m_M}}{\varepsilon^2},dp_{M,1},\ldots,dp_{M,m_M})\\&\times\mathcal{V}^*(\frac{u_{1,1}}{\varepsilon^2},\ldots,\frac{u_{1,n_1}}{\varepsilon^2},dq_{1,1},\ldots,dq_{1,n_1})\ldots\mathcal{V}^*(\frac{u_{N,1}}{\varepsilon^2},\ldots,\frac{u_{N,n_N}}{\varepsilon^2},dq_{N,1},\ldots,dq_{N,n_N}),\end{aligned}\tag{2.30}$$

and

$$h_{M,i}=\varepsilon^{-\alpha d/2}\hat{\phi}_0(\xi-\frac{p_{i,1}+\ldots+p_{i,m_i}}{\varepsilon^\alpha}),\quad h^*_{N,j}=\varepsilon^{-\alpha d/2}\hat{\phi}^*_0(\xi-\frac{q_{j,1}+\ldots+q_{j,n_j}}{\varepsilon^\alpha}).$$

Previously, we have dealt with the expectation of a product of centered Gaussians. For $\mathcal{I}_{M,N}$, however, each factor $\mathcal{V}$, defined in (2.28) is a centered product of Gaussians rather than a product of centered Gaussians. The rules for evaluating the expectation of such objects are recalled in Lemma A.1 in the Appendix. Recall that we have defined the sets

$$A_i=\{s_{i,1},\ldots,s_{i,m_i}\},\quad B_j=\{u_{j,1},\ldots,u_{j,n_j}\},\ \text{with } i=1,\ldots,M,\ j=1,\ldots,N.$$

Given a pairing $\mathcal{F}$, we decompose $\{A_i,B_j:i=1,\ldots,M,j=1,\ldots,N\}$ into connected components according to the interaction between the s,u variables. Let $N_s(\mathcal{F})$ be the size of smallest connected component, then by Lemma A.1 we have

$$\mathbb{E}\{\mathcal{I}_{M,N}\}=\sum_{\mathcal{F}:N_s(\mathcal{F})\geq 2}\prod_{(v_l,v_r)\in\mathcal{F}}(2\pi)^{-d}e^{-\mathfrak{g}(w_l)|v_l-v_r|/\varepsilon^2}\delta(w_l+w_r)\hat{R}(w_l)dw_ldw_r.\tag{2.31}$$

In particular, it is clear that $\mathbb{E}\{\mathcal{I}_{M,N}\}\leq\mathbb{E}\{I_{M,N}\}$ and

$$\mathbb{E}\{I_{M,N}\}-\mathbb{E}\{\mathcal{I}_{M,N}\}=\sum_{\mathcal{F}:N_s(\mathcal{F})=1}\prod_{(v_l,v_r)\in\mathcal{F}}(2\pi)^{-d}e^{-\mathfrak{g}(w_l)|v_l-v_r|/\varepsilon^2}\delta(w_l+w_r)\hat{R}(w_l)dw_ldw_r.\tag{2.32}$$

Comparing (2.7) and (2.8), to (2.29) and (2.31), we see that

$$\varepsilon^{-\alpha d(M+N)/2}\mathbb{E}\{\mathscr{F}_{m_1,\varepsilon}\ldots\mathscr{F}_{m_M,\varepsilon}\mathscr{F}^*_{n_1,\varepsilon}\ldots\mathscr{F}^*_{n_N,\varepsilon}\}.$$

has exactly the same form as

$$\mathbb{E}\{F_{m_1,\varepsilon}\ldots F_{m_M,\varepsilon}F^*_{n_1,\varepsilon}\ldots F^*_{n_N,\varepsilon}\}\tag{2.33}$$

if we replace $\xi \to \varepsilon^\alpha \xi$ and impose the constraint $N_s(\mathcal{F}) \geq 2$ in (2.33). Therefore, we can follow the same proof for Lemmas 2.2, 2.3 and obtain

$$\lim_{\varepsilon\to 0} \varepsilon^{-\alpha d(M+N)/2} \mathbb{E}\{\mathscr{F}_{m_1,\varepsilon} \ldots \mathscr{F}_{m_M,\varepsilon} \mathscr{F}^*_{n_1,\varepsilon} \ldots \mathscr{F}^*_{n_N,\varepsilon}\} = \sum_{\sigma: N_s(\mathcal{F}_\sigma)\geq 2} \lim_{\varepsilon\to 0} J^\varepsilon_{m_1,\ldots,n_N^*}(\sigma, \mathcal{F}_\sigma, \varepsilon^\alpha \xi, F),$$

where we recall $J^\varepsilon_{m_1,\ldots,n_N^*}$ is defined in (2.22).

We should note that in the proof of Lemma 2.2, for $g_{n,\varepsilon} = F_{n,\varepsilon}$, we used the fact that $\xi \neq 0$ so that

$$\varepsilon^{-\alpha d/2} |\hat{\phi}_0(\xi/\varepsilon^\alpha)| \leq C,$$

and actually goes to zero as $\varepsilon \to 0$. At this step, the analysis for $\mathscr{F}_{n,\varepsilon}$ can not proceed this way, as we have replaced $\xi \mapsto \varepsilon^\alpha \xi$. Instead, we use the condition $N_s(\mathcal{F}) \geq 2$, which implies that after computing moments, all the h factors in (2.29) take the form

$$h_{M,i} = \varepsilon^{-\alpha d/2} \hat{\phi}_0\big(\xi - \frac{P}{\varepsilon^\alpha}\big), \text{ and } h^*_{N,j} = \varepsilon^{-\alpha d/2} \hat{\phi}_0\big(\xi - \frac{Q}{\varepsilon^\alpha}\big),$$

for some $P, Q \neq 0$. If P or Q were to be zero, then A_i or B_j is not connected with any other set, which would imply $N_s(\mathcal{F}) = 1$. As P and Q are not zero, we only need to perform the same change of variables as in the proof of Lemma 2.2.

We may now follow the same proof as for Lemma 2.4 to obtain

$$\lim_{\varepsilon\to 0} \varepsilon^{-\alpha d(M+N)/2} \mathbb{E}\{\mathscr{F}_{m_1,\varepsilon} \ldots \mathscr{F}_{m_M,\varepsilon} \mathscr{F}^*_{n_1,\varepsilon} \ldots \mathscr{F}^*_{n_N,\varepsilon}\} = \sum_{\sigma: N_s(\mathcal{F}_\sigma)\geq 2, N_c(\mathcal{F}_\sigma)\leq 2} \lim_{\varepsilon\to 0} J^\varepsilon_{m_1,\ldots,n_N^*}(\sigma, \mathcal{F}_\sigma, \varepsilon^\alpha \xi, F). \tag{2.34}$$

Since $N_s(\mathcal{F}_\sigma) \geq 2$ and $N_c(\mathcal{F}_\sigma) \leq 2$, we have

$$N_s(\mathcal{F}_\sigma) = N_c(\mathcal{F}_\sigma) = 2,$$

that is, all connected components corresponding to $\mathcal{F}_\sigma$ contain two sets, which implies $M + N$ is even.

3 Homogenization of the low frequencies

We now prove Theorem 1.1. To show that

$$\psi_\varepsilon(t, \xi) \to \hat{\phi}_0(\xi) e^{-\frac{1}{2}D(0)t} \text{ in probability,}$$

we only need to verify the following result.

Proposition 3.1. *As $\varepsilon \to 0$, we have*

$$\mathbb{E}\{\psi_\varepsilon(t,\xi)\} \to \hat{\phi}_0(\xi) e^{-\frac{1}{2}D(0)t}, \tag{3.1}$$

and

$$\mathbb{E}\{|\psi_\varepsilon(t,\xi)|^2\} \to |\hat{\phi}_0(\xi)|^2 e^{-\mathrm{Re} D(0)t}. \tag{3.2}$$

Proof. By Lemma 2.1, we have

$$\mathbb{E}\{\psi_\varepsilon(t,\xi)\} = \sum_{n=0}^{\infty} \mathbb{E}\{f_{n,\varepsilon}(t,\xi)\}.$$

Lemma 2.2 ensures that we only need to compute

$$\lim_{\varepsilon\to 0} \mathbb{E}\{f_{n,\varepsilon}(t,\xi)\},$$

when $n = 2k$ for some $k \in \mathbb{N}$. By Lemma 2.3, we have

$$\lim_{\varepsilon\to 0} \mathbb{E}\{f_{n,\varepsilon}(t,\xi)\} = \sum_{\sigma} \lim_{\varepsilon\to 0} J_n^\varepsilon(\sigma, \mathcal{F}_\sigma, \xi, f). \tag{3.3}$$

It is straightforward to see that

$$\begin{aligned}
&J_n^\varepsilon(\sigma, \mathcal{F}_\sigma, \xi, f) \\
=&\frac{\hat{\phi}_0(\xi)}{(i\varepsilon)^{2k}} \int_{\sigma_{2k}(t)} ds \int_{\mathbb{R}^{kd}} \prod_{(v_l,v_r)\in\mathcal{F}_\sigma} (2\pi)^{-d} e^{-\mathfrak{g}(w_l)|v_l-v_r|/\varepsilon^2} \hat{R}(w_l) e^{i(|\varepsilon^\alpha\xi|^2 - |\varepsilon^\alpha\xi - w_l|^2)\frac{|v_l-v_r|}{2\varepsilon^2}} dw \\
\to&\hat{\phi}_0(\xi)(-1)^k \frac{t^k}{k!} \left(\int_{\mathbb{R}^d} \frac{\hat{R}(p)}{(\mathfrak{g}(p) + \frac{i}{2}|p|^2)} \frac{dp}{(2\pi)^d} \right)^k = \hat{\phi}_0(\xi) \frac{(-tD(0)/2)^k}{k!},
\end{aligned} \tag{3.4}$$

and thus

$$\lim_{\varepsilon\to 0} \mathbb{E}\{\psi_\varepsilon(t,\xi)\} = \sum_{k=0}^{\infty} \lim_{\varepsilon\to 0} \mathbb{E}\{f_{2k,\varepsilon}(t,\xi)\} = \sum_{k=0}^{\infty} \lim_{\varepsilon\to 0} J_{2k}^\varepsilon(\sigma, \mathcal{F}_\sigma, \xi, f) = \hat{\phi}_0(\xi) e^{-\frac{1}{2}D(0)t},$$

which is (3.1).

Since

$$\mathbb{E}\{|\psi_\varepsilon(t,\xi)|^2\} = \sum_{m,n=0}^{\infty} \mathbb{E}\{f_{m,\varepsilon}(t,\xi) f_{n,\varepsilon}^*(t,\xi)\},$$

by a similar discussion as in the proof of (3.1), we have

$$\lim_{\varepsilon\to 0} \mathbb{E}\{f_{m,\varepsilon}(t,\xi) f_{n,\varepsilon}^*(t,\xi)\} = \sum_{\sigma} \lim_{\varepsilon\to 0} J_{m,n^*}^\varepsilon(\sigma, \mathcal{F}_\sigma, \xi, f) \tag{3.5}$$

In addition, Lemma 2.4 shows that

$$\sum_{\sigma: N_c(\sigma)\geq 2} \lim_{\varepsilon\to 0} J_{m,n^*}^\varepsilon(\sigma, \mathcal{F}_\sigma, \xi, f) = 0, \tag{3.6}$$

so we are left with

$$\sum_{\sigma: N_c(\sigma)=1} \lim_{\varepsilon\to 0} J_{m,n^*}^\varepsilon(\sigma, \mathcal{F}_\sigma, \xi, f).$$

However, $N_c(\sigma) = 1$ implies there is no interaction between $f_{m,\varepsilon}(t,\xi)$ and $f^*_{n,\varepsilon}(t,\xi)$, so $m = 2k_1$, and $n = 2k_2$ are both even. The number of possible permutations is

$$\frac{(k_1+k_2)!}{k_1!k_2!},$$

and by the same calculation for (3.4), we have

$$\begin{aligned}\sum_{\sigma:N_c(\sigma)=1} \lim_{\varepsilon\to 0} J^\varepsilon_{m,n^*}(\sigma,\mathcal{F}_\sigma,\xi,f) =& \frac{(k_1+k_2)!}{k_1!k_2!}|\hat{\phi}_0(\xi)|^2(-1)^{k_1+k_2}\frac{t^{k_1+k_2}}{(k_1+k_2)!}\left(\frac{D(0)}{2}\right)^{k_1}\left(\frac{D^*(0)}{2}\right)^{k_2}\\ =&|\hat{\phi}_0(\xi)|^2\frac{(-tD(0)/2)^{k_1}}{k_1!}\frac{(-tD^*(0)/2)^{k_2}}{k_2!}\end{aligned} \tag{3.7}$$

Therefore, we have

$$\lim_{\varepsilon\to 0}\mathbb{E}\{|\psi_\varepsilon(t,\xi)|^2\} = \sum_{k_1,k_2=0}^{\infty}|\hat{\phi}_0(\xi)|^2\frac{(-tD(0)/2)^{k_1}}{k_1!}\frac{(-tD^*(0)/2)^{k_2}}{k_2!} = |\hat{\phi}_0(\xi)|^2 e^{-\mathrm{Re}D(0)t}, \tag{3.8}$$

which is (3.2). □

4 The high frequencies

In this section, we prove Theorem 1.2.

Convergence of the mean

We first show the convergence of $\mathbb{E}\{\Psi_\varepsilon(t,\xi)\}$ for fixed $t > 0$ and $\xi \neq 0$.

Lemma 4.1. *We have*

$$\mathbb{E}\{\Psi_\varepsilon(t,\xi)\} \to 0 \text{ as } \varepsilon \to 0.$$

Proof. By Lemmas 2.2, 2.3, we only need to show that

$$\lim_{\varepsilon\to 0} J^\varepsilon_n(\sigma,\mathcal{F}_\sigma,\xi,F) = 0,$$

when $n = 2k$. It is straightforward to see that

$$\begin{aligned}&J^\varepsilon_n(\sigma,\mathcal{F}_\sigma,\xi,F) = \varepsilon^{-\alpha d/2}\hat{\phi}_0(\frac{\xi}{\varepsilon^\alpha})\\ &\times\frac{1}{(i\varepsilon)^{2k}}\int_{\sigma_{2k}(t)}ds\int_{\mathbb{R}^{kd}}\prod_{(v_l,v_r)\in\mathcal{F}_\sigma}(2\pi)^{-d}e^{-\mathfrak{g}(w_l)|v_l-v_r|/\varepsilon^2}\hat{R}(w_l)e^{i(|\varepsilon^\alpha\xi|^2-|\varepsilon^\alpha\xi-w_l|^2)\frac{|v_l-v_r|}{2\varepsilon^2}}dw.\end{aligned}$$

Since $\xi \neq 0$, we have

$$\varepsilon^{-\alpha d/2}\hat{\phi}_0(\xi/\varepsilon^\alpha) \to 0 \text{ as } \varepsilon \to 0,$$

thus

$$|J^\varepsilon_n(\sigma,\mathcal{F}_\sigma,\xi,F)| \le C\varepsilon^{-\alpha d/2}\hat{\phi}_0(\xi/\varepsilon) \to 0 \text{ as } \varepsilon \to 0,$$

and the proof is complete. □

Convergence of the variance

Next, we look at the second moment.

Lemma 4.2. *We have*

$$\mathbb{E}\{|\Psi_\varepsilon(t,\xi)|^2\} \to \widehat{W}_\delta(t,\xi) \text{ as } \varepsilon \to 0.$$

The proof of Lemma 4.2 is very similar to [4, Proposition 3.12], and since Lemmas 5.1 and 5.2 below follow the same blueprint, we will provide the details here for the convenience of the reader.

Proof. By Lemmas 2.1-2.4, we only need to consider $J^\varepsilon_{m,n^*}(\sigma,\mathcal{F}_\sigma,\xi,F)$ for fixed $m,n\in\mathbb{N}$ (in the present case, we automatically have $N_c(\mathcal{F}_\sigma)\leq 2$). We write

$$A=\{s_1,\ldots,s_m\}, \quad B=\{u_1,\ldots,u_n\},$$

with $m+n=2k$ for some $k\in\mathbb{N}$. According to the pairing $\mathcal{F}_\sigma$, $\{A,B\}$ is decomposed into connected components. If A,B are "separate", we have two factors of $\varepsilon^{-\alpha d/2}|\hat{\phi}_0|(\xi/\varepsilon^\alpha)$ coming from the initial conditions, so by the same argument as in the proof of Lemma 4.1, we have

$$J^\varepsilon_{m,n^*}(\sigma,\mathcal{F}_\sigma,\xi,F)\to 0 \text{ as } \varepsilon\to 0.$$

Therefore, we only need to consider σ such that $A\leftrightarrow B$.

For a permutation σ of $A\cup B$, the simple diagram $\mathcal{F}_\sigma$ corresponds to

$$A\cup B=\{v_1^+,v_1^-,\ldots,v_k^+,v_k^-\},$$

with

$$v_1^+\geq v_1^- \ldots \geq v_k^+ \geq v_k^-,$$

and (v_i^+,v_i^-) forming a pair, $i=1,\ldots,k$. Since $A\leftrightarrow B$, there exists at least one pair such that v_i^+ and v_i^- come from different sets, and we call such pair a crossing edge between A and B. Assuming the total number of crossing edges is $N_{cr}(\sigma)\geq 1$, the interval $[0,t]$ is decomposed into $N_{cr}+1$ parts according to the position of those crossing edges, which we denote by

$$r_1^+\geq r_1^-\geq \ldots \geq r_{N_{cr}}^+ \geq r_{N_{cr}}^-,$$

with $r_i^\pm = v_j^\pm$ for some j, and with the convention where $r_0^-=t$ and $r_{N_{cr}+1}^+=0$. We further denote by $\mathcal{E}_{s,i}$, $i=0,\ldots,N_{cr}$, the set of edges between the vertices r_i^- and r_{i+1}^+ that are of the form (s_j,s_{j+1}), and by $\mathcal{E}_{u,i}$ the set of edges between r_i^- and r_{i+1}^+ that are of the form (u_j,u_{j+1}). The corresponding sets of indices are denoted by

$$\mathcal{A}_i=\{j:(v_j^+,v_j^-)\in\mathcal{E}_{s,i}, j=1,\ldots,k\},$$

and

$$\mathcal{B}_i=\{j:(v_j^+,v_j^-)\in\mathcal{E}_{u,i}, j=1,\ldots,k\},$$

with $i=0,\ldots,N_{cr}$. For a non-crossing edge (v_j^+,v_j^-), we denote $\tau_j=1$ if v_j^+,v_j^- are $s-$variables and $\tau_j=-1$ if they are $u-$variables.

Recall that

$$
\begin{aligned}
&J^{\varepsilon}_{m,n^*}(\sigma,\mathcal{F}_\sigma,\xi,F)\\
=&\frac{1}{(i\varepsilon)^m(-i\varepsilon)^n}\int_{\sigma_{2k}(t)}dsdu\int_{\mathbb{R}^{2kd}}\prod_{(v_l,v_r)\in\mathcal{F}_\sigma}(2\pi)^{-d}e^{-\mathfrak{g}(w_l)|v_l-v_r|/\varepsilon^2}\delta(w_l+w_r)\hat{R}(w_l)dw_ldw_r\\
&\times e^{iG_m(\xi,s^{(m)},p^{(m)})/\varepsilon^2}e^{-iG_n(\xi,u^{(n)},q^{(n)})/\varepsilon^2}\varepsilon^{-\alpha d}\hat{\phi}_0(\frac{\xi-p_1-\ldots-p_m}{\varepsilon^\alpha})\hat{\phi}_0^*(\frac{\xi-q_1-\ldots-q_n}{\varepsilon^\alpha}),
\end{aligned}
\tag{4.1}
$$

where v_l,v_r are the vertices of a given pair, and w_l,w_r are the corresponding p,q variables, that is, $w_l=p_i$ if $v_l=s_i$ and $w_l=-q_i$ if $v_l=u_i$. For a crossing edge $(v_l,v_r)=(r_i^+,r_i^-)$, the relevant p,q variables equal to each other due to $\delta(p-q)$, and we denote the corresponding $w_l=P_i$, with the convention that $P_0=0$. We also define $\mathfrak{s}_i=1$ if r_i^+ is $s-$variable and $\mathfrak{s}_i=-1$ if r_i^+ is $u-$variable. With the above notation, we have

$$
\begin{aligned}
&J^{\varepsilon}_{m,n^*}(\sigma,\mathcal{F}_\sigma,\xi,F)\\
=&\frac{1}{(i\varepsilon)^m(-i\varepsilon)^n}\int_{\sigma_{2k}(t)}dsdu\int_{\mathbb{R}^{kd}}dw\prod_{j=1}^{k}\frac{\hat{R}(w_j)}{(2\pi)^d}e^{-\mathfrak{g}(w_j)\frac{v_j^+-v_j^-}{\varepsilon^2}}\prod_{j=1}^{N_{cr}}e^{i\mathfrak{s}_j(|\xi-\ldots-P_{j-1}|^2-|\xi-\ldots-P_j|^2)\frac{r_j^+-r_j^-}{2\varepsilon^2}}\\
&\times\prod_{j=0}^{N_{cr}}\prod_{l\in\mathcal{A}_j\cup\mathcal{B}_j}e^{i\tau_l(|\xi-\ldots-P_j|^2-|\xi-\ldots-P_j-w_l|^2)\frac{v_l^+-v_l^-}{2\varepsilon^2}}\frac{1}{\varepsilon^{\alpha d}}|\hat{\phi}_0|(\frac{\xi-P_1-\ldots-P_{N_{cr}}}{\varepsilon^\alpha})^2.
\end{aligned}
\tag{4.2}
$$

Here, we have integrated out the variables w_r in (4.1), and changed the notation $w_l\mapsto w_j$. To get rid of the extra factor $\varepsilon^{-\alpha d}$, we change variables as before. Replacing

$$P_{N_{cr}}\mapsto\xi-P_1-\ldots-P_{N_{cr}-1}-\varepsilon^\alpha P_{N_{cr}},$$

and rewriting the terms in (4.2) associated with $P_{N_{cr}}$ using the new variable, we obtain

$$
\begin{aligned}
J^{\varepsilon}_{m,n^*}(\sigma,\mathcal{F}_\sigma,\xi,F)=&\frac{1}{(i\varepsilon)^m(-i\varepsilon)^n}\int_{\sigma_{2k}(t)}dsdu\int_{\mathbb{R}^{kd}}dw\prod_{j:w_j\neq P_{N_{cr}}}\frac{\hat{R}(w_j)}{(2\pi)^d}e^{-\mathfrak{g}(w_j)\frac{v_j^+-v_j^-}{\varepsilon^2}}\\
&\times\frac{\hat{R}(\xi-P_1-\ldots-P_{N_{cr}-1}-\varepsilon^\alpha P_{N_{cr}})}{(2\pi)^d}e^{-\mathfrak{g}(\xi-P_1-\ldots-P_{N_{cr}-1}-\varepsilon^\alpha P_{N_{cr}})\frac{r^+_{N_{cr}}-r^-_{N_{cr}}}{\varepsilon^2}}\\
&\times\prod_{j=1}^{N_{cr}-1}\left(e^{i\mathfrak{s}_j(|\xi-\ldots-P_{j-1}|^2-|\xi-\ldots-P_j|^2)\frac{r_j^+-r_j^-}{2\varepsilon^2}}\right)e^{i\mathfrak{s}_{N_{cr}}(|\xi-\ldots-P_{N_{cr}-1}|^2-|\varepsilon^\alpha P_{N_{cr}}|^2)\frac{r^+_{N_{cr}}-r^-_{N_{cr}}}{2\varepsilon^2}}\\
&\times\prod_{j=0}^{N_{cr}-1}\left(\prod_{l\in\mathcal{A}_j\cup\mathcal{B}_j}e^{i\tau_l(|\xi-\ldots-P_j|^2-|\xi-\ldots-P_j-w_l|^2)\frac{v_l^+-v_l^-}{2\varepsilon^2}}\right)\prod_{l\in\mathcal{A}_{N_{cr}}\cup\mathcal{B}_{N_{cr}}}e^{i\tau_l(|\varepsilon^\alpha P_{N_{cr}}|^2-|\varepsilon^\alpha P_{N_{cr}}-w_l|^2)\frac{v_l^+-v_l^-}{2\varepsilon^2}}\\
&\times|\hat{\phi}_0(P_{N_{cr}})|^2.
\end{aligned}
$$

Now, we freeze $r_1^-\geq r_2^-\geq\ldots\geq r^-_{N_{cr}}$, integrate out the other time variables and send $\varepsilon\to0$ to obtain

$$
\begin{aligned}
&J^{\varepsilon}_{m,n^*}(\sigma,\mathcal{F}_\sigma,\xi,F)\\
\to&(-1)^{n-k}\int_{\Delta_{N_{cr}}(t)}dv\int_{\mathbb{R}^{kd}}dw\prod_{j=1}^{N_{cr}-1}\frac{1}{(2\pi)^d}\frac{\hat{R}(P_j)}{\mathfrak{g}(P_j)-i\mathfrak{s}_j(|\xi-\ldots-P_{j-1}|^2-|\xi-\ldots-P_j|^2)/2}\\
&\times\prod_{j=0}^{N_{cr}-1}\prod_{l\in\mathcal{A}_j\cup\mathcal{B}_j}\frac{1}{(2\pi)^d}\frac{\hat{R}(w_l)}{\mathfrak{g}(w_l)-i\tau_l(|\xi-\ldots-P_j|^2-|\xi-\ldots-P_j-w_l|^2)/2}\\
&\times\frac{1}{(2\pi)^d}\frac{\hat{R}(\xi-P_0-\ldots-P_{N_{cr}-1})}{\mathfrak{g}(\xi-\ldots-P_{N_{cr}-1})-i\mathfrak{s}_{N_{cr}}|\xi-\ldots-P_{N_{cr}-1}|^2/2}\prod_{l\in\mathcal{A}_{N_{cr}}\cup\mathcal{B}_{N_{cr}}}\frac{1}{(2\pi)^d}\frac{\hat{R}(w_l)}{\mathfrak{g}(w_l)+i\tau_l|w_l|^2/2}\\
&\times|\hat{\phi}_0(P_{N_{cr}})|^2\prod_{j=0}^{N_{cr}}\frac{(v_j-v_{j+1})^{|\mathcal{A}_j|+|\mathcal{B}_j|}}{(|\mathcal{A}_j|+|\mathcal{B}_j|)!}.
\end{aligned}
$$

Here, we have changed the notation $r_i^- \mapsto v_i$, with $v_0=t, v_{N_{cr}+1}=0$. Next, we integrate out w_l except for $P_1,\ldots,P_{N_{cr}}$, so that

$$
\begin{aligned}
&J^{\varepsilon}_{m,n^*}(\sigma,\mathcal{F}_\sigma,\xi,F)\\
\to&(-1)^{n-k}\int_{\Delta_{N_{cr}}(t)}dv\int_{\mathbb{R}^{N_{cr}d}}dP\prod_{j=1}^{N_{cr}-1}\frac{1}{(2\pi)^d}\frac{\hat{R}(P_j)}{g(P_j)-i\mathfrak{s}_j(|\xi-\ldots-P_{j-1}|^2-|\xi-\ldots-P_j|^2)/2}\\
&\times\prod_{j=0}^{N_{cr}-1}(D(\xi-\ldots-P_j)/2)^{|\mathcal{A}_j|}(D^*(\xi-\ldots-P_j)/2)^{|\mathcal{B}_j|}(D(0)/2)^{|\mathcal{A}_{N_{cr}}|}(D^*(0)/2)^{|\mathcal{B}_{N_{cr}}|}\\
&\times\frac{\hat{R}(\xi-\ldots-P_{N_{cr}-1})}{g(\xi-\ldots-P_{N_{cr}-1})-i\mathfrak{s}_{N_{cr}}|\xi-\ldots-P_{N_{cr}-1}|^2/2}|\hat{\phi}_0(P_{N_{cr}})|^2\prod_{j=0}^{N_{cr}}\frac{(v_j-v_{j+1})^{|\mathcal{A}_j|+|\mathcal{B}_j|}}{(|\mathcal{A}_j|+|\mathcal{B}_j|)!}
\end{aligned}
\tag{4.3}
$$

Therefore, we have

$$
\lim_{\varepsilon\to 0}\mathbb{E}\{|\Psi_\varepsilon(t,\xi)|^2\}=\sum_{m,n=0}^{\infty}\sum_{\sigma:N_{cr}\geq 1}\lim_{\varepsilon\to 0}J^{\varepsilon}_{m,n^*}(\sigma,\mathcal{F}_\sigma,\xi,F)
$$

with

$$
\lim_{\varepsilon\to 0}J^{\varepsilon}_{m,n^*}(\sigma,\mathcal{F}_\sigma,\xi,F)
$$

given by the RHS of (4.3). It is clear that $n-N_{cr}$ is even, so that $(-1)^{n-k}=(-1)^{k-N_{cr}}$ and we also note that

$$
k-N_{cr}=\sum_{i=0}^{N_{cr}}(|\mathcal{A}_i|+|\mathcal{B}_i|).
$$

When those crossing edges and $|\mathcal{A}_j|,|\mathcal{B}_j|$ are fixed for $j=0,\ldots,N_{cr}$ (so the RHS of (4.3) is fixed), the total number of possible permutations is

$$
\prod_{j=0}^{N_{cr}}\frac{(|\mathcal{A}_j|+|\mathcal{B}_j|)!}{|\mathcal{A}_j|!|\mathcal{B}_j|!}.
$$

Now, we can sum over all permutations when N_{cr} is fixed, denoted by $\sigma_{N_{cr}}$, and integrate in $P_{N_{cr}}$ and obtain

$$
\begin{aligned}
&\sum_{\sigma_{N_{cr}}} \lim_{\varepsilon\to 0} J^{\varepsilon}_{m,n^*}(\sigma, \mathcal{F}_\sigma, t, \xi) \\
=&\|\hat{\phi}_0\|_2^2 \sum_{|\mathcal{A}_1|,|\mathcal{B}_1|=0}^{\infty} \cdots \sum_{|\mathcal{A}_{N_{cr}}|,|\mathcal{B}_{N_{cr}}|=0}^{\infty} \int_{\Delta_{N_{cr}}(t)} dv \int_{\mathbb{R}^{(N_{cr}-1)d}} dP \prod_{j=0}^{N_{cr}} \frac{(-(v_j - v_{j+1}))^{|\mathcal{A}_j|+|\mathcal{B}_j|}}{|\mathcal{A}_j|!|\mathcal{B}_j|!} \\
&\times \left(\prod_{j=1}^{N_{cr}-1} \mathrm{Re}D(P_j, \xi - \ldots - P_{j-1}) \right) \mathrm{Re}D(\xi - \ldots - P_{N_{cr}-1}, \xi - \ldots - P_{N_{cr}-1}) \\
&\times \left(\prod_{j=0}^{N_{cr}-1} (D(\xi - \ldots - P_j)/2)^{|\mathcal{A}_j|} (D^*(\xi - \ldots - P_j)/2)^{|\mathcal{B}_j|} \right) (D(0)/2)^{|\mathcal{A}_{N_{cr}}|} (D^*(0)/2)^{|\mathcal{B}_{N_{cr}}|}.
\end{aligned} \tag{4.4}
$$

After the summation, we get

$$
\begin{aligned}
\sum_{\sigma_{N_{cr}}} \lim_{\varepsilon\to 0} J^{\varepsilon}_{m,n^*}(\sigma, \mathcal{F}_\sigma, t, \xi) = \|\hat{\phi}_0\|_2^2 \int_{\Delta_{N_{cr}}(t)} dv \int_{\mathbb{R}^{(N_{cr}-1)d}} dP \left(\prod_{j=0}^{N_{cr}-1} e^{-(v_j - v_{j+1})\mathrm{Re}D(\xi - \ldots - P_j)} \right) \\
\times e^{-v_{N_{cr}}\mathrm{Re}D(0)} \left(\prod_{j=1}^{N_{cr}-1} \mathrm{Re}D(P_j, \xi - \ldots - P_{j-1}) \right) \mathrm{Re}D(\xi - \ldots - P_{N_{cr}-1}, \xi - \ldots - P_{N_{cr}-1}),
\end{aligned} \tag{4.5}
$$

which can also be written as

$$
\begin{aligned}
\sum_{\sigma_{N_{cr}}} \lim_{\varepsilon\to 0} J^{\varepsilon}_{m,n^*}(\sigma, \mathcal{F}_\sigma, t, \xi) =&\|\hat{\phi}_0\|_2^2 \int_{\Delta_{N_{cr}}(t)} dv \int_{\mathbb{R}^{N_{cr}d}} dP \left(\prod_{j=0}^{N_{cr}} e^{-(v_j - v_{j+1})\mathrm{Re}D(\xi - \ldots - P_j)} \right) \\
&\times \left(\prod_{j=1}^{N_{cr}} \mathrm{Re}D(P_j, \xi - \ldots - P_{j-1}) \right) \delta(\xi - P_1 - \ldots - P_{N_{cr}}).
\end{aligned} \tag{4.6}
$$

Thus, we have

$$
\begin{aligned}
\lim_{\varepsilon\to 0} \mathbb{E}\{|\Psi_\varepsilon(t,\xi)|^2\} =&\|\hat{\phi}_0\|_2^2 \sum_{N_{cr}=1}^{\infty} \int_{\Delta_{N_{cr}}(t)} dv \int_{\mathbb{R}^{N_{cr}d}} dP \left(\prod_{j=0}^{N_{cr}} e^{-(v_j - v_{j+1})\mathrm{Re}D(\xi - \ldots - P_j)} \right) \\
&\times \left(\prod_{j=1}^{N_{cr}} \mathrm{Re}D(P_j, \xi - \ldots - P_{j-1}) \right) \delta(\xi - P_1 - \ldots - P_{N_{cr}}) = \widehat{W}_\delta(t,\xi).
\end{aligned} \tag{4.7}
$$

The proof of Lemma 4.2 is complete.

Convergence of the higher order moments

In this section, we consider convergence of the general moments

$$\mathbb{E}\{\Psi_\varepsilon(t,\xi)^M (\Psi^*_\varepsilon(t,\xi))^N\},$$

for arbitrary $M, N \in \mathbb{N}$. By Lemma 2.1, we can write

$$\mathbb{E}\{\Psi_\varepsilon(t,\xi)^M (\Psi^*_\varepsilon(t,\xi))^N\} = \sum_{m_1,\ldots,n_N=0}^{\infty} \mathbb{E}\{g_{m_1,\varepsilon}\ldots g_{m_M,\varepsilon} g^*_{n_1,\varepsilon}\ldots g^*_{n_N,\varepsilon}\} \tag{4.8}$$

with $g_{n,\varepsilon}(t,\xi) = F_{n,\varepsilon}(t,\xi)$. As for the variance, we only need to consider

$$\lim_{\varepsilon\to 0} J^\varepsilon_{m_1,\ldots,n^*_N}(\sigma, \mathcal{F}_\sigma, \xi, F),$$

for fixed $m_1, \ldots, n_N$ and σ such that $N_c(\mathcal{F}_\sigma) \le 2$. Recall that (2.22) gives

$$\begin{aligned}
&|J^\varepsilon_{m_1,\ldots,n^*_N}(\sigma, \mathcal{F}_\sigma, \xi, F)| \\
\le& \frac{1}{\varepsilon^{2k}} \int_{\sigma_{2k}(t)} dsdu \int_{\mathbb{R}^{2kd}} \prod_{(v_l,v_r)\in\mathcal{F}_\sigma} (2\pi)^{-d} e^{-\mathfrak{g}(w_l)|v_l - v_r|/\varepsilon^2} \delta(w_l + w_r)\hat{R}(w_l) dw_l dw_r \prod_{i=1}^{M} |h_{M,i}| \prod_{j=1}^{N} |h^*_{N,j}|.
\end{aligned}$$

As before, we denote

$$A_i = \{s_{i,1}, \ldots, s_{i,m_i}\}, \quad B_j = \{u_{j,1}, \ldots, u_{j,n_j}\}, \text{ with } i = 1, \ldots, M,\ j = 1, \ldots, N.$$

The pairing $\mathcal{F}_\sigma$ decomposes

$$\{A_i, B_j : i = 1, \ldots, M, j = 1, \ldots, N\}$$

into the connected components. If there exists a component of size one, that is, $N_s(\sigma) = 1$, then, as in the proof of Lemma 4.1, we have a factor of

$$\varepsilon^{-\alpha d/2} |\hat{\phi}_0|(\xi/\varepsilon^\alpha),$$

coming from the corresponding initial condition, which implies that

$$J^\varepsilon_{m_1,\ldots,n^*_N}(\sigma, \mathcal{F}_\sigma, \xi, F) \to 0$$

as $\varepsilon \to 0$ since $\xi \ne 0$.

Thus, we only need to consider the case when

$$N_s(\sigma) = N_c(\sigma) = 2.$$

For any $S_1, S_2 \in \{A_i, B_j : i = 1, \ldots, M, j = 1, \ldots, N\}$ such that $S_1 \leftrightarrow S_2$, the following lemma shows that S_1, S_2 can not be both of type-A or type-B.

Lemma 4.3. *Fix σ and assume $N_c(\sigma) = 2$. If there exists a pair $S_1, S_2 \in \{A_i : i = 1, \ldots, M\}$ or $S_1, S_2 \in \{B_j : j = 1, \ldots, N\}$ such that $S_1 \leftrightarrow S_2$, then*

$$\lim_{\varepsilon\to 0} J^\varepsilon_{m_1,\ldots,n^*_N}(\sigma, \mathcal{F}_\sigma, \xi, F) = 0.$$

Proof. Let us assume that $S_1 = A_{i_1}, S_2 = A_{i_2}$ – the proof for the other case is identical. Then we can write

$$|J^{\varepsilon}_{m_1,\dots,n^*_N}(\sigma,\mathcal{F}_\sigma,\xi,F)| \leq \frac{1}{\varepsilon^{2k}}\int_{\sigma_{2k}(t)} dsdu \int_{\mathbb{R}^{2kd}} \prod_{(v_l,v_r)\in\mathcal{F}_\sigma} e^{-\mathfrak{g}(w_l)|v_l-v_r|/\varepsilon^2}\delta(w_l+w_r)\frac{\hat{R}(w_l)}{(2\pi)^d}dw_l dw_r$$
$$\times |h_{M,i_1}h_{M,i_2}| \prod_{i=1,i\neq i_1,i_2}^{M} |h_{M,i}| \prod_{j=1}^{N} |h^*_{N,j}|.$$

Since $A_{i_1} \leftrightarrow A_{i_2}$ and $N_{cr}(\sigma) = 2$, after integrating in w_r, we have

$$h_{M,i_1} = \varepsilon^{-\alpha d/2}\hat{\phi}_0(\frac{\xi - P}{\varepsilon^\alpha}), \text{ and } h_{M,i_2} = \varepsilon^{-\alpha d/2}\hat{\phi}_0(\frac{\xi + P}{\varepsilon^\alpha}),$$

for some variable

$$P = \sum_j p_{i_1,j} \neq 0,$$

where the range of j in the summation depends on σ. Now we only need to pick some $p_{i_1,j}$ and change this variable so that $(\xi - P)\varepsilon^\alpha \mapsto P$, which leads to

$$|h_{M,i_1}h_{M,i_2}|dp_{i_1,j} \mapsto \varepsilon^{-\alpha d/2}|\hat{\phi}_0(P)|\varepsilon^{-\alpha d/2}|\hat{\phi}_0(\frac{2\xi}{\varepsilon^\alpha} - P)|\varepsilon^{\alpha d}dP = |\hat{\phi}_0(P)\hat{\phi}_0(\frac{2\xi}{\varepsilon^\alpha} - P)|dP. \tag{4.9}$$

Then we perform the change of variables as in the proof of Lemma 2.2 for

$$\prod_{i=1,i\neq i_1,i_2}^{M} |h_{M,i}| \prod_{j=1}^{N} |h^*_{N,j}|,$$

and in the end obtain

$$\begin{aligned}
&|J^{\varepsilon}_{m_1,\dots,n^*_N}(\sigma,\mathcal{F}_\sigma,\xi,F)| \\
\leq&\frac{C^{M+N}}{\varepsilon^{2k}}\int_{[0,t]^{2k}} dsdu \int_{\mathbb{R}^{kd}} \prod_{(v_l,v_r)\in\mathcal{F}_{\sigma,1}} e^{-\mathfrak{g}(w_l)|v_l-v_r|/\varepsilon^2}\frac{\hat{R}(w_l)}{(2\pi)^d} \prod_{(v_l,v_r)\in\mathcal{F}_{\sigma,2}} (2\pi)^{-d}e^{-\mathfrak{g}(z_l)|v_l-v_r|/\varepsilon^2}\hat{R}(z_l) \\
&\times |\hat{\phi}_0(\tilde{w})\hat{\phi}_0(\frac{2\xi}{\varepsilon^\alpha} - \tilde{w})| \prod_{(v_l,v_r)\in\tilde{\mathcal{F}}_{\sigma,2}} |\hat{\phi}_0(w_l)|dwd\tilde{w}.
\end{aligned}$$

Here, as previously, z_l denotes some momentum variables – we will not need their precise form, while $(v_l, v_r) \in \mathcal{F}_{\sigma,1}$ denotes the pairings not affected by the change of variables, and $(v_l, v_r) \in \mathcal{F}_{\sigma,2}$ denotes the affected pairings. Finally, $\tilde{\mathcal{F}}_{\sigma,2}$ corresponds to the affected pairings when we change variables for

$$\prod_{i=1,i\neq i_1,i_2}^{M} |h_{M,i}| \prod_{j=1}^{N} |h^*_{N,j}|,$$

as in the proof of aforementioned Lemma 2.2. We have also changed the notation $P \mapsto \tilde{w}$. Now, after the temporal integration we can apply dominated convergence theorem to obtain

$$J^{\varepsilon}_{m_1,\dots,n^*_N}(\sigma,\mathcal{F}_\sigma,\xi,F) \to 0,$$

due to the factor $\hat{\phi}_0(\frac{2\xi}{\varepsilon^\alpha} - \tilde{w})$. $\square$

By the above discussion, the nontrivial contribution of $J^\varepsilon_{m_1,\ldots,n^*_N}(\sigma, \mathcal{F}_\sigma, \xi, F)$ as $\varepsilon \to 0$ comes only from the cases when $M = N$, the permutation σ is such that

$$N_s(\sigma) = N_c(\sigma) = 2,$$

and all connected components contain both type-A and type-B sets. Let $\Sigma(m_1, \ldots, n^*_N)$ be the set of such permutations. For $\sigma \in \Sigma(m_1, \ldots, n^*_N)$, we have $A_i \leftrightarrow B_{\tilde{i}}, i = 1, \ldots, M$, where $\{\tilde{1}, \ldots, \tilde{M}\}$ is a permutation of $\{1, \ldots, M\}$. We denote the set of σ corresponding to a given $\{\tilde{1}, \ldots, \tilde{M}\}$ by $\Sigma_{\{\tilde{1},\ldots,\tilde{M}\}}(m_1, \ldots, n^*_N)$. It is straightforward to check that

$$\sum_\sigma 1_{\sigma\in\Sigma_{\{\tilde{1},\ldots,\tilde{M}\}}(m_1,\ldots,n^*_N)} J^\varepsilon_{m_1,\ldots,n^*_N}(\sigma, \mathcal{F}_\sigma, \xi, F) = \sum_{\sigma_{m_1,n^*_{\tilde{1}}}} \cdots \sum_{\sigma_{m_M,n^*_{\tilde{M}}}} \prod_{i=1}^M J^\varepsilon_{m_i,n^*_{\tilde{i}}}(\sigma_{m_i,n^*_{\tilde{i}}}, \mathcal{F}_{\sigma_{m_i,n^*_{\tilde{i}}}}, \xi, F),$$

where $\sigma_{m_i,n^*_{\tilde{i}}}$ denotes the permutation of $A_i \cup B_{\tilde{i}}$ which keeps $A_i \leftrightarrow B_{\tilde{i}}$. Now, we can write

$$\begin{aligned}
&\lim_{\varepsilon\to 0} \mathbb{E}\{\Psi_\varepsilon(t,\xi)^M (\Psi^*_\varepsilon(t,\xi))^N\} \\
&= \sum_{m_1,\ldots,n_N=0}^\infty \sum_{\{\tilde{1},\ldots,\tilde{M}\}} \sum_\sigma 1_{\sigma\in\Sigma_{\{\tilde{1},\ldots,\tilde{M}\}}(m_1,\ldots,n^*_N)} \lim_{\varepsilon\to 0} J^\varepsilon_{m_1,\ldots,n^*_N}(\sigma, \mathcal{F}_\sigma, \xi, F) \\
&= \sum_{m_1,\ldots,n_N=0}^\infty \sum_{\{\tilde{1},\ldots,\tilde{M}\}} \sum_{\sigma_{m_1,n^*_{\tilde{1}}}} \cdots \sum_{\sigma_{m_M,n^*_{\tilde{M}}}} \prod_{i=1}^M \lim_{\varepsilon\to 0} J^\varepsilon_{m_i,n^*_{\tilde{i}}}(\sigma_{m_i,n^*_{\tilde{i}}}, \mathcal{F}_{\sigma_{m_i,n^*_{\tilde{i}}}}, \xi, F) \\
&= \sum_{\{\tilde{1},\ldots,\tilde{M}\}} \prod_{i=1}^M \left(\sum_{m_i,n^*_{\tilde{i}}=0}^\infty \sum_{\sigma_{m_i,n^*_{\tilde{i}}}} \lim_{\varepsilon\to 0} J^\varepsilon_{m_i,n^*_{\tilde{i}}}(\sigma_{m_i,n^*_{\tilde{i}}}, \mathcal{F}_{\sigma_{m_i,n^*_{\tilde{i}}}}, \xi, F) \right) = M!\widehat{W}_\delta(t,\xi)^M.
\end{aligned}$$

Here, the last equality comes from Lemma 4.2:

$$\lim_{\varepsilon\to 0} \mathbb{E}\{|\Psi_\varepsilon(t,\xi)|^2\} = \sum_{m_i,n^*_{\tilde{i}}=0}^\infty \sum_{\sigma_{m_i,n^*_{\tilde{i}}}} \lim_{\varepsilon\to 0} J^\varepsilon_{m_i,n^*_{\tilde{i}}}(\sigma_{m_i,n^*_{\tilde{i}}}, \mathcal{F}_{\sigma_{m_i,n^*_{\tilde{i}}}}, \xi, F) = \widehat{W}_\delta(t,\xi).$$

To summarize, we have shown that

$$\lim_{\varepsilon\to 0} \mathbb{E}\{\Psi_\varepsilon(t,\xi)^M (\Psi^*_\varepsilon(t,\xi))^N\} = 1_{M=N} M!\widehat{W}_\delta(t,\xi)^M,$$

for arbitrary $M, N \in \mathbb{N}$. The proof of Theorem 1.2 is complete.

5 The fluctuation analysis

In this section, we prove Theorems 1.3 and 1.4.

Pointwise fluctuation

We begin with Theorem 1.3. Recall that the corrector can be written as

$$\mathcal{U}_\varepsilon(t,\xi) = \varepsilon^{-\alpha d/2} \sum_{n=0}^{\infty} \mathscr{F}_{n,\varepsilon}(t,\xi),$$

and we have previously shown that

$$\lim_{\varepsilon\to 0} \varepsilon^{-\alpha d(M+N)/2} \mathbb{E}\{\mathscr{F}_{m_1,\varepsilon} \dots \mathscr{F}_{m_M,\varepsilon} \mathscr{F}^*_{n_1,\varepsilon} \dots \mathscr{F}^*_{n_N,\varepsilon}\} = \sum_{\sigma: N_s(\mathcal{F}_\sigma)=N_c(\mathcal{F}_\sigma)=2} \lim_{\varepsilon\to 0} J^\varepsilon_{m_1,\dots,n_N^*}(\sigma, \mathcal{F}_\sigma, \varepsilon^\alpha \xi, F), \tag{5.1}$$

when $M+N=2K$ for some $K\in\mathbb{N}$. Let us define

$$\tilde{\Sigma}(m_1,\dots,n_N^*) = \{\sigma : N_s(\mathcal{F}_\sigma) = N_c(\mathcal{F}_\sigma) = 2\}.$$

The constraint

$$N_s(\mathcal{F}_\sigma) = N_c(\mathcal{F}_\sigma) = 2$$

forms pairings over vertices

$$\{C_l : l = 1,\dots,M+N\} = \{A_i, B_j : i = 1,\dots,M, j = 1,\dots,N\}, \tag{5.2}$$

or equivalently the set

$$\{m_i, n_j^* : i = 1,\dots,M, j = 1,\dots,N\}. \tag{5.3}$$

We write

$$\tilde{\Sigma}(m_1,\dots,n_N^*) = \bigcup_{\mathfrak{p}} \tilde{\Sigma}_{\mathfrak{p}}(m_1,\dots,n_N^*),$$

where $\tilde{\Sigma}_{\mathfrak{p}}(m_1,\dots,n_N^*)$ is the set of permutations corresponding to a given pairing $\mathfrak{p}$ over (5.3). Then we can write

$$\sum_{\sigma: N_s(\mathcal{F}_\sigma)=N_c(\mathcal{F}_\sigma)=2} \lim_{\varepsilon\to 0} J^\varepsilon_{m_1,\dots,n_N^*}(\sigma, \mathcal{F}_\sigma, \varepsilon^\alpha \xi, F) = \sum_{\mathfrak{p}} \sum_{\sigma\in\tilde{\Sigma}_{\mathfrak{p}}(m_1,\dots,n_N^*)} \lim_{\varepsilon\to 0} J^\varepsilon_{m_1,\dots,n_N^*}(\sigma, \mathcal{F}_\sigma, \varepsilon^\alpha \xi, F).$$

For a given $\mathfrak{p}$, we assume that pairs have the form $(\mathfrak{p}(l), \mathfrak{p}(\tilde{l}))$ with $l = 1,\dots,K$, where

$$\{\mathfrak{p}(l), \mathfrak{p}(\tilde{l}) : l = 1,\dots,K\} = \{m_i, n_j^* : i = 1,\dots,M, j = 1,\dots,N\}. \tag{5.4}$$

It is straightforward to check that

$$\begin{aligned} &\sum_{\sigma\in\tilde{\Sigma}_{\mathfrak{p}}(m_1,\dots,n_N^*)} J^\varepsilon_{m_1,\dots,n_N^*}(\sigma, \mathcal{F}_\sigma, \varepsilon^\alpha \xi, F) \\ &= \sum_{\sigma(\mathfrak{p}(1),\mathfrak{p}(\tilde{1}))} \dots \sum_{\sigma(\mathfrak{p}(K),\mathfrak{p}(\tilde{K}))} \prod_{l=1}^{K} J^\varepsilon_{\mathfrak{p}(l),\mathfrak{p}(\tilde{l})}(\sigma(\mathfrak{p}(l),\mathfrak{p}(\tilde{l})), \mathcal{F}_{\sigma(\mathfrak{p}(l),\mathfrak{p}(\tilde{l}))}, \varepsilon^\alpha \xi, F), \end{aligned} \tag{5.5}$$

where $\sigma(\mathfrak{p}(l),\mathfrak{p}(\tilde{l}))$ denotes the permutation of $C_i \cup C_j$ such that $C_i \leftrightarrow C_j$ if $\mathfrak{p}(l),\mathfrak{p}(\tilde{l})$ corresponds to C_i, C_j. Now, we have

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\mathbb{E}\{\mathcal{U}_\varepsilon(t,\xi)^M(\mathcal{U}_\varepsilon^*(t,\xi))^N\} = \sum_{m_1,\dots,n_N=0}^{\infty}\sum_{\mathfrak{p}}\sum_{\sigma\in\tilde{\Sigma}_{\mathfrak{p}}(m_1,\dots,n_N^*)}\lim_{\varepsilon\to 0}J^\varepsilon_{m_1,\dots,n_N^*}(\sigma,\mathcal{F}_\sigma,t,\varepsilon^\alpha\xi)\\
&=\sum_{m_1,\dots,n_N=0}^{\infty}\sum_{\mathfrak{p}}\sum_{\sigma(\mathfrak{p}(1),\mathfrak{p}(\tilde{1}))}\cdots\sum_{\sigma(\mathfrak{p}(K),\mathfrak{p}(\tilde{K}))}\prod_{l=1}^{K}\lim_{\varepsilon\to 0}J^\varepsilon_{\mathfrak{p}(l),\mathfrak{p}(\tilde{l})}(\sigma(\mathfrak{p}(l),\mathfrak{p}(\tilde{l})),\mathcal{F}_{\sigma(\mathfrak{p}(l),\mathfrak{p}(\tilde{l}))},\varepsilon^\alpha\xi,F) \qquad (5.6)\\
&=\sum_{\mathfrak{p}}\prod_{l=1}^{K}\left(\sum_{\mathfrak{p}(l),\mathfrak{p}(\tilde{l})=0}^{\infty}\sum_{\sigma(\mathfrak{p}(l),\mathfrak{p}(\tilde{l}))}\lim_{\varepsilon\to 0}J^\varepsilon_{\mathfrak{p}(l),\mathfrak{p}(\tilde{l})}(\sigma(\mathfrak{p}(l),\mathfrak{p}(\tilde{l})),\mathcal{F}_{\sigma(\mathfrak{p}(l),\mathfrak{p}(\tilde{l}))},\varepsilon^\alpha\xi,F)\right).
\end{aligned}$$

Therefore, it is clear that we only need to compute

$$\sum_{m,n=0}^{\infty}\sum_{\sigma}\lim_{\varepsilon\to 0}J^\varepsilon_{m,n^*}(\sigma,\mathcal{F}_\sigma,\varepsilon^\alpha\xi,F) \text{ and } \sum_{m,n=0}^{\infty}\sum_{\sigma}\lim_{\varepsilon\to 0}J^\varepsilon_{m,n}(\sigma,\mathcal{F}_\sigma,\varepsilon^\alpha\xi,F)$$

to obtain $\lim_{\varepsilon\to 0}\mathbb{E}\{\mathcal{U}_\varepsilon(t,\xi)^M(\mathcal{U}_\varepsilon^*(t,\xi))^N\}$. The following lemmas combine to conclude the proof of Theorem 1.3.

The first lemma deals with the "complex-conjugate" moments.

Lemma 5.1. *We have*

$$\sum_{m,n=0}^{\infty}\sum_{\sigma}\lim_{\varepsilon\to 0}J^\varepsilon_{m,n^*}(\sigma,\mathcal{F}_\sigma,\varepsilon^\alpha\xi,F) = \widehat{W}_{\delta,s}(t,0).$$

Proof. Following the proof of Lemma 4.2 with ξ replaced by $\varepsilon^\alpha\xi$, we obtain

$$\begin{aligned}
&\sum_{m,n=0}^{\infty}\sum_{\sigma}\lim_{\varepsilon\to 0}J^\varepsilon_{m,n^*}(\sigma,\mathcal{F}_\sigma,t,\varepsilon^\alpha\xi) \to \|\hat{\phi}_0\|_2\sum_{N_{cr}=1}^{\infty}\int_{\Delta_{N_{cr}}(t)}dv\int_{\mathbb{R}^{N_{cr}d}}dP\\
&\times\left(\prod_{j=0}^{N_{cr}}e^{-(v_j-v_{j+1})\mathrm{Re}D(-P_0-\dots-P_j)}\right)\left(\prod_{j=1}^{N_{cr}}\mathrm{Re}D(P_j,-P_0-\dots-P_{j-1})\right)\delta(-P_1-\dots-P_{N_{cr}}).
\end{aligned}$$

The RHS equals to $\widehat{W}_{\delta,s}(t,0)$, which completes the proof. □

The second lemma address the "non-conjugated" moments.

Lemma 5.2.

$$\sum_{m,n=0}^{\infty}\sum_{\sigma}\lim_{\varepsilon\to 0}J^\varepsilon_{m,n}(\sigma,\mathcal{F}_\sigma,\varepsilon^\alpha\xi,F) = \mathcal{W}_\alpha(t,\xi).$$

Proof. We use the same notation in the proof of Lemma 4.2. Recall that

$$
\begin{aligned}
&J^{\varepsilon}_{m,n}(\sigma,\mathcal{F}_\sigma,\varepsilon^\alpha\xi,F)\\
=&\frac{1}{(i\varepsilon)^{m+n}}\int_{\sigma_{2k}(t)}dsdu\int_{\mathbb{R}^{2kd}}\prod_{(v_l,v_r)\in\mathcal{F}_\sigma}(2\pi)^{-d}e^{-\mathfrak{g}(w_l)|v_l-v_r|/\varepsilon^2}\delta(w_l+w_r)\hat{R}(w_l)dw_ldw_r\\
&\times e^{iG_m(\varepsilon^\alpha\xi,s^{(m)},p^{(m)})/\varepsilon^2}e^{iG_n(\varepsilon^\alpha\xi,u^{(n)},q^{(n)})/\varepsilon^2}\varepsilon^{-\alpha d}\hat{\phi}_0(\frac{\varepsilon^\alpha\xi-p_1-\ldots-p_m}{\varepsilon^\alpha})\hat{\phi}_0(\frac{\varepsilon^\alpha\xi-q_1-\ldots-q_n}{\varepsilon^\alpha}).
\end{aligned}
$$

We only need to consider σ such that the number of crossing edges $N_{cr}\geq 1$. For each crossing edge $(r_i^+,r_i^-), i=1,\ldots,N_{cr}$, we denote the $p-$variable by P_i. After the integration of the delta functions, we obtain

$$
\begin{aligned}
&J^{\varepsilon}_{m,n}(\sigma,\mathcal{F}_\sigma,\varepsilon^\alpha\xi,F)=\frac{1}{(i\varepsilon)^{m+n}}\int_{\sigma_{2k}(t)}dsdu\int_{\mathbb{R}^{kd}}dw\\
&\times\prod_{j=1}^{k}\frac{\hat{R}(w_j)}{(2\pi)^d}e^{-\mathfrak{g}(w_j)\frac{v_j^+-v_j^-}{\varepsilon^2}}\prod_{j=1}^{N_{cr}}e^{i(|\varepsilon^\alpha\xi-\mathfrak{s}_j(P_0+\ldots+P_{j-1})|^2-|\varepsilon^\alpha\xi-\mathfrak{s}_j(P_0+\ldots+P_j)|^2)\frac{r_j^+-r_j^-}{2\varepsilon^2}}\\
&\times\prod_{j=0}^{N_{cr}}\left(\prod_{l\in\mathcal{A}_j}e^{i(|\varepsilon^\alpha\xi-\ldots-P_j|^2-|\varepsilon^\alpha\xi-\ldots-P_j-w_l|^2)\frac{v_l^+-v_l^-}{2\varepsilon^2}}\prod_{l\in\mathcal{B}_j}e^{i(|\varepsilon^\alpha\xi+\ldots+P_j|^2-|\varepsilon^\alpha\xi+\ldots+P_j-w_l|^2)\frac{v_l^+-v_l^-}{2\varepsilon^2}}\right)\\
&\times\prod_{j=1}^{N_{cr}}e^{-i(|P_j|^2+2P_j\cdot(P_0+\ldots+P_{j-1}))\frac{r_j^-}{\varepsilon^2}}\frac{1}{\varepsilon^{\alpha d}}\hat{\phi}_0(\frac{\varepsilon^\alpha\xi-P_0-\ldots-P_{N_{cr}}}{\varepsilon^\alpha})\hat{\phi}_0(\frac{\varepsilon^\alpha\xi+P_0+\ldots+P_{N_{cr}}}{\varepsilon^\alpha}).
\end{aligned}
\tag{5.7}
$$

Compared to (4.2), the key difference is that we get an extra factor with a large phase:

$$
\prod_{j=1}^{N_{cr}}e^{-i(|P_i|^2+2P_i\cdot(P_0+\ldots+P_{i-1}))r_j^-/\varepsilon^2}.
$$

To get rid of the factor $\varepsilon^{-\alpha d}$, we change the variable

$$
P_{N_{cr}}\mapsto -P_0-\ldots-P_{N_{cr}-1}+\varepsilon^\alpha P_{N_{cr}}.
$$

Rewriting the terms in (5.7) associated with $P_{N_{cr}}$ using the new variable gives

$$
\begin{aligned}
&J^{\varepsilon}_{m,n}(\sigma,\mathcal{F}_\sigma,\varepsilon^\alpha\xi,F)=\frac{1}{(i\varepsilon)^{2k}}\int_{\sigma_{2k}(t)}dsdu\int_{\mathbb{R}^{kd}}dw\\
&\prod_{j:w_j\neq P_{N_{cr}}}\frac{\hat{R}(w_j)}{(2\pi)^d}e^{-\mathfrak{g}(w_j)\frac{v_j^+-v_j^-}{\varepsilon^2}}\frac{\hat{R}(-P_0-\ldots-P_{N_{cr}-1}+\varepsilon^\alpha P_{N_{cr}})}{(2\pi)^d}e^{-\mathfrak{g}(-P_0-\ldots-P_{N_{cr}-1}+\varepsilon^\alpha P_{N_{cr}})\frac{r^+_{N_{cr}}-r^-_{N_{cr}}}{\varepsilon^2}}\\
&\times\prod_{j=1}^{N_{cr}-1}\left(e^{i(|\varepsilon^\alpha\xi-\mathfrak{s}_j(P_0+\ldots+P_{j-1})|^2-|\varepsilon^\alpha\xi-\mathfrak{s}_j(P_0+\ldots+P_j)|^2)\frac{r_j^+-r_j^-}{2\varepsilon^2}}\right)\\
&\times e^{i(|\varepsilon^\alpha\xi-\mathfrak{s}_{N_{cr}}(P_0+\ldots+P_{N_{cr}-1})|^2-|\varepsilon^\alpha\xi-\mathfrak{s}_{N_{cr}}\varepsilon^\alpha P_{N_{cr}}|^2)\frac{r^+_{N_{cr}}-r^-_{N_{cr}}}{2\varepsilon^2}}\\
&\times\prod_{j=0}^{N_{cr}-1}\left(\prod_{l\in\mathcal{A}_j}e^{i(|\varepsilon^\alpha\xi-\ldots-P_j|^2-|\varepsilon^\alpha\xi-\ldots-P_j-w_l|^2)\frac{v_l^+-v_l^-}{2\varepsilon^2}}\prod_{l\in\mathcal{B}_j}e^{i(|\varepsilon^\alpha\xi+\ldots+P_j|^2-|\varepsilon^\alpha\xi+\ldots+P_j-w_l|^2)\frac{v_l^+-v_l^-}{2\varepsilon^2}}\right)\\
&\times\prod_{l\in\mathcal{A}_{N_{cr}}}e^{i(|\varepsilon^\alpha\xi-\varepsilon^\alpha P_{N_{cr}}|^2-|\varepsilon^\alpha\xi-\varepsilon^\alpha P_{N_{cr}}-w_l|^2)\frac{v_l^+-v_l^-}{2\varepsilon^2}}\prod_{l\in\mathcal{B}_{N_{cr}}}e^{i(|\varepsilon^\alpha\xi+\varepsilon^\alpha P_{N_{cr}}|^2-|\varepsilon^\alpha\xi+\varepsilon^\alpha P_{N_{cr}}-w_l|^2)\frac{v_l^+-v_l^-}{2\varepsilon^2}}\\
&\times\prod_{j=1}^{N_{cr}-1}e^{-i(|P_j|^2+2P_j\cdot(P_0+\ldots+P_{j-1}))\frac{r_j^-}{\varepsilon^2}}\hat{\phi}_0(\xi-P_{N_{cr}})\hat{\phi}_0(\xi+P_{N_{cr}})\\
&\times e^{-i(|-P_0-\ldots-P_{N_{cr}-1}+\varepsilon^\alpha P_{N_{cr}}|^2+2(-P_0-\ldots-P_{N_{cr}-1}+\varepsilon^\alpha P_{N_{cr}})\cdot(P_0+\ldots+P_{N_{cr}-1}))\frac{r^-_{N_{cr}}}{\varepsilon^2}}.
\end{aligned}\tag{5.8}
$$

If we freeze $r_1^-,\ldots,r^-_{N_{cr}},P_1,\ldots,P_{N_{cr}}$, integrate out the other variables, and send $\varepsilon\to0$, we see that

$$\lim_{\varepsilon\to0}|J^{\varepsilon}_{m,n}(\sigma,\mathcal{F}_\sigma,\varepsilon^\alpha\xi,F)-H^{\varepsilon}_{m,n}(\sigma,\mathcal{F}_\sigma,\varepsilon^\alpha\xi,F)|=0,$$

with

$$
\begin{aligned}
&H^{\varepsilon}_{m,n}(\sigma,\mathcal{F}_\sigma,\varepsilon^\alpha\xi,F)=\frac{1}{(-1)^k}\int_{\Delta_{N_{cr}}(t)}dv\int_{\mathbb{R}^{N_cd}}dP\\
&\left(\prod_{j=1}^{N_{cr}-1}D(P_j,-P_0-\ldots-P_{j-1})/2\right)D(-P_0-\ldots-P_{N_{cr}-1},-P_0-\ldots-P_{N_{cr}-1}))/2\\
&\times\left(\prod_{j=0}^{N_{cr}-1}(D(P_0+\ldots+P_j)/2)^{|\mathcal{A}_j|+|\mathcal{B}_j|}\frac{(v_j-v_{j+1})^{|\mathcal{A}_j|+|\mathcal{B}_j|}}{(|\mathcal{A}_j|+|\mathcal{B}_j|)!}\right)(D(0)/2)^{|\mathcal{A}_{N_{cr}}|+|\mathcal{B}_{N_{cr}}|}\\
&\times\frac{(v_{N_{cr}}-v_{N_{cr}+1})^{|\mathcal{A}_{N_{cr}}|+|\mathcal{B}_{N_{cr}}|}}{(|\mathcal{A}_{N_{cr}}|+|\mathcal{B}_{N_{cr}}|)!}\prod_{j=1}^{N_{cr}-1}e^{-i(|P_j|^2+2P_j\cdot(P_0+\ldots+P_{j-1}))\frac{v_j}{\varepsilon^2}}\hat{\phi}_0(\xi-P_{N_{cr}})\hat{\phi}_0(\xi+P_{N_{cr}})\\
&\times e^{-i(|-P_0-\ldots-P_{N_{cr}-1}+\varepsilon^\alpha P_{N_{cr}}|^2+2(-P_0-\ldots-P_{N_{cr}-1}+\varepsilon^\alpha P_{N_{cr}})\cdot(P_0+\ldots+P_{N_{cr}-1}))\frac{v_{N_{cr}}}{\varepsilon^2}}.
\end{aligned}\tag{5.9}
$$

Here, we used the property

$$D(\xi)=D(-\xi)\text{ and }D(p,\xi)=D(-p,-\xi).$$

We will consider separately the cases $N_{cr} \geq 2$ and $N_{cr} = 1$.

Multiple scattering $N_{cr} \geq 2$. When $N_{cr} \geq 2$, we have at least one oscillatory phase in (5.9), since

$$\begin{aligned}&\prod_{j=1}^{N_{cr}-1} e^{-i(|P_j|^2+2P_j\cdot(P_0+\ldots+P_{j-1}))\frac{v_j}{\varepsilon^2}}\\&\times e^{-i(|-P_0-\ldots-P_{N_{cr}-1}+\varepsilon^\alpha P_{N_{cr}}|^2+2(-P_0-\ldots-P_{N_{cr}-1}+\varepsilon^\alpha P_{N_{cr}})\cdot(P_0+\ldots+P_{N_{cr}-1}))\frac{v_{N_{cr}}}{\varepsilon^2}} = e^{-i|P_1|^2\frac{v_1}{\varepsilon^2}}X\end{aligned} \tag{5.10}$$

with $|X| = 1$ and independent of v_1. For the integral in v, we have

$$\begin{aligned}&|\int_{\Delta_{N_{cr}}(t)} dv \prod_{j=0}^{N_{cr}} (v_j - v_{j+1})^{|\mathcal{A}_j|+|\mathcal{B}_j|} e^{-i|P_1|^2\frac{v_1}{\varepsilon^2}} X|\\ \leq &C \int_{\Delta_{N_{cr}-1}(t)} \prod_{j=2}^{N_{cr}} dv_j | \int_{v_2}^t (t-v_1)^{|\mathcal{A}_0|+|\mathcal{B}_0|}(v_1-v_2)^{|\mathcal{A}_1|+|\mathcal{B}_1|} e^{-i|P_1|^2\frac{v_1}{\varepsilon^2}} dv_1|\end{aligned} \tag{5.11}$$

for some C. Applying the Riemann-Lebesgue lemma gives

$$| \int_{v_2}^t (t-v_1)^{|\mathcal{A}_0|+|\mathcal{B}_0|}(v_1-v_2)^{|\mathcal{A}_1|+|\mathcal{B}_1|} e^{-i|P_1|^2\frac{v_1}{\varepsilon^2}} dv_1| \to 0, \tag{5.12}$$

provided that $P_1 \neq 0$. Thus, by the dominated convergence theorem, we obtain

$$\int_{\Delta_{N_{cr}}(t)} dv \prod_{j=0}^{N_{cr}} (v_j - v_{j+1})^{|\mathcal{A}_j|+|\mathcal{B}_j|} e^{-i|P_1|^2\frac{v_1}{\varepsilon^2}} X \to 0, \tag{5.13}$$

when $P_1 \neq 0$, which implies

$$H^\varepsilon_{m,n}(\sigma, \mathcal{F}_\sigma, \varepsilon^\alpha\xi, F) \to 0, \text{ as } \varepsilon \to 0,$$

if $N_{cr} \geq 2$.

Single scattering $N_{cr} = 1$. When $N_{cr} = 1$, (5.9) simplifies to

$$\begin{aligned}H^\varepsilon_{m,n}(\sigma, \mathcal{F}_\sigma, \varepsilon^\alpha\xi, F) &= \frac{D(0,0)}{2(-1)^k} \int_0^t dv \int_{\mathbb{R}^d} dP \hat{\phi}_0(\xi - P)\hat{\phi}_0(\xi + P) e^{-i|\varepsilon^\alpha P|^2\frac{v}{\varepsilon^2}}\\ &\times \left(\frac{D(0)}{2}\right)^{|\mathcal{A}_0|+|\mathcal{B}_0|+|\mathcal{A}_1|+|\mathcal{B}_1|} \frac{(t-v)^{|\mathcal{A}_0|+|\mathcal{B}_0|}}{(|\mathcal{A}_0|+|\mathcal{B}_0|)!}\frac{v^{|\mathcal{A}_1|+|\mathcal{B}_1|}}{(|\mathcal{A}_1|+|\mathcal{B}_1|)!}.\end{aligned} \tag{5.14}$$

If $\alpha \in (0,1)$, we have a large phase factor $e^{i|P|^2 v/\varepsilon^{2-2\alpha}}$, so for the same reason as for $N_{cr} \geq 2$, we have

$$H^\varepsilon_{m,n}(\sigma, \mathcal{F}_\sigma, \varepsilon^\alpha\xi, F) \to 0,$$

which implies

$$J^\varepsilon_{m,n}(\sigma, \mathcal{F}_\sigma, \varepsilon^\alpha\xi, F) \to 0.$$

If $\alpha = 1$, we have

$$\begin{aligned} H^{\varepsilon}_{m,n}(\sigma, \mathcal{F}_\sigma, \varepsilon^\alpha \xi, F) = & \frac{D(0,0)}{2(-1)^k} \int_0^t dv \int_{\mathbb{R}^d} dP \hat{\phi}_0(\xi - P)\hat{\phi}_0(\xi + P) e^{-i|P|^2 v} \\ & \times \left(\frac{D(0)}{2}\right)^{|\mathcal{A}_0|+|\mathcal{B}_0|+|\mathcal{A}_1|+|\mathcal{B}_1|} \frac{(t-v)^{|\mathcal{A}_0|+|\mathcal{B}_0|}}{(|\mathcal{A}_0|+|\mathcal{B}_0|)!} \frac{v^{|\mathcal{A}_1|+|\mathcal{B}_1|}}{(|\mathcal{A}_1|+|\mathcal{B}_1|)!}, \end{aligned} \tag{5.15}$$

which is $\varepsilon-$independent. Following the argument we used in the proof of Lemma 4.2, we have

$$\begin{aligned} \sum_\sigma \lim_{\varepsilon \to 0} J^{\varepsilon}_{m,n}(\sigma, \mathcal{F}_\sigma, \varepsilon^\alpha \xi, F) = & \sum_{\sigma : N_{cr}=1} \lim_{\varepsilon \to 0} H^{\varepsilon}_{m,n}(\sigma, \mathcal{F}_\sigma, \varepsilon^\alpha \xi, F) \\ = & - D(0,0) e^{-D(0)t} \int_0^t dv \int_{\mathbb{R}^d} dP \hat{\phi}_0(\xi - P)\hat{\phi}_0(\xi + P) e^{-i|P|^2 v}. \end{aligned} \tag{5.16}$$

Finally, if $\alpha > 1$, similarly, we have

$$\begin{aligned} \sum_\sigma \lim_{\varepsilon \to 0} J^{\varepsilon}_{m,n}(\sigma, \mathcal{F}_\sigma, \varepsilon^\alpha \xi, F) = & \sum_{\sigma : N_{cr}=1} \lim_{\varepsilon \to 0} H^{\varepsilon}_{m,n}(\sigma, \mathcal{F}_\sigma, \varepsilon^\alpha \xi, F) \\ = & - D(0,0) e^{-D(0)t} t \int_{\mathbb{R}^d} dP \hat{\phi}_0(\xi - P)\hat{\phi}_0(\xi + P). \end{aligned} \tag{5.17}$$

The proof of Lemma 5.2 is complete.

Remark 5.3. The proof shows that only single scattering contributes to the "non-conjugated" moments when $\alpha \geq 1$. This is similar to the result obtained for heat equation [1, Theorem 2], where the single scattering constitutes the whole random corrector. For Schrödinger equation, the situation is different, as multiple scatterings show up in "complex-conjugated" moments as in the proof of Lemma 5.1.

Correlation of the fluctuations

Here, we prove Theorem 1.4. Recall that we look at the behavior of

$$W_\varepsilon(t,x,\xi) = \int_{\mathbb{R}^d} \mathcal{U}_\varepsilon(t, \xi + \frac{\varepsilon^\beta \eta}{2}) \mathcal{U}^*_\varepsilon(t, \xi - \frac{\varepsilon^\beta \eta}{2}) e^{i\eta \cdot x} \frac{d\eta}{(2\pi)^d}. \tag{5.18}$$

To prove the convergence of

$$\langle W_\varepsilon(t), \varphi \rangle = \int_{\mathbb{R}^d} W_\varepsilon(t,x,\xi)\varphi^*(x,\xi) dx d\xi$$

in probability, it suffices to show the convergence of

$$\mathbb{E}\{\langle W_\varepsilon(t), \varphi \rangle\}$$

and

$$\mathbb{E}\{|\langle W_\varepsilon(t), \varphi \rangle|^2\}.$$

Given that

$$\mathbb{E}\{\langle W_\varepsilon(t),\varphi\rangle\} = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^{3d}} \mathbb{E}\{\mathcal{U}_\varepsilon(t,\xi+\frac{\varepsilon^\beta\eta}{2})\mathcal{U}^*_\varepsilon(t,\xi-\frac{\varepsilon^\beta\eta}{2})\}e^{i\eta\cdot x}\varphi^*(x,\xi)d\eta dx d\xi$$

and

$$\begin{aligned}\mathbb{E}\{|\langle W_\varepsilon(t),\varphi\rangle|^2\} = \int_{\mathbb{R}^{6d}} &\mathbb{E}\{\mathcal{U}_\varepsilon(t,\xi_1+\frac{\varepsilon^\beta\eta_1}{2})\mathcal{U}^*_\varepsilon(t,\xi_1-\frac{\varepsilon^\beta\eta_1}{2})\mathcal{U}^*_\varepsilon(t,\xi_2+\frac{\varepsilon^\beta\eta_2}{2})\mathcal{U}_\varepsilon(t,\xi_2-\frac{\varepsilon^\beta\eta_2}{2})\}\\ &\times e^{i\eta_1\cdot x_1}\varphi^*(x_1,\xi_1)e^{-i\eta_2\cdot x_2}\varphi(x_2,\xi_2)\frac{d\eta dx d\xi}{(2\pi)^{2d}},\end{aligned}$$

we first prove the following two results.

Lemma 5.4. *If $\alpha+\beta=2$ and $\alpha\in(0,2]$, then as $\varepsilon\to 0$,*

$$\begin{aligned}&\mathbb{E}\{\mathcal{U}_\varepsilon(t,\xi+\frac{\varepsilon^\beta\eta}{2})\mathcal{U}^*_\varepsilon(t,\xi-\frac{\varepsilon^\beta\eta}{2})\}\\ \to &\sum_{N_{cr}=1}^\infty \int_{\Delta_{N_{cr}}(t)} dv \int_{\mathbb{R}^{N_{cr}d}} dP \left(\prod_{j=0}^{N_{cr}} e^{-(v_j-v_{j+1})\mathrm{Re}D(-P_0-\ldots-P_j)}\right)\left(\prod_{j=1}^{N_{cr}}\mathrm{Re}D(P_j,-P_0-\ldots-P_{j-1})\right)\\ &\times\delta(-P_1-\ldots-P_{N_{cr}})\prod_{j=1}^{N_{cr}}e^{iP_j\cdot\eta v_j}\left(1_{\alpha\in(0,2)}\|\hat\phi_0\|_2^2+1_{\alpha=2}\int_{\mathbb{R}^d}\hat\phi_0(\xi+\frac{\eta}{2}-p)\hat\phi_0^*(\xi-\frac{\eta}{2}-p)dp\right).\end{aligned}$$

Lemma 5.5. *If $\xi_1\neq\xi_2$, $\alpha+\beta=2$ and $\alpha\in(0,1)$, then*

$$\begin{aligned}&\lim_{\varepsilon\to0}\mathbb{E}\{\mathcal{U}_\varepsilon(t,\xi_1+\frac{\varepsilon^\beta\eta_1}{2})\mathcal{U}^*_\varepsilon(t,\xi_1-\frac{\varepsilon^\beta\eta_1}{2})\mathcal{U}^*_\varepsilon(t,\xi_2+\frac{\varepsilon^\beta\eta_2}{2})\mathcal{U}_\varepsilon(t,\xi_2-\frac{\varepsilon^\beta\eta_2}{2})\}\\ =&\lim_{\varepsilon\to0}\mathbb{E}\{\mathcal{U}_\varepsilon(t,\xi_1+\frac{\varepsilon^\beta\eta_1}{2})\mathcal{U}^*_\varepsilon(t,\xi_1-\frac{\varepsilon^\beta\eta_1}{2})\}\mathbb{E}\{\mathcal{U}^*_\varepsilon(t,\xi_2+\frac{\varepsilon^\beta\eta_2}{2})\mathcal{U}_\varepsilon(t,\xi_2-\frac{\varepsilon^\beta\eta_2}{2})\}.\end{aligned}$$

The assumption $\alpha+\beta=2$ in Lemmas 5.4 and 5.5 matches the kinetic scaling. To see this, recall that

$$\mathcal{U}_\varepsilon(t,\xi)=\varepsilon^{-\alpha d/2}(\psi_\varepsilon(t,\xi)-\mathbb{E}\{\psi_\varepsilon(t,\xi)\}),$$

and

$$\psi_\varepsilon(t,\xi)=\varepsilon^{\alpha d}\hat\phi(t/\varepsilon^2,\varepsilon^\alpha\xi)e^{i|\varepsilon^\alpha\xi|^2t/2\varepsilon^2}.$$

If we let

$$\mathscr{U}(t,x)=\phi(t,x)-\mathbb{E}\{\phi(t,x)\},$$

then the Wigner transform written in physical domain is

$$\int_{\mathbb{R}^d}\mathscr{U}(\frac{t}{\varepsilon^2},\frac{x}{\varepsilon^{\alpha+\beta}}-\frac{y}{2\varepsilon^\alpha})\mathscr{U}^*(\frac{t}{\varepsilon^2},\frac{x}{\varepsilon^{\alpha+\beta}}+\frac{y}{2\varepsilon^\alpha})e^{i\xi\cdot y}dy,$$

that is, we need $\alpha+\beta=2$ so that the propagation speed is of order one. Note the compensated phase factor from the compensation

$$e^{i\xi\cdot\eta t\varepsilon^{2\alpha+\beta-2}},$$

disappears in the limit when choosing $\alpha+\beta=2$.

Proof of Lemma 5.4. We will use the representation

$$\mathcal{U}_\varepsilon(t,\xi)=\varepsilon^{-\alpha d/2}\sum_{n\geq 1}\mathscr{F}_{n,\varepsilon}(t,\xi),$$

so we only need to consider

$$\mathbb{E}\{\varepsilon^{-\alpha d}\mathscr{F}_{m,\varepsilon}(t,\xi_1)\mathscr{F}^*_{n,\varepsilon}(t,\xi_{-1})\},$$

with

$$\xi_1=\xi+\frac{\varepsilon^\beta\eta}{2},\quad \xi_{-1}=\xi-\frac{\varepsilon^\beta\eta}{2}.$$

Compared to (4.2), we need to change ξ to $\varepsilon^\alpha\xi_1$ or $\varepsilon^\alpha\xi_{-1}$ (the factor ε^α comes from the fact that we are looking at the low frequency regime). Using the notations in the proof of Lemma 4.2, we obtain

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\sum_{m,n\geq 1}\mathbb{E}\{\varepsilon^{-\alpha d}\mathscr{F}_{m,\varepsilon}(t,\xi_1)\mathscr{F}^*_{n,\varepsilon}(t,\xi_{-1})\}\\
=&\lim_{\varepsilon\to 0}\sum_{\sigma:N_{cr}\geq 1}\frac{(-1)^n}{(i\varepsilon)^{m+n}}\int_{\sigma_{2k}(t)}dsdu\int_{\mathbb{R}^{kd}}dw\prod_{j=1}^k\frac{\hat{R}(w_j)}{(2\pi)^d}e^{-\mathfrak{g}(w_j)\frac{v_j^+-v_j^-}{\varepsilon^2}}\\
&\times\prod_{j=1}^{N_{cr}}e^{i\mathfrak{s}_j(|\varepsilon^\alpha\xi_{\mathfrak{s}_j}-\ldots-P_{j-1}|^2-|\varepsilon^\alpha\xi_{\mathfrak{s}_j}-\ldots-P_j|^2)\frac{r_j^+-r_j^-}{2\varepsilon^2}}\\
&\times\prod_{j=0}^{N_{cr}}\left(\prod_{l\in\mathcal{A}_j\cup\mathcal{B}_j}e^{i\tau_l(|\varepsilon^\alpha\xi_{\tau_l}-\ldots-P_j|^2-|\varepsilon^\alpha\xi_{\tau_l}-\ldots-P_j-w_l|^2)\frac{v_l^+-v_l^-}{2\varepsilon^2}}\right)\prod_{j=1}^{N_{cr}}e^{iP_j\cdot\eta r_j^-\varepsilon^{\alpha+\beta-2}}\\
&\times\frac{1}{\varepsilon^{\alpha d}}\hat{\phi}_0(\xi_1-\frac{P_1+\ldots+P_{N_{cr}}}{\varepsilon^\alpha})\hat{\phi}_0^*(\xi_{-1}-\frac{P_1+\ldots+P_{N_{cr}}}{\varepsilon^\alpha}).
\end{aligned}\tag{5.19}$$

Apart from the change $\xi\mapsto\varepsilon^\alpha\xi_{\pm 1}$, the key difference between (5.19) and (4.2) is the extra phase factor

$$\prod_{j=1}^{N_{cr}}e^{iP_j\cdot\eta r_j^-\varepsilon^{\alpha+\beta-2}}$$

due to $\eta\neq 0$. Since $\alpha+\beta=2$, this phase factor becomes

$$\prod_{j=1}^{N_{cr}}e^{iP_j\cdot\eta r_j^-\varepsilon^{\alpha+\beta-2}}\mapsto\prod_{j=1}^{N_{cr}}e^{iP_j\cdot\eta r_j^-},$$

and we only need to follow the proof of Lemma 4.2 to obtain

$$
\begin{aligned}
&\lim_{\varepsilon\to 0}\sum_{m,n\geq 1}\mathbb{E}\{\varepsilon^{-\alpha d}\mathscr{F}_{m,\varepsilon}(t,\xi_1)\mathscr{F}^*_{n,\varepsilon}(t,\xi_{-1})\}\\
=&\sum_{N_{cr}=1}^{\infty}\int_{\Delta_{N_{cr}}(t)}dv\int_{\mathbb{R}^{N_{cr}d}}dP\left(\prod_{j=0}^{N_{cr}}e^{-(v_j-v_{j+1})\mathrm{Re}D(-P_0-\ldots-P_j)}\right)\left(\prod_{j=1}^{N_{cr}}\mathrm{Re}D(P_j,-P_0-\ldots-P_{j-1})\right)\\
&\times\delta(-P_1-\ldots-P_{N_{cr}})\prod_{j=1}^{N_{cr}}e^{iP_j\cdot\eta v_j}\left(1_{\alpha\in(0,2)}\|\hat{\phi}_0\|_2^2+1_{\alpha=2}\int_{\mathbb{R}^d}\hat{\phi}_0(\xi+\frac{\eta}{2}-p)\hat{\phi}_0^*(\xi-\frac{\eta}{2}-p)dp\right).
\end{aligned}
\tag{5.20}
$$

The last factor comes from

$$
\int_{\mathbb{R}^d}\hat{\phi}_0(\xi+\frac{\varepsilon^\beta\eta}{2}-p)\hat{\phi}_0^*(\xi-\frac{\varepsilon^\beta\eta}{2}-p)dp,
$$

and the assumption of $\beta=2-\alpha$. This finishes the proof. □

Proof of Lemma 5.5. The proof is similar to the case when we show the convergence of

$$
\mathbb{E}\{\mathcal{U}_\varepsilon(t,\xi)^M(\mathcal{U}_\varepsilon^*(t,\xi))^N\}\text{ for }M,N\in\mathbb{N}.
$$

The only difference is that ξ is replaced by $\xi_1\pm\frac{\varepsilon^\beta\eta_1}{2}$ and $\xi_2\pm\frac{\varepsilon^\beta\eta_2}{2}$. First, by following the proof of (5.6), we have

$$
\begin{aligned}
&\lim_{\varepsilon\to 0}\mathbb{E}\{\mathcal{U}_\varepsilon(t,\xi_1+\frac{\varepsilon^\beta\eta_1}{2})\mathcal{U}_\varepsilon^*(t,\xi_1-\frac{\varepsilon^\beta\eta_1}{2})\mathcal{U}_\varepsilon^*(t,\xi_2+\frac{\varepsilon^\beta\eta_2}{2})\mathcal{U}_\varepsilon(t,\xi_2-\frac{\varepsilon^\beta\eta_2}{2})\}\\
=&\lim_{\varepsilon\to 0}\mathbb{E}\{\mathcal{U}_\varepsilon(t,\xi_1+\frac{\varepsilon^\beta\eta_1}{2})\mathcal{U}_\varepsilon^*(t,\xi_1-\frac{\varepsilon^\beta\eta_1}{2})\}\mathbb{E}\{\mathcal{U}_\varepsilon^*(t,\xi_2+\frac{\varepsilon^\beta\eta_2}{2})\mathcal{U}_\varepsilon(t,\xi_2-\frac{\varepsilon^\beta\eta_2}{2})\}\\
+&\lim_{\varepsilon\to 0}\mathbb{E}\{\mathcal{U}_\varepsilon(t,\xi_1+\frac{\varepsilon^\beta\eta_1}{2})\mathcal{U}_\varepsilon^*(t,\xi_2+\frac{\varepsilon^\beta\eta_2}{2})\}\mathbb{E}\{\mathcal{U}_\varepsilon^*(t,\xi_1-\frac{\varepsilon^\beta\eta_1}{2})\mathcal{U}_\varepsilon(t,\xi_2-\frac{\varepsilon^\beta\eta_2}{2})\}\\
+&\lim_{\varepsilon\to 0}\mathbb{E}\{\mathcal{U}_\varepsilon(t,\xi_1+\frac{\varepsilon^\beta\eta_1}{2})\mathcal{U}_\varepsilon(t,\xi_2-\frac{\varepsilon^\beta\eta_2}{2})\}\mathbb{E}\{\mathcal{U}_\varepsilon^*(t,\xi_2+\frac{\varepsilon^\beta\eta_2}{2})\mathcal{U}_\varepsilon^*(t,\xi_1-\frac{\varepsilon^\beta\eta_1}{2})\}=I_1+I_2+I_3,
\end{aligned}
$$

and to complete the proof, we only need to show $I_2=I_3=0$.

To study the limit of I_2, we take, for example,

$$
\mathbb{E}\{\mathcal{U}_\varepsilon(t,\xi_1+\varepsilon^\beta\eta_1/2)\mathcal{U}_\varepsilon^*(t,\xi_2+\varepsilon^\beta\eta_2/2)\}.
$$

We may follow the proof of Lemma 5.4 and obtain a phase factor

$$
\prod_{j=1}^{N_{cr}}e^{iP_j\cdot(\xi_1-\xi_2+\varepsilon^\beta(\eta_1-\eta_2)/2)r_j^-\varepsilon^{\alpha-2}},
$$

as in (5.19). Since $\xi_1\neq\xi_2$, the assumption that $\alpha\in(0,2)$ ensures that we have a large phase for multiple scattering; for single scattering, after change of variable $P_1\mapsto\varepsilon^\alpha P_1$, we get a factor

$$
e^{iP_1\cdot(\xi_1-\xi_2+\varepsilon^\beta(\eta_1-\eta_2)/2)r_j^-\varepsilon^{2\alpha-2}},
$$

so we have a large phase if $\alpha \in (0,1)$. In the end, we only need to follow the proof of Lemma 5.2 to conclude that $I_2 = 0$.

For I_3, take, for example,

$$\mathbb{E}\{\mathcal{U}_\varepsilon(t, \xi_1 + \varepsilon^\beta \eta_1/2)\mathcal{U}_\varepsilon(t, \xi_2 - \varepsilon^\beta \eta_2/2)\}.$$

As in the proof of Lemma 5.2, the corresponding phase factor becomes

$$\prod_{j=1}^{N_{cr}} e^{-i(|P_j|^2 - P_j\cdot(\varepsilon^\alpha \xi_1 - \varepsilon^\alpha \xi_2 + \varepsilon^{\alpha+\beta}(\eta_1+\eta_2)/2 - 2(P_0+\ldots+P_{j-1}))r_j^-/\varepsilon^2},$$

as in (5.7). The rest of discussion is the same, that is when $\alpha \in (0,1)$, there is always a large phase, which implies $I_3 = 0$. $\square$

Now we can discuss the limit of W_ε. We use $\mathcal{F}_x, \mathcal{F}_\xi$ to denote the Fourier transform in x, ξ variable respectively. First, by the dominated convergence theorem, we have

$$\lim_{\varepsilon\to 0} \mathbb{E}\{\langle W_\varepsilon(t), \varphi\rangle\} = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^{2d}} \lim_{\varepsilon\to 0} \mathbb{E}\{\mathcal{U}_\varepsilon(t, \xi + \frac{\varepsilon^\beta \eta}{2})\mathcal{U}_\varepsilon^*(t, \xi - \frac{\varepsilon^\beta \eta}{2})\}(\mathcal{F}_x\varphi)^*(\eta, \xi) d\eta d\xi. \tag{5.21}$$

Using Lemma 5.4, we need to discuss the following two cases.

Case 1: $\alpha + \beta = 2, \alpha \in (0,2)$. Using (5.20), we integrate η, ξ in (5.21) to obtain

$$\begin{aligned}
&\lim_{\varepsilon\to 0} \mathbb{E}\{\langle W_\varepsilon(t), \varphi\rangle\} = \|\hat{\phi}_0\|_2^2 \sum_{N_{cr}=1}^{\infty} \int_{\Delta_{N_{cr}}(t)} dv \int_{\mathbb{R}^{N_{cr}d}} dP \\
&\times \left(\prod_{j=0}^{N_{cr}} e^{-(v_j - v_{j+1})\mathrm{Re}D(-P_0-\ldots-P_j)}\right)\left(\prod_{j=1}^{N_{cr}} \mathrm{Re}D(P_j, -P_0 - \ldots - P_{j-1})\right) \\
&\times \delta(-P_1 - \ldots - P_{N_{cr}})(\mathcal{F}_\xi\varphi)^*(-\sum_{j=1}^{N_{cr}} P_j v_j, 0) = \int_{\mathbb{R}^{2d}} \bar{W}_{\delta,s}(t,x,0)\varphi^*(x,\xi) dx d\xi,
\end{aligned}$$

with

$$\begin{aligned}
\bar{W}_{\delta,s}(t,x,\xi) =& \|\hat{\phi}_0\|_2^2 \sum_{N_{cr}=1}^{\infty} \int_{\Delta_{N_{cr}}(t)} dv \int_{\mathbb{R}^{N_{cr}d}} dP \left(\prod_{j=0}^{N_{cr}} e^{-(v_j - v_{j+1})\mathrm{Re}D(\xi-P_0-\ldots-P_j)}\right) \\
&\times \left(\prod_{j=1}^{N_{cr}} \mathrm{Re}D(P_j, \xi - P_0 - \ldots - P_{j-1})\right)\delta(\xi - P_1 - \ldots - P_{N_{cr}})\delta(x - \xi t + \sum_{j=1}^{N_{cr}} P_j v_j).
\end{aligned}$$

Clearly, we have

$$\bar{W}_{\delta,s}(t,x,\xi) = \bar{W}_\delta(t,x,\xi) - \|\hat{\phi}_0\|_2^2 \delta(\xi)\delta(x) e^{-\mathrm{Re}D(0)t},$$

which consists of the scattering component of the transport equation (1.28) with the initial condition

$$\bar{W}_\delta(0,x,\xi) = \|\hat{\phi}_0\|_2^2 \delta(\xi)\delta(x).$$

Case 2: $\alpha = 2, \beta = 0$. By a similar discussion, we have

$$\lim_{\varepsilon \to 0} \mathbb{E}\{\langle W_\varepsilon(t), \varphi \rangle\} = \int_{\mathbb{R}^{2d}} \bar{W}_{\delta,s}(t,x,0)\varphi^*(x,\xi) dx d\xi \tag{5.22}$$

with

$$\bar{W}_{\delta,s}(t,x,\xi) = \bar{W}_\delta(t,x,\xi) - (2\pi)^d \delta(\xi)|\phi_0(x)|^2 e^{-\mathrm{Re}D(0)t},$$

and $\bar{W}_\delta(t,x,\xi)$ solving (1.28) with initial condition $\bar{W}_\delta(0,x,\xi) = (2\pi)^d\delta(\xi)|\phi_0(x)|^2$.

By Lemma 5.5, if we further assume $\alpha \in (0,1)$, we have

$$\lim_{\varepsilon \to 0} \mathbb{E}\{|\langle W_\varepsilon(t), \varphi \rangle|^2\} = |\lim_{\varepsilon \to 0} \mathbb{E}\{\langle W_\varepsilon(t), \varphi \rangle\}|^2, \tag{5.23}$$

which implies $\langle W_\varepsilon(t), \varphi \rangle$ converges in probability.

A Moments of product of Gaussians

The following result is standard, we present a proof for the sake of convenience. We assume that

$$\{N_{ij} : i = 1, \ldots, m, j = 1, \ldots, M_i\}$$

are zero-mean real (complex) Gaussian random variables, and write

$$\mathbb{E}\{\prod_{i=1}^m \prod_{j=1}^{M_i} N_{ij}\} = \sum_{\mathcal{F}} \prod_{((i,j),(\tilde{i},\tilde{j})) \in \mathcal{F}} \mathbb{E}\{N_{ij} N_{\tilde{i}\tilde{j}}\}, \tag{A.1}$$

where $\sum_{\mathcal{F}}$ extends over all pairings formed over vertices $\{(i,j) : i = 1, \ldots, m, j = 1, \ldots, M_i\}$. We set

$$A_i = \{(i,j) : j = 1, \ldots, M_i\}.$$

For a given pairing $\mathcal{F}$ and $i \neq \tilde{i}$, we say that A_i is connected to $A_{\tilde{i}}$, and denote this by $A_i \leftrightarrow A_{\tilde{i}}$, if there exist $j, \tilde{j}$ such that $((i,j),(\tilde{i},\tilde{j})) \in \mathcal{F}$. In this way, the set $\{A_i : i = 1, \ldots, m\}$ is decomposed into connected components, and we denote the size of the smallest component by $N_s(\mathcal{F})$.

Lemma A.1. *For each $i = 1, \ldots, m$, let $X_i = \prod_{j=1}^{M_i} N_{ij}$, then we have*

$$\mathbb{E}\{\prod_{i=1}^m (X_i - \mathbb{E}\{X_i\})\} = \sum_{\mathcal{F}: N_s(\mathcal{F}) \geq 2} \prod_{((i,j),(\tilde{i},\tilde{j})) \in \mathcal{F}} \mathbb{E}\{N_{ij} N_{\tilde{i}\tilde{j}}\} \tag{A.2}$$

Proof. We write

$$\mathbb{E}\{\prod_{i=1}^m (X_i - \mathbb{E}\{X_i\})\} = \mathbb{E}\{X_1 \prod_{i=2}^m (X_i - \mathbb{E}\{X_i\})\} - \mathbb{E}\{X_1\}\mathbb{E}\{\prod_{i=2}^m (X_i - \mathbb{E}\{X_i\})\}, \tag{A.3}$$

and note that every term in the expansion of

$$X_1 \prod_{i=2}^{m} (X_i - \mathbb{E}\{X_i\})$$

is a product of zero-mean Gaussians (with possible multiplicative constant), so when taking expectation, we follow the rule of computing joint moments of zero-mean Gaussians. For any pairing such that A_1 is not connected to any $A_i, i \neq 1$, we have a cancellation from the corresponding term in

$$\mathbb{E}\{X_1\}\mathbb{E}\{\prod_{i=2}^{m} (X_i - \mathbb{E}\{X_i\})\}.$$

Thus, we can write

$$\mathbb{E}\{\prod_{i=1}^{m} (X_i - \mathbb{E}\{X_i\})\} = \mathbb{E}_1\{X_1 \prod_{i=2}^{m} (X_i - \mathbb{E}\{X_i\})\}, \tag{A.4}$$

where $\mathbb{E}_1$ stands for the expectation with the summation over those $\mathcal{F}$ such that $A_1 \leftrightarrow A_i$ for some $i \neq 1$. Following a similar procedure for $X_2 - \mathbb{E}\{X_2\}$, we have

$$\mathbb{E}\{\prod_{i=1}^{m} (X_i - \mathbb{E}\{X_i\})\} = \mathbb{E}_{1,2}\{X_1 X_2 \prod_{i=3}^{m} (X_i - \mathbb{E}\{X_i\})\}, \tag{A.5}$$

with $\mathbb{E}_{1,2}$ stands for the expectation with the summation over those $\mathcal{F}$ such that $A_1 \leftrightarrow A_i$ for some $i \neq 1$ and $A_2 \leftrightarrow A_i$ for some $i \neq 2$. In the end, we obtain

$$\mathbb{E}\{\prod_{i=1}^{m} (X_i - \mathbb{E}\{X_i\})\} = \mathbb{E}_{1,\ldots,m}\{\prod_{i=1}^{m} X_i\}, \tag{A.6}$$

where we only take the expectation with the summation over those $\mathcal{F}$ such that for all $i = 1, \ldots, m$, $A_i \leftrightarrow A_j$ with some $j \neq i$, and these are exactly the pairings with $N_s(\mathcal{F}) \geq 2$. $\Box$

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