Lausanne lecture notes: Branching Brownian motion, parabolic equations and traveling waves

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Contents

The overall plan

The preliminary overall plan for these lectures is as follows.

Lecture 1. A basic introduction to the Fisher-KPP and its connection to branching Brownian motion. Voting models on the BBM genealogical trees and other semilinear parabolic partial differential equations.

Lecture 2. The Bramson shift and long time behavior and convergence to a traveling wave for the solutions to the Fisher-KPP type equations and other semi-linear parabolic PDE. This is a very classical subject, we present a new approach, developed recently in [1] and [2]. Some applications of the Bramson shift for particular initial conditions for the Fisher-KPP equation to the asymptotics of the extremal process of BBM, based on the results of [76].

Lecture 3. Some of the algebraic properties of the pushmi-pullyu fronts and their connections to reactive conservation laws type partial differential equations [2]. The long time behavior of the pushmi-pullyu fronts for the Burgers-FKPP equation [4].

Lecture 4. The shape defect function and the rates of convergence of the solutions to traveling waves [1].

A note on references: we do not attempt to be exhaustive in our references, please consult the original papers we cite for further references. We apologize in advance for the omissions.

1 Lecture 1: BBM and semi-linear parabolic equations

1.1 Overview of the lecture

In this lecture, we will discuss representations for the solutions to semlinear parabolic equations

$$u_t = \Delta u + f(u), \tag{1.1.1}$$ {{23aug806}}

in terms of the branching Brownian motion. Such representation was first discovered by McKean for a special class of nonlinearities $f(u)$, now known as the McKean nonlinearities. In particular, they belong to the wider class of the Fisher-KPP nonlinearities that goes back to the original papers by Fisher [42] and Kolmogorov, Petrovskii and Piskunov [57] that both appeared in 1937. McKean's representation for $u(t,x)$ uses only the positions $X_1(t),\dots,X_{N_t}(t)$ of the BBM particles present at the time t. Recently, Etheridge, Freeman and Penington [39, 40] presented a new and surprising interpretation of the solutions to (1.1.1) with the Allen-Cahn nonlinearity $f(u) = u(1-u)(u-1/2)$ that is neither of the McKean nor of the Fisher-KPP class, in terms of the ternary BBM. Their representation uses both the positions $X_1(t),\dots,X_{N_t}(t)$ and also a voting model on the Galton-Watson tree of the BBM. We discuss in this lecture these representations and the extensions of the Etheredige-Freeman-Penington voting model to a large class of nonlinearities $f(u)$ in (1.1.1). We also present some explicit voting models for some "basic" nonlinearities, such as $f(u) = u^m - u^n$.

The material is mostly based on [3] where an interested reader can find more details and examples, as well as some of the omitted proofs.

1.2 Branching Brownian motion

Models involving branching particles appear very naturally in the context of biological invasions in ecology, as well as in SIR-type models of epidemics. A simple and common process of this type is the binary branching Brownian motion. It is described as follows. A single particle starts at a position $x \in \mathbb{R}^d$ at $t = 0$ and performs a standard Brownian motion. The particle carries an exponential clock at a rate $\beta > 0$, which simply means that the clock rings at a random time $\tau > 0$, with

$$\mathbb{P}(\tau > t) = e^{-\beta t}. \tag{1.2.1}$$ {{sep2702}}

We will often, but not always, assume that the branching rate $\beta = 1$. At the time τ, when the clock rings, the particle splits into two particles that we will refer to as the children, and the original particle is sometimes called the parent. The original particle is removed at the branching event. The two children perform independent standard Brownian motions for $t > \tau$, both of them starting at the position of the branching event. Each of the children carries its own exponential clock, and when the corresponding clock rings, the particle splits into two, and the process continues. Thus, at each time $t > 0$ we have a collection of particles $x_1(t), \dots, x_{N_t}(t)$. Here, N_t is the random number of particles present at the time t.

The above model is usually referred to as a binary Brownian motion since the number of children is limited to two. A simple modification is a process where the particles may produce a random number k of children at each birth event, with the corresponding probabilities p_k, such that

$$\sum_{k=1}^{\infty} p_k = 1. \tag{1.2.2}$$ {{sep2704}}

Then, the average number of off-spring is

$$\bar{N} = \sum_{k=1}^{\infty} k p_k. \tag{1.2.3}$$ {{sep2706}}

We will usually assume that only finitely many p_k are not zero, for simplicity, though this is by no means necessary. In particular, some interesting effects happen if p_k decay slowly as $k \to +\infty$.

To summarize, a branching Brownian motion is characterized by (i) the exponential clock rate $\beta > 0$, (ii) the probabilities p_k to have k children at each branching event, and, finally, (iii) the diffusivity $\sigma > 0$ at which the Brownian motions run.

Remark 1.2.1. All Brownian motions will always have diffusivity $\sigma = \sqrt{2}$.

The total number $N(t)$ of particles present at the time $t > 0$ can be thought of as a pure birth process – $N(t)$ can go up but not down. As a warm-up, let us prove the following.

Proposition 1.2.2. *Let $N(t)$ be the number of particles present in the binary BBM with the exponential clock as in (1.2.1), then*

$$\mathbb{E}(N(t)) = \exp((\bar{N} - 1)t). \tag{1.2.4}$$ {{sep2708}}

Proof. Let us fix some time $t > 0$. Then, we can write the following renewal relation for the expected total number of particles

$$\mathbb{E}(N(t)) = 1 \cdot \mathbb{P}(\tau > t) + \sum_{k=2}^{\infty} k p_k \int_0^t \mathbb{E}(N(t-s))\mathbb{P}(\tau_1 \in ds) = e^{-t} + \bar{N} \int_0^t \mathbb{E}(N(t-s))e^{-s} ds. \tag{1.2.5}$$ {{sep2718}}

Here, the first term in the right side accounts for the case when there is no branching until the time t, and the second for the event when the first branching happens in an interval $[s, s+ds]$ with some $0 < s < t$ and that k children are produced at the first branching. This recursive relation relies crucially on the independence of the off-spring of a given parent. Hence, the function

$$u(t) = \mathbb{E}(N(t)), \tag{1.2.6}$$ {{aug304}}

satisfies an integral equation

$$u(t) = e^{-t} + \bar{N}\int_0^t u(t-s)e^{-s}ds, \tag{1.2.7}$$ {{sep2720}}

and (1.2.5) follows. □

As a side remark, (1.2.7) can be written as an ODE

$$\frac{du}{dt} = (\bar{N}-1)u, \quad u(0) = 1. \tag{1.2.8}$$ {{aug302}}

This is maybe the simplest example of an interpretation of the solutions to a differential equation in terms of a branching Brownian motion. As there is no importance to the spatial positions of the particles, this is, of course, simply a Galton-Watson process. Observe that if our starting point is the ODE (1.2.8), with a given $\bar{N} > 0$, then we can choose any BBM to represent $u(t)$ by (1.2.6), as long as we choose the probabilities p_k so that (1.2.3) holds. This non-uniqueness of a stochastic model behind a deterministic differential equation is a generic phenomenon – there are often many representations for the solutions and it is not always clear which one is the "best" or "natural".

1.2.1 Linear parabolic equations and branching Brownian motion

Let us next explain how we can obtain a probabilistic interpretation for a parabolic equation with a constant zero-order term:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + mu, \tag{1.2.9}$$ {{oct430}}

with some $m \in \mathbb{R}$ fixed. Let us assume that the BBM starts at $t = 0$ at the position x and denote the locations of the BBM particles at the time $t > 0$ by $X_1(t), \dots, X_{N_t}(t)$. Here and below, unless specified otherwise, we will assume that the BBM exponential clock rate $\beta = 1$. Given a bounded function $g(x)$, consider

$$u(t,x) = \mathbb{E}_x \sum_{k=1}^{N_t} g(X_k(t)). \tag{1.2.10}$$ {{oct432}}

For example, to get the expected number of particles, as in (1.2.6), we simply set $g(x) \equiv 1$.

In order to get an equation for $u(t,x)$ let us write a renewal relation, using the independence of the off-spring particles. Looking at the first branching event gives the identity

$$\begin{aligned} u(t,x) &= \mathbb{E}_x(g(B_t))\mathbb{P}(\tau_1 > t) + \sum_{k=1}^{\infty} kp_k \int_0^t \mathbb{E}_x(u(t-s,B_s))\mathbb{P}(\tau_1 \in ds) \\ &= \mathbb{E}_x(g(B_t))e^{-t} + \sum_{k=1}^{\infty} kp_k \int_0^t \mathbb{E}_x(u(t-s,B_s))e^{-s}ds. \end{aligned} \tag{1.2.11}$$ {{oct436}}

Here, as usual, the notation $\mathbb{E}_x$ means that the Brownian motion B_s starts at the position $x \in \mathbb{R}^n$ at the time $t = 0$. Once again, the first term in the right side accounts for the event when there was no branching until the time $t > 0$, and the second for the event that the first branching happened in a time interval $[s, s+ds]$ with some $0 < s < t$. Note that the function

$$v(t,x) = \mathbb{E}_x(g(B_t))$$

is the solution to the heat equation

$$\frac{\partial v}{\partial t} = \Delta v, \tag{1.2.12}$$

with the initial condition $v(0,x) = g(x)$. Hence, it can be written as

$$v(t,x) = [e^{t\Delta} g(\cdot)](x). \tag{1.2.13}$$
{{oct802}}

In addition, for $0 < s < t$ fixed, the function

$$w(\tau,x) = \mathbb{E}_x(u(t-s, B_\tau)) \tag{1.2.14}$$

is the solution to the heat equation

$$\frac{\partial w}{\partial \tau} = \Delta w, \tag{1.2.15}$$

with the initial condition $w(0,x) = u(t-s,x)$. That is, we have

$$w(\tau,x) = [e^{\tau\Delta} u(t-s,\cdot)](x). \tag{1.2.16}$$
{{oct502}}

It follows that

$$\mathbb{E}_x(u(t-s,B_s)) = w(s,x) = [e^{s\Delta} u(t-s,\cdot)](x). \tag{1.2.17}$$
{{oct439}}

Hence, (1.2.11) has the form

$$u(t,x) = e^{-t}[e^{t\Delta} g(\cdot)](x) + \bar{N}\int_0^t [e^{s\Delta} u(t-s,\cdot)](x) e^{-s} ds. \tag{1.2.18}$$
{{oct433}}

This is simply the Duhamel formula for the initial value problem

$$\begin{aligned} &\frac{\partial u}{\partial t} = \Delta u + (\bar{N}-1)u, \\ &u(0,x) = g(x). \end{aligned} \tag{1.2.19}$$
{{oct434}}

This is exactly (1.2.9), with $m = \bar{N} - 1$.

1.2.2 McKean's interpretation of the Fisher-KPP equation

The Fisher-KPP equation

$$\frac{\partial u}{\partial t} = \Delta u + u - u^2 \tag{1.2.20}$$
{{16jul1124}

was introduced in the classical papers by Fisher [42] and Kolmogorov, Petrovskii and Piskunov [57] as a very basic PDE model of spreading in 1937. It was extensively studied and used, without any connection to the probability theory, in many applications where spreading is relevant, from biology to flame propagation. Then, in 1975 Henry McKean in [65] discovered a direct link between the binary branching Brownian motion and the Fisher-KPP equation that we now describe.

Given a bounded function $g(x)$, we consider a functional of the branching Brownian motion not of the additive form (1.2.10) but multiplicative:

$$v(t,x) = \mathbb{E}_x\Big(\prod_{k=1}^{N_t} g(X_k(t))\Big). \tag{1.2.21}$$
{{oct435}}

In order to get an equation for $v(t,x)$ let us again write a renewal relation, very similar to what we have seen in (1.2.11). Because of the product structure of (1.2.21) and since the children at each branching event behave independently, looking at the first branching event gives

$$\begin{aligned} v(t,x) &= \mathbb{E}_x(g(B_t))\mathbb{P}(\tau_1 > t) + \sum_{k=1}^\infty p_k \int_0^t \mathbb{E}_x\Big(\big[v(t-s,B_s)\big]^k\Big)\mathbb{P}(\tau_1 \in ds) \\ &= \mathbb{E}_x(g(B_t))e^{-t} + \sum_{k=1}^\infty p_k \int_0^t \mathbb{E}_x\Big(\big[v(t-s,B_s)\big]^k\Big) e^{-s} ds. \end{aligned} \tag{1.2.22}$$
{{oct437}}

Hence, as in (1.2.17), we have

$$\mathbb{E}_x\big[v(t-s,B_s)\big]^k = [e^{s\Delta}v^k(t-s,\cdot)](x). \tag{1.2.23}$$ {{oct506}}

Recall also (1.2.13):

$$\mathbb{E}_x(g(B_t)) = [e^{t\Delta}g(\cdot)](x). \tag{1.2.24}$$ {{oct804}}

Now, (1.2.22) becomes

$$\begin{aligned} v(t,x) &= [e^{t\Delta}g(\cdot)](x)e^{-t} + \sum_{k=1}^{\infty} p_k \int_0^t [e^{s\Delta}v^k(t-s,\cdot)](x)e^{-s}ds \\ &= [e^{t\Delta}g(\cdot)](x)e^{-t} + \int_0^t [e^{s\Delta}F(v(t-s,\cdot))](x)e^{-s}ds. \end{aligned} \tag{1.2.25}$$ {{oct440}}

This is the Duhamel representation for the initial value problem

$$\begin{aligned} &\frac{\partial v}{\partial t} = \Delta v - v + F(v), \\ &v(0,x) = g(x). \end{aligned} \tag{1.2.26}$$ {{oct441}}

Here, the nonlinearity $F(v)$ is given by the generating function for the branching process:

$$F(v) = \sum_{k=1}^{\infty} p_k v^k. \tag{1.2.27}$$ {{oct442}}

From the PDE point of view, it is often convenient to use instead the function

$$u(t,x) = 1 - v(t,x) = 1 - \mathbb{E}_x\Big(\prod_{k=1}^{N_t} g(X_k(t))\Big). \tag{1.2.28}$$

It satisfies the initial value problem

$$\begin{aligned} &\frac{\partial u}{\partial t} = \Delta u + f(u), \\ &u(0,x) = 1 - g(x). \end{aligned} \tag{1.2.29}$$ {{oct443}}

Here, we have defined

$$f(u) = 1 - u - F(1-u) = 1 - u - \sum_{k=1}^{\infty} p_k(1-u)^k. \tag{1.2.30}$$ {{oct514}}

In the case of the purely binary branching, when $p_2 = 0$ and all other $p_k = 0$, the function $f(u)$ takes the form

$$f(u) = 1 - u - (1-u)^2 = u(1-u).$$

Then, (1.2.29) becomes the classical Fisher-KPP equation (1.2.20):

$$\begin{aligned} &\frac{\partial u}{\partial t} = \Delta u + u - u^2, \\ &u(0,x) = 1 - g(x). \end{aligned} \tag{1.2.31}$$ {{oct445}}

1.2.3 Some basic properties of the Fisher-KPP and McKean nonlinearities

To understand what kind of parabolic equations are related to the branching Brownian motion by the McKean probabilistic interpretation, let us mention some very basic general properties of the nonlinearities $f(u)$ of the form (1.2.30):

$$f(u) = 1 - u - \sum_{k=1}^{\infty} p_k (1-u)^k. \qquad (1.2.32) \quad \{\{\texttt{oct516}\}\}$$

First, because

$$\sum_{k=1}^{\infty} p_k = 1, \qquad (1.2.33) \quad \{\{\texttt{oct522}\}\}$$

we have

$$f(0) = f(1) = 0, \quad f(u) > 0 \text{ for } 0 < u < 1. \qquad (1.2.34) \quad \{\{\texttt{oct518}\}\}$$

Second, $f(u)$ is concave, so that, in particular it satisfies the so-called Fisher-KPP condition

$$f(u) \le f'(0)u, \quad \text{for all } u \in (0,1). \qquad (1.2.35) \quad \{\{\texttt{oct520}\}\}$$

This property of $f(u)$ will be very important in the discussion of the long time behavior of the solutions to (1.2.29).

The above conditions are, clearly, not sufficient for a nonlinearity $f(u)$ to be of the BBM origin but give a good idea of the special properties of nonlinearities in that class.

1.2.4 Not all Fisher-KPP nonlinearities are of the McKean type

As we have noted, the McKean nonlinearities belong to the class of Fisher-KPP nonlinearities, in the sense that they all satisfy (1.2.34)-(1.2.35). On the other hand, it is also well known that solutions to parabolic equations (1.2.29) with an FKPP type $f(u)$ enjoy many special properties that we will discuss later in these notes. However, while the McKean nonlinearities lie in the FKPP class, they form a very special sub-class of that set which excludes many natural examples.

To explain the above point, we first write the McKean nonlinearities in (1.2.32) in the form

$$f(u) = 1 - u - \sum_{k=1}^{\infty} p_k (1-u)^k = \sum_{k=1}^{N} p_k((1-u) - (1-u)^k) = \lambda(u - A(u)), \qquad (1.2.36) \quad \{\{\texttt{mar708}\}\}$$

with

$$\lambda = \sum_{k=1}^{N} k p_k - 1 = \sum_{k=2}^{N} (k-1) p_k = f'(0) > 0, \qquad (1.2.37) \quad \{\{\texttt{aug406}\}\}$$

and the function $A(u)$ defined by

$$\lambda A(u) = \sum_{k=1}^{N} p_k \Big((1-u)^k - 1 + ku \Big). \qquad (1.2.38) \quad \{\{\texttt{aug408}\}\}$$

One can immediately check, using the definition (1.2.37) of λ and (1.2.38), that $A(u)$ is non-negative and convex on $[0,1]$, and

$$A(0) = 0, \qquad A(1) = 1, \qquad A'(0) = 0. \qquad (1.2.39) \quad \{\{\texttt{mar718bis}$$

It follows, in particular, that the function $A(u)$ is increasing on $[0,1]$. In the case of the classical Fisher-KPP equation (1.2.20) with $f(u) = u - u^2$, we have $\lambda = 1$ and $A(u) = u^2$.

We may further write $f(u)$ in the form

$$f(u) = \lambda u(1 - \alpha(u)), \tag{1.2.40}$$ {{mar702}}

with

$$\alpha(u) = \frac{A(u)}{u} = \frac{\beta}{\lambda}\sum_{k=2}^{N} p_k \alpha_k(u). \tag{1.2.41}$$ {{mar704}}

The coefficients $\alpha_k(u)$ are

$$\alpha_k(u) = \frac{(1-u)^k - 1 + ku}{u} = k - \frac{1-(1-u)^k}{u} = k - (1 + (1-u) + \cdots + (1-u)^{k-1}). \tag{1.2.42}$$ {{mar706}}

Note that each term $\alpha_k(u)$ is increasing and concave on $[0,1]$ with $\alpha_k(0) = 0$ and $\alpha_k(1) = k - 1$. It follows from (1.2.37) and (1.2.40)-(1.2.42) that if $f(u)$ is a McKean nonlinearity then

$$\alpha(0) = 0, \quad \alpha(1) = 1, \tag{1.2.43}$$ {{mar716}}

and $\alpha(u)$ is increasing and concave.

As we have mentioned, the standard example of the Fisher-KPP nonlinearity is $f(u) = u - u^2$, which is in the McKean class, as can be seen by setting $p_2 = 1$ and $p_k = 0$ for $k > 2$. However, even the original example

$$f(u) = u(1-u)^2 = u - 2u^2 + u^3 \tag{1.2.44}$$ {{aug2312}}

in the KPP paper [57] is not of that form because the function

$$A(u) = 2u^2 - u^3 \tag{1.2.45}$$

is not convex for $u \in (2/3, 1)$. Or, consider the functions $f_n(u) = u - u^n$ that are of the Fisher-KPP type but the corresponding functions $\alpha_n(u) = u^{n-1}$ are convex and not concave. This means that they are also not of the McKean type. In short, McKean's connection between semi-linear parabolic equations and branching Brownian motion, while incredibly elegant, does not cover all polynomial Fisher-KPP nonlinearities.

1.3 Voting schemes for semi-linear equations

We now describe a different probabilistic interpretation that covers all semilinear parabolic equations with polynomial nonlinearities, via a branching Brownian motion. The presentation is based on our recent paper [3]. We should also mention a very remarkable thesis of O'Dowd [81] where some of these results were also obtained. The approach of [3] is very much inspired by the arguments in the papers by Etherdige, Freeman and Penington [39, 40] for the Allen-Cahn nonlinearity. We first recall that connection in Section 1.3.1. Then, we discuss the random outcome and random threshold voting models that allow us to give a probabilistic interpretation to solutions to equations of the form

$$u_t = \Delta u + f(u), \tag{1.3.1}$$ {{23aug402}}

with any polynomial nonlinearity $f(u)$ such that

$$f(0) = f(1) = 0. \tag{1.3.2}$$ {{aug2625}}

This is done in Sections 1.3.2 and 1.3.5, with the main results given in Theorems 1.3.2 and 1.3.3. Section 1.3.2 also describes an interpretation of the standard heat equation, in terms of a BBM and an unbiased voting model, formulated in Proposition 1.3.1. Finally, in Section 1.3.6 we drop the assumption (1.3.2) and describe a recursive procedure on the genealogical tree that gives a BBM-interpretation for the solutions to any equation of the form (1.3.1) with a polynomial non-linearity $f(u)$. The result is described in Theorem 1.3.4.

1.3.1 The Etheridge-Freeman-Penington model for the Allen-Cahn equation

An alternative connection between semilinear parabolic equations and branching Brownian motion to McKean's was pointed out in a beautiful paper by Etheridge, Freeman, and Penington [39] (see also [36, 58, 78]). One of the main points of the approach of [39, 40] is to consider functionals that depend not just on the locations of the BBM particles, as was done by McKean in [65], but also on the structure of the (random) genealogical tree that results from the branching. These ideas also go back to Sznitman [89]. There is a natural way to associate a random genealogical tree $\mathcal{T}(t)$ to each realization of the BBM running on a time interval $0 \le s \le t$. Each vertex of the tree corresponds to a branching event, while each of the edges coming out of a vertex represents an offspring particle born at that branching event. The root of the tree $\mathcal{T}(t)$ represents the original particle that started at the time $s = 0$ at a position $x \in \mathbb{R}^d$. We refer to [39, 40] for a formal definition of $\mathcal{T}(t)$.

Before introducing a generalization of their ideas, let us recall the example of [39]. Consider a ternary branching Brownian motion starting at the time $t = 0$ at a point $x \in \mathbb{R}^d$ – each branching event produces three children, with probability one. The process is run until a time $t > 0$, with the BBM particles at the time $t > 0$ located at the positions $X_1(t), \dots, X_{N_t}(t)$. Then, each of the youngest generation particles $X_j(t)$, $j = 1, \dots, N_t$, "votes" 0 or 1, with the probabilities

$$\mathbb{P}(V_j = 1) = g(X_j(t)) \quad \text{and} \quad \mathbb{P}(V_j = 0) = 1 - g(X_j(t)). \tag{1.3.3}$$ {{oct616}}

Here, $g(x)$ is a prescribed function such that $0 \le g(x) \le 1$ for all $x \in \mathbb{R}$, and V_j is the vote of the particle $X_j(t)$, $j = 1, \dots, N_t$. This produces the votes of the youngest generation of particles. Next, we go back up the ternary branching tree $\mathcal{T}(t)$, with the rule that each parent accepts the vote of the majority of its three children. In this way, we obtain the votes of all particles on the genealogical tree.

Let V_{orig} be the resulting vote of the original ancestral particle that started at $t = 0$ at the position x, and consider the function

$$u(t, x) = \mathbb{P}_x(V_{\text{orig}} = 1). \tag{1.3.4}$$ {{oct544}}

We now derive an equation for $u(t, x)$ using a similar approach to (1.2.22). There are exactly two possible ways in which the original ancestor can vote 1: either all three of its children voted 1 or two of them voted 1 and one voted 0. In the latter case, there are three choices of the particle that voted 0. If there has been no branching before the time t then the only particle present is the original ancestor, and it takes the vote by itself. This gives the renewal identity

$$\begin{aligned} u(t, x) &= \mathbb{E}_x\big[g(B_t)\big]\mathbb{P}(\tau_1 > t) \\ &\quad + \int_0^t \mathbb{E}_x(u^3(t - s, B_s) + 3u^2(t - s, B_s)(1 - u(t - s, B_s))\mathbb{P}(\tau_1 \in ds) \\ &= \mathbb{E}_x[g(B_t)]e^{-\beta t} + \beta \int_0^t \mathbb{E}_x(u^3(t - s, B_s) + 3u^2(t - s, B_s)(1 - u(t - s, B_s))e^{-\beta s} ds \\ &= e^{(\Delta - \beta)t} g(x) + \beta \int_0^t e^{(\Delta - \beta)s}\big[u^3(t - s, \cdot) + 3u^2(1 - u)(t - s, \cdot)\big](x) ds. \end{aligned} \tag{1.3.5}$$ {{oct545}}

A simple computation shows that, miraculously, this simplifies to

$$u^3 + 3u^2(1-u) - u = 3u^2 - 2u^3 - u = u(3u - 2u^2 - 1) = u(1-u)(2u-1). \tag{1.3.6}$$

We deduce that the function $u(t,x)$ defined by (1.3.4) satisfies the Allen-Cahn equation

$$\begin{aligned} &\frac{\partial u}{\partial t} = \Delta u + \beta u(1-u)(2u-1), \\ &u(0,x) = g(x). \end{aligned} \tag{1.3.7}$$

{{oct547}}

This equation is probably the most standard example of a semi-linear parabolic equation that does not have a McKean connection to BBM and, prior to [39], was believed to have no probabilistic interpretation, to the best of our knowledge. Note that the nonlinearity

$$f(u) = u(1-u)(2u-1) \tag{1.3.8}$$

{{oct827}}

does not satisfy the Fisher-KPP properties we have discussed in Section 1.2.3. Indeed, it is not even non-negative for $u \in (0,1)$ but rather changes its sign. Thus, the Allen-Cahn equation (1.3.7) does not have an interpretation in terms of a McKean functional. The voting scheme idea of [39] adds a genuinely new aspect here and dramatically broadens the class of equations that have an interpretation in terms of the BBM.

Let us also mention another, more geometric,interpretation of the above voting rule. Assume, for simplicity that the initial condition $g(x) = \mathbb{1}(x \le 0)$ is a step function at the origin. Then, we have $V_{orig} = 1$ if and only if one can find a sub-collection $\{Y_k(t)\}$ of the youngest generation of particles, $X_j(t)$, $j = 1, \dots, N_t$, that forms a binary sub-tree of the full ternary tree $\mathcal{T}(t)$, and such that all $Y_k(t) \le 0$. This is further discussed in [64].

1.3.2 Random outcome probabilistic voting models

The voting procedure of [39] that we have described above is deterministic, in the sense that once the genealogical tree $\mathcal{T}(t)$ and the votes of the youngest generation particles $X_1(t), \dots, X_{N_t}(t)$ are fixed, the vote $V_{\rm orig}$ of the original particle is completely determined. However, one can also randomize the voting process itself, in at least two clear ways that we now discuss. We consider a general BBM, with the probabilities p_k to produce k offspring particles at each branching event, with $2 \le k \le N$. As before, we denote by $\mathcal{T}(t)$ the genealogical tree produced by branching events on the time interval $0 \le s \le t$, and by p_k the probability that a parent produces exactly k children at a given branching event.

Let us fix a continuous function $g(x)$ such that $0 \le g(x) \le 1$ for all $x \in \mathbb{R}^d$ and run a branching Brownian motion starting at a position $x \in \mathbb{R}^d$ until a time $t > 0$. At the time t, each of the BBM particles $X_1(t), \dots, X_{N_t}(t)$ votes randomly 0 or 1, with the probability to vote 1 given by

$$\mathbb{P}(\mathrm{Vote}(X_k(t)) = 1) = g(X_k(t)), \quad \text{for each } 1 \le k \le N_t. \tag{1.3.9}$$

{{mar1702}}

In a difference with [39], we also fix a collection of probabilities $0 \le \alpha_{kn} \le 1$, with $0 \le k \le n$, and $n \ge 2$, such that

$$\alpha_{0n} = 0, \quad \alpha_{nn} = 1, \quad \text{for all } n \ge 2. \tag{1.3.10}$$

{{aug2314}}

Given the votes of the particles that are present at the time t, we propagate the vote up the genealogical tree $\mathcal{T}(t)$ as follows. If a parent particle has n children and k out of its n children voted 1, then the parent particle votes 1 with the probability α_{kn}. That is, the vote of the parent is no longer a deterministic function of the votes of its children. Using this rule to go up the tree all the way to the root produces the vote $\mathrm{Vote}_{\mathrm{orig}}$ of the original ancestor particle, and we can, as before, define

$$u(t,x) = \mathbb{P}_x(\mathrm{Vote}_{\mathrm{orig}} = 1). \qquad (1.3.11) \quad \{\{\mathtt{mar1704}\}\}$$

If there was no branching event until the time t, so that $N_t = 1$, then the vote of the original particle is 1 with the probability $g(X_1(t))$. Note that while we allow α_{kn} to be different from 0 and 1, we do impose (1.3.10) which says that if all children voted unanimously, then the parent accepts the vote of the children. We refer to the above as a random outcome voting model.

Similarly to (1.3.5), one can write a renewal equation for the function $u(t,x)$ defined in (1.3.11):

$$\begin{aligned}
u(t,x) &= \mathbb{E}(g(x+B_t))\mathbb{P}(\tau > t) \\
&\quad + \int_0^t \mathbb{E}\Big(\sum_{n=2}^N p_n \sum_{k=0}^n \binom{n}{k} \alpha_{kn} u^k(t-s, x+B_s)(1-u(t-s,x+B_s))^{n-k}\Big)\mathbb{P}(\tau \in ds) \\
&= \mathbb{E}(g(x+B_t))e^{-\beta t} \\
&\quad + \int_0^t \mathbb{E}\Big(\sum_{n=2}^N p_n \sum_{k=0}^n \binom{n}{k} \alpha_{kn} u^k(t-s, x+B_s)(1-u(t-s,x+B_s))^{n-k}\Big)\beta e^{-\beta s} ds \\
&= e^{(\Delta-\beta)t} g(x) + \beta \int_0^t \Big(\sum_{n=2}^N p_n \sum_{k=0}^n \binom{n}{k} \alpha_{kn} e^{(\Delta-\beta)s} u^k(t-s,\cdot)(1-u(t-s,\cdot))^{n-k}(x)\Big)ds.
\end{aligned} \qquad (1.3.12) \quad \{\{\mathtt{mar1706}\}\}$$

The first term on the right in (1.3.12) comes from the event that there was no branching until the time t. The second accounts for the first branching event happening at a time $t \in [s, s+ds]$, with $0 < s < t$. The binomial coefficient counts the number of possibilities to choose the k children who voted 1 out of the n children. Note that (1.3.12) is the Duhamel formulation of the initial value problem

$$\begin{aligned} u_t &= \Delta u + f(u), \\ u(0,x) &= g(x), \end{aligned} \qquad (1.3.13) \quad \{\{\mathtt{mar1708}\}\}$$

with the nonlinearity

$$f(u) = \beta \sum_{n=2}^N p_n \sum_{k=0}^n \binom{n}{k} \alpha_{kn} u^k (1-u)^{n-k} - \beta u = \beta \sum_{n=2}^N p_n \Big(\sum_{k=0}^n \binom{n}{k} \alpha_{kn} u^k (1-u)^{n-k} - u\Big). \qquad (1.3.14) \quad \{\{\mathtt{mar1710}\}\}$$

As we have seen in the Etheridge-Freeman-Penington example, unlike in the McKean interpretation, the nonlinearities produced in this way need not be of the Fisher-KPP type. The advantage of the voting models is precisely in providing a probabilistic interpretation for nonlinear parabolic equations not accessible by the McKean formula.

1.3.3 The standard heat equation and unbiased voting

We now consider some concrete examples of parabolic equations coming from probabilistic voting models, starting with the standard heat equation. Let us first note an elementary identity: for any $n \geq 1$ we have

$$\sum_{k=0}^n \binom{n}{k} \frac{k}{n} u^k (1-u)^{n-k} = u. \qquad (1.3.15) \quad \{\{\mathtt{jan2814}\}\}$$

To see why (1.3.15) holds, we re-write the left side as

$$\begin{aligned}\sum_{k=0}^{n}\frac{n!}{k!(n-k)!}\frac{k}{n}u^k(1-u)^{n-k}&=\sum_{k=1}^{n}\frac{(n-1)!}{(k-1)!(n-k)!}u^k(1-u)^{n-k}\\&=\sum_{k=0}^{n-1}\frac{(n-1)!}{k!(n-1-k)!}u^{k+1}(1-u)^{n-1-k}=u\sum_{k=0}^{n-1}\binom{n-1}{k}u^k(1-u)^{n-1-k}\\&=u(u+1-u)^{n-1}=u.\end{aligned} \tag{1.3.16}$$

{{jan2816}}

Using (1.3.15) in the representation (1.3.14) for $f(u)=0$, we see that taking the probabilities

$$\alpha_{kn}=\frac{k}{n} \tag{1.3.17}$$

{{mar1716}}

in the above voting scheme leads to the standard heat equation: (1.3.13) becomes

$$\begin{aligned}&u_t=\Delta u,\\&u(0,x)=g(x).\end{aligned} \tag{1.3.18}$$

{{mar1718}}

That is, consider any branching Brownian motion, regardless of the branching probabilities p_k, and introduce the voting scheme such that a parent with n children, out of which k voted 1, votes 1 with the "unbiased" probability $\alpha_{kn}=k/n$. Then, the function $u(t,x)$, the probability that the original ancestor particle votes 1, is the solution to the standard heat equation. To the best of our knowledge, even this very simple and intuitive probabilistic interpretation of the heat equation is new. Let us summarize this result as follows.

Proposition 1.3.1. *Let $g(x)$ be a continuous function that satisfies $0\le g(x)\le 1$ for all $x\in\mathbb{R}^d$. Consider the random outcome voting model with the voting probabilities $\alpha_{kn}=k/n$, $0\le k\le n$, for any branching Brownian motion. Then, the function $u(t,x)=\mathbb{P}_x(V_{\rm orig}=1)$ is the solution to the initial value problem*

$$\begin{aligned}&u_t=\Delta u,\quad t>0,\ x\in\mathbb{R}^d,\\&u(0,x)=g(x),\quad x\in\mathbb{R}^d.\end{aligned} \tag{1.3.19}$$

{{aug2626}}

1.3.4 Representing general nonlinearities

Let now $f(u)$ be a polynomial of degree N:

$$f(u)=\sum_{k=0}^{N}f_ku^k, \tag{1.3.20}$$

{{aug2404}}

that vanishes at $u=0$ and $u=1$:

$$f(0)=f(1)=0. \tag{1.3.21}$$

{{aug2406}}

Our goal is to find $\beta>0$ and α_{kN}, $0\le k\le N$ such that

$$\alpha_{0N}=0,\quad \alpha_{NN}=1,\quad 0\le\alpha_{kN}\le 1,\quad \text{for all } 1\le k\le N-1, \tag{1.3.22}$$

{{aug2408}}

and so that representation (1.3.14) holds for $f(u)$. We set all $p_n=0$ except for $p_N=1$, so that each branching event produces exactly N offspring. We look for α_{kN} in the form

$$\alpha_{kN}=\frac{k}{N}+\mu_k\quad \text{for any } 1\le k\le N-1, \tag{1.3.23}$$

{{aug2410}}

with μ_k to be chosen later. Recalling (1.3.14) and (1.3.15), we see that we need to have

$$f(u) = \beta \sum_{k=1}^{N-1} \binom{N}{k} \mu_k u^k (1-u)^{N-k}. \tag{1.3.24}$$ {{aug2412}}

Recall that the Bernstein polynomials

$$B_{k,N}(u) = \binom{N}{k} u^k (1-u)^{N-k}, \quad k = 0, \dots, N, \tag{1.3.25}$$ {{aug2414}}

form a basis for the vector space of polynomials of degree at most N. Therefore, $f(u)$ has a representation

$$f(u) = \sum_{k=0}^{N} b_k[f] B_{k,N}(u), \tag{1.3.26}$$ {{aug2416}}

with some coefficients $b_k[f]$. Note that

$$b_0[f] = f(0), \quad b_N[f] = f(1). \tag{1.3.27}$$ {{aug2418}}

We deduce from (1.3.21) and (1.3.27) that

$$b_0[f] = b_N[f] = 0. \tag{1.3.28}$$ {{aug2420}}

Next, comparing (1.3.24) and (1.3.26), we see that for (1.3.24) to hold, we need to have

$$\mu_k = \frac{b_k[f]}{\beta}, \quad \text{for all } 1 \le k \le N-1. \tag{1.3.29}$$ {{aug2422}}

It remains to choose $\beta > 0$ so that α_{kN} given by (1.3.23) satisfy (1.3.22). Note that, since we have set $\mu_0 = \mu_N = 0$, we automatically have $\alpha_{0N} = 0$ and $\alpha_{NN} = 1$. The rest of the conditions in (1.3.22) translates into

$$0 \le \frac{k}{N} + \mu_k \le 1, \quad \text{for all } 1 \le k \le N-1. \tag{1.3.30}$$ {{aug2423}}

We see from (1.3.29) that this is equivalent to

$$0 \le \frac{k}{N} + \frac{b_k[f]}{\beta} \le 1, \quad \text{for all } 1 \le k \le N-1. \tag{1.3.31}$$ {{aug2424}}

This condition holds as long as we choose $\beta > 0$ sufficiently large, so that

$$\beta \ge N \max_k |b_k[f]|. \tag{1.3.32}$$ {{aug2425}}

Therefore, we have proved the following.

Theorem 1.3.2. *Let $f(u)$ be a polynomial of degree N such that $f(0) = f(1) = 0$. Then, there exists a random outcome voting model representation in terms of a purely N-ary branching Brownian motion for the solution to the initial value problem (1.3.13) with the initial condition $g(x)$ that is continuous and satisfies $0 \le g(x) \le 1$ for all $x \in \mathbb{R}^d$.*

We note that Theorem 1.3.2 has also been observed in [81], although with different terminology and presentation.

1.3.5 A random threshold voting model

An alternative probabilistic voting model has been recently suggested in [64]. For simplicity of notation, let us fix the number N of offspring produced at each branching event and consider a purely N-ary BBM. Then, at each branching event we choose a number $L \in \{1, \dots, N\}$, with the probability $\zeta_k = \mathbb{P}(L = k)$ so that

$$\sum_{k=1}^{N} \zeta_k = 1. \tag{1.3.33}$$ {{aug2427}}

Thus, an integer $L(\nu)$ is assigned separately to each vertex ν of the genealogical tree $\mathcal{T}(t)$. The voting is done as follows. As before, at the time t the youngest generation of particles votes according to (1.3.9). The difference is in the way the votes are propagated up the genealogical tree. A parent at a vertex ν votes 1 if and only if at least $L(\nu)$ of its children voted 1. We refer to this process as a random threshold voting model.

The same argument as in (1.3.12) shows that the function

$$u(t, x) = \mathbb{P}_x(\mathrm{Vote}_{\mathrm{orig}} = 1) \tag{1.3.34}$$ {{aug2429bis

is a solution to the initial value problem

$$\begin{aligned} u_t &= \Delta u + G(u), \\ u(0, x) &= g(x), \end{aligned} \tag{1.3.35}$$ {{aug2430}}

with the nonlinearity

$$G(u) = \beta \sum_{j=0}^{N} \zeta_j \sum_{k=j}^{N} \binom{N}{k} u^k (1-u)^{N-k} - \beta u = \beta \sum_{k=0}^{N} \binom{N}{k} u^k (1-u)^{N-k} \sum_{j=0}^{k} \zeta_j - \beta u. \tag{1.3.36}$$ {{jun2114}}

A simple observation is that if we start with the random threshold voting model and set

$$\alpha_{kN} = \sum_{j=0}^{k} \zeta_j, \tag{1.3.37}$$ {{jun2118}}

in (1.3.14), then the nonlinearities $f(u)$ in (1.3.14), coming from the random outcome model with the probabilities α_{kN}, and $G(u)$ in (1.3.36) are the same. Note that (1.3.33) implies that $0 \le \alpha_{kN} \le 1$ and $\alpha_{NN} = 1$, so α_{kN} satisfy the assumptions that we needed in Section 1.3.2.

On the other hand, given a random outcome voting model of Section 1.3.2, with a collection of probabilities α_{kN} that additionally have the property that the probabilities α_{kN} are increasing in k, then we can obtain a random threshold model by setting

$$\beta_{kN} = \alpha_{kN} - \alpha_{k-1,N}. \tag{1.3.38}$$ {{jun2120}}

Note that

$$\sum_{k=0}^{N} \beta_{kN} = \alpha_{NN} = 1. \tag{1.3.39}$$ {{jun2156}}

Monotonicity of α_{kN} in k is a natural assumption, as it says that the larger number of children voted 1 the higher the probability that the parent votes 1. Moreover, it is easy to see that given a polynomial nonlinearity $f(u)$ satisfying (1.3.21), we can always find a collection of probabilities α_{kN} that is increasing in k and so that (1.3.14) holds. To see that, let us take a nonlinearity $f(u)$ that is a polynomial of degree N such that $f(0) = f(1) = 0$. The construction of the probabilities α_{kN} in the argument leading to Theorem 1.3.2 produces α_{kN} that are increasing in k as long as the branching rate β satisfies the condition

$$\beta \geq 2N \max_k |b_k[f]|, \tag{1.3.40}$$ {{aug2429}}

that is slightly stronger than (1.3.32). This is because if α_{kN} are given by (1.3.23) and (1.3.29), then

$$\begin{aligned} \alpha_{k+1,N} &= \frac{k+1}{N} + \mu_k = \frac{k+1}{N} + \frac{b_{k+1}[f]}{\beta} = \frac{1}{N} + \alpha_{k,N} + \frac{b_{k+1}[f] - b_k[f]}{\beta} \\ &\geq \frac{1}{N} + \alpha_{k,N} - \frac{2}{\beta} \max_k |b_k[f]| \geq \alpha_{k,N}. \end{aligned} \tag{1.3.41}$$ {{aug2428}}

We have proved the following.

Theorem 1.3.3. *Let $f(u)$ be a polynomial of degree N such that $f(0) = f(1) = 0$. Then, there exists a random threshold voting model representation in terms of a purely N-ary branching Brownian motion for the solution to the initial value problem (1.3.13) with the initial condition $g(x)$ that is continuous and satisfies $0 \leq g(x) \leq 1$ for all $x \in \mathbb{R}^d$.*

1.3.6 Recursive up the tree propagation models for other nonlinearities

The random outcome and random threshold voting models apply to equations of the form

$$u_t = \Delta u + f(u), \tag{1.3.42}$$ {{23aug410}}

with nonlinearities $f(u)$ such that

$$f(0) = f(1) = 0. \tag{1.3.43}$$ {{aug2602}}

The reason for this restriction is that the above assumption guarantees that if the initial condition $u(0, x) = g(x)$ satisfies $0 \leq g(x) \leq 1$ for all $x \in \mathbb{R}^d$, then

$$0 < u(t, x) < 1 \text{ for all } t > 0 \text{ and } x \in \mathbb{R}^d. \tag{1.3.44}$$ {{aug2629}}

Thus, it is conceivable that $u(t, x)$ can be interpreted as a probability of some event. For a general polynomial $f(u)$ that does not satisfy (1.3.43), solutions to (1.3.42) do not necessarily satisfy (1.3.44), so there is no reason to expect that they can be interpreted as a probability. However, we can replace the voting model interpretation by a recursive propagation up the genealogical tree $\mathcal{T}(t)$ of the branching Brownian motion that we now describe.

Let

$$f(u) = f_0 + f_1 u + \cdots + f_N u^N \tag{1.3.45}$$ {{aug2604}}

be a polynomial of degree N. Consider the corresponding symmetric polynomial of N variables

$$S_N(u_1, \ldots, u_N) = f_0 + \frac{f_1}{N}(u_1 + \cdots + u_N) + f_2 \binom{N}{2}^{-1} \sum_{k \neq j} u_k u_j + \cdots + f_N \prod_{i=1}^{N} u_i, \tag{1.3.46}$$ {{aug2606}}

so that

$$f(u) = S_N(u, \ldots, u). \tag{1.3.47}$$ {{aug2610}}

To build a solution to (1.3.42) with $f(u)$ as above, we run a purely N-ary BBM, with an exponential clock running at the rate $\beta = 1$, until a time $t > 0$. The particles $X_1, \ldots, X_{N_t}$ that are present at the time t are assigned the random values $u_k = g(X_k(t))$. Here, $g(x)$ is a given continuous function. Then, we propagate the values up the genealogical tree $\mathcal{T}(t)$ by assigning to each parent the value

$$u_{\text{parent}} = S_N(u_1, \dots, u_N) + \frac{u_1 + \dots + u_N}{N}. \tag{1.3.48}$$ {{aug2608}}

Here, $u_1, \dots, u_N$ are the values that have been previously assigned to the N children of the parent under consideration. This recursive procedure allows us to define the value u_{orig} at the root of the tree, the original particle that was present at the time $t = 0$ at the position $x \in \mathbb{R}^d$. We set

$$u(t,x) = \mathbb{E}_x[u_{\text{orig}}]. \tag{1.3.49}$$ {{aug2612}}

Once again, the renewal argument using the independence of the offspring particles, nearly identical to that in (1.3.12)-(1.3.13), shows that $u(t,x)$ satisfies the initial value problem

$$\begin{aligned} &u_t = \Delta u + f(u), \\ &u(0,x) = g(x). \end{aligned} \tag{1.3.50}$$ {{aug2614}}

This gives the following.

Theorem 1.3.4. *Let $f(u)$ be a polynomial of degree N. Then, there exists a recursive up the tree propagation representation in terms of a purely N-ary branching Brownian motion for the solution to the initial value problem (1.3.50) with the initial condition $g(x)$ that is continuous and bounded.*

We point out that this model is deterministic and so it is not a special case of either of the two random voting models above.

1.4 Examples of voting models

In this section, we consider some examples of voting models, beyond the heat equation and the Allen-Cahn equation we have considered above.

1.4.1 Random outcome voting models for the McKean nonlinearities

Let us go back to the McKean type nonlinearities of the form

$$f(u) = \beta\Big(1 - u - \sum_{k=2}^{N} p_k(1-u)^k\Big), \tag{1.4.1}$$ {{23aug412}}

with

$$\sum_{k=2}^{N} p_k = 1. \tag{1.4.2}$$ {{23aug414}}

First, we note the elementary identity:

$$1-(1-u)^n = (1-u+u)^n - (1-u)^n = \sum_{k=0}^{n} \binom{n}{k} u^{n-k}(1-u)^k - (1-u)^n = \sum_{k=0}^{n-1} \binom{n}{k} u^{n-k}(1-u)^k. \tag{1.4.3}$$ {{aug2914}}

This allows us to write a nonlinearity of the form (1.4.1), using (1.4.2) and (1.4.3), as

$$
\begin{aligned}
f(u) &= \beta\Big(1-u-\sum_{n=2}^{N} p_n(1-u)^n\Big) = \beta\sum_{n=2}^{N} p_n\Big(1-(1-u)^n-u\Big)\\
&= \beta\sum_{n=2}^{N} p_n\Big(\sum_{k=0}^{n-1}\binom{n}{k}\,u^{n-k}(1-u)^k-u\Big).
\end{aligned}
\tag{1.4.4}
$$

{{aug2402}}

Comparing to (1.3.14), we see that this corresponds to the random outcome voting model that is not really random: we have $\alpha_{0n}=0$ for all $n\geq 2$, and

$$
\alpha_{kn}=1,\quad \text{for all } 1\leq k\leq n. \tag{1.4.5}
$$

{{aug2433}}

Therefore, the McKean nonlinearities come from a very simple voting rule: the parent particle votes 1 if and only if at least one of its children voted 1. This, of course, agrees with the familiar interpretation of the probability distribution of the maximum of BBM in terms of the solution to the Fisher-KPP equation [20, 26].

1.4.2 Uniformly biased voting models

Next, we introduce a uniform bias in the random outcome voting model (1.3.17) we have obtained for the standard heat equation. A parent with n children, out of which k voted 1, now votes 1 with a "uniformly biased" probability

$$
\alpha_{kn}=\frac{(1+\gamma)k}{n},\quad 0\leq k\leq n-1,\quad \alpha_{nn}=1. \tag{1.4.6}
$$

{{mar1720}}

Here, $\gamma\geq 0$ is a parameter measuring the "bias" toward voting 1 versus voting 0. As we need to have $\alpha_{kn}\leq 1$ for all $1\leq k\leq n-1$, the bias $\gamma>0$ needs to satisfy

$$
\gamma\leq\frac{n}{n-1}-1=\frac{1}{n-1}. \tag{1.4.7}
$$

{{mar1727bis

In particular, if γ is fixed, only finitely many p_n may be non-zero. Using expression (1.4.6) for α_{kn} in (1.3.14) and recalling (1.3.15) gives the corresponding nonlinearity as

$$
\begin{aligned}
f(u) &= \beta\sum_{n=2}^{N} p_n\Big(\sum_{k=0}^{n}\binom{n}{k}\,\alpha_{kn}u^k(1-u)^{n-k}-u\Big)\\
&= \beta\sum_{n=2}^{N} p_n\Big(\sum_{k=0}^{n}\binom{n}{k}\,\frac{(1+\gamma)k}{n}u^k-\gamma u^n-u\Big)=\beta\gamma\sum_{n=2}^{N} p_n(u-u^n).
\end{aligned}
\tag{1.4.8}
$$

{{mar1721}}

Taking $\gamma=\beta^{-1}$ gives

$$
f(u)=u-A(u),\quad A(u)=\sum_{k=2}^{N} p_k u^k. \tag{1.4.9}
$$

{{mar1722}}

As we have seen in Section 1.2.4, these nonlinearities are of the Fisher-KPP type but do not have a McKean representation. Instead, such nonlinearities come from voting models with a uniform bias toward voting 1, as in (1.4.6). They lead to convex functions $A(u)$ such that $\alpha(u)=A(u)/u$ is also convex, which is impossible for the McKean type.

1.4.3 Group voting models

The nonlinearities of the form (1.4.8) have the property that $f'(0) \neq 0$ except in the trivial cases when $\beta = 0$ or $\gamma = 0$. In order to obtain a voting model representation for nonlinearities with derivatives that vanish at $u = 0$, it is convenient to consider voting with a "group-based" bias. Let us fix some $m > 1$ and assume that branching can only happen into $n > m$ children; that is, $p_k = 0$ for all $k \leq m$. The voting scheme is as follows: if a parent has n children, of which k vote 1, and $k < m$, then the parent votes 1 with the unbiased probability

$$\alpha_{n,m}^{(k)} = \frac{k}{n}, \quad \text{if } 0 \leq k < m, \tag{1.4.10}$$ {{jan2725}}

as in (1.3.17). However, if $m \leq k \leq n-1$, so that one can choose a group of m out of n children that all voted 1, then the parent votes 1 with the biased probability

$$\alpha_{n,m}^{(k)}(\gamma) = \frac{k}{n} + \gamma \binom{k}{m} \binom{n}{m}^{-1}, \quad \text{if } m \leq k < n-1, \tag{1.4.11}$$ {{mar1724}}

and, finally, if all children voted 1, then

$$\alpha_{n,m}^{(k)} = 1, \quad \text{if } k = n. \tag{1.4.12}$$ {{mar1725}}

Thus, the bias for the parent to vote 1 relative to the unbiased probability k/n is proportional to the ratio of the number $\binom{k}{m}$ of m-tuples such that all particles in the m-tuple voted 1 to the total number $\binom{n}{m}$ of m-tuples of the n children. This is a generalization of the bias in the voting model (1.4.6) for the nonlinearity $u - u^n$, where the m-tuple is simply a single particle. This leads to the nonlinearity

$$\begin{aligned} f(u) &= \beta \sum_{n=m+1}^{N} p_n \Big(\sum_{k=0}^{n} \binom{n}{k} \alpha_{nm}^{(k)}(\gamma) u^k (1-u)^{n-k} - u \Big) \\ &= \beta\gamma \sum_{n=m+1}^{N} p_n \sum_{k=m}^{n-1} \binom{n}{k} \binom{k}{m} \binom{n}{m}^{-1} u^k (1-u)^{n-k}. \end{aligned} \tag{1.4.13}$$ {{mar1727}}

Here, we used (1.3.15) and (1.4.10)-(1.4.12). We now proceed to simplify the right side of (1.4.13). Expanding the term $(1-u)^{n-k}$ gives

$$\begin{aligned} f(u) &= \beta\gamma \sum_{n=m+1}^{N} p_n \sum_{k=m}^{n-1} \binom{n}{k} \binom{k}{m} \binom{n}{m}^{-1} u^k \Big(\sum_{q=k}^{n} \binom{n-k}{q-k} (-1)^{q-k} u^{q-k} \Big) \\ &= \beta\gamma \sum_{n=m+1}^{N} p_n \sum_{k=m}^{n-1} \sum_{q=k}^{n} \frac{n!}{k!(n-k)!} \frac{k!}{m!(k-m)!} \frac{m!(n-m)!}{n!} \frac{(n-k)!}{(q-k)!(n-q)!} (-1)^{q-k} u^q \\ &= \beta\gamma \sum_{n=m+1}^{N} p_n \sum_{k=m}^{n-1} \sum_{q=k}^{n} \frac{(n-m)!}{(k-m)!(q-k)!(n-q)!} (-1)^{q-k} u^q \\ &= \beta\gamma \sum_{n=m+1}^{N} p_n \sum_{k=m}^{n-1} \sum_{q=k}^{n-1} \frac{(n-m)!}{(k-m)!(q-k)!(n-q)!} (-1)^{q-k} u^q \\ &\quad + \beta\gamma \sum_{n=m+1}^{N} p_n \sum_{k=m}^{n-1} \frac{(n-m)!}{(k-m)!(n-k)!} (-1)^{n-k} u^n = f_1(u) + f_2(u). \end{aligned} \tag{1.4.14}$$ {{mar1728}}

In order to simplify this expression for $f(u)$, we use the identity

$$\sum_{k=0}^{n-1} \frac{(-1)^k n!}{(n-k)!\ k!} = (-1)^{n+1}, \tag{1.4.15}$$ {{jan2624}}

that holds for all $n \geq 1$ and can be obtained by expanding $(1-1)^n$. This allows us to write

$$\begin{aligned}
f_2(u) &= \beta\gamma \sum_{n=m+1}^{N} p_n \sum_{k=m}^{n-1} \frac{(n-m)!}{(k-m)!(n-k)!}(-1)^{n-k}u^n \\
&= \beta\gamma \sum_{n=m+1}^{N} p_n \sum_{k=0}^{n-1-m} \frac{(n-m)!}{k!(n-k-m)!}(-1)^{n-k-m}u^n \\
&= \beta\gamma \sum_{n=m+1}^{N} p_n(-1)^{n-m}(-1)^{n-m+1}u^n = -\beta\gamma \sum_{n=m+1}^{N} p_n u^n.
\end{aligned} \tag{1.4.16}$$ {{mar1729}}

For the first term in the right side of (1.4.14), we can write

$$\begin{aligned}
f_1(u) &= \beta\gamma \sum_{n=m+1}^{N} p_n \sum_{k=m}^{n-1}\sum_{q=k}^{n-1} \frac{(n-m)!}{(k-m)!(q-k)!(n-q)!}(-1)^{q-k}u^q \\
&= \beta\gamma \sum_{n=m+1}^{N} p_n \sum_{q=m}^{n-1}\sum_{k=m}^{q} \frac{(n-m)!}{(k-m)!(q-k)!(n-q)!}(-1)^{q-k}u^q \\
&= \beta\gamma \sum_{n=m+1}^{N} p_n \sum_{q=m}^{n-1} \frac{(n-m)!}{(n-q)!}(-1)^q u^q \sum_{k=m}^{q} \frac{1}{(k-m)!(q-k)!}(-1)^k \\
&= \beta\gamma \sum_{n=m+1}^{N} p_n \sum_{q=m}^{n-1} \frac{(n-m)!}{(n-q)!}(-1)^{q+m} u^q \sum_{\ell=0}^{q-m} \frac{1}{\ell!(q-\ell-m)!}(-1)^\ell.
\end{aligned} \tag{1.4.17}$$ {{mar1730}}

The last sum is, up to a $(q-m)!$ factor, a binomial expansion:

$$\sum_{\ell=0}^{q-m} \frac{1}{\ell!(q-\ell-m)!}(-1)^\ell = \frac{1}{(q-m)!}(1-1)^{q-m}. \tag{1.4.18}$$ {{mar1731}}

Thus, the only nontrivial term in f_1 occurs when $q = m$, leading to

$$f_1(u) = \beta\gamma \sum_{n=m+1}^{N} p_n u^m = \beta\gamma u^m. \tag{1.4.19}$$ {{mar1732}}

Combining (1.4.14), (1.4.16) and (1.4.19) gives

$$f(u) = \beta\gamma \sum_{n=m+1}^{N} p_n(u^m - u^n). \tag{1.4.20}$$ {{mar1733}}

Thus, the group voting models lead to this simple class of nonlinearities that vanish at least quadratically at the origin.

2 Lecture 2: The Bramson shift and its applications

2.1 Overview of the lecture

We will discuss in this lecture the long time behavior of the solutions to the Fisher-KPP type equations and their convergence to traveling waves. This question was first studied in the original papers by Fisher [42] and Kolmogorov, Petrovskii and Piskunov [57]. While Fisher did not present a rigorous proof, KPP's fascinating paper essentially discovered the intersection number type argument and used it to prove the convergence. They also obtained a rough estimate

$$m(t) = c_* t + o(t), \quad \text{as } t \to +\infty, \tag{2.1.1}$$ {{23aug902}}

for the front location of the solutions to

$$u_t = u_{xx} + f(u), \tag{2.1.2}$$ {{23aug906}}

with $f(u) = u(1-u)^2$. We recall their method and result in Theorem 2.3.3 below. Before that, we review some basic facts about traveling waves for reaction-diffusion equations with positive nonlinearities. We also introduce the notion of steepness comparison of the solutions that will keep reappearing in these notes. The definition we present is due to Giletti and Matano [50] but the ideas really go back as far as the original KPP paper.

Next important progress came in the papers by Bramson [27, 28] who used McKean's connection between the Fisher-KPP equation and BBM to improve the front asymptotics (2.1.1) to

$$m(t) = 2t - \frac{3}{2}\log t + x_0 + o(1), \quad \text{as } t \to +\infty. \tag{2.1.3}$$ {{23aug904}}

Bramson's result, described in Theorem 2.4.1, applies to equations of the form (2.1.2) with a McKean type nonlinearity $f(u)$. The logarithmic correction in (2.1.3) is surprisingly important and shows up in many other models involving log-correlated random fields. Some examples of random processes in this class are briefly discussed in Section 2.4.1 below, without doing this issue any justice, see references mentioned in that section. The recent PDE proofs of Bramson's asymptptics and further extensions can be found in [22, 23, 51, 53, 79, 80].

In the second part of the lecture, we will use these results to deduce some properties of the extremal process of the branching Brownian motion. More precisely, it turns out that if we take the initial condition for (2.1.2) to be small: $u_0(x) = \varepsilon\phi(x)$, with a fixed function $\phi(x)$, then the behavior of the term $x_0(\varepsilon)$ in Bramson's asymptotics (2.1.3) can tell us a lot about the limiting extremal process of BBM. This part of the lecture is based on [75]. However, the idea that Bramson's shift for well chosen initial conditions contains information about the extremal process of BBM is due to [29, 30].

2.2 Traveling waves for positive nonlinearities

We first recall some basic facts about traveling waves for

$$u_t = u_{xx} + f(u), \ t > 0, \ x \in \mathbb{R}, \tag{2.2.1}$$ {{oct1610}}

with a non-negative $f(u)$:

$$f(0) = f(1) = 0, \quad f(u) > 0 \quad \text{for all } 0 < u < 1. \tag{2.2.2}$$ {{apr2902}}

Equation (2.2.1) has special solutions, called traveling waves. These are solutions to (2.2.1) of the form

$$u(t,x) = U_c(x - ct). \tag{2.2.3}$$ {{oct1612}}

In order for such $u(t,x)$ to be a solution to (2.2.1), the function $U_c(x)$ has to satisfy the ODE

$$-cU_c' = U_c'' + f(U_c). \tag{2.2.4}$$ {{oct1614}}

In addition, we will require that $U_c(x)$ satisfy the following boundary conditions at infinity

$$U_c(x) \to 1, \quad \text{as } x \to -\infty, \quad U_c(x) \to 0, \quad \text{as } x \to -\infty. \tag{2.2.5}$$ {{oct1608}}

Here is the key result on the existence of traveling waves for positive nonlinearities.

Proposition 2.2.1. *Assume that $f(u)$ satisfies the positivity assumption (2.2.2). Then, there exists $c_* > 0$ such that equation (2.2.4) has positive solutions $U_c(x) > 0$ that satisfy the boundary conditions (2.2.5) if and only if $c \geq c_*$. Moreover, if, in addition to (2.2.2), $f(u)$ satisfies the Fisher-KPP condition*

$$f(0) = f(1) = 0, \quad f(u) > 0, \ f(u) \leq f'(0)u, \quad \textit{for all } 0 < u < 1, \tag{2.2.6}$$ {{23aug910}}

then

$$c_* = 2\sqrt{f'(0)}. \tag{2.2.7}$$ {{nov231}}

For the Fisher-KPP nonlinearities, this result goes back to the original papers [42, 57], and for general positive nonlinearities it was proved in [52]. In this lecture, we will mostly assume that $f(u)$ is of the Fisher-KPP type.

An important role below will be played by the traveling wave profile $U_*(x) = U_{c_*}(x)$ that corresponds to the minimal speed c_*, and its asymptotics as $x \to +\infty$. Let us set

$$\lambda_c = \frac{c - \sqrt{c^2 - 4f'(0)}}{2}, \quad \text{for } c > c_*, \quad \lambda_* = \frac{c_*}{2} = \sqrt{f'(0)}. \tag{2.2.8}$$ {{oct1620}}

Proposition 2.2.2. *Let $f(u)$ satisfy the Fisher-KPP assumption (2.2.6). If $c > c_*$ then there exists a constant $A_1 > 0$ so that*

$$U_c(x) \sim A_1 e^{-\lambda_c x}, \quad \textit{as } x \to +\infty, \tag{2.2.9}$$ {{oct1618}}

and there exists $A_2 > 0$ so that

$$U_*(x) \sim A_2 x e^{-\lambda_* x}, \quad \textit{as } x \to +\infty. \tag{2.2.10}$$ {{oct1622}}

Proposition 2.2.2 says that for $c > c_*$ the traveling waves $U_c(x)$ have a "purely exponential" decay as $x \to +\infty$ but the minimal speed traveling wave $U_*(x)$ has an extra factor of x in front of the exponential. This turns out to be surprisingly important in the long time evolution of the solutions to (2.2.1).

The constants A_1 and A_2 change if we shift the wave $U(x) \to U(x - x_0)$: for example, if $U_*(x)$ has asymptotics (2.2.10), then

$$U_*(x - x_0) \sim A_2(x - x_0)e^{-\lambda_*(x - x_0)} \sim A_2[x_0] x e^{-\lambda_* x}, \quad \text{as } x \to +\infty, \tag{2.2.11}$$ {{nov1030}}

with

$$A_2[x_0] = A_2 e^{\lambda_* x_0}. \tag{2.2.12}$$

Unless stated otherwise, we will fix the translation of the wave by requiring that $U_*(x)$ has the asymtptotics

$$U_*(x) \sim x e^{-\lambda_* x}, \quad \text{as } x \to +\infty, \tag{2.2.13}$$ {{nov1031}}

with the pre-factor equal to 1.

2.3 Convergence to a traveling wave in shape

We now consider the long time convergence of the shape of the solutions to a semilinear parabolic equation with a step function initial condition

$$\begin{aligned} &u_t = u_{xx} + f(u),\ t > 0,\ x \in \mathbb{R}, \\ &u(0, x) = \mathbb{1}(x \le 0). \end{aligned} \tag{2.3.1}$$ {{oct1602}}

One may, of course, consider more general initial conditions than in (2.3.1) with very similar results but it is a bit simpler to study convergence to a traveling wave for the step function initial condition. We will assume that $f(u)$ satisfies (2.2.2) but we really only need the assumption

$$f(0) = f(1) = 0 \tag{2.3.2}$$ {{23aug912}}

in the analysis below.

2.3.1 The steepness comparison

We will use the notion of steepness of the solution. While such arguments date back to the original KPP paper [57], a very nice introduction is in a recent paper by Giletti and Matano [50]. Let us first consider smooth monotonically decreasing functions $u(x)$, $x \in \mathbb{R}$, such that

$$\lim_{x\to-\infty} u(x) = 1, \quad \lim_{x\to+\infty} u(x) = 0. \tag{2.3.3}$$ {{jul1504}}

Given two such functions u_1 and u_2, we say that u_1 is steeper than u_2 if

$$|u_1'(u_1^{-1}(z))| > |u_2'(u_2^{-1}(z))|, \quad \text{for all } z \in (0, 1). \tag{2.3.4}$$ {{jul1506}}

In other words, the graph of $u_1(x)$ is steeper than the graph of $u_2(x)$ when compared at each fixed level $z \in (0, 1)$, rather than at a fixed point $x \in \mathbb{R}$. This notion is translation invariant; if u_1 is steeper than u_2, it is also steeper than any translate $u_2(\cdot + h)$, with a fixed $h \in \mathbb{R}$.

The differentiability or even continuity of u_1 and u_2 are not really needed for the steepness comparison. To avoid the slope comparison in (2.3.4), we say that u_1 is steeper than u_2 if for any two translates $\tilde{u}_1(x) = u_1(x - \ell_1)$ and $\tilde{u}_2(x) = u_2(x - \ell_2)$ of u_1 and u_2 there exists x_0 such that $\tilde{u}_1(x) > \tilde{u}_2(x)$ for $x < x_0$ and $\tilde{u}_1(x) < \tilde{u}_2(x)$ for $x > x_0$. Here, we still assume that u_1 and u_2 obey the boundary conditions (2.3.3). For smooth functions, the two definitions are equivalent.

We first claim that faster traveling waves are steeper than the slow ones.

Proposition 2.3.1. *Assume that $f(u)$ satisfies the positivity assumption (2.2.2) and let $U_c(x)$ be a traveling wave solution to (2.3.1) with $c > c_*$. Then, $U_*(x)$ is steeper than $U_c(x)$.*

We leave the proof to the reader. There are two main steps in the proof: first, one needs to show that no two translates of $U_*(x)$ and $U_c(x)$ can touch each other, and, second, use expression (2.2.8) for the exponential decay rates of the two traveling waves, to show that $U_*(x) < U_c(x)$ as $x \to +\infty$.

The next key observation is that equation (2.3.1) preserves the steepness ordering of the solutions.

Proposition 2.3.2. *Let $u_1(t, x)$ and $u_2(t, x)$ be the solutions to (2.3.1) with the corresponding initial conditions u_{10} and u_{20} that satisfy (2.3.3). If u_{10} is steeper than u_{20}, then $u_1(t, \cdot)$ is steeper than $u_2(t, \cdot)$ for all $t > 0$.*

This result was essentially proved for the classical Fisher-KPP equation in the original KPP paper [57]. Let us stress that the proof below does not use the positivity assumption (2.2.2). We also mention that there is a corresponding result by Bachmann [19] for random walks with log-concave jump laws. Without the log-concavity assumption on the jump laws, the steepness comparison principle does not hold for random walks.

The proof of Proposition 2.3.2

Let $u_1(t,x)$ and $u_2(t,x)$ be the solutions to (2.3.1) with monotonically decreasing initial conditions u_{10}, u_{20} that satisfy (2.3.3), such that u_{10} is steeper than u_{20}. First, we note that since the initial conditions are decreasing, both $u_1(t,x)$ and $u_2(t,x)$ are decreasing and have the left and right limits as in (2.3.3), so the steepness comparison makes sense.

To show that $u_1(t,\cdot)$ is steeper than $u_2(t,\cdot)$ for any $t>0$, consider the difference

$$w(t,x;k_0) = u_1(t,x) - u_2(t,x+k_0),$$

for a fixed $k_0 \in \mathbb{R}$. The function $w(t,x;k_0)$ satisfies

$$w_t = w_{xx} + g(t,x)w, \quad g(t,x) = \frac{f(u_1(t,x)) - f(u_2(t,x+k_0))}{u_1(t,x) - u_2(t,x+k_0)}, \tag{2.3.5}$$ {{jul1606}}

with the initial condition

$$w(0,x;k_0) = u_{10}(x) - u_{20}(x+k_0). \tag{2.3.6}$$ {{jul1608}}

Note that if $f(u)$ is twice differentiable, then the function $g(t,x)$ is differentiable and uniformly bounded.

Since u_{10} is steeper than u_{20}, it is also steeper than $u_{20}(\cdot + k_0)$. Therefore, there exists x_0 so that

$$w(0,x;k_0) > 0 \text{ for all } x < x_0,$$

and

$$w(0,x;k_0) < 0 \text{ for all } x > x_0.$$

Since $w(t,x;k_0)$ is a solution to the parabolic equation (2.3.5), the strong maximum principle implies that $w(t,x;k_0)$ has exactly one zero $y(t;k_0)$ for all $t>0$, so that $w(t,x;k_0)>0$ for all $x<y(t;k_0)$ and $w(t,x;k_0)<0$ for all $x>y(t;k_0)$, with $y(0;k_0)=x_0$. In addition, we have $w_x(t,y(t;k_0))<0$, which translates into

$$\partial_x u_1(t,y(t;k_0)) < \partial_x u_2(t,y(t;k_0)). \tag{2.3.7}$$ {{jul1610}}

Since this is true for all $k_0 \in \mathbb{R}$, it follows that $u_1(t,\cdot)$ is steeper than $u_2(t,\cdot)$. $\Box$

2.3.2 Convergence in shape

We now establish convergence of the solution in shape to a traveling wave, originally proved in the KPP paper [57]. We normalize the minimal speed traveling wave by $U_*(0)=1/2$.

Theorem 2.3.3. *Suppose that $f(u)$ satisfies (2.2.2), and let $u(t,x)$ be the solution to (2.3.1) with the initial condition $u_{\rm in}$ that satisfies (2.3.3) and is steeper than the minimal speed traveling wave $U_*(x)$. Then, there exists a function $m(t)$ such that*

$$\frac{dm(t)}{dt} \to c_*, \quad \text{as } t \to +\infty, \tag{2.3.8}$$ {{nov206}}

and

$$u(t,x+m(t)) \to U_*(x) \quad \text{as } t\to+\infty, \text{ uniformly on } \mathbb{R}. \tag{2.3.9}$$ {{dec130bis

The steepness assumption on the initial condition is really not necessary and is only made to shorten the proof. The result holds for a large class of initial conditions that decay sufficiently fast as $x \to +\infty$. For example, one may assume that $u_{\rm in}(x)$ is non-negative everywhere and is compactly supported on the right: there exists L_0 so that $u_{\rm in}(x) = 0$ for all $x \geq L_0$. A typical example one may have in mind is the step function initial condition $u_0(x) = \mathbb{1}(x \leq 0)$.

Proof. It suffices to consider the initial condition $u(0,x) = \mathbb{1}(x \leq 0)$. This is because if we take any other solution $v(t,x)$ to (2.3.1) with the initial condition $v_{\rm in}(x)$ that is steeper than $U_*(x)$, then $v(t,x)$ is steeper than $U_*(x)$ and is less steep than $u(t,x)$, for any $t > 0$. Hence, convergence in shape of $u(t,x)$ to $U_*(x)$ will imply the corresponding convergence of $v(t,x)$, also to $U_*(x)$. Note that for any $\tau > 0$, the function

$$u^{(\tau)}(t,x) = u(t+\tau,x)$$

is the solution to (2.3.1) with initial condition $u^{(\tau)}(0,x) = u(\tau,x)$ that is less steep than the step function $u(0,x) = \mathbb{1}(x \leq 0)$. It follows that for any $t > 0$ and $\tau > 0$ the function $u(t,\cdot)$ is steeper than $u(t+\tau,\cdot)$. In addition, $u(t,\cdot)$ is steeper than the minimal speed traveling wave $U_*(x)$ for all $t > 0$. This is because $u(0,x) = \mathbb{1}(x \leq 0)$ is steeper than $U_*(x)$ and the solution to (2.3.1) with the initial condition $U_*(x)$ is $U_*(x - c_* t)$, which has the same shape as $U_*(x)$. Hence, if for each $z \in (0,1)$ and $t > 0$, we let $x(t,z)$ be the unique point such that $u(t,x(t,z)) = z$, then, the function

$$E(t,z) = u_x(t,x(t,z)) \leq 0, \tag{2.3.10}$$ {{jul1612}}

is increasing in t for all $z \in (0,1)$, and

$$E(t,z) \leq \bar{E}(z) := U_*'(U_*^{-1}(z)). \tag{2.3.11}$$ {{dec132}}

Let now $m(t)$ be the position such that $u(t,m(t)) = 1/2$ for all $t > 0$, and consider the translate

$$\tilde{u}(t,x) = u(t,x+m(t)),$$

as well as the corresponding inverse $\xi(t,z)$ defined by

$$\tilde{u}(t,\xi(t,z)) = z, \quad \text{for } 0 < v < 1.$$

Observe that $\xi(t,1/2) = 0$ for all $t > 0$, simply because $\tilde{u}(t,0) = 1/2$ by construction. We see from (2.3.10)-(2.3.11) that the function $E(t,z)$ is negative and increasing in time. Thus, it has a limit

$$E(t,z) \to E_\infty(z) \leq \bar{E}(z) < 0, \quad \text{as } t \to +\infty. \tag{2.3.12}$$ {{dec926}}

Hence

$$\frac{\partial \xi(t,z)}{\partial z} = \frac{1}{E(t,z)} \to \frac{1}{E_\infty(z)}, \quad \text{as } t \to +\infty,$$

and

$$\xi(t,z) = \int_{1/2}^{z} \frac{\partial \xi(t,z')}{\partial z'} dz' \to \int_{1/2}^{z} \frac{dz'}{E_\infty(z')} := \xi_\infty(z). \tag{2.3.13}$$ {{dec310}}

As a consequence, the function $\tilde{u}(t,x)$ also converges uniformly on compact sets to a limit $\tilde{u}_\infty(x)$:

$$\tilde{u}(t,x) \to \tilde{u}_\infty(x) \text{ as } t \to +\infty, \tag{2.3.14}$$ {{dec314}}

with $\tilde{u}_\infty(x)$ determined by

$$\xi_\infty(\tilde{u}_\infty(x)) = x. \tag{2.3.15}$$ {{apr902}}

Moreover, due to (2.3.12), we have

$$|\xi_\infty(z)| = \int_{1/2}^{v} \frac{dvz}{|E_\infty(z')|} \leq \int_{1/2}^{z} \frac{dv'}{|\bar{E}(z')|} := \bar{\xi}(z). \tag{2.3.16}$$ {{apr904}}

This yields the correct behavior of the limits $x \to \pm\infty$:

$$\tilde{u}_\infty(-\infty) = 1, \quad \tilde{u}_\infty(+\infty) = 0. \qquad (2.3.17) \quad \texttt{\{\{dec318\}\}}$$

Furthermore, as $u(t,x)$ is strictly decreasing in x and $u_x(t,m(t)) < 0$, the function $m(t)$ is differentiable in t, with

$$\dot{m}(t) = -\frac{u_t(t,m(t))}{u_x(t,m(t))} = -\frac{u_t(t,m(t))}{\tilde{u}_x(t,0)}. \qquad (2.3.18) \quad \texttt{\{\{nov302\}\}}$$

Hence, $\tilde{u}(t,x)$ satisfies

$$\tilde{u}_t - \dot{m}(t)\tilde{u}_x = \tilde{u}_{xx} + f(\tilde{u}). \qquad (2.3.19) \quad \texttt{\{\{dec312\}\}}$$

By the parabolic regularity theory, the numerator in the very right side of (2.3.18) is bounded. Moreover, since $\tilde{u}$ converges to $\tilde{u}_\infty(x)$ that is steeper than $U_*(x)$, the denominator is bounded away from zero and converges to $\partial_x \tilde{u}_\infty(0) \neq 0$. It follows that $\dot{m}(t)$ is bounded uniformly in t. In addition, because $u(t',x)$ is less steep than $u(t,x)$ for $t' > t$, we know that $\tilde{u}_t(t,x) \to 0$ as $t \to +\infty$. Then, passing to the limit $t \to +\infty$ in (2.3.19), we deduce that there exists $c \in \mathbb{R}$ such that $\dot{m}(t) \to c$ as $t \to +\infty$ and $\tilde{u}_\infty(x)$ satisfies

$$-c\partial_x \tilde{u}_\infty = \partial_x^2 \tilde{u}_\infty + f(\tilde{u}_\infty). \qquad (2.3.20) \quad \texttt{\{\{dec316\}\}}$$

We see from (2.3.20) that $\tilde{u}_\infty(x)$ is a traveling wave solution to (2.3.1) moving with the speed c. It remains to show that $c = c_*$. The key point is that the steepness comparison argument above applies to any traveling wave solution to

$$-cU_c' = U_c'' + f(U_c). \qquad (2.3.21) \quad \texttt{\{\{apr906\}\}}$$

In other words, if we set

$$E_c(z) = U_c'(U_c^{-1}(z)), \quad \text{for } 0 < z < 1,$$

then we know that

$$E_\infty(z) \le E_c(z),$$

for any U_c that satisfies (2.3.21) with some $c \ge c_*$. Therefore, the limit $\tilde{u}_\infty(x)$ is the traveling wave that is the steepest among all traveling wave solutions. Proposition 2.3.1 implies that

$$\tilde{u}_\infty(x) = U_*(x)$$

is the minimal speed traveling wave. This finishes the proof of Theorem 2.3.3. $\square$

2.4 The Bramson shift and convergence to a wave

Theorem 2.3.3 says nothing about the location $m(t)$ of the front of the solution to the Fisher-KPP equation,

$$\begin{aligned} &u_t - u_{xx} = f(u), \quad t > 0, \ x \in \mathbb{R}, \\ &u(0,x) = u_0(x), \end{aligned} \qquad (2.4.1) \quad \texttt{\{\{e2.1\}\}}$$

except for the rough asymptotics (2.3.8)

$$m(t) = c_* t + o(t), \quad \text{as } t \to +\infty. \qquad (2.4.2) \quad \texttt{\{\{mar2202\}\}}$$

Here and below we assume that $u_0(x)$ is a compact perturbation of the step function $\mathbb{1}(x \le 0)$. More precisely, we assume that $0 \le u_0(x) \le 1$ for all $x \in \mathbb{R}$ and that there exist $L_1 \le L_0$ such that $u_0(x) = 1$ for all $x < L_1$ and $u_0(x) = 0$ for all $x > L_0$. More general initial conditions can be considered but they do need to decay faster than e^{-x} as $x \to +\infty$: see [22, 23] for the sharp conditions on $u_0(x)$ that are needed.

Fisher has already made an informal argument in [42] that the $o(t)$ term in (2.4.2) is of the order $O(\log t)$. More precisely, he claimed that it equals to the leading order to $(1/2)\log t$. An important series of papers by Bramson [27], [28] proved the following corrected version of Fisher's prediction.

Theorem 2.4.1. *Suppose that $f(u)$ satisfies the Fisher-KPP property (2.2.2). There is a constant x_∞, depending on u_0, such that*

$$m(t) = c_* t - \frac{3}{2\lambda_*} \log t - x_\infty + o(1), \ \ \textit{as } t \to +\infty, \tag{2.4.3}$$ {{23aug602}}

with $\lambda_* = c_*/2$.

This theorem has the following interpretation in terms of the branching Brownian motion. Let

$$M(t) = \max_{1 \le k \le N_t} X_k(t), \tag{2.4.4}$$

be the running maximum of the BBM that started at $x = 0$ at the time $t = 0$. Theorem 2.4.1 says that there exists x_∞ such that

$$\mathbb{P}\Big(M(t) > c_* t - \frac{3}{2\lambda_*} \log t - x_\infty + x\Big) \to U_*(x), \quad \text{as } t \to +\infty. \tag{2.4.5}$$ {{23aug621}}

Bramson's proof of Theorem 2.4.1 used probabilistic arguments coming from the connection to branching Brownian motion and applied only to the McKean nonlinearities. Here, we will briefly describe a simple and reasonably robust proof of Theorem 2.4.1 from [53, 79] that works for all Fisher-KPP nonlineariies and does not use the intersection number argument. These ideas are further developed to study the refined asymptotics of the solutions in [51, 80]. A completely different and totally mind boggling even if not rigorous approach to these further corrections was developed in [22, 23]. Previous PDE results on the Bramson shift are [59, 92], while there are many probabilistic papers addressing the maximum of a branching Brownian motion, or a branching random walk: see, for example, [5, 7, 8, 19, 29, 30, 35, 66, 84].

2.4.1 The Bramson correction and other log-correlated random fields

Very surprisingly, the $3/2$ factor in front of the $\log t$ term in (2.4.3) is much more than it seems. We are not going to discuss this issue in any detail in these lectures but let us just give a couple of examples of this phenomenon, with some further references.

Maximum of independent Gaussians. First, let discuss a case when the asymptotics in (2.4.5) has the factor $1/2$ predicted by Fisher and not Bramson's $3/2$. This happens in a simplified model with no correlations between particles. Let us think of $N \in \mathbb{N}$ as an analog of the time variable for the BBM and take a large number M of independent particles $Y_1, \dots, Y_M$, with M that depends on N. In order to mimic the concept that N is the time of the BBM, we assume that Y_k are mean zero Gaussian random variables with variance N. Thus, we can think of each Y_k as a snapshot of a Brownian motion at the time $t = N$. To make sure that the number M of these variables also mimics BBM, we assume that it grows exponentially in N, as is the case for BBM. We take $M = 2^N$, to ensure that M is an integer. The key difference with the BBM is that here, for an infinitely greater simplicity, we assume that all Y_k are simply independent. Let us consider the maximal particle

$$\bar{M}_N = \max(Y_1, \dots, Y_M). \tag{2.4.6}$$

In particular, we are interested in the median location m_N such that

$$\mathbb{P}(\bar{M}_N > m_N) = 1/2. \tag{2.4.7}$$ {{oct402}}

In this simple model, since all particles are identical, this can be analyzed by writing

$$\mathbb{P}(\bar{M}_N < y) = \big(\mathbb{P}(Y_1 < y)\big)^{2^N} = (1 - \mathbb{P}(Y_1 > y))^{2^N}. \tag{2.4.8}$$ {{sep2829}}

A straightforward if lengthy calculation shows that

$$m_N = c_* N - \frac{1}{2\lambda_*}\log N + x_0 + o(1), \quad \text{as } N \to +\infty, \tag{2.4.9}$$ {{oct412}}

with the corresponding c_* and $\lambda_* = c_*/2$ coming from the fact that $M = 2^N$ and not $\exp(N)$. This is exactly Fisher's prediction – the coefficient here is $1/2$ not Bramson's $3/2$. That is, Bramson's asymptotics comes from the correlations built into the positions of BBM. They come from the genealogical tree structure and thus have a logarithmic nature. It turns out that Bramson's asymptotics for the extremal values of such log-correlated fields are quite ubiquitous.

Maxima of the Riemann zeta function. The first example of the Bramson-like behavior concerns the maxima of the Riemann zeta function

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s} \tag{2.4.10}$$

on the critical line. The Lindelöf hypothesis says that for any $\varepsilon > 0$ we have

$$\zeta(1/2 + iT) = o(T^{\varepsilon}), \quad \text{as } T \to +\infty. \tag{2.4.11}$$

To the best of my knowledge, the best result in this direction is by Bourgain [25] who showed that

$$\zeta(1/2 + iT) = o(T^{13/84+\varepsilon}). \tag{2.4.12}$$ {{23aug604}}

Fyodorov, Hiary and Keating, in a series of papers [43, 44], considered the following related question. Consider the Riemann zeta function on the critical line, and pick an interval of length 1, unfiormly at random, on an interval of the form $[T, 2T]$, with some $T \gg 1$. That is, we consider

$$\tilde{\zeta}(t) = \max_{|t-s|\le 1} \log|\zeta(1/2 + is)|, \tag{2.4.13}$$

with t chosen uniformly on $[T, 2T]$. We are interested in

$$u(T, x) = \mathrm{Prob}\big[\tilde{\zeta}(t) > x\big]. \tag{2.4.14}$$

They conjectured that

$$u\big(T, x + \log\log T - \frac{3}{4}\log\log\log T\big) \sim F(x), \tag{2.4.15}$$ {{23aug606}}

with a function $F(x)$ that has the same asymptotics as $x \to +\infty$ as the Fisher-KPP traveling wave:

$$F(y) \sim Aye^{-2y}, \quad \text{as } y \to +\infty. \tag{2.4.16}$$ {{23aug608bi

After a simple rescaling $x \to x/2$, (2.4.15) is exactly the same asymptotics as in (2.4.3), and the profile asymptotics (2.4.16) is exactly the same as (2.2.10) for the Fisher-KPP traveling wave!

As far as the rigorous the results in this direction, Arguin, Belius, Bourgade, Radziwill, and Soundararajan proved in [10] that for any $\varepsilon > 0$,

$$\text{Prob}\Big\{(1-\varepsilon)\log\log T < |\tilde{\zeta}(t)| \le (1+\varepsilon)\log\log T\Big\} \to 1, \quad \text{as } T \to +\infty. \tag{2.4.17}$$ {{23aug610}}

This is the speed asymptotics, exactly as (2.4.2) of the original KPP paper.

The most recent result, to the best of my knowledge, is by Arguin, Bourgade, and Radziwill [11] who proved a Bramson-like upper bound

$$u(T, x + \log\log T - \frac{3}{4}\log\log\log T + y) \le Cye^{-2y}. \tag{2.4.18}$$ {{23aug608}}

It seems that no such lower bound is known. Further references for these questions for the Riemann zeta function are, among others, [12, 14].

The characteristic polynomial of random unitary matrices. The second example concerns the circular β ensembles of random unitary matrices. Consider random unitary matrices, with the eigenvalue distribution on $\mathbb{S}^n$ given by

$$C_{n,\beta}\prod_{1\le j<k\le n}\left|e^{i\theta_j} - e^{i\theta_k}\right|^\beta d\theta_1\dots d\theta_n, \tag{2.4.19}$$ {{23aug612}}

and let $P_n(z)$ be the corresponding characteristic polynomial. The Fyodorov-Hiary-Keating conjecture for $\beta = 2$ is that

$$\max_{z\in\mathbb{S}^1}\log|P_n(z)| = \log n - \frac{3}{4}\log\log n + X_n. \tag{2.4.20}$$ {{23aug614}}

Here, X_n is conjectured to have a limit and Fisher-KPP traveling wave-like law, as in (2.4.18). The best result, as far as I know, is by Chhaibi, Madaule, and Najnudel [33] who capture both the speed and the Bramson correction but not the $O(1)$ term:

$$\max_{z\in\mathbb{S}^1}\text{Re}\log P_n(z) = \sqrt{\frac{2}{\beta}}\Big(\log n - \frac{3}{4}\log\log n + O(1)\Big). \tag{2.4.21}$$

Previous results in this direction are [9, 67, 82].

The advantage of the branching Brownian motion compared to the other examples in this class, known as the log-correlated processes, is that analytic techniques are available to study the Fisher-KPP equation that allow to prove such results in what seems to be a much simpler way, and also make conjectures for the other processes in the log-correlated class. This makes BBM a very interesting special case of the log-correlated processes.

2.4.2 Strategy of the proof of Theorem 2.4.1

We now very briefly discuss the PDE strategy of the proof of the Bramson correction. For simplicity, we assume that the nonlinearity is $f(u) = u - u^2$ but the proof outlined below only relies on the Fisher-KPP property (2.2.2) of $f(u)$. There is a separate question of what happens when $f(u)$ does not of the Fisher-KPP type – this will be addressed later in these notes.

Consider the Cauchy problem (2.4.1)

$$u_t - u_{xx} = u - u^2, \quad x \in \mathbb{R}, \quad t > 1, \tag{2.4.22}$$

and proceed with a sequence of changes of variables. We first go into the moving frame:

$$x \mapsto x - 2t + r\log t,$$

with $r \in \mathbb{R}$ to be determined, leading to

$$u_t - u_{xx} - \Big(2 - \frac{r}{t}\Big)u_x = u - u^2. \tag{2.4.23}$$
{{e2.0}}

Next, we take out the exponential factor: set

$$u(t,x) = e^{-x}v(t,x) \tag{2.4.24}$$

so that $v(t,x)$ satisfies

$$v_t - v_{xx} - \frac{r}{t}(v - v_x) + e^{-x}v^2 = 0, \qquad x \in \mathbb{R}, \quad t > 1. \tag{2.4.25}$$
{{e2.6}}

We note that for $x \to +\infty$, the term $e^{-x}v^2$ in (2.4.25) is negligible, while for $x \to -\infty$ the same term will create a large absorption and force the solution to be close to zero. For this reason, the linear Dirichlet problem

$$\begin{aligned} &z_t - z_{xx} - \frac{r}{t}(z - z_x) = 0, \quad x > 0,\\ &z(t,0) = 0, \end{aligned} \tag{2.4.26}$$
{{sep1810}}

is a reasonable approximation for (2.4.25) for $x \gg 1$. The philosophy of the proof is that if we choose the correct reference frame, where the front is located then $u(t,x)$ should remain of the size $O(1)$ as $t \to +\infty$, for $x \sim O(1)$. Then, so should be $v(t,x)$ and $z(t,x)$. Our goal is to find $r \in \mathbb{R}$ so that $z(t,x)$ would remain $O(1)$ as $t \to +\infty$.

If we "naively" drop the term of the size $O(1/t)$ in (2.4.26), we obtain the heat equation on the half line

$$\begin{aligned} &\tilde z_t - \tilde z_{xx} = 0, \quad x > 0,\\ &\tilde z(t,0) = 0. \end{aligned} \tag{2.4.27}$$
{{23aug618}}

Its solution has the long time asymptotics

$$\tilde z(t,x) \sim \frac{Cx}{t^{3/2}}e^{-x^2/(4t)}, \quad \text{as } t \to +\infty. \tag{2.4.28}$$
{{23aug616}}

With some technical work, one can show that the solution to (2.4.26) has the long time behavior

$$z(t,x) \sim \frac{Cx}{t^{3/2-r}}e^{-x^2/(4t)}, \quad \text{as } t \to +\infty. \tag{2.4.29}$$
{{23aug620}}

As we want to keep $z(t,x)$ of the size $O(1)$ as $t \to +\infty$, we are forced to take $r = 3/2$. This is the analytical reason behind the Bramson' correction.

Making the connection between the approximate Dirichlet problem (2.4.26) and the original problem (2.4.23) more precise and quantitative is a key to the proof of Theorem 2.4.1. We omit the technical details that can be found in [53, 79] but the key step is the following lemma.

Lemma 2.4.2. *There exists a constant $r_\infty > 0$ with the following property. For any $\gamma > 0$ and $\varepsilon > 0$ we can find T_ε so that for all $t > T_\varepsilon$ and $x_\gamma = t^\gamma$ we have*

$$|u(t,x_\gamma) - r_\infty x_\gamma e^{-x_\gamma}e^{-x_\gamma^2/(4t)}| \le \varepsilon x_\gamma e^{-x_\gamma}e^{-x_\gamma^2/(6t)}. \tag{2.4.30}$$
{{sep2902}}

Therefore, we have

$$u(t,x) = r_\infty x e^{-x} e^{-x^2/(4t)} + l.o.t. \quad \text{for } x \sim t^\gamma. \tag{2.4.31}$$ {{nov2302}}

This determines the unique translation: if we accept that $u(t,x)$ converges to a translate $U(x - x_\infty)$ of $U_*(x)$, then for large x (in the moving frame) we have

$$u(t,x) \sim U_*(x - x_\infty) \sim x e^{-x + x_\infty}. \tag{2.4.32}$$ {{sep1812}}

Comparing this with (2.4.31), we infer that

$$x_\infty = \log r_\infty.$$

This argument, however, assumes that the two approximations are both valid for $x \sim O(t^\gamma)$. This is quite delicate, the details can be found in [53, 79].

2.5 The Bramson shift for small initial conditions and the asymptotic properties of BBM

It turns out that by analyzing the constant shift x_∞ in the Bramson asymptotics (2.4.3) in Theorem 2.4.1 for some special initial conditions $u_0(x)$, we can deduce interesting conclusions about the extremal process of the branching Brownian motion. To be specific, we will focus on the binary branching Brownian motion. The results described below come from [76]. That paper contains both the law of large numbers for the extremal process of BBM and as limit for its the fluctuations. Here, we will only describe the law of large numbers, as it is much less technical.

2.5.1 The Laplace transform of the point process for the branching Brownian motion

Let again $u(t,x)$ be the solution to

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + u - u^2, \tag{2.5.1}$$ {{20apr1402}}

with the initial condition $g(x)$ such that $0 \le g(x) \le 1$ for all $x \in \mathbb{R}$, and $g(x)$ is compactly supported on the right – there exists L_0 such that $g(x) = 0$ for all $x \ge L_0$. Theorem 2.4.1 says that there exists a constant $\widehat{s}[g]$, the Bramson shift corresponding to the initial condition g, such that

$$u(t, x + m(t)) \to U_*(x + \widehat{s}[g]) \text{ as } t \to +\infty, \tag{2.5.2}$$ {{20apr1414}}

with

$$m(t) = 2t - \frac{3}{2}\log t. \tag{2.5.3}$$ {{20apr1416}}

Note that we have chosen the sign of $\widehat{s}[g]$ in (2.5.2) in the way that makes the shift positive for "small" initial conditions that we will consider later. The traveling wave $U_*(x)$ is normalized so that its asymptotics as $x \to +\infty$ is

$$U_*(x) \sim x e^{-x}, \quad \text{as } x \to +\infty, \tag{2.5.4}$$ {{23aug914}}

with the pre-factor in (2.5.4) equal to 1.

We now recall how the Laplace transform of the point process of the branching Brownian motion can be connected to the Fisher-KPP equation using the McKean representation of the solutions.

Let $X_1(t), \dots, X_{N_t}(t)$ be the locations of the BBM particles at a time $t > 0$, and consider the corresponding point process

$$\mathcal{E}(t, y; x) = \sum_{k=1}^{N_t} \delta(x + y - X_k(t)), \tag{2.5.5}$$ {{oct527}}

understood as a measure in the x-variable. Note that we centered the process at a location y. For the moment, we assume that the BBM starts at the position $x = 0$ at $t = 0$.

The Laplace functional of the point process $\mathcal{E}(t, y; \cdot)$ is

$$\Psi(\phi)(t, y) = \mathbb{E}_0 \exp\Big(- \int \phi(x) d\mathcal{E}(t, y; x)\Big) = \mathbb{E}_0 \exp\Big(- \sum_{k=1}^{N_t} \phi(X_k(t) - y)\Big). \tag{2.5.6}$$ {{oct528}}

Here, $\phi(x)$ is a non-negative bounded test function. The subscript 0 in (2.5.6) refers to the starting point of the branching Brownian motion. A simple but important observation is that (2.5.6) can be written as

$$\Psi(\phi)(t, y) = \mathbb{E}_0 \exp\Big(- \sum_{k=1}^{N_t} \phi(X_k(t) - y)\Big) = \mathbb{E}_0\Big(\prod_{k=1}^{N_t} g(X_k(t) - y)\Big), \tag{2.5.7}$$ {{oct529}}

with

$$g(x) = e^{-\phi(x)}. \tag{2.5.8}$$ {{oct530}}

Combining with what we have done in Section 1.2.2, we conclude that if we let $u(t, x)$ be the solution to the initial value problem

$$\begin{aligned} &\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + u - u^2, \\ &u(0, x) = 1 - e^{-\phi(x)}, \end{aligned} \tag{2.5.9}$$ {{oct531}}

then the Laplace functional is given by

$$\Psi(\phi)(t, y) = 1 - u(t, -y). \tag{2.5.10}$$ {{oct532}}

Thus, the Laplace transform of the point process of the branching Brownian motion can be directly computed in terms of a solution of the Fisher-KPP equation with a suitable initial condition. In particular, to find the shift y that would make the limit of $Z_k(t) = X_k(t) - y$ non-trivial, we need to find the locations where $u(t, y)$ is neither close to 0 nor to 1. These are, of course, near the Bramson position $m(t)$ given by (2.5.3).

2.5.2 The limiting extremal process of BBM and its connection to the Bramson shift

Motivated by the above discussion, let $x_1(t) \ge x_2(t) \ge \ldots \ge x_{N_t}(t)$ be the ordered positions of the BBM particles at time t, and consider the BBM measure seen from $m(t)$:

$$\mathcal{X}_t = \sum_{k \le N_t} \delta_{m(t) - x_k(t)}. \tag{2.5.11}$$ {{20jun1820}

It was shown in [5, 7, 8, 30] that there exists a point process $\mathcal{X}$ so that we have

$$\mathcal{X}_t \Rightarrow \mathcal{X} = \sum_k \delta_{\chi_k} \quad \text{as } t \to +\infty, \tag{2.5.12}$$

with $\chi_1 \leq \chi_2 \leq \ldots$, so that χ_1 corresponds to the maximal particle in the BBM, χ_2 to the second largest, and so on. We call $\mathcal{X}$ the extremal process of BBM. One can see from the representation (2.5.7), (2.5.8), and (2.5.9) for the Laplace transform of $\mathcal{X}_t$, together with the Bramson asymptotics (2.5.2) that the Laplace transform of the extremal process is

$$\mathbb{E}\left[e^{-\mathcal{X}(\psi)}\right] = 1 - U_*(\widehat{s}[\widehat{\psi}]), \quad \widehat{\psi} = 1 - e^{-\psi}. \tag{2.5.13}$$
{{eq:6_2_1}}

Note that if $\psi(x)$ is compactly supported on the right then so is $\widehat{\psi}(x)$, so that the Bramson shift $\widehat{s}[\widehat{\psi}]$ is well defined.

There is also a conditional version of (2.5.13), in terms of the derivative martingale introduced in [66]:

$$Z_t = \sum_{k \leq N_t} (2t - x_k(t)) e^{-(2t - x_k(t))} \to Z \quad \text{as } t \to +\infty, \ \mathbb{P}\text{-a.s.} \tag{2.5.14}$$
{{20apr1424}

One can show that the Laplace transform of Z can be interpreted in terms of the profile $U_*(x)$ of the minimal speed Fisher-KPP traveling wave: for each $y \in \mathbb{R}$ we have

$$\mathbb{E}\left[e^{-Ze^{-y}}\right] = 1 - U_*(y). \tag{2.5.15}$$
{{20apr1434}

Comparing (2.5.13) and (2.5.15) gives the identity

$$\mathbb{E}\left[e^{-\mathcal{X}(\psi)}\right] = \mathbb{E}\left[e^{-Ze^{-\widehat{s}[\widehat{\psi}]}}\right], \quad \widehat{\psi}(x) = 1 - e^{-\psi(x)}. \tag{2.5.16}$$
{{20apr2204}

The results in [5], [8] and in Appendix C of [30] imply the conditional version of (2.5.16):

$$\mathbb{E}\left[e^{-\mathcal{X}(\psi)}|Z\right] = e^{-Ze^{-\widehat{s}[\widehat{\psi}]}}. \tag{2.5.17}$$
{{20apr2204a

In principle, (2.5.17) completely characterizes the conditional distribution of the measure $\mathcal{X}$ in terms of its conditional Laplace transform. However, the Bramson shift is a very implicit function of the initial condition, and making the direct use of (2.5.17) is by no means straightforward. Our goal here is to make use of this connection to obtain some properties of the extremal process $\mathcal{X}$.

2.5.3 From the rescaled extremal BBM process to small initial conditions

Let us illustrate what kind of results on the Bramson shift we may need on the example of the asymptotic growth of $\mathcal{X}$, conjectured in [30], and proved in [35] using purely probabilistic tools. In order to relate this result to the Bramson shift and the realm of PDE, we can do the following. Consider the shifted and rescaled version of the measure $\mathcal{X}$:

$$Y_n(dx) = n^{-1} e^{-n} \mathcal{X}_n(dx), \quad \mathcal{X}_n = \sum_k \delta_{\chi_k - n}, \tag{2.5.18}$$

so that

$$\frac{1}{ne^n} \mathcal{X}((-\infty, n]) = Y_n((-\infty; 0]). \tag{2.5.19}$$
{{20apr1428}

We may analyze the conditional on Z Laplace transform of Y_n using (2.5.17): given a non-negative function $\phi_0(x)$ compactly supported on the right, we have

$$\mathbb{E}\left[e^{-Y_n(\phi_0)}|Z\right] = \mathbb{E}\left[e^{-n^{-1}e^{-n}\mathcal{X}_n(\phi_0)}|Z\right] = \mathbb{E}\left[e^{-\mathcal{X}(\phi_n)}|Z\right] = \exp\big\{-Ze^{-\widehat{s}[\psi_n]}\big\}, \tag{2.5.20}$$
{{20apr1430}

with

$$\phi_n(x) = n^{-1}e^{-n}\phi_0(x-n), \quad \psi_n(x) = 1 - \exp\{-\phi_n(x)\}. \tag{2.5.21}$$ {{20apr1436}

For $n \gg 1$, which is the limit we are interested in, the function $\psi_n(x)$ is small: it is of the size $O(n^{-1}e^{-n})$, as is $\phi_n(x)$. Thus, (2.5.20) relates the understanding of the conditional on Z weak limit of Y_n to the asymptotics of the Bramson shift for small initial conditions for the Fisher-KPP equation (2.5.1), and this is the strategy exploited in [76] to obtain limit theorems for the process $\mathcal{X}$. Let us stress that a connection between the limiting statistics of BBM and the Bramson shift for small initial conditions was already made in [30], though with a slightly different objective in mind, and in a rather different way.

2.5.4 The Bramson shift for small initial conditions: rough asymptotics

We now state the results for the Bramson shift of the solutions to the Fisher-KPP equation

$$\frac{\partial u_\varepsilon}{\partial t} = \frac{\partial^2 u_\varepsilon}{\partial x^2} + u_\varepsilon - u_\varepsilon^2, \tag{2.5.22}$$ {{jul610}}

with a small initial condition

$$u_\varepsilon(0,x) = \varepsilon\phi_0(x), \tag{2.5.23}$$ {{jul612}}

that we will need for studying the limiting behavior of $\mathcal{X}$. Here, $\varepsilon \ll 1$ is a small parameter, and the function $\phi_0(x)$ is non-negative, bounded and compactly supported on the right: there exists $L_0 \in \mathbb{R}$ such that $\phi_0(x) = 0$ for $x \geq L_0$. We will use the notation $x_\varepsilon = \widehat{s}[\varepsilon\phi_0]$ for the Bramson shift of $\varepsilon\phi_0$:

$$|u_\varepsilon(t, x + m(t)| \to U(x + x_\varepsilon) \to 0 \text{ as } t \to +\infty. \text{ uniformly on compact intervals in } x, \tag{2.5.24}$$ {{jul614}}

We chose the sign of x_ε in (2.5.24) so that $x_\varepsilon > 0$ for $\varepsilon > 0$ sufficiently small. The following proposition gives the asymptotic behavior for x_ε for small $\varepsilon > 0$ that is sufficiently precise to recover the law of large numbers.

Proposition 2.5.1. *Under the above assumptions on ϕ_0, we have*

$$|x_\varepsilon - \log\varepsilon^{-1} + \log\log\varepsilon^{-1} + \log\bar{c}| \to 0 \text{ as } \varepsilon \downarrow 0, \tag{2.5.25}$$ {{jun2702}}

with

$$\bar{c} = \frac{1}{\sqrt{4\pi}}\int_{-\infty}^{\infty} e^x\phi_0(x)dx. \tag{2.5.26}$$ {{jul620}}

We refer to (2.5.25) as "rough asymptotics" simply because we would need its refinement to understand the fluctuations of the extremal process.

2.5.5 The law of large numbers for the extremal process for BBM

An immediate corollary of Proposition 2.5.1 is the following. We set

$$\mu(dx) = \frac{1}{\sqrt{4\pi}}e^x\,dx, \tag{2.5.27}$$ {{20mar3102}

so that

$$\bar{c} = \mu(\phi_0). \tag{2.5.28}$$ {{20mar3118}

Theorem 2.5.2. *Conditionally on Z, we have*

$$Y_n(dx) \underset{n\to\infty}{\longrightarrow} Z\mu(dx) \quad \textit{in probability.} \tag{2.5.29}$$ {{20mar3022}

In other words, $Y_n(dx)$ looks like an exponential shifted by $\log Z$ to the left. We can reformulate Theorem 2.5.2 as follows: consider the measures $\mathcal{X}_n^*$ shifted by $\log Z$:

$$\mathcal{X}_n^* \equiv \sum_k \delta_{\chi_k + \log Z}, \tag{2.5.30}$$ {{20mar3026}

and

$$Y_n^*(dx) = n^{-1} e^{-n} \mathcal{X}_n^*(dx). \tag{2.5.31}$$ {{20mar3024}

Corollary 2.5.3. *We have*

$$Y_n^*(dx) \underset{n\to\infty}{\longrightarrow} \mu(dx), \quad \text{in } \mathcal{M}_v^+ \text{ in probability.} \tag{2.5.32}$$ {{20mar3028}

2.5.6 The Bramson shift for small initial conditions: fine asymptotics

Let us briefly mention that in order to obtain results on the fluctuations of $Y_n(dx)$ around $Z\mu(dx)$, we will need a finer asymptotics for the shift x_ε than in Proposition 2.5.1. Let us define the constant

$$\bar{c}_1 = \frac{1}{\sqrt{4\pi}} \int_{-\infty}^{\infty} x e^x \phi_0(x) dx, \tag{2.5.33}$$ {{20apr104}}

that depends on the initial condition ϕ_0, as does $\bar{c}$ in (2.5.26), and universal constants

$$g_\infty = \int_0^1 e^{z^2/4} \int_z^\infty e^{-y^2/4} dy dz - 2 \int_1^\infty e^{z^2/4} \int_z^\infty \frac{1}{y^2} e^{-y^2/4} dy, \tag{2.5.34}$$ {{20apr1440}

and

$$m_1 = \frac{3}{2} g_\infty + k_0 + \frac{1}{2}, \tag{2.5.35}$$ {{20apr102}}

that do not depend on ϕ_0. Here, k_0 is the constant that appears in the asymptotics

$$U(x) \sim (x + k_0) e^{-x}, \quad \text{as } x \to +\infty. \tag{2.5.36}$$

The following theorem allows us to obtain convergence in law of the fluctuations of Y_n.

Theorem 2.5.4. *Under the above assumptions on ϕ_0, we have the asymptotics*

$$x_\varepsilon = \log \varepsilon^{-1} - \log\log \varepsilon^{-1} - \log \bar{c} - \frac{2 \log\log \varepsilon^{-1}}{\log \varepsilon^{-1}} - \Big(m_1 - \log \bar{c} + \frac{\bar{c}_1}{\bar{c}} \Big) \frac{1}{\log \varepsilon^{-1}} + O\Big(\frac{1}{(\log \varepsilon^{-1})^{1+\gamma}} \Big), \tag{2.5.37}$$ {{19jun1102}

as $\varepsilon \downarrow 0$, with some $\gamma > 0$.

The first two terms in (2.5.25) and (2.5.37) have been predicted in [30] in addressing a different BBM question, using an informal Tauberian type argument that we were not able to make rigorous. The proof of this theorem does not seem to be directly related to the arguments of [30] but the general approach to the statistics of BBM via the Bramson shift asymptotics for small initial conditions comes from [30].

We will not discuss in detail the implications of the expansion (2.5.37) but simply say that it can be used to show that the fluctuations of the properly rescaled and re-shifted extremal BBM process converge to a 1-stable variable. This is proved by finding the Laplace transform of that process and identifying it is the Laplace transform of the 1-stable process. The term $\log\log \varepsilon^{-1}/\log \varepsilon^{-1}$ in (2.5.37) is absolutely indispensable here. We should also mention that the term of the order $1/\log \varepsilon^{-1}$ leads to a small extra shift that is additional to $\log Z$.

2.5.7 Outline of the proof of the asymptptocs of the Bramson shift for small initial conditions

We now explain how the asymptotics for the Bramson shift for the solutions with small initial conditions comes about. We only consider the level of precision in Proposition 2.5.1.

The solution asymptotics in self-similar variables

Proposition 2.5.1 is a consequence of the following two steps. The first result connects the Bramson shift of a solution to the Fisher-KPP equation with a small initial condition to the asymptotics of the solution to a problem in the self-similar variables with an initial condition shifted far to the right.

Proposition 2.5.5. *Let r_ℓ be the solution to*

$$\frac{\partial r_\ell}{\partial \tau} - \frac{\eta}{2}\frac{\partial r_\ell}{\partial \eta} - \frac{\partial^2 r_\ell}{\partial \eta^2} - r_\ell + \frac{3}{2}e^{-\tau/2}\frac{\partial r_\ell}{\partial \eta} + e^{3\tau/2-\eta\exp(\tau/2)}r_\ell^2 = 0, \quad \tau > 0, \quad \eta \in \mathbb{R}, \tag{2.5.38}$$

with the initial condition $r_\ell(0,\eta) = \psi_0(\eta - \ell)$, where $\psi_0(\eta) = e^\eta \phi_0(\eta)$. Then, for each $\ell > 0$, the function $r_\ell(\tau,\eta)$ has the asymptotics

$$r_\ell(\tau,\eta) \sim r_\infty(\ell)\eta e^{-\eta^2/4}, \text{ as } \tau \to +\infty, \text{ for } \eta > 0. \tag{2.5.39}$$ {{may1012}}

Furthermore, the Bramson shift that appears in Proposition 2.5.1 is given by

$$x_\varepsilon = \log \varepsilon^{-1} - \log r_\infty(\ell_\varepsilon), \text{ with } \ell_\varepsilon = \log \varepsilon^{-1}. \tag{2.5.40}$$ {{19jun1210}

The second result, at the core of the proof of Proposition 2.5.1, describes the asymptotics of $r_\infty(\ell)$ for large ℓ.

Proposition 2.5.6. *The function $r_\infty(\ell)$ satisfies the following asymptotics:*

$$r_\infty(\ell) = \bar{c}\ell + O(\log \ell), \quad \text{as } \ell \to +\infty, \tag{2.5.41}$$ {{jun2502}}

with the constant $\bar{c}$ as in (2.5.26).

To prove Theorem 2.5.4, we needs to refine Proposition 2.5.6 to the following.

Proposition 2.5.7. *The function $r_\infty(\ell)$ satisfies the following asymptotics:*

$$r_\infty(\ell) = \bar{c}\ell + 2\bar{c}\log \ell + m_1\bar{c} + \bar{c}_1 - \bar{c}\log\bar{c} + O(\ell^{-\delta}), \tag{2.5.42}$$ {{jan810bis}

with the constants $\bar{c}$, $\bar{c}_1$ and m_1 as in (2.5.26), (2.5.33) and (2.5.35).

Using (2.5.40), we obtain from Proposition 2.5.7 that

$$\begin{aligned} x_\varepsilon &= \ell_\varepsilon - \log r_\infty(\ell_\varepsilon) = \ell_\varepsilon - \log\Big(\bar{c}\ell_\varepsilon + 2\bar{c}\log \ell_\varepsilon + m_1\bar{c} - \bar{c}\log\bar{c} + \bar{c}_1 + O(\ell_\varepsilon^{-\delta})\Big) \\ &= \ell_\varepsilon - \log \ell_\varepsilon - \log \bar{c} - 2\frac{\log \ell_\varepsilon}{\ell_\varepsilon} - \frac{m_1}{\ell_\varepsilon} + \frac{\log \bar{c}}{\ell_\varepsilon} - \frac{\bar{c}_1}{\bar{c}\ell_\varepsilon} + O(\ell_\varepsilon^{-1-\delta}), \end{aligned} \tag{2.5.43}$$ {{19jun1208}

which proves Theorem 2.5.4.

Of course, Proposition 2.5.6 in an immediate consequence of Proposition 2.5.7 but its proof is much simpler so we only outline that.

2.5.8 Reduction to the self-similar variables

The conclusion of Proposition 2.5.5 follows from a series of changes of variables that we now describe. We first go into the moving frame, writing solution to (2.5.22)-(2.5.23) as

$$u_\varepsilon(t,x) = \tilde{u}_\varepsilon(t, x - 2t + \frac{3}{2}\log(t+1)). \tag{2.5.44}$$ {{jun2008}}

The function $\tilde{u}_\varepsilon(t,x)$ satisfies

$$\frac{\partial \tilde{u}_\varepsilon}{\partial t} - \Big(2 - \frac{3}{2(t+1)}\Big)\frac{\partial \tilde{u}_\varepsilon}{\partial x} = \frac{\partial^2 \tilde{u}_\varepsilon}{\partial x^2} + \tilde{u}_\varepsilon - \tilde{u}_\varepsilon^2. \tag{2.5.45}$$

Next, we take out the exponential decay factor, writing

$$\tilde{u}_\varepsilon(t,x) = e^{-x} z_\varepsilon(t,x), \tag{2.5.46}$$ {{jun2062}}

which gives

$$\frac{\partial z_\varepsilon}{\partial t} - \frac{3}{2(t+1)}\Big(z_\varepsilon - \frac{\partial z_\varepsilon}{\partial x}\Big) = \frac{\partial^2 z_\varepsilon}{\partial x^2} - e^{-x} z_\varepsilon^2. \tag{2.5.47}$$

As (2.5.47) is a perturbation of the standard heat equation, it is helpful to pass to the self-similar variables:

$$z_\varepsilon(t,x) = v_\varepsilon(\log(t+1), \frac{x}{\sqrt{t+1}}). \tag{2.5.48}$$ {{jun2066}}

The function $v_\varepsilon(\tau,\eta)$ is the solution of

$$\frac{\partial v_\varepsilon}{\partial \tau} - \frac{\eta}{2}\frac{\partial v_\varepsilon}{\partial \eta} - \frac{\partial^2 v_\varepsilon}{\partial \eta^2} - \frac{3}{2} v_\varepsilon + \frac{3}{2} e^{-\tau/2}\frac{\partial v_\varepsilon}{\partial \eta} + e^{\tau - \eta \exp(\tau/2)} v_\varepsilon^2 = 0, \tag{2.5.49}$$

with the initial condition

$$v_\varepsilon(0,\eta) = \varepsilon e^{\eta}\phi_0(\eta). \tag{2.5.50}$$ {{jul602}}

In order to get rid of the pre-factor ε in the initial condition (2.5.50), and also to adjust the zero-order term in (2.5.49), it is convenient to represent $v_\varepsilon(\tau,\eta)$ as

$$v_\varepsilon(\tau,\eta) = \varepsilon v_1(\tau,\eta) e^{\tau/2}. \tag{2.5.51}$$ {{jun2070}}

Here, $v_1(\tau,\eta)$ is the solution of

$$\frac{\partial v_1}{\partial \tau} - \frac{\eta}{2}\frac{\partial v_1}{\partial \eta} - \frac{\partial^2 v_1}{\partial \eta^2} - v_1 + \frac{3}{2} e^{-\tau/2}\frac{\partial v_1}{\partial \eta} + \varepsilon e^{3\tau/2 - \eta \exp(\tau/2)} v_1^2 = 0, \tag{2.5.52}$$

with the initial condition

$$v_1(0,\eta) = e^{\eta}\phi_0(\eta). \tag{2.5.53}$$ {{jun2080}}

The next, and last, in this chain of preliminary transformations is to eliminate the pre-factor ε in the last term in (2.5.52). We choose

$$\beta(\tau) = e^{-\tau/2}\log\varepsilon, \tag{2.5.54}$$ {{jun2090}}

so that

$$\varepsilon e^{3\tau/2 - \eta \exp(\tau/2)} = e^{3\tau/2 - (\eta - \beta(\tau))\exp(\tau/2)}, \tag{2.5.55}$$ {{jun2088}}

and make a change of the spatial variable:

$$v_1(\tau,\eta) = r_\varepsilon(\tau,\eta-\beta(\tau)). \tag{2.5.56}$$ {{jun2092}}

The function r_ε satisfies:

$$\frac{\partial r_\varepsilon}{\partial \tau} - \frac{\eta}{2}\frac{\partial r_\varepsilon}{\partial \eta} - \frac{\partial^2 r_\varepsilon}{\partial \eta^2} - r_\varepsilon + \frac{3}{2}e^{-\tau/2}\frac{\partial r_\varepsilon}{\partial \eta} + e^{3\tau/2-\eta\exp(\tau/2)}r_\varepsilon^2 = 0, \tag{2.5.57}$$

with the initial condition

$$r_\varepsilon(0,\eta) = \psi_0(\eta-\ell_\varepsilon), \tag{2.5.58}$$ {{19jul2504}}

with ℓ_ε as in (2.5.40), and

$$\psi_0(\eta) = e^\eta\phi_0(\eta). \tag{2.5.59}$$ {{19jul2502}}

This, with a slight abuse of notation, is exactly (2.5.38). Note that r_ε depends on ε only through ℓ_ε as it appears in the initial condition. We will interchangeably, with some abuse of notation use $r_\varepsilon(t,x)$ and $r_{l_\varepsilon}(t,x)$ for the same object.

As far as the asymptotics of $r_\varepsilon(\tau,\eta)$ and its connection to the Bramson shift are concerned, it was shown in [79] that there exists a constant $v_\infty(\varepsilon) > 0$ so that the solution $v_\varepsilon(\tau,\eta)$ of (2.5.49) has the asymptotics

$$v_\varepsilon(\tau,\eta) \sim v_\infty(\varepsilon)\eta e^{-\eta^2/4}e^{\tau/2}, \text{ as } \tau\to+\infty, \text{ for } \eta>0. \tag{2.5.60}$$ {{may1002}}

and the Bramson shift is given by

$$x_\varepsilon = -\log v_\infty(\varepsilon). \tag{2.5.61}$$ {{may1004}}

The corrseponding long-time asymptotics for the function $v_1(\tau,\eta)$, the solution to (2.5.52) is

$$v_1(\tau,\eta) \sim \tilde{v}_\infty(\varepsilon)\eta e^{-\eta^2/4}, \text{ as } \tau\to+\infty, \text{ for } \eta>0, \tag{2.5.62}$$ {{may1006}}

with

$$\tilde{v}_\infty(\varepsilon) = \varepsilon v_\infty(\varepsilon), \tag{2.5.63}$$ {{may1008}}

and the asymptotics for r_ε is

$$r_\varepsilon(\tau,\eta) = v_1(\tau,\eta+\beta(\tau)) \sim \tilde{v}_\infty(\varepsilon)(\eta+\beta(\tau))e^{-(\eta+\beta(\tau)^2/4} \sim \tilde{v}_\infty(\varepsilon)\eta e^{-\eta^2/4}, \text{ as } \tau\to+\infty, \text{ for } \eta>0, \tag{2.5.64}$$ {{may1012bis

so that

$$r_\infty(\ell_\varepsilon) = \tilde{v}_\infty(\varepsilon) = \varepsilon v_\infty(\varepsilon), \tag{2.5.65}$$ {{may1014}}

and the Bramson shift is

$$x_\varepsilon = -\log v_\infty(\varepsilon) = \log\varepsilon^{-1} - \log r_\infty(\ell_\varepsilon). \tag{2.5.66}$$ {{jun2510}}

This finishes the proof of Proposition 2.5.5.

2.5.9 Connection to the linear Dirichlet problem

Instead of giving the proof of Proposition 2.5.6, let us recall the intuition that leads to the long-time asymptotics (2.5.39) for the solution of (2.5.38), and also explain how the asymptotics (2.5.41) comes about. The key point is that we may think of (2.5.38) as a linear equation with the factor

$$e^{3\tau/2-\eta\exp(\tau/2)}r_\ell(\tau,\eta) \tag{2.5.67}$$ {{nov802}}

in the last term in its right side playing the role of an absorption coefficient. Disregarding our lack of information about $r_\ell(\tau,\eta)$ that enters (2.5.67), we expect that when $\tau \gg 1$ this term is "extremely large" for $\eta < 0$ and "extremely small" for $\eta > 0$. Thinking again of (2.5.38) as a linear equation for $r_\ell(\tau,\eta)$, the former means that $r_\ell(\tau,\eta)$ is very small for $\eta < 0$, while the latter indicates that $r_\ell(\tau,\eta)$ essentially solves a linear problem for $\eta > 0$. The drift term in (2.5.38) with the pre-factor $e^{-\tau/2}$ is also very small at large times. Thus, if we take some $T \gg 1$, then for $\tau \geq T$, a good approximation to (2.5.38) is the linear Dirichlet problem

$$\begin{aligned}
&\frac{\partial\zeta_\ell}{\partial\tau} - \frac{\eta}{2}\frac{\partial\zeta_\ell}{\partial\eta} - \frac{\partial^2\zeta_\ell}{\partial\eta^2} - \zeta_\ell = 0, \quad \tau > T,\ \eta > 0\\
&\zeta_\ell(\tau,0) = 0,\\
&\zeta_\ell(T,\eta) = r_\ell(T,\eta).
\end{aligned} \tag{2.5.68}$$ {{may2414}}

In other words, one would solve the full nonlinear problem on the whole line only until a large time $T \gg 1$, and for $\tau > T$ simply solve the linear Dirichlet problem (2.5.68). It is easy to see that

$$\bar{\zeta}(\eta) = \eta e^{-\eta^2/4}, \tag{2.5.69}$$ {{may2416}}

is a steady solution to (2.5.68). In addition, the operator

$$\mathcal{L}u = \frac{\partial^2 u}{\partial\eta^2} + \frac{\eta}{2}\frac{\partial u}{\partial\eta} + u, \quad \eta > 0, \tag{2.5.70}$$ {{may2420}}

with the Dirichlet boundary condition at $\eta = 0$ has a discrete spectrum. It follows that $\zeta_\ell(\tau,\eta)$ has the long time asymptotics

$$\zeta_\ell(\tau,\eta) \sim \zeta_\infty(\ell)\eta e^{-\eta^2/4}, \quad \tau \to +\infty. \tag{2.5.71}$$ {{may2418}}

As the integral

$$\int_0^\infty \eta\zeta_\ell(\tau,\eta)d\eta = \int_0^\infty \eta\zeta_\ell(T,\eta)d\eta \tag{2.5.72}$$ {{may2422}}

is conserved, the coefficient $\zeta_\infty(\ell)$ is determined by the relation

$$\zeta_\infty(\ell)\int_0^\infty \eta^2 e^{-\eta^2/4}d\eta = \int_0^\infty \eta\zeta_\ell(T,\eta)d\eta, \tag{2.5.73}$$ {{may2424}}

so that

$$\zeta_\infty(\ell) = \frac{1}{\sqrt{4\pi}}\int_0^\infty \eta\zeta_\ell(T,\eta)d\eta = \frac{1}{\sqrt{4\pi}}\int_0^\infty \eta r_\ell(T,\eta)d\eta. \tag{2.5.74}$$ {{may2426}}

As we expect $\zeta_\ell(\tau,\eta)$ and $r_\ell(\tau,\eta)$ to be close if T is sufficiently large, we should have an approximation

$$\zeta_\infty(\ell) \approx r_\infty(\ell), \tag{2.5.75}$$ {{may2502}}

if $T \gg 1$. This, in turn, implies that

$$r_\infty(\ell) = \lim_{\tau\to+\infty} \frac{1}{\sqrt{4\pi}} \int_0^\infty \eta r_\ell(\tau,\eta)d\eta. \tag{2.5.76}$$ {{may2504}}

This informal argument is made rigorous in [79].

The limit in the right side of (2.5.76) is an implicit functional of the initial conditions for the nonlinear problem (2.5.38), and the evolution of the solution in the initial time layer, before the linear approximation kicks in, is difficult to control, so that there is no explicit expression for $r_\infty(\ell)$. In the present setting, however, the initial condition $r_\ell(0,\eta)$ in (2.5.38) is shifted to the right by $\ell \gg 1$. Therefore, at small times the solution is concentrated at $\eta \gg 1$, a region where the factor in front of the nonlinear term in (2.5.38)

$$\exp\left(\frac{3\tau}{2} - \eta e^{\tau/2}\right) \ll 1 \tag{2.5.77}$$ {{may2506}}

is very small even for $\tau = O(1)$. Hence, solutions to the nonlinear equation (2.5.38) with the initial conditions (2.5.58) should be well approximated, to the leading order, by the linear problem

$$\frac{\partial \tilde r_\ell}{\partial \tau} - \frac{\eta}{2}\frac{\partial \tilde r_\ell}{\partial \eta} - \frac{\partial^2 \tilde r_\ell}{\partial \eta^2} - \tilde r_\ell + \frac{3}{2}e^{-\tau/2}\frac{\partial \tilde r_\ell}{\partial \eta} = 0, \quad \tilde r_\ell(0,\eta) = r_\ell(0,\eta), \tag{2.5.78}$$

even for small times. However, the solution "does not yet know" for "small" τ that there is a large dissipative term in the nonlinear equation, or the Dirichlet boundary condition in the linear version, and evolves "as if (2.5.78) is posed for $\eta \in \mathbb{R}$". This leads to exponential growth in τ until the solution spreads sufficiently far to the left, close to $\eta = 0$ and "discovers" the Dirichlet boundary condition (or the nonlinearity in the full nonlinear version). During this "short time" evolution we have

$$\frac{d}{d\tau}\int \eta \tilde r_\ell(\tau,\eta)d\eta = \frac{3}{2}e^{-\tau/2}\int \tilde r_\ell(\tau,\eta)d\eta. \tag{2.5.79}$$ {{may2510}}

Unlike the first moment, the total mass in the right side does not grow as ℓ gets larger – the shift of the initial condition to the right increases the first moment but not the mass. Thus, the first moment of $r_\ell(\tau,\eta)$ will only change by a factor that is $o(1)$ during the "short time" evolution, so that it is conserved to the leading order in ℓ. The "long time" evolution following this initial time layer is well approximated by the linear Dirichlet problem (2.5.68) that preserves the first moment. Thus, altogether, the first moment will not change to the leading order if $\ell \gg 1$ is large, so that

$$\lim_{\tau\to+\infty}\int_0^\infty \eta r_\ell(\tau,\eta)d\eta = (1+o(1))\int_0^\infty \eta r_\ell(0,\eta)d\eta, \quad \text{as } \varepsilon \to 0, \tag{2.5.80}$$ {{may2514}}

which leads to the explicit expression for $r_\infty(\ell)$ in terms of the initial first moment:

$$\begin{aligned} r_\infty(\ell) &= (1+o(1))\frac{1}{\sqrt{4\pi}}\int_0^\infty \eta r_\ell(0,\eta)d\eta = (1+o(1))\frac{1}{\sqrt{4\pi}}\int_0^\infty \eta e^{\eta-\ell}\phi_0(\eta-\ell)d\eta \\ &= (1+o(1))\frac{1}{\sqrt{4\pi}}\int_{-\ell}^\infty (\eta+\ell)e^{\eta}\phi_0(\eta)d\eta = (1+o(1))\frac{\ell}{\sqrt{4\pi}}\int_{-\infty}^\infty e^{\eta}\phi_0(\eta)d\eta \\ &= \bar c(1+o(1))\ell, \end{aligned} \tag{2.5.81}$$ {{may2512}}

which is (2.5.41). This very informal argument is behind the reason why we can describe the Bramson shift so explicitly for $\varepsilon \ll 1$, which corresponds to $\ell \gg 1$. The rest of the proof of Proposition 2.5.6 formalizes this argument by providing matching upper and lower bounds on the limit in the right side of (2.5.76).

3 Lecture 3: Pushmi-pullyu nonlinearities and the Burgers-FKPP equation

3.1 Overview of the lecture

Let us recall that for the Fisher-KPP nonlinearities $f(u)$ the front of the solutions to

$$u_t = u_{xx} + f(u), \tag{3.1.1}$$

is located at the position

$$m(t) = c_* t - \frac{3}{2\lambda_*} t + x_0 + o(1), \quad \text{as } t \to +\infty, \tag{3.1.2}$$ {{23aug1002}

with $c_* = 2\sqrt{f'(0)}$ and $\lambda_* = c_*/2$. For BBM this is the median location of the maximal particle, as it is either proved or conjectured to be for other log-correlated random processes. The question we discuss in this lecture is how far one can stray from the Fisher-KPP or McKean nonlinearity to keep Bramson's asymptotics (3.1.2). For instance, consider a voting model for at ternary BBM, where at each node of the genealogical tree one takes the maximum voting with probability p and the majority voting with the probability $1 - p$. How close to 1 does p have to be so that the media asymptotics is still given by (3.1.2) and when does it transition that to the Allen-Cahn pushded regime with

$$m(t) = c_p t + x_0 + o(1), \quad \text{as } t \to +\infty, \tag{3.1.3}$$ {{23aug1004}

with the corresponding speed $c_p > 0$.

To understand this issue, we will consider nonlinearities of the form

$$f(u) = (u - A(u))(1 + \chi A'(u)). \tag{3.1.4}$$ {{23aug1006}

Here, $A(u)$ is a convex increasing function such that $A(0) = 0$ and $A(1) = 1$. A typical example to keep in mind is $A(u) = u^n$. When $\chi \geq 0$ sufficiently close to zero, the function $f(u)$ is of the Fisher-KPP class. This property is lost at some $\chi_{FKPP} \in (0, 1)$. However, we show that Bramson's asymptotics (3.1.2) holds for all $0 \leq \chi < 1$. At $\chi = 1$ we see what we call the "pushmi-pullyu" transition and the front is located at

$$m(t) = c_* t - \frac{1}{2\lambda_*} t + x_0 + o(1), \quad \text{as } t \to +\infty, \tag{3.1.5}$$ {{23aug1008}

same as what we have seen for the maximum of independent Brownian motion. For $\chi > 1$ one see the "pushed" asymptotics as in (3.1.3).

Interestingly, the pushmi-pullyu case $\chi = 1$ is closely related to the Burgers-FKPP equation

$$u_t + \beta(A(u))_x = u_x x + u - A(u). \tag{3.1.6}$$

Its solution exhibit the same pulled to pushed transition at $\beta = 1$. The proofs rely on several miracles we do not understand and a relative entropy argument. As an interesting twist, the entropy is taken relative to a super-solution rather than to a solution, as is typically done.

The final introductory point is that here we will first see the shape defect function that will play a key role in the next lecture as a way to quantify the convergence rates to traveling waves.

3.2 The traveling wave profile and shape defect functions

3.2.1 The traveling wave profile function

As we have discussed, traveling waves for an equation of the form

$$u_t = u_{xx} + f(u), \quad t > 0, \quad x \in \mathbb{R}, \tag{3.2.1}$$ {{23aug716}}

are solutions to

$$-cU_c' = U_c'' + f(U_c), \quad U_c(-\infty) = 1, \; U_c(+\infty) = 0. \tag{3.2.2}$$ {{23aug704}}

We recall that if

$$f(0) = f(1) = 0, \quad f'(0) > 0, \tag{3.2.3}$$ {{23aug706}}

there is $c_* > 0$ so that traveling waves for (3.2.1) exist for all $c \geq c_*$, see Proposition 2.2.1. We will assume that (3.2.3) holds throughout this lecture.

Recall that for all $c \geq c_*$ the traveling wave profiles $U_c(x)$ are monotonically decreasing in x. Thus, for each $c \geq c_*$ there exists a function $\eta_c(u)$, so that $U_c(x)$ satisfies an ordinary differential equation

$$-U_c' = \eta_c(U_c), \quad U_c(-\infty) = 1, \quad U_c(+\infty) = 0. \tag{3.2.4}$$ {{23aug712}}

We will refer to $\eta_c(u)$ as the traveling wave profile function. While it is defined very implicitly in terms of the nonlinearity $f(u)$, it turns out to be surprisingly useful.

One can check that for each $c \geq c_*$, the function $\eta_c(u)$ is continuously differentiable for $u \in [0, 1]$ and satisfies

$$\eta_c(0) = \eta_c(1) = 0, \; \eta_c'(0) > 0, \quad \eta_c(u) > 0, \quad \text{for all } u \in (0, 1). \tag{3.2.5}$$ {{23aug708}}

Comparing to (3.2.3), we see that $\eta_c(u)$ looks a little bit like $f(u)$ itself. An elementary computation using (3.2.2) and (3.2.4) connects the functions $f(u)$ and $\eta_c(u)$ by

$$f(u) = \eta_c(u)(c - \eta_c'(u)), \quad \text{for all } u \in (0, 1). \tag{3.2.6}$$ {{23aug710}}

3.2.2 Purely exponentially decaying waves

We now discuss how some properties of traveling waves can be restated in terms of the traveling wave profile function $\eta_c(u)$. We begin with their tail asymptotics. Traveling waves generally have the asymptotics

$$U_c(x) \sim (A_c x + B_c) e^{-\lambda_c x}, \tag{3.2.7}$$ {{23aug714}}

with the exponent λ_c given by

$$\lambda_c = \frac{c - \sqrt{c^2 - 4f'(0)}}{2} = \eta_c'(0), \tag{3.2.8}$$ {{23aug718}}

and some $A_c \geq 0$ and $B_c \in \mathbb{R}$. It is well known that the presence or absence of the pre-factor x in front of the exponential in (3.2.7) is very important. For reasons that will become clear later, this factor determines whether the solutions to (3.2.1) are "pulled", in the sense that their evolution is dominated by the tail behavior far ahead of the front, or are "pushed", in the sense that their long time behavior is dominated by the region near the front. However, the pulled or pushed nature of the wave is often difficult to predict just from the nonlinearity $f(u)$ itself. The introduction of the wave profile function allows to do this, albeit somewhat implicitly. More precisely, the next statement characterizes the nonlinearities $f(u)$ for which the decay is purely exponential, so that $A_c = 0$ in (3.2.7).

Proposition 3.2.1. *Let $c \geq c_*$ and $f \in C^1[0,1]$ be of the form*

$$f(u) = \eta_c(u)(c - \eta_c'(u)), \tag{3.2.9}$$ {{dec2844}}

with a function $\eta_c(u)$ such that

$$\eta_c(0) = \eta_c(1) = 0, \quad \eta_c'(0) > 0, \quad \eta_c(u) > 0, \; \eta_c'(u) < c, \quad \textit{for } 0 < u < 1. \tag{3.2.10}$$ {{23aug722}}

If the function $\eta_c(u)$ is $C^{1,\alpha}([0,1])$ with some $\alpha \in (0,1)$, then the traveling wave $U_c(x)$ has the purely exponential asymptotics

$$U_c(x) \sim Be^{-\lambda_c x}, \quad \textit{as } x \to +\infty, \tag{3.2.11}$$ {{23aug727}}

with some $B > 0$.

Note that for a given smooth nonlinearity $f(u)$, the function $\eta(u) = \eta_{c_*}(u)$ is $C^1([0,1])$ but it need not be in $C^{1,\alpha}([0,1])$ in general. In particular, it is the case for the Fisher-KPP nonlinearities for which the wave asymptotics has the form (3.2.7) with $A_c > 0$. Then, the function $\eta(u)$ behaves at $u = 0$ as

$$\eta(u) = \lambda_c\Big(u + \frac{u}{\log u}\Big) + \dots, \quad \text{as } u \to 0, \tag{3.2.12}$$ {{23aug728}}

and is continuously differentiable but not $C^{1,\alpha}$. This will be very important later, when we discuss the convergence rates for the solutions to (3.2.1) to a traveling wave. It turns out that the Bramson logarithmic correction $(3/2)\log t$ in Theorem 2.4.1 comes exactly from the term $u/\log u$ in (3.2.12). This will be the subject of the next lecture.

3.2.3 When is the minimal speed given by the Fisher-KPP formula

It is well known that for the Fisher-KPP type nonlinearities the traveling wave minimal speed is given by the Fisher-KPP formula

$$c_*[f] = 2\sqrt{f'(0)}. \tag{3.2.13}$$ {{dec2829}}

However, the Fisher-KPP condition is not necessary for (3.2.13) to hold. A well-known example from [52], also discussed in detail in [74], is the nonlinearity

$$f(u) = u(1-u)(1+au), \tag{3.2.14}$$ {{dec2831}}

with $a > 0$. This nonlinearity satisfies the FKPP property for all $0 \leq a \leq 1$. However, its minimal wave speed satisfies (3.2.13) for all $0 \leq a \leq 2$, as can be shown by a phase plane analysis. A generalization of this example:

$$f(u) = u(1-u^n)(1+au^n), \tag{3.2.15}$$ {{dec2840}}

was considered in [38]. This nonlinearity also has the Fisher-KPP property for all $0 \leq a \leq 1$ but satisfies (3.2.13) in the much larger range $0 \leq a \leq n+1$ [38]. We will refer to the nonlinearities in (3.2.15) as Hadeler-Rothe nonlinearities. They present a very nice playground, to study the transition from the pulled to pushed behavior that happens at $a = n+1$.

A natural question is for which other nonlinearities the Fisher-KPP formula for the speed holds. Here is a sufficient condition.

Proposition 3.2.2. *Assume that $f(u)$ satisfies (3.2.3) and, in addition, there is $\bar\eta(u) \in C^1[0,1]$ that satisfies*

$$\bar\eta(0) = \bar\eta(1) = 0, \quad \bar\eta'(0) = 1, \quad \bar\eta(u) > 0, \; \bar\eta'(u) < 2, \quad \textit{for } 0 < u < 1, \tag{3.2.16}$$ {{dec2830}}

and such that

$$f(u) \leq f'(0)\bar{\eta}(u)(2-\bar{\eta}'(u)), \quad \text{for all } 0 \leq u \leq 1. \tag{3.2.17}$$ {{dec925}}

Then, the minimal speed $c_*[f]$ is

$$c_*[f] = 2\sqrt{f'(0)}. \tag{3.2.18}$$ {{dec2842}}

The advantage of Proposition 3.2.2 is that it allows to use explicit functions $\eta(u)$ to verify the validity of the Fisher-KPP formula. For example, if we take $\bar{\eta}(u) = u(1-u)$, then assumption (3.2.17) becomes

$$f(u) \leq u(1-u)(1+2u). \tag{3.2.19}$$

In particular, it holds for nonlinearities of the form (3.2.14) exactly in the range $0 \leq a \leq 2$, without any need for a phase plane analysis. On the other hand, for

$$\bar{\eta}(u) = u(1-u^n), \tag{3.2.20}$$

the assumption (3.2.17) becomes

$$f(u) \leq u(1-u^n)(1+(n+1)u^n). \tag{3.2.21}$$ {{dec2841}}

Nonlinearities of the form (3.2.15) satisfy (3.2.21) in the range $0 \leq a \leq n+1$. We immediately conclude that in that range the minimal traveling wave speed is $c_* = 2$.

3.2.4 Semi-FKPP and pushmi-pullyu nonlinearities

We now introduce a generalization of the Hadeler-Rothe nonlinearities (3.2.15). Let us start with a Fisher-KPP nonlinearity of the form

$$\eta(u) = u - A(u), \tag{3.2.22}$$ {{mar720}}

with an increasing convex function $A(u)$ such that

$$A(0) = 0, \quad A(1) = 1, \quad A'(0) = 0. \tag{3.2.23}$$ {{mar718}}

Note that

$$\eta'(0) = 1. \tag{3.2.24}$$ {{nar802}}

Warning. We will always assume below that $A(u)$ satisfies (3.2.23) and is increasing and convex, unless otherwise specified.

We will be interested in the nonlinearities of the form

$$f(u) = \lambda^2(u - A(u))(1+\chi A'(u)), \tag{3.2.25}$$ {{23aug1010}

with some $\lambda > 0$ and $\chi \geq 0$. It is easy to see that there exists $\chi_{FKPP} \in (0,1)$ so that such nonlinearities satisfy the Fisher-KPP property

$$f(0) = f(1) = 0, \ f(u) > 0 \text{ and } f(u) \leq f'(0)u, \quad \text{for all } 0 \leq u \leq 1, \tag{3.2.26}$$ {{23aug729}}

as long as $0 \leq \chi \leq \chi_{FKPP}$.

We now make a couple of definitions concerning $\chi > \chi_{FKPP}$. First, we say that a function $f(u)$ is of the semi-FKPP type if

$$f(u) \leq f'(0)(u - A(u))(1+\chi A'(u)), \tag{3.2.27}$$ {{mar722}}

with $0 \le \chi < 1$, and an increasing convex function $A(u)$ that satisfies (3.2.23).

Second, a function $f(u)$ is a pushmi-pullyu nonlinearity if it has the form (3.2.25) with $\chi = 1$:

$$f(u) = \lambda^2 \eta(u)(2 - \eta'(u)) = \lambda^2 (u - A(u))(1 + A'(u)), \tag{3.2.28}$$ {{mar721}}

with $A(u)$ as above, and some $\lambda > 0$. Note that $\lambda^2 = f'(0)$.

We will see that the solutions to

$$u_t = u_{xx} + f(u) \tag{3.2.29}$$ {{dec2802}}

with semi-FKPP nonlinearities lead to the Bramson asymptotics in Theorem 2.4.1, even though $f(u)$ is not of the Fisher-KPP type if $\chi_{FKPP} < \chi < 1$. In other words, the Bramson asymptotics is not restricted to equations of the Fisher-KPP type. However, in the pushmi-pullyu case $\chi = 1$ the behavior of the solutions changes drastically.

3.2.5 Convergence in shape and the shape defect function

As we have seen in Theorem 2.3.3, the solution $u(t,x)$ to the initial value problem

$$u_t = u_{xx} + f(u), \tag{3.2.30}$$ {{23aug702}}

with the initial condition $u(0,x) = \mathbb{1}(x \le 0)$, converges in shape to a minimal speed traveling wave. That is, there exists a reference frame $m(t)$ such that

$$|u(t, x + m(t)) - U_*(x)| \to 0, \quad \text{as } t \to +\infty, \text{ uniformly in } x \in \mathbb{R}. \tag{3.2.31}$$ {{mar808}}

Here, $U_*(x)$ is the traveling wave solution to (3.2.30), with $c = c_*$, normalized so that $U_*(0) = 1/2$.

The proof of Theorem 2.3.3 relied crucially on the steepness comparison in Proposition 2.3.2: the evolution by (3.2.30) preserves the steepness comparison of the initial conditions. The proof of that proposition relied on an intersection number argument that is not quantitative, as are most of the arguments relying on a version of the comparison principle.

An interesting way to quantify the idea of steepness is in terms of what we will call the shape defect function. Let $u(t,x)$ be a solution to (3.2.30) with $f(u)$ written in the form (3.2.6):

$$f(u) = \eta_c(u)(c - \eta_c'(u)), \tag{3.2.32}$$ {{23aug723}}

and $\eta_c(u)$ as in (3.2.10) and some $c \ge c_*[f]$. We define the shape defect function as

$$w(t,x) = -u_x(t,x) - \eta_c(u(t,x)). \tag{3.2.33}$$ {{23aug724}}

Note that if $u(t,x) = U_c(t,x)$ then $w(t,x) \equiv 0$ because the traveling wave $U_c(x)$ satisfies (3.2.4). Thus, in a sense, the shape function measures the distance between the solution and the traveling wave. However, we also have $w(t,x) \equiv 0$ if $u(t,x) \equiv 0$ or $u(t,x) \equiv 1$, so one needs to be careful in using this notion.

Recall that under assumption (3.2.3), traveling waves exist for all $c \ge c_*$ and one can define the shape defect function relative to any traveling wave with a speed $c \ge c_*$. Unless specified otherwise, we will always assume below that $c = c_*$.

The definition of the shape defect function means that $w(t,x) > 0$ for all $x \in \mathbb{R}$ if and only if $u(t,x)$ is steeper than the traveling wave profile. A direct computation shows that the shape defect function satisfies

$$
\begin{aligned}
w_t - w_{xx} &= -u_{xt} + u_{xxx} - \eta_c'(u)u_t + \eta_c'(u)u_{xx} + \eta_c''(u)u_x^2 \\
&= -(\eta_c(u)(c-\eta_c'(u)))_x - \eta_c'(u)(u_{xx} + \eta_c(u)(c-\eta_c'(u))) + \eta_c'(u)u_{xx} + \eta_c''(u)u_x^2 \\
&= -\eta_c'(u)(c-\eta_c'(u))u_x + \eta_c(u)\eta_c''(u)u_x - \eta_c(u)\eta_c'(u)(c-\eta_c'(u)) + \eta_c''(u)u_x^2 \\
&= \eta_c''(u)u_x(u_x+\eta_c(u)) - \eta_c'(u)(c-\eta_c'(u))(u_x+\eta_c(u)) = -(\eta_c''(u)u_x - \eta_c'(u)(c-\eta_c'(u)))w \\
&= (\eta_c''(u)(w+\eta_c(u)) + \eta_c'(u)(c-\eta_c'(u)))w.
\end{aligned}
\tag{3.2.34}
$$ {{dec2120}}

Hence, if at $t=0$ we know that $w(0,x)\geq 0$ for all $w\in\mathbb{R}$, then $w(t,x)>0$ for all $t>0$ and $x\in\mathbb{R}$. This is a robust way to see the preservation of steepness property: if the initial condition $u(0,x)$ is steeper than a traveling wave, it remains steeper than the wave for all $t>0$. In particular, this quantifies the proof of Proposition 2.3.2, at least when u_2 is a traveling wave.

3.2.6 The shape defect function and the energy functional

Another interesting observation is that the shape defect function provides an energy for the reaction-diffusion equation (3.2.30). Again, we use the representation (3.2.32) for $f(u)$ and write (3.2.30) in the moving frame:

$$
u_t - cu_x = u_{xx} + \eta_c(u)(c-\eta_c'(u)). \tag{3.2.35}
$$ {{23aug725}}

Consider the energy functional

$$
\mathcal{E}_c(u) = \frac{1}{2}\int_{\mathbb{R}} e^{cx}(u_x+\eta_c(u))^2dx = \frac{1}{2}\int_{\mathbb{R}} e^{cx}w^2(x)dx, \tag{3.2.36}
$$ {{dec1802}}

with $w(x)$ defined by (3.2.33). Let us compute

$$
\begin{aligned}
\frac{\delta\mathcal{E}_c}{\delta u} &= -\frac{\partial}{\partial x}\Big(e^{cx}(u_x+\eta_c(u))\Big) + e^{cx}(u_x+\eta_c(u))\eta_c'(u) \\
&= e^{cx}\big(-u_{xx} - \eta_c'(u)u_x - cu_x - c\eta_c(u) + u_x\eta_c'(u) + \eta_c(u)\eta_c'(u)\big) \\
&= -e^{cx}\big(u_{xx} + cu_x + \eta_c(u)(c-\eta_c'(u))\big).
\end{aligned}
\tag{3.2.37}
$$ {{dec1804}}

Therefore, equation (3.2.35) has a variational formulation

$$
\frac{\partial u}{\partial t} = -e^{-cx}\frac{\delta\mathcal{E}_c}{\delta u}. \tag{3.2.38}
$$ {{dec1808}}

As a consequence, it follows that if $u(t,x)$ is a solution to (3.2.35), then

$$
\frac{d\mathcal{E}_c(t)}{dt} \leq 0, \tag{3.2.39}
$$ {{mar816}}

with a strict inequality unless $u=U_c(x)$, $u\equiv 0$ or $u\equiv 1$. This does not by itself imply convergence to a traveling wave in shape because one needs to rule out the limits $u\equiv 0$ or $u\equiv 1$ but is a fun observation nevertheless and gives a new light for the reason "why" solutions to (3.2.30) converge to a traveling wave.

Alternative variational formulations for reaction-diffusion equations have been previously introduced in [47, 68, 72, 73, 83]. The energy functional considered in those papers is

$$
\tilde{\mathcal{E}}_c[u] = \int_{\mathbb{R}} e^{cx}\Big(\frac{1}{2}u_x^2 + F(u)\Big)dx. \tag{3.2.40}
$$ {{23aug726}}

Here, $F(u)$ is the anti-derivative of $(-f(u))$: $F'(u) = -f(u)$. One issue with the functional $\tilde{\mathcal{E}}_c[u]$ is that it is not defined for $u(t,x) = U_c(x)$ unless $2\lambda_c > c$. This restricts its use to bistable nonlinearities that have $f'(0) < 0$ and the decay rate of the wave is given by

$$\lambda_c = \frac{c + \sqrt{c^2 - 4f'(0)}}{2}, \tag{3.2.41}$$ {{mar812}}

with the plus sign in the numerator. In our case, the functional $\mathcal{E}_c[u]$ vanishes if $u(t,x) = U_c(x)$ and is thus well-defined. In addition, it coincides with $\tilde{\mathcal{E}}_c[u]$ for sufficiently rapidly decaying solutions. To see this, let us set

$$N_c(u) = \int_0^u \eta_c(u')du', \tag{3.2.42}$$

and write

$$\begin{aligned}
\mathcal{E}_c[u] &= \frac{1}{2}\int_{\mathbb{R}} e^{cx}(u_x + \eta_c(u))^2 dx = \frac{1}{2}\int_{\mathbb{R}} e^{cx}(u_x^2 + 2u_x\eta_c(u) + \eta_c^2(u))dx \\
&= \frac{1}{2}\int_{\mathbb{R}} e^{cx}(u_x^2 + 2(N_c(u))_x + \eta_c^2(u))dx = \frac{1}{2}\int_{\mathbb{R}} e^{cx}(u_x^2 - 2cN_c(u) + \eta_c^2(u))dx \\
&= \int_{\mathbb{R}} e^{2x}\Big(\frac{1}{2}u_x^2 + V_c(u)\Big)dx.
\end{aligned} \tag{3.2.43}$$

Here, we have defined

$$V_c(u) = -2N_c(u) + \frac{1}{2}\eta_c^2(u). \tag{3.2.44}$$

However, $V_c(u)$ is an anti-derivative of $(-f(u))$:

$$V_c'(u) = -2\eta_c(u) + \eta_c(u)\eta_c'(u) = -\eta_c(u)(2 - \eta_c'(u)) = -f(u). \tag{3.2.45}$$

This agrees with (3.2.40), so that $\mathcal{E}_c[u]$ coincides with $\tilde{\mathcal{E}}_c[u]$ when both are defined. We are not going to pursue this direction but this approach seems to make the variational tools of [47, 83] available for a larger class than the bistable equations considered in the aforementioned papers and, in particular, for the Fisher-KPP type equations.

3.3 Spreading for semi-FKPP and pushmi-pullyu nonlinearities

Let now consider the long time behavior of the solutions to

$$u_t = \Delta u + f(u), \tag{3.3.1}$$ {{may416}}

with the initial condition $u(0,x) = \mathbb{1}(x \le 0)$. We assume that $f(u)$ has the form

$$f(u) = f'(0)(u - A(u))(1 + \chi A'(u)). \tag{3.3.2}$$ {{may406}}

Recall that f is if the Fisher-KPp type for $0 \le \chi \le \chi_{FKPP}$, of the semi-FKPP type for $0 \le \chi < 1$, and of the pushmi-pullyu type for $\chi = 1$. Here is a generalization of Theorem 2.4.1.

Theorem 3.3.1. *Let f be a nonlinearity of the form (3.3.2) with $\chi \in [0,1]$ and an increasing and convex function $A(u)$ such that $A(0) = 0$, $A(1) = 1$ and $A'(0) = 0$. Then there is $m(t)$, given below, so that*

$$\lim_{t\to\infty} \sup_{|x|\le L} |u(t, x + m(t)) - U_*(x)| = 0. \tag{3.3.3}$$

(i) If f is a semi-FKPP type nonlinearity, that is, $0 \le \chi < 1$ in (3.3.2), then there exists $x_0 \in \mathbb{R}$ so that

$$m(t) = 2t - \frac{3}{2}\log t + x_0 + o(1), \quad \text{as } t \to +\infty. \tag{3.3.4}$$

{{mar1620}}

(ii) If f is a pushmi-pullyu type nonlinearity, that is, $\chi = 1$ in (3.3.2), then there exists $x_1 \in \mathbb{R}$ so that $m(t)$ has the asymptotics

$$m(t) = 2t - \frac{1}{2}\log t + x_1 + o(1), \quad \text{as } t \to +\infty. \tag{3.3.5}$$

{{mar1620bis

The asymptotics (3.3.4) for semi-FKPP type nonlinearities is exactly the same as for the Fisher-KPP nonlinearities that we have seen in Theorem 2.4.1. However, in the pushmi-pullyu case, the logarithmic correction changes from Bramson's $3/2$ to $1/2$ that we have seen for the maximum of independent Gaussians. That is, the front location at the transition from the pulled to pushed case behaves as a maximum of independent Gaussians, for reasons that we do not really understand.

The pushmi-pullyu asymptotics (3.3.5) has been predicted in [38, 62, 91] using formal matched asymptotics for the situations when the minimal speed traveling wave has a purely exponential decay, as in (3.2.11):

$$U_*(x) \sim Be^{-\lambda_* x}, \quad \text{as } x \to +\infty. \tag{3.3.6}$$

{{mar1622}}

To the best of our knowledge, the only rigorous result in this direction is the asymptotics

$$m(t) = 2t - \frac{1}{2}\log t + o(\log t), \quad \text{as } x \to +\infty, \tag{3.3.7}$$

{{mar1623}}

obtained in [49] by a careful gluing of sub- and super-solutions, a very different approach from what we describe here.

One very interesting and completely open question is to understand the transition from Bramson's $3/2\log t$ to $1/2\log t$ correction that is typical for systems of independent Gaussians in other log-correlated systems, or to explain this transition in terms of the voting schemes or other probabilistic tools.

The full proof of Theorem 3.3.1 is beyond the scope of these lectures and can be found in [2]. Below, we will highlight some interesting aspects of the proof: how the pushmi-pullyu nonlinearities are related to the reactive conservation laws, introduce the miracle of a weighted Hopf-Cole transform and then describe in more detail the proof of the corresponding result for the Burgers-FKPP equation. We also refer the reader to the recent paper [37] for a fascinating analysis of the pushed-pulled transition.

3.3.1 Connection to the reactive conservation laws for pushmi-pullyu nonlinearities

There is an interesting connection between reaction-diffusion equations and reactive conservation laws provided by the shape defect function that plays a key role in the proof of Theorem 3.3.1. Let us assume that the nonlinearity $f(u)$ is of the pushmi-pullyu type:

$$f(u) = \bar\eta(u)(2 - \bar\eta'(u)), \tag{3.3.8}$$

{{jan510}}

and the function $\bar\eta(u)$ as in (3.2.16):

$$\bar\eta(0) = \bar\eta(1) = 0, \quad \bar\eta'(0) = 1, \quad \bar\eta(u) > 0, \ \bar\eta'(u) < 2, \quad \text{for } 0 < u < 1, \tag{3.3.9}$$

{{jan512}}

so that $c_*[f] = 2$. We also let $U(x)$ be the corresponding minimal speed traveling wave profile, the solution to (3.2.4):

$$-U' = \bar\eta(U), \quad U(-\infty) = 1, \quad U(+\infty) = 0, \tag{3.3.10}$$

{{jan506}}

and

$$-2U' = U'' + f(U), \quad U(-\infty) = 1, \quad U(+\infty) = 0. \tag{3.3.11}$$ {{jan514}}

Let us set

$$A(u) = u - \bar{\eta}(u), \tag{3.3.12}$$ {{mar822}}

and write

$$-2U' + (A(U))' - U'' = -2U' + U' - (\bar{\eta}(U))' - U'' = -U' = \bar{\eta}(U) = U - A(U). \tag{3.3.13}$$ {{dec933}}

Thus, apart from (3.3.11), the solution to (3.3.10) is also a traveling wave solution to the reactive conservation law

$$u_t + (A(u))_x = u_{xx} + u - A(u). \tag{3.3.14}$$ {{dec934}}

In other words, if $f(u)$ is of the pushmi-pullyu type, then the reactive conservation law (3.3.14) and the reaction-diffusion equation equation

$$u_t = u_{xx} + f(u), \tag{3.3.15}$$ {{mar818}}

with

$$f(u) = (u - A(u))(1 + A'(u)), \tag{3.3.16}$$ {{mar820}}

have exactly the same minimal speed traveling wave profiles. It would be very interesting to have an explanation of this phenomenon.

3.3.2 Comparison to reactive conservation laws

The connection between the reactive conservation law (3.3.14) and the reaction-diffusion equation equation (3.3.15) goes beyond the common traveling wave profile. Let $f(u)$ be a pushmi-pullyu nonlinearity of the form (3.3.16), and $u(t,x)$ be the solution to

$$u_t = u_{xx} + f(u). \tag{3.3.17}$$ {{dec2112}}

As usual, we assume that $A(u)$ satisfies (3.2.23) and is increasing and convex

$$A'(u) \ge 0, \quad A''(u) \ge 0, \quad \text{for all } u \in [0,1]. \tag{3.3.18}$$ {{dec2110}}

We claim that if the shape defect function is non-negative:

$$w(t,x) = -u_x(t,x) - \bar{\eta}(u(t,x)) \ge 0, \quad \text{for all } x \in \mathbb{R} \text{ and } t \ge 0, \tag{3.3.19}$$ {{dec2108}}

then $u(t,x)$ is a sub-solution to the reactive conservation law (3.3.14):

$$u_t + (A(u))_x \le u_{xx} + u - A(u). \tag{3.3.20}$$ {{dec2102}}

Here, $\bar{\eta}(u)$ and $A(u)$ are related by (3.3.12). Let us recall that we have shown that (3.3.19) holds at all times $t > 0$ as long as it is satisfied at $t = 0$. Thus, (3.3.19) is simply a restriction on the initial condition.

To show that (3.3.20) holds, let us write

$$\begin{aligned} u_t + A'(u)u_x - u_{xx} - u + A(u) &= \bar{\eta}(u)(2 - \bar{\eta}'(u)) + A'(u)u_x - u + A(u) \\ &= (u - A(u))(1 + A'(u)) + A'(u)(-w - \bar{\eta}(u)) - u + A(u) \\ &= u - A(u) + A'(u)u - A(u)A'(u) - wA'(u) - A'(u)(u - A(u)) - u + A(u) \\ &= -wA'(u) \le 0, \end{aligned} \tag{3.3.21}$$ {{dec2106}}

because of (3.3.18) and (3.3.19).

In the general case, if (3.3.19) does not hold, so that if the shape defect function is not positive everywhere, a solution $u(t,x)$ to (3.3.17) satisfies the forced reactive conservation law

$$u_t + (A(u))_x = u_{xx} + u - A(u) - A'(u)w, \qquad w = -u_x - u + A(u). \tag{3.3.22}$$ {{dec2114}}

One can show in many situations that the $w(t,x) \to 0$ as $t \to +\infty$ even if (3.3.19) is not satisfied. We will avoid this technical complication for the sake of simplicity of presentation and to shorten some of the arguments.

A "good" explanation as to why the solution to the reaction-diffusion equation is a sub-solution to the reactive conservation law is also lacking at the moment.

3.3.3 The weighted Hopf-Cole transform

Another tool needed for the proof of Theorem 3.3.1 we would like to highlight is the weighted Hopf-Cole transform for pushmi-pullyu and semi-FKPP nonlinearities. Recall that a crucial step in the proof of Theorem 2.4.1 was the reduction to a linear Dirichlet boundary problem on a half line, discussed in Section 2.4.2. A direct attempt at doing something similar fails when $f(u)$ is not of the Fisher-KPP type. The weighted Hopf-Cole transform is a way to remedy that failure.

Let us consider a reaction-diffusion equation with a semi-FKPP nonlinearity

$$u_t - 2u_x = u_{xx} + (u - A(u))(1 + \chi A'(u)), \tag{3.3.23}$$ {{dec1431}}

with $0 \le \chi \le 1$. The function $A(u)$ satisfies the familiar assumptions (3.2.23) and (3.3.18). Let us set

$$\alpha(u) = \frac{A(u)}{u}, \tag{3.3.24}$$

and assume that $\alpha(u)$ is also convex. This is the case, for example, if

$$A(u) = \sum_k p_k u^k, \tag{3.3.25}$$

with

$$\sum_k p_k = 1. \tag{3.3.26}$$

We have obtained such nonlinearities previously using the voting models with a uniform bias.

We now apply the weighted Hopf-Cole transform

$$v(t,x) = \exp\Big(x + \sqrt{\chi}\int_x^\infty \alpha(u(t,y))dy\Big)u(t,x) \tag{3.3.27}$$

to (3.3.23). Our goal is to show that it $v(t,x)$ a sub-solution to the heat equation:

$$v_t - v_{xx} \le 0. \tag{3.3.28}$$ {{23aug729bi

This differential inequality is essential in using the aforementioned approximation by the linear Dirichlet problem, as in the proof of Theorem 2.4.1.

Unfortunately, (3.3.28) is proved by a long calculation. We use a notation

$$\Gamma := x + \sqrt{\chi}\int_x^\infty \alpha(u(t,y))dy$$

for short, and utilize the following computations

$$v_x = e^{\Gamma} u_x + (1 - \sqrt{\chi}\alpha(u)) e^{\Gamma} u, \tag{3.3.29}$$

$$v_{xx} = e^{\Gamma} u_{xx} + 2(1 - \sqrt{\chi}\alpha(u)) e^{\Gamma} u_x - \sqrt{\chi}\alpha'(u) e^{\Gamma} u u_x + (1 - \sqrt{\chi}\alpha(u))^2 e^{\Gamma} u, \tag{3.3.30}$$

and

$$\begin{aligned} v_t &= e^{\Gamma} u_t + \Big(\sqrt{\chi}\int_x^{\infty} \alpha'(u(t,y)) u_t(t,y) dy\Big) e^{\Gamma} u \\ &= e^{\Gamma} u_t + \Big(\sqrt{\chi}\int_x^{\infty} \alpha'(u(t,y))(u_{yy} + \eta(u)(1 + \chi A'(u)) + 2u_y) dy\Big) e^{\Gamma} u = e^{\Gamma} u_t \\ &+ e^{\Gamma} u\Big(- \sqrt{\chi}\alpha'(u) u_x - \sqrt{\chi}\int_x^{\infty} \alpha''(u) u_y^2 dy - 2\sqrt{\chi}\alpha(u) + \sqrt{\chi}\int_x^{\infty} \alpha'(u)\eta(u)(1 + \chi A'(u)) dy\Big). \end{aligned} \tag{3.3.31}$$

Next, we write

$$\begin{aligned} & e^{-\Gamma}\big(v_t - v_{xx}\big) = u_t - u_{xx} - 2u_x \\ & + u\Big(- \sqrt{\chi}\alpha'(u) u_x - \sqrt{\chi}\int_x^{\infty} \alpha''(u) u_y^2 dy - 2\sqrt{\chi}\alpha(u) + \sqrt{\chi}\int_x^{\infty} \alpha'(u)\eta(u)(1 + \chi A'(u)) dy\Big) \\ & + 2\sqrt{\chi}\alpha(u) u_x + \sqrt{\chi}\alpha'(u) u u_x - (1 - \sqrt{\chi}\alpha(u))^2 u, \end{aligned} \tag{3.3.32}$$

which is

$$\begin{aligned} & e^{-\Gamma}\big(v_t - v_{xx}\big) = \eta(u)(1 + \chi A'(u)) - (1 - \sqrt{\chi}\alpha(u))^2 u - 2\sqrt{\chi}\alpha(u) u + 2\sqrt{\chi}\alpha(u) u_x \\ & + u\Big(- \sqrt{\chi}\int_x^{\infty} \alpha''(u) u_y^2 dy + \sqrt{\chi}\int_x^{\infty} \alpha'(u)\big(\eta(u)(1 + \chi A'(u))\big) dy\Big). \end{aligned} \tag{3.3.33}$$ {{dec1504}}

Because by assumptions, we have

$$\alpha''(u) \geq 0 \text{ and } \alpha'(u) \geq 0 \ \text{ for all } 0 \leq u \leq 1. \tag{3.3.34}$$ {{dec1502}}

Then, we can estimate two integrals in (3.3.33) since

$$\begin{aligned} -\sqrt{\chi}\int_x^{\infty} \alpha''(u)(-u_y)(-u_y) dy &\leq -\chi\int_x^{\infty} \alpha''(u)\eta(u)(-u_y) dy \\ &= -\chi\alpha'(u)\eta(u) - \chi\int_x^{\infty} \alpha'(u)\eta'(u) u_y dy, \end{aligned} \tag{3.3.35}$$ {{dec1506}}

and

$$\begin{aligned} \sqrt{\chi}\int_x^{\infty} \alpha'(u)\big(\eta(u)(1 + bA'(u))\big) dy &\leq -\int_x^{\infty} \alpha'(u)(1 + \chi A'(u)) u_y dy \\ &= \alpha(u) - \chi\int_x^{\infty} \alpha'(u)(1 - \eta'(u)) u_y dy \\ &= \alpha(u) + \chi\alpha(u) + \chi\int_x^{\infty} \alpha'(u)\eta'(u) u_y dy. \end{aligned} \tag{3.3.36}$$ {{dec1510}}

Combining (3.3.35) and (3.3.36) gives

$$\begin{aligned} & u\Big(- \sqrt{\chi}\int_x^{\infty} \alpha''(u) u_y^2 dy + \sqrt{\chi}\int_x^{\infty} \alpha'(u)\big(\eta(u)(1 + \chi A'(u))\big) dy\Big) \\ & = \chi\alpha(u) u + \alpha(u) u - \chi\alpha'(u)\eta(u) u. \end{aligned} \tag{3.3.37}$$

Going back to (3.3.33), we obtain

$$\begin{aligned}
e^{-\Gamma}\big(v_t - v_{xx}\big) &\le \eta(u)(1+\chi A'(u)) - u - \chi\alpha^2(u)u + 2\sqrt{\chi}\alpha(u)u_x + (\chi+1)\alpha(u)u - \chi\alpha'(u)\eta(u)u\\
&\le \chi\eta(u)A'(u) - \chi\alpha^2(u)u - 2\chi\alpha(u)\eta(u) + \chi\alpha(u)u - \chi\alpha'(u)\eta(u)u\\
&= \chi u\big((1-\alpha(u))(\alpha(u)+u\alpha'(u)) + \alpha^2(u) - \alpha(u) - \alpha'(u)(u-u\alpha(u))\big) = 0,
\end{aligned} \tag{3.3.38}$$

{{dec1516}}

and (3.3.28) follows. Isn't this a miracle? There is absolutely no explanation at the moment as to why this computation works.

The boundary conditions for the linearized problem. The function $\alpha(u) = A(u)/u$ has $\alpha(1) = 1$. Thus, as long as $\chi < 1$, the function $v(t,x)$ defined by the weighted Hopf-Cole transform tends to zero as $x \to -\infty$. This means that the linearized half-line toy problem is

$$\begin{aligned}
v_t - v_{xx} = 0,&\quad x > 0,\\
v(t,0) = 0.&
\end{aligned} \tag{3.3.39}$$

{{23aug1014}

This is exactly the problem we have seen in the proof of Theorem 2.4.1: compare this to (2.4.27). In particular, the asymptotics (2.4.28):

$$v(t,x) \sim \frac{Cx}{t^{3/2}} e^{-x^2/(4t)}, \quad \text{as } t \to +\infty. \tag{3.3.40}$$

{{23aug1018}

leads to Bramson's $(3/2)\log t$ correction.

On the other hand, if $\beta = 2$, then we can only expect $v(t,x)$ to remain positive and bounded as $x \to -\infty$. Thus, the "correct" version of the linearized problem is not with the Dirichlet boundary condition as in (3.3.39) but the Neumann one

$$\begin{aligned}
v_t - v_{xx} = 0,&\quad x > 0,\\
v_x(t,0) = 0.&
\end{aligned} \tag{3.3.41}$$

{{23aug1016}

The solution to (3.3.41) has the asymptotics

$$v(t,x) \sim \frac{C}{t^{1/2}} e^{-x^2/(4t)}, \quad \text{as } t \to +\infty. \tag{3.3.42}$$

{{23aug1020}

This gives the pushmi-pullyu correction $(1/2)\log t$. These differences reflect the different behaviors in Theorem 3.4.1. Here, we are actually hiding a serious difficulty: to show that the linearized Dirichlet half-line problem is a good approximation to the full nonlinear problem for the semi-FKPP nonlinearities is much harder than in the Fisher-KPP case and requires yet another computational miracle that we do not totally understand. The details can be found in [2].

3.4 The pushmi-pullyu fronts for the Burgers-FKPP equation

A crucial part in the proof of Theorem 3.3.1 is played by the corresponding result for the reactive conservation laws that we will discuss now, on the particular example of the long time behavior of the solutions to the Burgers-FKPP equation

$$u_t + \beta u u_x = u_{xx} + u - u^2, \quad t > 0,\ x \in \mathbb{R}. \tag{3.4.1}$$

{{burgerskpp

Here, $\beta \in \mathbb{R}$ is a parameter that measures the strength of the advection effect. When $\beta = 2$, this is exactly the reactive conservation law (3.3.14) with $A(u) = u^2$, hence its connection to the pushmi-pullyu reaction-diffusion equations. The relevance of this type of nonlinear advection-reaction-diffusion model in biological and chemical applications is discussed in Murray's book [74].

Our main interest is in the study of the transition from the "pulled" to "pushed" nature of the Burgers-FKPP equation that happens at $\beta_c = 2$ and its effect on the long time behavior of the solutions.

3.4.1 The pulled to pushed fronts transition in the Burgers-FKPP equation

The behavior of traveling waves for (3.4.1) already illustrates the change in behavior at $\beta_c = 2$. For a given $\beta \in \mathbb{R}$, the Burgers-FKPP equation (3.4.1) admits traveling wave solutions for all $c \geq c_*(\beta)$, with the minimal speed

$$c_* = \begin{cases} 2, & \text{if } \beta \leq 2, \\ \dfrac{\beta}{2} + \dfrac{2}{\beta}, & \text{if } \beta \geq 2. \end{cases} \tag{3.4.2}$$ {{speed-bis}}

The minimal speed traveling wave ϕ_β satisfies

$$-c_*\phi_\beta' + \beta\phi_\beta\phi_\beta' = \phi_\beta'' + \phi_\beta - \phi_\beta^2, \quad \phi_\beta(-\infty) = 1, \quad \text{and} \quad \phi_\beta(+\infty) = 0. \tag{3.4.3}$$ {{nov2514}}

It happens that the traveling wave profile for $\beta \geq 2$ is explicit. Indeed, one can check by direct computation that

$$\phi_\beta(x) = \frac{1}{1 + e^{\beta x/2}}, \quad \text{for } \beta \geq 2. \tag{3.4.4}$$ {{jul1502}}

On the other hand, when $\beta < 2$, the profile of the minimal speed traveling wave is, to the best of our knowledge, not explicit, and the asymptotics of ϕ_β as $x \to +\infty$ are no longer purely exponential, being given by

$$\phi_\beta(x) \sim (Ax + B)e^{-x}, \quad \text{as } x \to +\infty, \text{ for } \beta < 2, \tag{3.4.5}$$ {{jul1602}}

with some $A > 0$ and $B \in \mathbb{R}$ that depend on β. This was shown, for instance, in [74] by a phase plane analysis.

3.4.2 The large time behavior of the solutions

The main result on the spreading of the solutions to (3.4.1) is the following analog of Theorem 3.3.1.

Theorem 3.4.1. *Let $u(t,x)$ be the solution to (3.4.1) with $u_{\text{in}}(x) = \mathbb{1}(x \leq 0)$. Then, for each $\beta \leq 2$, there exists a constant x_∞ that depends on β so that*

$$\lim_{t \to +\infty} u(t, x + m_\beta(t)) = \phi_\beta(x), \tag{3.4.6}$$ {{nov2516}}

with the function $m_\beta(t)$ given by

$$m_\beta(t) = 2t - \frac{3}{2}\log(t+1) - x_\infty + o(1), \ \text{ as } t \to +\infty, \tag{3.4.7}$$ {{nov2518}}

if $\beta < 2$, and for $\beta = 2$ by

$$m_{\beta=2}(t) = 2t - \frac{1}{2}\log(t+1) - x_\infty + o(1), \ \text{ as } t \to +\infty. \tag{3.4.8}$$ {{nov2520}}

For $\beta > 2$, there exists $\omega > 0$, which depends on β but not on u_{in}, and $K > 0$, which depends both on β and u_{in}, such that

$$\sup_{x \in \mathbb{R}} |u(t,x) - \phi_\beta(x - c_*t - x_\infty)| < Ke^{-\omega t}. \tag{3.4.9}$$ {{nov2522}}

This result reflects the different nature of the Burgers-FKPP fronts we have discussed above for various values of $\beta \in \mathbb{R}$. For $\beta < 2$, the solution is pulled and the front location has the same asymptotics (3.4.7) as for the standard Fisher-KPP equation. For $\beta > 2$, the solution is pushed and the exponential-in-time convergence to the traveling wave (3.4.9) agrees with what is seen for pushed fronts. The new asymptotics (3.4.8) for the "pushmi-pullyu" solutions at $\beta = \beta_c$ is different from both of these cases.

The case $\beta > 2$ falls into the category of pushed fronts, and the proof follows the classical strategy of [86, 87, 88], with appropriate modifications. So we focus on $\beta \le 2$. We will only provide some snippets from the proof.

3.4.3 Convergence in shape

The first step is to establish convergence of the solution in shape to a traveling wave.

Proposition 3.4.2. *Let $u(t,x)$ be the solution to (3.4.1) with the initial condition $u_{\rm in} \in \mathcal{W}$ that is steeper than the minimal speed traveling wave $\phi_\beta(x)$, or with $u_{\rm in}(x) = \mathbb{1}(x \le 0)$. Then, there exists a function $m_\beta(t)$ such that $m_\beta'(t) \to c_*(\beta)$ as $t \to +\infty$ and*

$$u(t, x + m_\beta(t)) \to \phi_\beta(x) \quad \text{as } t \to +\infty, \text{ uniformly on } \mathbb{R}. \tag{3.4.10}$$ {{dec130}}

Here, $\phi_\beta(x)$ is a solution to (3.4.3) with the minimal speed $c_* = c_*(\beta)$.

This is proved very similarly to Theorem 2.3.3.

3.4.4 The weighted Hopf-Cole transform

Let us recall that the standard Burgers equation

$$u_t + \beta u u_x = u_{xx} \tag{3.4.11}$$ {{dec323}}

can be linearized by means of the Hopf-Cole transform. Namely, if u is a solution to (3.4.11) then the function

$$v(t,x) = \exp\Big(\frac{\beta}{2}\int_x^{+\infty} u(t,y)dy\Big) \tag{3.4.12}$$ {{dec324}}

satisfies the heat equation

$$v_t = v_{xx}. \tag{3.4.13}$$ {{dec325}}

The second simple observation is that if $\widehat{u}(t,x)$ is the solution to the standard Fisher-KPP equation in a frame moving with the speed $c_* = 2$:

$$\widehat{u}_t - 2\widehat{u}_x = \widehat{u}_{xx} + \widehat{u} - \widehat{u}^2, \tag{3.4.14}$$ {{dec326}}

then the function

$$v(t,x) = e^x \widehat{u}(t,x) \tag{3.4.15}$$ {{dec328}}

satisfies

$$v_t = v_{xx} - e^{-x} v^2. \tag{3.4.16}$$ {{jul1908}}

The nonlinear term in (3.4.16) is negligible for x very large and positive but plays the role of a large absorption for x very negative. Therefore, the solution to (3.4.16) should be well approximated by the solution of the heat equation on a half-line $x > 0$ with the Dirichlet boundary condition:

$$\begin{aligned} &v_t = v_{xx}, \quad x > 0, \\ &v(t,0) = 0. \end{aligned} \tag{3.4.17}$$ {{jul1910}}

This simple idea is what is driving the convergence to a traveling wave proofs in [51, 53, 79, 80].

The weighted Hopf-Cole transform that we discuss below allows us to adapt this intuition to the Burgers-FKPP equation (3.4.1) with $\beta \le 2$, and also shows why the transition from pulled to pushed fronts happens at $\beta = 2$.

We will consider the solution to (3.4.1) in the reference frame

$$\tilde{u}(t,x) = u(t, x + m_\beta(t)), \qquad (3.4.18)$$ {{dec414}}

centered at

$$m_\beta(t) = 2t - \frac{r(\beta)}{2}\log(t+1). \qquad (3.4.19)$$ {{dec416}}

Here, we take

$$r(\beta) = \begin{cases} 3, & \text{if } \beta < 2, \\ 1, & \text{if } \beta = 2, \end{cases} \qquad (3.4.20)$$ {{jul1904}}

in accordance with the different behavior in Theorem 3.4.1 in these two cases. In the above reference frame, (3.4.1) takes the form

$$\tilde{u}_t - \Big(2 - \frac{r(\beta)}{2(t+1)}\Big)\tilde{u}_x + \beta\tilde{u}\tilde{u}_x = \tilde{u}_{xx} + \tilde{u} - \tilde{u}^2. \qquad (3.4.21)$$ {{dec330}}

Motivated by (3.4.12) and (3.4.15), we introduce the weighted Hopf-Cole transform

$$v(t,x) = \exp(\Gamma(t,x))\tilde{u}(t,x), \quad \Gamma(t,x) = x + \frac{\beta}{2}\int_x^{+\infty}\tilde{u}(t,y)dy, \qquad (3.4.22)$$ {{dec332}}

that is a combination of (3.4.12) and (3.4.15).

The boundary conditions. Note that, as long as $\beta < 2$, the function $v(t,x)$ tends to zero as $x \to -\infty$, and if $\beta = 2$, then we can only expect it to remain positive and bounded as $x \to -\infty$. Furthermore, if $\beta > 2$ then $v(t,x)$ should blow up as $x \to -\infty$. These differences reflect the three different behaviors in Theorem 3.4.1.

Proposition 3.4.3. *Let $u(t,x)$ be the solution to (3.4.1) with $\beta \le 2$ and the initial condition $u(0,x)$ as in Theorem 3.4.1. Then, the function $v(t,x)$ defined in (3.4.22) satisfies the differential inequality*

$$v_t - v_{xx} + \frac{r(\beta)}{2(t+1)}(v_x - v) \le 0. \qquad (3.4.23)$$ {{dec404}}

Sketch of the argument. A lengthy but straightforward computation shows that the function $v(t,x)$ satisfies an equation of the form

$$v_t - v_{xx} + \frac{r(\beta)}{2(t+1)}(v_x - v) = -G(t,x;\tilde{u})v, \qquad (3.4.24)$$ {{dec333}}

where

$$G(t,x;\tilde{u}) = \tilde{u}(t,x) - \frac{\beta}{2}\int_x^{+\infty}\tilde{u}(t,y)(1-\tilde{u}(t,y))dy. \qquad (3.4.25)$$ {{dec334}}

For the wave, by an explicit computation:

$$G(t,x;\phi_\beta) = 0. \qquad (3.4.26)$$

Steepness comparison implies $G(t,x,\tilde{u}) \ge 0$.

The end of the proof for $\beta < 2$. With (3.4.23) in hand, because we also know that $v(t,x) \to 0$ as $x \to -\infty$ if $\beta < 2$, we are able to construct upper and lower barriers in the self-similar variables for the linearized equation for $v(t,x)$ on the half line, and then the convergence in the tail implies the convergence in the bulk due to the pulled-front nature of the dynamics, as in [79]. Interestingly, this last step also utilizes the assumption that the initial condition, and hence the solution, is steeper than the minimal speed traveling wave, in an explicit quantitative way. Qualitatively, the case $\beta < 2$ is similar to the standard Fisher-KPP equation, and the weighted Hopf-Cole transform gives a tool to see that. However, the repeated use of the steepness comparison is something new in this argument for Burgers-FKPP equation.

3.4.5 The critical case $\beta_c = 2$

Let us now discuss the ingredients of the the proof of Theorem 3.4.1 in the critical case $\beta = 2$, which is remarkably different from the approach for the standard Fisher-KPP equation. The first key observation is that when $\beta = 2$, the Burgers-FKPP equation (3.4.1) has a special structure: the function

$$p(t,x) = e^x \widehat{u}(t,x), \tag{3.4.27}$$

{{jul1510}}

satisfies a spatially inhomogeneous conservation law:

$$p_t + (e^{-x}p^2)_x = p_{xx}. \tag{3.4.28}$$

{{intro:para

An immediate consequence of (3.4.28) is a conservation law for the exponential moment of $\widehat{u}(t,x)$:

$$\int p(t,x)dx = \int e^x \widehat{u}(t,x)dx = \int e^x u_{\rm in}(x)dx, \quad \text{for all } t > 0. \tag{3.4.29}$$

{{aug1210}}

Note that the exponential moment in (3.4.29) is infinite for the traveling wave which is an extra technical difficulty.

An upper bound for the shift. The conservation law (3.4.28) eventually leads to a lower bound for $m_2(t)$ of the form

$$m_2(t) \geq 2t - \frac{1}{2}\log t + O(1), \quad \text{as } t \to +\infty. \tag{3.4.30}$$

{{aug1312}}

Here is the reason. Let us write fix $m_2(t)$ by $u(t, m_2(t)) = 1/2$, set $m_2(t) = 2t - \mu(t)$. Our goal is to show that $\mu(t) \leq 1/2 \log t + C$. Consider the three regions (note that we are in the frame $x \to x - 2t$):

$$L = \{x < -\mu(t)\}, \quad M = \{-\mu(t) \leq x \leq N\sqrt{t}\}, \quad R = \{x > N\sqrt{t}\}. \tag{3.4.31}$$

{{21may2522}

The mass of $p(t,x)$ in region L is small because $p(x) \leq e^x$. In the middle, use steepness comparison to TW:

$$\begin{aligned}\int_M p(t,x)dx &= \int_{-\mu(t)}^{N\sqrt{t}} e^x \widehat{u}(t,x)dx \leq \int_{-\mu(t)}^{N\sqrt{t}} \frac{e^x}{1+e^{x+\mu(t)}}dx \\ &= e^{-\mu(t)} \int_0^{N\sqrt{t}+\mu(t)} \frac{e^x}{1+e^x}dx \leq CNe^{-\mu(t)}\sqrt{t}.\end{aligned} \tag{3.4.32}$$

{{21may2524}

And to the right we can use a moment bound

$$\int \left(e^{mx} + e^{-mx}\right)p(t,x)dx \leq Ce^{Cm^2 t}, \tag{3.4.33}$$

{{21apr1406}

with $m = 1/\sqrt{t}$. Together, these bounds will lead to a contradiction if $\mu(t) \gg (1/2)\log t$.

A lower bound for the shift

A matching upper bound for $m_2(t)$ is related to the behavior of $p(t,x)$. Note that, together with the explicit expression for the profile $\phi_2(x)$, the convergence to a traveling wave in shape yields, roughly,

$$p(t,0) \approx \exp\big(-(2t - m_2(t))\big). \tag{3.4.34}$$
{{aug1216}}

Thus, an upper bound of the form

$$m_2(t) \le 2t - \frac{1}{2}\log t + O(1), \quad \text{as } t \to +\infty, \tag{3.4.35}$$
{{aug1314}}

would follow from an L^∞-bound on $p(t,x)$ of the form

$$p(t,x) \le \frac{C}{\sqrt{t}}. \tag{3.4.36}$$
{{aug1218}}

Such decay, while natural to expect in view of (3.4.28), is not automatic for solutions of mass-conserving advection-diffusion equations, even if the advection is bounded.

The proof of (3.4.36) turns out to be rather intricate. While (3.4.28) looks like a degenerate viscous conservation law, we were unable to adapt the methods of [32] or [56] to (3.4.28) and instead take a different approach.

The first step is a relative entropy computation inspired by [34, 71] where it was used for linear advection-diffusion equations. An unusual twist is that we compute the relative entropy not with respect to another solution but to a super-solution to (3.4.28). This leads to a weighted dissipation inequality for the function

$$\varphi(t,x) = \frac{p(t,x)}{\rho(t,x)}, \qquad \text{where } \rho(t,x) = 1 - u(t, x+2t), \tag{3.4.37}$$
{{jul1512}}

of the form

$$\frac{d}{dt}\int \varphi^2(t,x)\rho(t,x)dx \le -2\int \varphi_x^2(t,x)\rho(t,x)dx. \tag{3.4.38}$$
{{jul1514}}

This comes from the following:

Proposition 3.4.4. *Let $v(t,x)$ be a smooth bounded function, $q(t,x)$ be a solution to*

$$q_t + (vq)_x = q_{xx}, \tag{3.4.39}$$
{{21may2540

and $\rho(t,x)$ be a super-solution to (3.4.39):

$$\rho_t + (v\rho)_x \ge \rho_{xx}, \tag{3.4.40}$$
{{21may2541

Then we have

$$\frac{d}{dt}\int \varphi^2(t,x)\rho(t,x)dx \le -2\int \varphi_x^2(t,x)\rho(t,x)dx. \tag{3.4.41}$$
{{21may2544

The fact that $\rho = 1 - u$ satisfies

$$\rho_t + (u\rho)_x \ge \rho_{xx} \tag{3.4.42}$$

is another computational miracle that relies on the steepness property of u.

The dissipation identity (3.4.38) is similar to that for the standard heat equation, where it takes the form

$$\frac{d}{dt}\int \varphi^2(t,x)dx \le -2\int \varphi_x^2(t,x)dx, \tag{3.4.43}$$
{{aug1118}}

that is, as in (3.4.38) but without the weight $\rho(t,x)$. In the latter case, (3.4.43) combined with the Nash inequality and a standard duality argument directly leads to the temporal decay rate $t^{-d/2}$ in $\mathbb{R}^d$. Here, the time-dependent weight $\rho(t,x)$ that appears in (3.4.38) is degenerate as $x \to -\infty$, so the standard Nash inequality can not be used. Instead, we obtain a Nash-type inequality for weighted spaces for a certain class of degenerate weights. The weights need to satisfy certain quantitative assumptions, and we need to verify that the dynamics do not take the weight $\rho(t,x)$, defined in (3.4.37), out of the class of the admissible weights or make the constants in the weighted Nash inequality degenerate as $t \to +\infty$. The details are in [4].

4 Lecture 4: Convergence rates to traveling waves

4.1 Overview of the lecture

As we have seen, the original KPP approach to the convergence in shape relies on the "soft" steepness comparison argument. Such proofs are very elegant but do not provide a rate of convergence. It turns our that the shape defect function gives a simple way to get the rates of convergence in the results such as Theorem 3.3.1.

We will consider in this lecture the long-time behavior of solutions to reaction-diffusion equations of the form

$$u_t = u_{xx} + f(u), \quad t > 0, \quad x \in \mathbb{R}, \tag{4.1.1}$$ {{e.rde}}

with a nonlinearity $f \in C^2([0,1])$ that satisfies

$$f(0) = f(1) = 0, \quad f'(0) > 0, \quad f(u) > 0 \text{ for } u \in (0,1). \tag{4.1.2}$$ {{e.f}}

In addition, we will normalize the nonlinearity so that

$$f'(0) = 1. \tag{4.1.3}$$ {{e.normaliz

This condition can be achieved by a simple space-time rescaling and is not an extra assumption on $f(u)$. Under these assumptions, there exists $c_* \geq 2\sqrt{f'(0)}$ such that (4.1.1) admits traveling wave solutions of the form $u(t,x) = U_c(x - ct)$ for all $c \geq c_*$. As before, we denote by $U_*(x)$ the traveling wave corresponding to the minimal speed c_*.

To be concrete and avoid some additional technicalities, we will consider the case where the initial condition for (4.1.1) is a step-function:

$$u_0(x) = u(0,x) = \mathbb{1}(x \leq 0). \tag{4.1.4}$$ {{mar2302}}

As we have mentioned before, this assumption may be greatly relaxed, as long as $u_0(x)$ is sufficiently rapidly decaying as $x \to +\infty$, see [22] for a recent detailed analysis of this issue. We have seen that the solution $u(t,x)$ to (4.1.1) converges to $U_*(x)$ in shape. That is, there exists a reference frame $m(t)$, known as the front location, such that

$$u(t, x + m(t)) - U_*(x) = o(1), \quad \text{as } t \to +\infty. \tag{4.1.5}$$ {{e.c021601}

Note that, strictly speaking, the front location is only defined up to an $o(1)$ term as $t \to +\infty$. Moreover, the KPP paper showed that the front location $m(t)$ has the asymptotics

$$m(t) = c_* t + o(t), \quad \text{as } t \to +\infty. \tag{4.1.6}$$ {{mar2306}}

Of course, for the Fisher-KPP nonlinearities, we have already seen that this can be improved to Bramson's asymptotics in Theorem 2.4.1. However, both (4.1.5) and (4.1.6) are fairly universal results and hold not only for the Fisher-KPP nonlinearities.

The goal of this lecture is to show a simple way to obtain the rates of convergence in (4.1.5). It turns out that (i) the shape defect function provides a very straightforward way to achieve this in many situations, and (ii) the rate of convergence in shape in (4.1.5) is actually controlled by the error in the approximation (4.1.6) of the front position. The presentation here is based on [1].

4.2 Front location and convergence rates in the pushed and pulled cases

The precise character of the $o(t)$ correction to the front location in (4.1.6) and the rate of the "convergence in shape" in (4.1.5) depend heavily on the profile of the nonlinearity $f(u)$, as neither can be easily obtained from the intersection number arguments.

The results quantifying these convergence rates and making the asymptotics of the front location $m(t)$ more precise than (4.1.6) are very different in what are known as the "pushed" and "pulled" regimes. Informally, front propagation is pushed if it is "bulk dominated" and is pulled if it is "tail dominated". For positive nonlinearities that satisfy (4.1.2)-(4.1.3) the spreading speed for the linearized problem

$$u_t = u_{xx} + u, \tag{4.2.1}$$ {{jun2210}}

is $c_{\text{lin}} = 2$. We will give a more refined definition below but for the moment the reader can think that propagation is pushed if $c_* > c_{\text{lin}} = 2$ and pulled if $c_* = c_{\text{lin}} = 2$. Contemporary arguments to establish convergence rates in the pushed case are spectral in nature, while, for pulled fronts, are motivated in great part by the connection to branching Brownian motion and typically use entirely different techniques.

When the front is pushed, so that $c_* > 2$, its location has the asymptotics

$$m(t) = c_* t + x_0 + o(1), \quad \text{as } t \to +\infty, \tag{4.2.2}$$ {{mar2308}}

with some $x_0 \in \mathbb{R}$. Moreover, the convergence rate in (4.1.5) is exponential [41, 86]:

$$|u(t, x + m(t)) - U_*(x)| \le c e^{-\omega t}, \tag{4.2.3}$$ {{mar2312}}

with some $\omega > 0$. The proofs of (4.2.2)-(4.2.3) in [41, 86] as well as the later extensions to other "pushed fronts" problems are based on spectral gap arguments and provide implicit estimates on the exponential rate $\omega > 0$ of convergence in (4.2.3).

On the other hand, when $f(u)$ is of the Fisher-KPP type, so that, in addition to (4.1.2), it satisfies

$$f(u) \le f'(0)u, \quad \text{for all } 0 < u < 1, \tag{4.2.4}$$ {{mar2310}}

the propagation is pulled and spreading is dominated by the region far ahead of the front. Under this assumption, when the normalization (4.1.3) is adopted, the minimal speed $c_* = c_{\text{lin}} = 2$ and the front location has the asymptotics

$$m(t) = 2t - \frac{3}{2} \log t + x_\infty + o(1), \quad \text{as } t \to +\infty, \tag{4.2.5}$$ {{mar2404}}

with some $x_\infty \in \mathbb{R}$, as we have seen in Theorem 2.4.1. However, unlike in the pushed case, where the front location asymptotics (4.2.2) was sufficient for the convergence rate estimate (4.2.3), obtaining a convergence rate in (4.1.5) for the Fisher-KPP nonlinearities required a much finer asymptotics than given by the Bramson result (4.2.5). To this end, Graham has improved in [51] the Bramson asymptotics for the Fisher-KPP nonlinearities to show that

$$m(t) = 2t - \frac{3}{2}\log t + x_0 - \frac{3\sqrt{\pi}}{\sqrt{t}} + \frac{9}{8}(5 - 6\log 2)\frac{\log t}{t} + \frac{x_1}{t} + o\Big(\frac{1}{t}\Big), \quad \text{as } t \to +\infty, \tag{4.2.6}$$ {{mar2408}}

with some $x_0, x_1 \in \mathbb{R}$. This confirmed a series of formal predictions in [23, 38], partly proved in [54, 80]. The "very fine" asymptotics in (4.2.6) allow Graham to obtain a convergence bound of the form

$$|u(t, x + m(t)) - U_*(x)| = O\Big(\frac{1}{t}\Big) \tag{4.2.7}$$ {{mar2410}}

after using an asymptotic expansion based on (4.2.6) that approximately solves (4.1.1). It was also shown in [51] that this rate can not be improved for the Fisher-KPP nonlinearities.

While the Bramson asymptotics (4.2.5) holds for all Fisher-KPP reactions, it does not hold for all nonlinearities that satisfy (4.1.2)-(4.1.3) for which $c_* = 2$. As we have seen in Theiorem 3.3.1, for the pushmi-pully nonlinearities $f(u)$ the front location asymptotics is not (4.2.5) but

$$m(t) = 2t - \frac{1}{2}\log t + x_0 + o(1), \quad \text{as } t \to +\infty. \tag{4.2.8}$$ {{jun2208}}

There are two important points to make before discussing the results of [1]. First, while convergence rates have been established in the Fisher-KPP and pushed cases, nothing quantitative is known for the intermediate cases; that is, pushmi-pullyu nonlinearities and pulled nonlinearities not satisfying the Fisher-KPP condition (4.2.4). Second, the arguments used to establish convergence rates in the Fisher-KPP and pushed regimes are quite different. This indicates the difficulty in closing the gap: establishing sharp rates in the transitional cases and developing a cohesive understanding of convergence rates in all cases.

4.2.1 The pushed, pulled and pushmi-pullyu regimes

Ww now define what we mean by the pushed, pulled and pushmi-pullyu regimes. The distinction between various regimes of propagation can not be made based solely on whenever the propagation speed is predicted by the linearization (4.2.1) or not. It turns out that it should be made based both on the propagation speed and the asymptotics behavior of the traveling wave as $x \to +\infty$. Let us, therefore, define terminology for the three classes roughly discussed above. We remind the reader that $f(u)$ satisfies (4.1.2)-(4.1.3).

- A traveling wave is **pushed** if $c_* > 2$.
- A traveling wave is **pulled** if $c_* = 2$ and there is some $A_0 > 0$ such that

$$U_*(x) = A_0 x e^{-x} + O(e^{-x}) \qquad \text{as } x \to \infty. \tag{4.2.9}$$ {{e.c062202}

- A traveling wave is **pushmi-pullyu** if $c_* = 2$ and there is $A_1 > 0$ such that

$$U_*(x) = A_1 e^{-x} + o(e^{-x}) \qquad \text{as } x \to \infty. \tag{4.2.10}$$ {{e.c062203}

A simple linearization argument shows that the two asymptotics in (4.2.9)-(4.2.10) are the only possibilities when $c_* = 2$, so the cases above are exhaustive. Intuitively, once the normalization (4.1.3) is fixed, "large" nonlinearities f correspond to pushed fronts, "small" ones correspond to pulled fronts, and the boundary case corresponds to pushmi-pullyu fronts.

The reason behind this classification goes back to the linearized Dirichlet half line problem we have seen in the proof of Theorem 2.4.1: in the pulled case, the solution to (4.1.1) is faithfully approximated as

$$u(t, x + 2t - \frac{3}{2}\log t) \sim e^{-x}z(t, x). \tag{4.2.11}$$

Here, $z(t, x)$ is the solution to (2.4.27):

$$\begin{aligned} &z_t - z_{xx} - \frac{3}{2t}z = 0, \quad x > 0, \\ &z(t, 0) = 0, \end{aligned} \tag{4.2.12}$$

{{23aug802}}

and has the long time asymptotics

$$z(t, x) \sim Cxe^{-x^2/(4t)}, \quad \text{as } t \to +\infty. \tag{4.2.13}$$

{{23aug804}}

Altogether, in the pulled regime $u(t, x)$ is approximated by

$$u(t, x) \sim Cxe^{-x}e^{-x^2/(4t)}, \quad \text{as } t \to +\infty. \tag{4.2.14}$$

{{23aug804bi

We see the pre-factor x, as in (4.2.9).

4.2.2 An informal statement of the results

Our interest here is to complete and unify the separate pictures for the pulled, pushed, and pushmi-pullyu cases described above. Despite very different approaches to the proof of convergence to the traveling wave in the pushed and pulled cases, one can see one common feature in the original KPP results (4.1.5)-(4.1.6) and in the pushed case (4.2.2)-(4.2.3). Namely, the obtained rate of convergence of $u(t, x)$ to $U_*(x)$ is much finer than the corresponding obtained rate of convergence for the front location. To see this, one needs to only compare (4.1.5) to (4.1.6) in the pulled case and (4.2.2) to (4.2.3) in the pushed case.

Here, we recover and explain this philosophy that "rough front location asymptotics gives a finer rate of convergence to a traveling wave." We introduce a novel approach to quantifying the convergence rate in (4.1.5) that provides one simple explanation both for the exponential and algebraic rates in the pushed and pulled cases, respectively. Roughly, we prove the following (cf. Theorem 4.3.1), under some technical assumptions:

$$|u(t, m(t) + \cdot) - U_*(\cdot)| = \begin{cases} O(t^{-1}) & \text{if } c_* = 2, \\ O\big(\exp\big(-\frac{(c_*^2-4)t}{4}\big)\big) & \text{if } c_* > 2. \end{cases} \tag{4.2.15}$$

{{e.rate}}

As we have mentioned, in the case $c_* = 2$, the convergence rate in (4.2.15) has been established in [51] for the Fisher-KPP nonlinearities based on the very fine asymptotics (4.2.6). The proof we describe here is completely different and avoids (4.2.6) altogether. For the other pulled and pushmi-pullyu cases the rate in (4.2.15) is, to the best of our knowledge, new, as is the explicit rate in the pushed case.

The proof of the convergence rates in (4.2.15) is based on the estimates for the shape defect function

$$w(t, x) = -u_x(t, x) - \eta(u(t, x)). \tag{4.2.16}$$

{{e.sdf_def}

This, in a sense, is a measure of the "distance in shape" between $u(t, x)$ and the profile $U_*(x)$. A major advantage here is that we do not *a priori* need to know *which* shift of U_* is the closest one in order to use w to obtain bounds on $u(t, x) - U_*(x)$. Imprecisely, one finds that

$$w = O(\varepsilon) \quad \text{if and only if} \quad u = U_* + O(\varepsilon) \tag{4.2.17}$$

{{e.w_eps}}

where the second inequality holds up to the appropriate shift. The main idea is to estimate $w(t,x)$ directly through its evolution equation

$$w_t - w_{xx} = w(Q(u) + \eta''(u)w), \tag{4.2.18}$$ {{e.sdf}}

where

$$Q(u) = \eta'(u)(c_* - \eta'(u)) + \eta(u)\eta''(u) \qquad \text{for all } u \in (0,1), \tag{4.2.19}$$ {{e.Q}}

and use that information to read off the rate of convergence of $u(t,x)$ to the traveling wave profile $U_*(x)$. As we see below, the nonlinearity $Q(u)$ satisfies

$$Q(0) = f'(0) = 1 \tag{4.2.20}$$ {{jun2302}}

and, for a large class of nonlinearities, we also have

$$Q(u) \le 1 \qquad \text{for all } u \in [0,1], \tag{4.2.21}$$ {{e.c062301}}

see Lemma 4.7.1.

A key informal observation is that if $u(t,x)$ is a solution to (4.1.1), there is a "phantom front" location $m_w(t)$ that is far behind the true front $m(t)$ and is where the shape defect function $w(t,x)$ "wants" to have its front. The phantom front location of w can be read off its equation (4.2.18). Surprisingly, the evolution of $w(t,x)$ in (4.2.18) turns out to be "Fisher-KPP-like," regardless of whether the solution $u(t,x)$ to (4.1.1) itself is of the pushed, pulled or pushmi-pullyu nature. This is the main and, to us, unexpected unifying element of all three cases. The simple reason behind this pulled nature of $w(t,x)$ is that, because of (4.2.20)-(4.2.21), ahead of the front it satisfies

$$w_t \le w_{xx} + w, \tag{4.2.22}$$ {{jun2304}}

which is exactly the same linearized problem as for the Fisher-KPP equation.

The second new key point is that the distance

$$D(t) = m(t) - m_w(t) \tag{4.2.23}$$ {{jun2012}}

between the true and the phantom fronts controls the rate of convergence in (4.2.15), once again, regardless of whether the front is pushed or pulled. More precisely, at an informal level, the main result of this paper is that the convergence rate in (4.2.15) comes from the estimate

$$|u(t,m(t)+\cdot)-U_*(\cdot)| \sim |w(t,m(t)+\cdot)| = |w(t,D(t)+m_w(t)+\cdot)| \sim \exp\Big(-D(t)-\frac{D^2(t)}{4t}\Big), \tag{4.2.24}$$ {{mar2602}}

where the first approximation follows from (4.2.17) and the second comes from the "Fisher-KPP like" nature of (4.2.22). In particular, this explains why one needs only "rough" asymptotics for $m(t)$ and $m_w(t)$ to get an "exponentially finer" convergence rate in (4.2.15). In order to pass from (4.2.24) to (4.2.15), we show that, as long as $f(u)$ satisfies (4.1.2)-(4.1.3) and some additional technical assumptions, the front location and the phantom front location have the following behavior as $t \to +\infty$:

$$\begin{aligned}
&m(t) = c_*t + O(1), && m_w(t) = 2t - \frac{3}{2}\log t + O(1), && \text{in the pushed case,}\\
&m(t) = 2t - \frac{1}{2}\log t + O(1), && m_w(t) = 2t - \frac{3}{2}\log t + O(1), && \text{in the pushmi-pullyu case,}\\
&m(t) = 2t - \frac{3}{2}\log t + O(1), && m_w(t) = 2t - \frac{5}{2}\log t + O(1), && \text{in the pulled case.}
\end{aligned} \tag{4.2.25}$$ {{mar2604}}

Using (4.2.24) and (4.2.25) leads directly to (4.2.15).

We have already discussed the asymptotics for $m(t)$ in (4.2.25) in all three cases in Theorem 3.3.1, and to a better precision than stated in (4.2.25). Our main goal here is to explain what the phantom front location $m_w(t)$ is, how (4.2.24) comes about, and how the asymptotics of $m_w(t)$ in (4.2.25) can be computed. We emphasize that, unlike [51, 80] that analyzed the Fisher-KPP case, we only use the $O(1)$-precise asymptotics for $m(t)$ and not anything finer to get the convergence rates in (4.2.15).

In all of the three cases in (4.2.25), the analysis of the phantom front location $m_w(t)$ for the shape defect function is based on typical techniques for the Fisher-KPP equations (pulled fronts). This leads to the surprising conclusion that, for a large class of nonlinearities, the convergence of the shifted solution $u(t, x + m(t))$ to $U_*(x)$ is a pulled phenomenon, regardless of the pushed, pulled, or pushmi-pullyu character of the spreading of $u(t, x)$ itself. The reader may notice that the phantom front asymptotics $m_w(t)$ in (4.2.25) has the Bramson form (4.2.5), which is a signature of the pulled fronts, precisely when $m(t)$ is *not* pulled. On the other hand, in the pulled case it is the front asymptotics $m(t)$ itself that has the Bramson asymptotics (4.2.5), while the phantom front position $m_w(t)$ has an extra $\log t$ delay relative to this location. This will be explained below. Of course, without such a delay between $m(t)$ and $m_w(t)$, we would have $D(t) = O(1)$ and (4.2.24) would be useless!

We hope to convince the reader that the scheme outlined above is exceedingly simple to put into practice, beyond the situations we consider in the present paper. Once one starts to work directly with the shape defect function $w(t, x)$ and has the intuition (4.2.24), the convergence proof is straightforward. In particular, the sometimes heavy computations, such as in the proof of Lemma 4.6.1 below, should not obfuscate this basic fact. We do not consider more general problems here because our interest is in the simplest possible presentation to illustrate the meaning behind the convergence rates.

4.3 Convergence rates for the Hadeler-Rothe nonlinearities

To fix the ideas in a simple setting, we will look in detail at the special class of the Hadeler-Rothe nonlinearities that we have already seen in (3.2.15).. They have the form

$$f(u) = (u - u^n)(1 + \chi n u^{n-1}), \tag{4.3.1}$$ {{e.HR}}

with some $n \geq 2$ and $\chi \geq 0$. The traveling waves for such nonlinearities were discussed in detail in [52, 74] for $n = 2$ and in [38] for $n > 2$. The classical Fisher-KPP nonlinearity $f(u) = u - u^2$ is a special case of (4.3.1) with $\chi = 0$ and $n = 2$.

It was shown in [38, 52, 74] for nonlinearities of the form (4.3.1) that there is a pushed-to-pulled transition at $\chi = 1$:

$$c_*(\chi) = \begin{cases} 2 & \text{if } 0 \leq \chi \leq 1, \\ \sqrt{\chi} + \frac{1}{\sqrt{\chi}} & \text{if } \chi \geq 1. \end{cases} \tag{4.3.2}$$ {{mar2622}}

Moreover, the traveling wave profile function is explicit for $\chi \geq 1$ and is given by

$$\eta(u) = \sqrt{\chi}(u - u^n), \tag{4.3.3}$$ {{mar2621}}

see [2, Proposition A.2]. Hence, when $\chi \geq 1$, the traveling waves have the purely exponential asymptotics (cf. (4.2.10)): there exists $\varepsilon, A_1 > 0$ so that

$$U_*(x) \sim A_1 e^{-\lambda_0 x} + O(e^{-(\lambda_0+\varepsilon)x}), \quad \text{as } x \to +\infty. \tag{4.3.4}$$ {{mar2610}}

When $0 \leq \chi < 1$, no such explicit expression is possible for $\eta(u)$ because U_* has the pulled asymptotics: there exists some $\varepsilon > 0$ and $A_0 > 0$ so that

$$U_*(x) \sim (A_0 x + B_0)e^{-\lambda_0 x} + O(e^{-(\lambda_0+\varepsilon)x}), \quad \text{as } x \to +\infty. \tag{4.3.5}$$
{{e.lambda}}

The decay rate $\lambda_0 > 0$ in (4.3.4) and (4.3.5) is the largest root of

$$c_*\lambda_0 = \lambda_0^2 + f'(0). \tag{4.3.6}$$
{{mar2814}}

Recalling (4.1.3), if $c_* = 2$, then $\lambda_0 = 1$. Let us mention that, after a spatial shift, we may assume that $B_0 = 0$, so that (4.3.5) becomes

$$U_*(x) \sim A_0 x e^{-\lambda_0 x} + O(e^{-(\lambda_0+\varepsilon)x}), \quad \text{as } x \to +\infty. \tag{4.3.7}$$
{{july1002}}

This is another natural normalization of the traveling wave.

The corresponding front location asymptotics for the solutions to (4.1.1) with a rapidly decaying initial condition was established is covered by Theorem 3.3.1: there exists x_0 that depends on the initial condition u_0, so that, as $t \to \infty$

$$\begin{aligned}
&m(t) = 2t - \frac{3}{2}\log t + x_0, && \text{for } 0 \leq \chi < 1 \text{ (the pulled case)},\\
&m(t) = 2t - \frac{1}{2}\log t + x_0, && \text{for } \chi = 1 \text{ (the pushmi-pullyu case)},\\
&m(t) = c_*(\chi)t + x_0, && \text{for } 1 < \chi \text{ (the pushed case)}.
\end{aligned} \tag{4.3.8}$$
{{mar2710}}

It is convenient to recall the asymptotic behavior of U_* as $x \to -\infty$ as well: there are $A_1, \varepsilon > 0$ so that

$$1 - U_*(x) \sim A_1 e^{\lambda_1 x} + O(e^{(\lambda_1+\varepsilon)x}), \quad \text{as } x \to -\infty. \tag{4.3.9}$$
{{mar2614}}

Here, λ_1 is the nonnegative root of

$$-c_*\lambda_1 = \lambda_1^2 + f'(1). \tag{4.3.10}$$
{{e.lambda_b

Notice that, due to (4.3.1), we have

$$\lambda_1 > 0 \qquad \text{since} \qquad f'(1) = -(n-1)(1+\chi n) < 0. \tag{4.3.11}$$
{{e.c062401}

4.3.1 The main result for the Hadeler-Rothe nonlinearities

In this section, we state the convergence rates in (4.2.15) for the Hadeler-Rothe nonlinearities of the form (4.3.1). For simplicity, we take an initial condition $u(0,x) = u_0(x)$ such that $0 \leq u_0(x) \leq 1$ for all $x \in \mathbb{R}$, and there exsts some $L_0 \in \mathbb{R}$, so that

$$u_0(x) = 0 \text{ if } x \geq L_0, \quad \text{and} \quad w_0(x) = w(0,x) \geq 0, \quad \text{for all } x \in \mathbb{R}. \tag{4.3.12}$$
{{e.u_0}}

The non-negativity assumption on $w(0,x)$ simply says that the initial condition $u_0(x)$ is "steeper" than $U_*(x)$. In particular, it follows from (4.3.12) that $u_0(x)$ is decreasing. The comparison principle and (4.2.18) yield that then $u(t,x)$ remains steeper than $U_*(x)$ for all $t > 0$, in the sense that

$$w(t,x) > 0, \quad \text{for all } t > 0,\ x \in \mathbb{R}. \tag{4.3.13}$$
{{e.w>0}}

A typical example of such initial condition is $u_0(x) = \mathbb{1}(x \leq 0)$. We believe that the non-negativity assumption on $w(0,x)$ can be relaxed by using results such as by Angenent in [6] or Roquejoffre in [85] to show that $w(t,x)$ "eventually" becomes nonnegative, at least on every compact set. We adopt this assumption to avoid the related technicalities.

Our main result for the Hadeler-Rothe nonlinearities is as follows.

Theorem 4.3.1. *Suppose that $u(t,x)$ solves* (4.1.1) *with a nonnegative initial condition $u_0(x)$ satisfying* (4.3.12)*. Assume that $f(u)$ is given by (4.3.1) with some $\chi \geq 0$ and $n \geq 2$. Let c_* be given by (4.3.2). Then there is $\sigma : [0,\infty) \to \mathbb{R}$ so that:*

(i) if $0 \leq \chi \leq 1$, then

$$\|u(t,\cdot+\sigma(t)) - U_*(\cdot)\|_{L^\infty} \leq \frac{C}{t}, \tag{4.3.14}$$

{{mar2806}}

(ii) if $\chi > 1$, then for any $\Lambda > 0$,

$$\|u(t,\cdot+\sigma(t)) - U_*(\cdot)\|_{L^\infty([-\Lambda,\infty))} \leq \frac{C_\Lambda}{\sqrt{t}} e^{-\frac{(c_*^2-4)}{4}t}. \tag{4.3.15}$$

{{e.c81901-b

As will be seen from the proof, convergence occurs in a (stronger) weighted L^∞-norm, but we opt for the simpler statement here.

The main ingredients in Theorem 4.3.1 are knowledge of the true front location $m(t)$ as well as the behavior of the functions $Q(u)$ and $\eta(u)$ in (4.2.21). In this sense, we use the form (4.3.1) in a rather weak way.

4.3.2 Discussion of the proof

A very useful observation is that, for the Hadeler-Rothe nonlinearities, (4.2.21) holds and the traveling wave profile function $\eta(u)$ is concave.

Proposition 4.3.2. *Assume that $f(u)$ has the form (4.3.1), then, for any $\chi \geq 0$ and $n \geq 2$,*

$$Q(u) \leq 1 \quad \text{and} \quad \eta''(u) \leq 0, \quad \text{for all } u \in (0,1). \tag{4.3.16}$$

A more precise version is stated in Lemma 4.4.6. Proposition 4.3.2 follows immediately from the explicit expression (4.3.3) for $\eta(u)$ when $\chi \geq 1$.

Proposition 4.3.2 is nearly enough to understand the phantom front $m_w(t)$ as we have, at highest order,

$$w_t \approx w_{xx} + w \tag{4.3.17}$$

{{jul1004}}

ahead of the front. Remarkably, this is exactly the same as the linearization for the classical Fisher-KPP equation

$$u_t = u_{xx} + u - u^2. \tag{4.3.18}$$

This would suggest that $m_w(t)$ should be given by the standard Bramson asymptotics (4.2.5) for the Fisher-KPP case. However, it has been observed that the Bramson shift may be sensitive to lower order terms ahead of the front for nonlinearities that are not better than Lipschitz near $u = 0$ [24]. In that case, (4.3.17) may be not a faithful approximation to (4.2.18). It is, thus, crucial to understand the regularity of η near $u = 0$. As a consequence, we consider two cases depending on this regularity.

The pushed and pushmi-pullyu cases: $\chi \geq 1$

Consider first the pushed and pushmi-pullyu cases, where η is given explicitly by (4.3.3) and is smooth at $u = 0$. In this case,

$$Q(u) = 1 - n(1 - 2\chi + \chi n)u^{n-1} - \chi n u^{2n-2} = 1 + O(u^{n-1}) \qquad \text{as } u \to 0. \tag{4.3.19}$$

{{mar2682}}

Recall that $n \geq 2$. Hence, we expect that, ahead of the front of $u(t,x)$, the shape defect function $w(t,x)$ does behave approximately as a solution to

$$w_t = w_{xx} + w, \tag{4.3.20}$$
{{mar2631}}

when $\chi \geq 1$. An informal consequence of [53] is that $w(t,x)$, being bounded and approximately satisfying (4.3.20) where it is small, "wants to have a front" at the location

$$m_w(t) = 2t - \frac{3}{2}\log t, \tag{4.3.21}$$
{{mar2633}}

and should have the approximate form

$$w(t, x + m_w(t)) \approx \exp\Big\{ - x - \frac{x^2}{4t} + (\text{lower order terms})\Big\}, \quad \text{for } x \gg 1. \tag{4.3.22}$$
{{mar2634}}

On the other hand, $w(t,x)$ is governed by $u(t,x)$, which has its front at the position $m(t) = c_* t$ in the pushed case $\chi > 1$, and at $m(t) = 2t - {}^1\!/\!_2 \log t$ in the pushmi-pullyu case $\chi = 1$, as in Theorem 3.3.1. Hence, we have, up to lower order terms

$$D(t) = m(t) - m_w(t) \approx \begin{cases} \log t & \text{if } \chi = 1, \\ (c_* - 2)t & \text{if } \chi > 1. \end{cases}, \tag{4.3.23}$$

According to (4.3.22), this produces

$$w(t, m(t)) = w(t, D(t) + m_w(t)) \approx \exp\Big\{ - D(t) - \frac{D^2(t)}{4t}\Big\}, \tag{4.3.24}$$
{{mar2635}}

which, along with (4.2.17), yields Theorem 4.3.1.

Let us note that the explicit form of η, beyond Proposition 4.3.2, is not needed here, because the key estimate used above, that is, the right hand side of (4.3.19), follows directly from the traveling wave asymptotics (4.3.4) and (4.3.26) below. Indeed, we can see that, whenever (4.3.4) holds, we have, for some $\alpha > 0$,

$$\eta(u) \sim u + O(u^{1+\alpha}). \tag{4.3.25}$$
{{e.eta_asym

See Lemma 4.4.5.

The pulled case: $0 \leq \chi < 1$

For $0 \leq \chi < 1$, we do not have an explicit expression for $\eta(u)$ or $Q(u)$. To understand the behavior of $Q(u)$ for $u \ll 1$ in this range of χ, we can, at least informally, deduce the behavior of η and its derivatives from (4.3.5).

Let us can write two useful identities involving η:

$$f(u) = \eta(u)(c_* - \eta'(u)) \qquad \text{and} \qquad \eta(u) = -U_*' \circ U_*^{-1}(u). \tag{4.3.26}$$
{{e.f_eta}}

From these, we immediately observe that

$$\eta \in C^\infty_{\mathrm{loc}}(0,1), \qquad \eta'(0) = \lambda_0, \qquad \text{and} \qquad \eta'(1) = -\lambda_1. \tag{4.3.27}$$
{{e.eta_C2}}

Both (4.3.26) and (4.3.27) hold for any f satisfying (4.1.2)-(4.1.3). The endpoint regularity is more subtle and is affected by the additional linear factor in (4.3.5) that is present in the pulled case. Indeed, from (4.3.5), it is straightforward to see that

$$\eta(u) \sim u + \frac{u}{\log u}, \qquad \text{as } u \to 0, \tag{4.3.28}$$
{{e.eta_asym

from which we formally deduce that

$$\eta'(u) \sim 1 + \frac{1}{\log u} \qquad \text{and} \qquad \eta''(u) \sim -\frac{1}{u \log^2 u} \qquad \text{as } u \to 0^+. \tag{4.3.29}$$ {{e.eta'}}

These are made precise in Lemma 4.4.4 below. Therefore, when $0 \le \chi < 1$, the function $Q(u)$ defined in (4.2.19) has the asymptotics

$$Q(u) \sim 1 - \frac{2}{\log^2 u}, \quad \text{as } u \to 0. \tag{4.3.30}$$ {{mar2630}}

Thus, a good approximation to $w(t,x)$ is by a solution to a modification of (4.3.20):

$$w_t - w_{xx} \approx w\Big(1 - \frac{2}{\log^2 u}\Big). \tag{4.3.31}$$ {{e.c051701}

Using, once again very informally, the main result of [24], we see that the shape defect function $w(t,x)$ "wants to have its front" at the location

$$m_w(t) = 2t - \frac{5}{2}\log t, \tag{4.3.32}$$ {{mar2636}}

while the front of $u(t,x)$ is at the Bramson position

$$m(t) = 2t - \frac{3}{2}\log t. \tag{4.3.33}$$

Thus, for $0 \le \chi < 1$, we have $D(t) = \log t$ and (4.3.24) again yields the $O(1/t)$ convergence rate in (4.2.15).

The above informal arguments indicate that, as we have already mentioned, the behavior of the shape defect function $w(t,x)$ is always a pulled phenomenon regardless of the pushed, pulled, or pushmi-pullyu spreading of $u(t,x)$ itself.

4.4 Estimates on the shape defect function

One of our main technical points o is that the proof of Theorem 4.3.1 requires understanding the front location asymptotics for $u(t,x)$ only up to $O(1)$ as $t \to +\infty$. For the Hadeler-Rothe nonlinearities we have the following consequence of Theorem 3.3.1.

Proposition 4.4.1. *Under the assumptions of Theorem 4.3.1, let the function $m(t)$ be given by (4.3.8). Then, we have*

$$\lim_{L\to\infty} \limsup_{t\to\infty} \sup_{x \ge m(t)+L} u(t,x) = 0 \qquad \text{and} \qquad \lim_{L\to\infty} \liminf_{t\to\infty} \inf_{x \le m(t)-L} u(t,x) = 1. \tag{4.4.1}$$ {{mar2802}}

The next lemma gives preliminary control on how quickly $u(t,x)$ tends to its limits as $x \to \pm\infty$.

Lemma 4.4.2. *With $m(t)$ as in Proposition 4.4.1 and $w(t,x)$ satisfying (4.3.13), there is $C > 0$ so that*

$$u(t, x + m(t)) \ge U_*(x + C) \quad \text{for all } x < 0, \qquad \text{and} \qquad u(t, x + m(t)) \le U_*(x - C) \quad \text{for all } x > 0. \tag{4.4.2}$$

By a simple ODE comparison argument using (4.2.16), and (4.3.13), we see that, for any x_1, x_2,

$$\text{if } u(t,x_1) = U_*(x_2) \quad \text{then} \quad u(t,x_1+x) \begin{cases} \le U(x_2+x) & \text{if } x > 0, \\ \ge U(x_2+x) & \text{if } x < 0. \end{cases} \tag{4.4.3}$$ {{e.c062701}

Then Lemma 4.4.2 follows directly from Proposition 4.4.1. The proof is omitted.

The main step allowing us to deduce the bounds in Theorem 4.3.1 is the following estimate on the shape defect function at the front location $m(t)$.

Theorem 4.4.3. *Suppose the assumptions of Theorem 4.3.1 hold. Let $m(t)$ and $\lambda_1 > 0$ be as in (4.3.8) and (4.3.10), respectively, and let $\varepsilon > 0$.*
(i) If $0 \le \chi < 1$, then

$$w(t,x+m(t)) \le \frac{C}{t}\Big((1+x)^2 e^{-x-\frac{x^2}{Ct}}\Big)\mathbb{1}(x \ge 0) + \frac{C_\varepsilon}{t} e^{(\lambda_1-\varepsilon)x}\mathbb{1}(x \le 0). \tag{4.4.4}$$

(ii) If $\chi = 1$ then

$$w(t,x+m(t)) \le \frac{C}{t}\Big((1+x) e^{-x-\frac{x^2}{Ct}}\Big)\mathbb{1}(x \ge 0) + \frac{C_\varepsilon}{t} e^{(\lambda_1-\varepsilon)x}\mathbb{1}(x \le 0). \tag{4.4.5}$$

(iii) If $\chi > 1$ and $x > L_0 - m(t)$ (recall L_0 from (4.3.12)) then

$$w(t,x+m(t)) \le \frac{C}{\sqrt{t}} \exp\left\{-\frac{c_*^2-4}{4}t - \frac{c_* x}{2} - \frac{x^2}{4t}\right\}, \tag{4.4.6}$$ {{e.w.pushed

with $c_* = c_*(\chi)$ given by (4.3.2).

We note that the ε in cases (i) and (ii) can almost certainly be removed with a more careful proof.

While the statements in Theorem 4.4.3(i)-(ii) for the pulled and pushmi-pullyu cases are slightly different, the proofs, postponed until **??**, are nearly identical. They are based on the intuition discussed in Section 4.3.2: the equation for $w(t,x)$ wants to spread slower than the equation for $u(t,x)$.

4.4.1 Deducing Theorem 4.3.1 from Theorem 4.4.3

Theorem 4.3.1 follows from Theorem 4.4.3 using two ingredients: (i) estimates on how $\eta(u)$ behaves near $u = 0$, and (ii) using an ODE argument on how the smallness of $w(t,x)$ shows that $u(t,x)$ is close to a traveling wave. We do not present the details of the second step, and for the first we simply list the result, to emphasize the difference between the pulled case and the pushmi-pullyu and pushed.

Lemma 4.4.4 (Asymptotics of $\eta(u)$ in the pulled case)**.** *Assume that $f \in C^2([0,1])$ and satisfies (4.1.2)-(4.1.3). Suppose that the profile $U_*(x)$ has the asymptotics (4.3.5) as $x \to +\infty$. Then there exists $C > 0$ so that, for $u \in (0, 1/100)$,*

(i) $$\Big|\eta(u) - \Big(u + \frac{u}{\log u}\Big)\Big| \le C\frac{u \log\log(1/u)}{\log^2(1/u)},$$

(ii) $$\Big|\eta'(u) - \Big(1 + \frac{1}{\log u}\Big)\Big| \le C\frac{\log\log(1/u)}{\log^2(1/u)},$$

(iii) $$\Big|\eta(u)\eta''(u) - \Big(\frac{-1}{\log^2 u}\Big)\Big| \le C\frac{\log\log(1/u)}{\log^3(1/u)}.$$

Lemma 4.4.5 (Asymptotics of η in the pushed and pushmi-pullyu cases)**.** *Assume that $f \in C^2([0,1])$ and satisfies (4.1.2)-(4.1.3). Suppose that the profile U_* has the asymptotics (4.3.4) as $x \to +\infty$. Then, there exist $\alpha > 0$ and $C > 0$ such that, for all $u \geq 0$,*

$$|\eta'(u) - \lambda_0| \leq Cu^\alpha. \tag{4.4.7}$$ {{mar2808}}

4.4.2 The properties of $Q(u)$ and $\eta(u)$

We state a critical lemma about the behavior of η and Q. This is the key and essentially only place where we use the form (4.3.1) of the Hadeler-Rothe nonlinearities $f(u)$.

Lemma 4.4.6. *Suppose the assumptions of Theorem 4.3.1 hold. Then*

$$\eta''(u) \leq 0 \quad \text{and} \quad Q(u) \leq 1, \quad \text{for all } u \in (0,1). \tag{4.4.8}$$ {{e.c051604}

Further, we have the refined bounds: letting

$$R(u) = 1 - Q(u(t,x)), \tag{4.4.9}$$ {{mar2902}}

for any $\delta_0, \delta_1 \in (0, 1/100)$ with δ_1 sufficiently small, there are $r_0 > 0$ and $r_1 > 0$ such that

$$R(u) \geq \begin{cases} r_0, & \text{if } \delta_0 \leq u \leq 1 - \delta_1, \\ 1 + r_1, & \text{if } u \geq 1 - \delta_1. \end{cases} \tag{4.4.10}$$ {{mar2812}}

Also $r_1 \to -f'(1) > 0$ as $\delta_1 \to 0$. If, additionally, $\chi \in [0,1)$, then we have

$$R(u) \geq \frac{2}{\log^2 u} - \frac{C \log\log 1/u}{\log^3 1/u}, \qquad \text{if } u \leq \delta_0. \tag{4.4.11}$$ {{e.c060503}

The constant C depends only on χ and n. The constants r_0 and r_1 depend on χ, n, δ_0, and δ_1.

Let us make two comments. First, the term $2/\log^2 u$ in (4.4.11) is crucial for the coefficient $5/2$ in the phantom front location

$$m_w(t) = 2t - \frac{5}{2}\log t \tag{4.4.12}$$ {{mar2912}}

that appears in (4.2.25) in the pulled case. Second, the form (4.3.1) of f is mainly used to prove the bound (4.4.8). Indeed, the estimate (4.4.11) follows directly from Lemma 4.4.4 and the definition (4.2.19) of Q.

4.5 The pushmi-pullyu case: the proof of Theorem 4.4.3(ii)

We begin with the pushmi-pullyu case $\chi = 1$. In that case, the front location is

$$m(t) = 2t - \frac{1}{2}\log t. \tag{4.5.1}$$

We recall the following estimate to the right of $m(t)$ when $\chi = 1$.

Lemma 4.5.1. *For any t sufficiently large and any L, we have*

$$w(t, x + m(t) - L) \leq \frac{C_L}{t}(x_+ + 1)e^{-x_+ - \frac{x_+^2}{Ct}}. \tag{4.5.2}$$

We omit this proof as it is essentially the same as [2, Lemma 6.6]. In view of Lemma 4.5.1, we need only consider the behavior of $w(t,x)$ behind the position $m(t) - L$. We do this via the construction of a super-solution. Changing to the moving frame

$$\widetilde{w}(t,x) = w(t, x + m(t) - L) \quad \text{and} \quad \tilde{u}(t,x) = u(t, x + m(t) - L), \tag{4.5.3}$$

and applying Lemma 4.4.6 to (4.2.18), we find, for any $\varepsilon > 0$,

$$\widetilde{w}_t - \Big(2 - \frac{1}{2t}\Big)\widetilde{w}_x \le \widetilde{w}_{xx} + (f'(1) + \varepsilon)\widetilde{w} \qquad \text{for } x < 0. \tag{4.5.4}$$

Above we have potentially increased L so that, by Proposition 4.4.1, $u > 1 - \delta_1$ with δ_1 as in Lemma 4.4.6 for $x < 0$.

We next remove an integrating factor. Let $\lambda_{1,\varepsilon}$ be the positive root of

$$-2\lambda = \lambda^2 + f'(1) + 2\varepsilon \tag{4.5.5}$$ {{e.c071701}

(cf. (4.3.10)), and let

$$z(t,x) = e^{-\lambda_{1,\varepsilon} x}\widetilde{w}(t,x), \tag{4.5.6}$$

we obtain the differential inequality

$$z_t - \Big(2(1 + \lambda_{1,\varepsilon}) - \frac{1}{2t}\Big)z_x \le z_{xx} - \frac{\lambda_{1,\varepsilon}}{2t}z - \varepsilon z \qquad \text{for } x < 0. \tag{4.5.7}$$ {{e.c60201}}

Before constructing a supersolution for (4.5.7), we note the following boundary conditions. First, due to Lemma 4.5.1, we have

$$\widetilde{w}(t,0) \le \frac{C_{L,\varepsilon}}{t}. \tag{4.5.8}$$

Second, due to Lemma 4.4.2 and parabolic regularity theory, we have, for any $x < 0$,

$$\widetilde{w}(t,x) = -\tilde{u}_x(t,x) - \eta(\tilde{u}(t,x)) \le C \sup_{(s,x)\in[t-1,t]\times[x-1,x+1]} (1 - \tilde{u}(s,x)) \le Ce^{\lambda_1 x}. \tag{4.5.9}$$ {{e.c051606}

As a result, if we can produce a supersolution $\overline{z}(t,x)$ for (4.5.7) defined for $t \ge T$ and $x \in [-\delta t, 0]$ that satisfies the boundary conditions

$$\overline{z}(t,0) \ge \frac{C}{t} \quad \text{and} \quad \overline{z}(t,-\delta t) \ge Ce^{-(\lambda_1 - \lambda_{1,\varepsilon})\delta t}, \quad \text{for } t \ge T, \tag{4.5.10}$$ {{e.c60203}}

and the initial condition at $t = T$

$$\inf_{x\in[-\delta T,0]} \overline{z}(t,x) \ge C, \tag{4.5.11}$$ {{mar2816}}

then we would conclude, via the comparison principle, that $\tilde{z}(t,x) \le \overline{z}(t,x)$ for $t \ge T$ and $x \in [-\delta t, 0]$. Let us note that $\lambda_{1,\varepsilon} < \lambda_1$ due to (4.5.5).

We define the function $\overline{z}(t,x)$ by

$$\overline{z}(t,x) = \frac{A}{t} \qquad \text{for } x < 0 \text{ and } t > T. \tag{4.5.12}$$

It is clearly possible to choose A, depending on L, δ and $T > 0$, so that the conditions in (4.5.10)-(4.5.11) are satisfied. It remains to check that $\overline{z}$ is a super-solution of (4.5.7). A direct computation yields, for any $x \in (-\delta t, 0)$,

$$\overline{z}_t - \Big(2(1 + \lambda_{1,\varepsilon}) - \frac{1}{2t}\Big)\overline{z}_x - \overline{z}_{xx} + \Big(\frac{\lambda_{1,\varepsilon}}{2t} + \varepsilon\Big)\overline{z} = \overline{z}\left(-\frac{1}{t} + \frac{\lambda_{1,\varepsilon}}{2t} + \varepsilon\right) > 0, \tag{4.5.13}$$

as long as we increase T if necessary. Hence, $\overline{z}$ is a super-solution for (4.5.7). We deduce that

$$\tilde{w}(t,x) \le \frac{A}{t} e^{\lambda_{1,\varepsilon} x}, \quad \text{for } t > T \text{ and } -\delta t \le x \le 0. \tag{4.5.14}$$

In view of (4.5.5), $\lambda_{1,\varepsilon} \nearrow \lambda_1$ as $\varepsilon \to 0$. Hence, the above is the desired bound for $x \in [-\delta t, 0]$. On the other hand, the bounds on $\tilde{w}$ for $x \le -\delta t$ follow directly from (4.5.9). This completes the proof of Theorem 4.4.3(ii). $\Box$

4.6 The pulled case: the proof of Theorem 4.4.3(i)

When $0 \le \chi < 1$ the front is located at the position

$$m(t) = 2t - \frac{3}{2}\log t. \tag{4.6.1}$$

{{mar2914}}

Exactly the same argument as in the proof of Theorem 4.4.3(ii) to control the behavior of $w(t,x)$ for $x < m(t)$ can be applied. Thus, we only need to control $w(t, x + m(t))$ for $x > 0$. This is done by the following.

Lemma 4.6.1. *Under the assumptions of Theorem 4.4.3(i), we have*

$$w(t, x + m(t)) \le \frac{C(x^2+1)}{t} e^{-x - \frac{x^2}{Ct}} \qquad \textit{for all } x > 0. \tag{4.6.2}$$

Before starting the proof, let us make the following comment. As discussed in the introduction, the convergence rate of $w(t,x)$ is controlled by the lag $D(t)$ of the phantom front $m_w(t)$ behind the true front $m(t)$, as in (4.2.23)-(4.2.24). When $0 \le \chi < 1$, the phantom front $m_w(t)$ is given by (4.4.12) and $m(t)$ in (4.6.1). On the other hand, the use of the naive linearization such as (4.3.20)

$$w_t \approx w_{xx} + w, \tag{4.6.3}$$

would produce an incorrect estimate $m_w(t) \sim 2t - (3/2)\log t$ which would lead to $D(t) \sim O(1)$, and a bound in the spirit of (4.3.24) on the convergence rate would be useless. Thus, the lag comes solely from the non-zero term $R(u)$ in (4.4.11). We have to use this estimate in an essential way to obtain any convergence rate in (4.2.15) in the pulled case, let alone a sharp one.

Proof. First, for L and $T > 0$ to be determined, we let

$$\widetilde{w}(t,x) = w(t, x + m(t) - L) = w(t, x + 2t - \tfrac{3}{2}\log(t+T) - L), \tag{4.6.4}$$

{{jun2102}}

and define $\tilde{u}$ similarly. Then, recalling Lemma 4.4.6, since $\eta''(u) \le 0$, we find

$$\widetilde{w}_t - \Big(2 - \frac{3}{2(t+T)}\Big)\widetilde{w}_x \le \widetilde{w}_{xx} + (1 - R(\tilde{u}))\widetilde{w}. \tag{4.6.5}$$

{{e.c60305}}

We remove an exponential,

$$z(t,x) = e^{x}\widetilde{w}(t,x) \tag{4.6.6}$$

{{jun2016}}

to obtain

$$z_t + \frac{3}{2(t+T)}(z_x - z) \le z_{xx} - zR(\tilde{u}). \tag{4.6.7}$$

{{e.c60304}}

We now define a supersolution to (4.6.5) for $t \ge 1$ and $x \in \mathbb{R}$ as follows. For $B \ge 1$ and $T \ge 1$ to be chosen, let

$$\zeta(t,x) = \theta(t)\Big(\frac{x+B}{B}\Big)^2 \exp\left\{4 - 2\sqrt{\theta(t)} - \frac{(x+B)^2}{4(t+T)}\left(1 - \frac{1}{8}\sqrt{\theta(t)}\right)\right\}, \tag{4.6.8}$$

{{jun2014}}

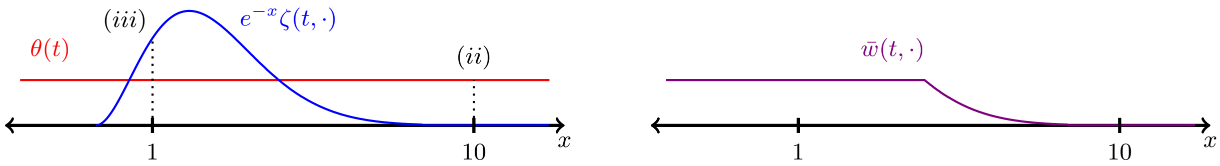

Figure 1: A depiction of the conditions (ii) and (iii) and their relationship to $\bar{w}$.

where we have defined

$$\theta(t) = \frac{T}{t+T}. \tag{4.6.9}$$

Let us set

$$\overline{w}(t,x) = \begin{cases} \theta(t) & \text{if } x \le 1, \\ \min\{\theta(t), e^{-x}\zeta(t,x)\} & \text{if } x \ge 1. \end{cases} \tag{4.6.10}$$

The proof of Lemma 4.6.1 will be finished if we show that $w(t,x) \le A\overline{w}(t,x)$, with some $A > 0$.

Before we proceed, let us explain where (4.6.8) comes from. First, from (4.3.31), we expect w to "look like" the solution of

$$\phi_t = \phi_{xx} + \phi\Big(1 - \frac{2}{\log^2 {}^1\!/\!_\phi}\Big). \tag{4.6.11}$$ {{jun2018}}

The traveling wave solution of this equation has the asymptotics $x^2 e^{-x}$ as $x \to +\infty$ [24], which motivates a multiplicative factor x^2 in (4.6.8), as we have already removed an exponential factor in (4.6.6). On the other hand, "far to the right," we should have a Gaussian behavior, which motivates the $\exp\{-{}^{x^2}\!/\!_{4t}\}$ type term in (4.6.8). In addition, as we have mentioned above, we expect the phantom front location $m_w(t)$ to be near the front location for (4.6.11), which is known to be at the position given by (4.4.12). Thus, the lag between the true and the phantom fronts is $D(t) \sim \log t$. Because of that, we expect $w \sim O({}^1\!/\!_t)$. This explains the multiplicative factor $\theta(t)$ in (4.6.8). The other terms in (4.6.8) are simply technical; in particular, the B and T factors allow to verify the supersolution condition and to "fit" $\bar{w}$ above w initially.

By the comparison principle applied to the differential linear inequality (4.6.7) for $z(t,x)$, we will have shown that

$$w(t,x) \le A\overline{w}(t,x), \quad \text{for } t \ge 1 \text{ and } x \in \mathbb{R}, \tag{4.6.12}$$ {{mar2820}}

with some $A > 0$, if we show the following:
(i) the initial comparison holds:

$$w(1,x) \le A\overline{w}(1,x) \text{ for all } x \in \mathbb{R}, \tag{4.6.13}$$ {{mar2821}}

(ii) the function $\overline{w}(t,x)$ has the form

$$\overline{w}(t,x) = e^{-x}\zeta(t,x) \text{ for } t \ge 1 \text{ and } x \ge 10, \tag{4.6.14}$$ {{mar2822}}

or, equivalently, we have $\theta(t) \ge e^{-x}\zeta(t,x)$ in the above region,
(iii) at $x = 1$ we have the opposite comparison

$$e^{-1}\zeta(t,1) \ge \theta(t) \text{ for all } t \ge 1, \tag{4.6.15}$$ {{mar2823}}

(iv) the function $\theta(t)$ is a super-solution to (4.6.5) for $t \ge 1$ and $x \le 10$, and
(v) the function $\zeta(t,x)$ is a super-solution to (4.6.7) for $t \ge 1$ and $x \ge 1$.

In particular, (4.6.14)-(4.6.15) are important because they allow us to make the matching between $\theta(t)$ and $e^{-x}\zeta(t,x)$ somewhere in the interval $(1,10)$ as the minimum of two super-solutions. This is crucial because, as $\zeta(t,x)$ vanishes at $x=-B$, it can not be a super-solution for $x<0$, and, as we will see, $\theta(t)$ is not a super-solution for $x>10$. This is depicted in Figure 1.

We now check conditions (i)-(v). The initial comparison (4.6.13) is easy to check using well-known bounds on parabolic equations. In particular, $w(t,x)$ is bounded, up to a large multiplicative constant, by a Gaussian in x, for each $t>0$ fixed. Hence, after increasing T, independent of all parameters, and increasing A, depending on L and B, the bound (4.6.13) must hold. Recall that L appears in the change of variables (4.6.4).

Next, we notice that (ii) is clear by observation if B is sufficiently large. Similarly, after increasing T (depending only on B), (iii) is also clear by observation.

To see that (iv) is satisfied requires us to increase L (independent of all parameters) and apply Proposition 4.4.1 with any δ_1 sufficiently small to find that

$$\tilde{u}(t,x) \geq 1-\delta_1 \qquad \text{for all } t\geq 1,\ x\leq 10. \tag{4.6.16}$$

Then, from Lemma 4.4.6, we have

$$1-R(\tilde{u}) \leq -r_1 \qquad \text{for all } t\geq 1,\ x\leq 10. \tag{4.6.17}$$

Thus, up to increasing T, depending only on $\delta_1>0$, we have

$$\theta_t - \Big(2-\frac{3}{2(t+T)}\Big)\theta_x - \theta_{xx} - (1-R(\tilde{u}))\theta \geq -\frac{T}{(t+T)^2} + r_1\frac{T}{t+T} > 0. \tag{4.6.18}$$

Therefore, (iv) holds.

We now check (v), which is a computationally tedious condition to verify, even though the computations are completely elementary. First, we compute:

$$\begin{aligned}\frac{\overline{z}_t + \frac{3}{2(t+T)}(\overline{z}_x - \overline{z}) - \overline{z}_{xx} + \overline{z}R(\tilde{u})}{\overline{z}} &= \frac{\dot{\theta}}{\theta} - \frac{\dot{\theta}}{\sqrt{\theta}} + \frac{(x+B)^2}{4(t+T)^2}(1-\frac{1}{8}\sqrt{\theta}) + \frac{1}{8}\frac{\dot{\theta}}{2\sqrt{\theta}}\frac{(x+B)^2}{4(t+T)} \\ &+ \frac{3}{2(t+T)}\Big(\frac{2}{x+B} - \frac{x+B}{2(t+T)}(1-\frac{1}{8}\sqrt{\theta}) - 1\Big) \\ &- \Big(\frac{2}{(x+B)^2} - \frac{5}{2(t+T)}(1-\frac{1}{8}\sqrt{\theta}) + \frac{(x+B)^2}{4(t+T)^2}(1-\frac{1}{8}\sqrt{\theta})^2\Big) + R(\tilde{u}).\end{aligned} \tag{4.6.19}$$

Noticing that $\dot{\theta}/\theta = -1/(t+T)$ and $\dot{\theta}/\sqrt{\theta} = -\sqrt{\theta}/(t+T)$, cancelling the obvious terms, and then grouping terms by the growth in x yields

$$\begin{aligned}&\frac{\overline{z}_t - \frac{3}{2(t+T)}(\overline{z}_x - \overline{z}) - \overline{z}_{xx} + \overline{z}R(\tilde{u})}{\overline{z}} \\ &= \Big(\frac{\sqrt{\theta}}{t+T} - \frac{3}{2(t+T)}\frac{1}{8}\sqrt{\theta}\Big) + \frac{(x+B)^2}{4(t+T)^2}\Big((1-\frac{1}{8}\sqrt{\theta}) - \frac{1}{2(t+T)}\Big)\frac{1}{8}\sqrt{\theta} \\ &\quad + \frac{3}{2(t+T)}\Big(\frac{2}{x+B} - \frac{x+B}{2(t+T)}(1-\frac{1}{8}\sqrt{\theta})\Big) - \frac{2}{(x+B)^2} + R(\tilde{u}).\end{aligned} \tag{4.6.20}$$

Since $\theta\leq 1$, we have, up to increasing T (independent of all parameters),

$$\begin{aligned}&\frac{\overline{z}_t - \frac{3}{2(t+T)}(\overline{z}_x - \overline{z}) - \overline{z}_{xx} + \overline{z}R(\tilde{u})}{\overline{z}} \\ &\geq \frac{\sqrt{\theta}}{2(t+T)} + \frac{(x+B)^2}{4(t+T)^2}\frac{\sqrt{\theta}}{16} + \frac{3}{2(t+T)}\Big(\frac{2}{x+B} - \frac{x+B}{2(t+T)}\Big) - \frac{2}{(x+B)^2} + R(\tilde{u}).\end{aligned} \tag{4.6.21}$$

Using Young's inequality and then increasing T (independent of all parameters), we arrive at

$$\begin{aligned}
&\frac{\overline{z}_t - \frac{3}{2(t+T)}(\overline{z}_x - \overline{z}) - \overline{z}_{xx} + \overline{z}R(\tilde{u})}{\overline{z}} \\
&\qquad \geq \frac{\sqrt{\theta}}{2(t+T)} + \frac{(x+B)^2}{8(t+T)^2}\frac{\sqrt{\theta}}{16} + \frac{3}{2(t+T)}\frac{2}{x+B} - \frac{72\sqrt{\theta}}{T(t+T)} - \frac{2}{(x+B)^2} + R(\tilde{u}) \\
&\qquad \geq \frac{\sqrt{\theta}}{4(t+T)} + \frac{(x+B)^2}{8(t+T)^2}\frac{\sqrt{\theta}}{16} + \frac{3}{2(t+T)}\frac{2}{x+B} - \frac{2}{(x+B)^2} + R(\tilde{u}).
\end{aligned} \tag{4.6.22}$$

{{e.c60401}}

At this point, we can see why the right hand side of (4.6.22) should be positive. Recall that, according to Lemma 4.4.6 (equation (4.4.10)), the term $R(\tilde{u}) \geq r_0 > 0$ when $\tilde{u}$ is not too small. Hence, it should dominate the next to last term in the right side of (4.6.22) in that region if B is large. On the other hand, for $\tilde{u}$ small, the term $R(\tilde{u})$ looks like $2/\log^2(\tilde{u})$, according to (4.4.11). Moreover, as $\tilde{u}(t,x) \approx U_*(x)$ and $U_*(x)$ has the asymptotics (4.3.5), we have $\log^2(\tilde{u}) \approx x^2$. Thus, once again, $R(\tilde{u})$ dominates the next to last term in the right side of (4.6.22).

We make the discussion above more precise. Let us fix $\delta_1 > 0$ as in Lemma 4.4.6. We claim that, up to increasing L (depending on δ_1), we have

$$\tilde{u}(t,x) \geq \begin{cases} 1 - \delta_1 & \text{if } x \leq L/2, \\ \frac{x+1}{C_L} e^{-x - \frac{C_L x^2}{t}} & \text{if } x \geq L/2, \end{cases} \tag{4.6.23}$$

{{e.c60402}}

for all $t \geq 1$, with a constant C_L that depends on L. The first alternative above is due to Proposition 4.4.1. The second alternative follows from [53, Proposition 3.1] and its proof, as well as an application of the comparison principle.

We first consider the "large" $\tilde{u}$ regime (and, thus, x "not too far on the right"). If $\tilde{u} \geq \delta_0$, then $R(\tilde{u}) \geq r_0$ due to (4.4.10) and we find

$$-\frac{2}{(x+B)^2} + R(\tilde{u}) \geq -\frac{2}{(x+B)^2} + r_0 > 0, \tag{4.6.24}$$

up to increasing B further if necessary so that $2/B^2 < r_0$. In particular, then we have, from (4.6.22),

$$\frac{\overline{z}_t - \frac{3}{2(t+T)}(\overline{z}_x - \overline{z}) - \overline{z}_{xx} + \overline{z}R(\tilde{u})}{\overline{z}} > 0, \quad \text{if } \tilde{u}(t,x) \geq \delta_0, \tag{4.6.25}$$

as desired.

Next we consider the "small" $\tilde{u}$ regime (and, thus, "large" x regime). Note that, by (4.6.23), if $\tilde{u} \leq \delta_0$, then

$$x \geq \min\left(\frac{1}{2}\log\frac{1}{C\delta_0}, \sqrt{\frac{t}{2C}\log\frac{1}{C\delta_0}}\right) \geq \sqrt{\frac{1}{2C}\log\frac{1}{C\delta_0}}. \tag{4.6.26}$$

{{e.c032901}}

In particular, this case is restricted to x that is very large, after possibly decreasing δ_0.

We begin by estimating $R(\tilde{u})$ using (4.4.11). For the quadratic term, we apply (4.6.23) to find

$$\frac{2}{(\log(\tilde{u}))^2} \geq \frac{2}{x^2}\frac{1}{(1 + \frac{Cx}{t} - \frac{\log x}{x} + \frac{\log C}{x})^2}. \tag{4.6.27}$$

Then, using that $(1+z)^{-2} \geq 1 - 2z$ for all $z \geq -1$, we obtain

$$\frac{2}{(\log(\tilde{u}))^2} \geq \frac{2}{x^2}\left(1 - 2\left(\frac{Cx}{t} - \frac{\log x}{x} + \frac{\log C}{x}\right)\right) = \frac{2}{x^2} - \frac{4C}{xt} + \frac{4\log x}{x^3} - \frac{4\log C}{x^3}. \tag{4.6.28}$$

A similar argument, using the inequality

$$(1-z)^{-3} \le 1 + Cz, \quad \text{for } 0 \le z \le 1/2, \tag{4.6.29}$$

yields a bound for the second term in $R(\tilde{u})$:

$$\begin{aligned} -\frac{C}{|\log(\tilde{u})|^3} &\ge -\frac{C}{x^3}\frac{1}{\left(1+\frac{Cx}{t}-\frac{\log x}{x}+\frac{\log C}{x}\right)^3} \ge -\frac{C}{x^3}\frac{1}{\left(1-\frac{\log x}{x}\right)^3} \\ &\ge -\frac{C}{x^3}\left(1+C\frac{\log x}{x}\right) = -\frac{C}{x^3}-\frac{C\log x}{x^4}. \end{aligned} \tag{4.6.30}$$

Using these in (4.6.22), we find

$$\begin{aligned} \frac{\overline{z}_t - \frac{3}{2(t+T)}(\overline{z}_x - \overline{z}) - \overline{z}_{xx} - \overline{z}R(\tilde{u})}{\overline{z}} &\ge \frac{\sqrt{\theta}}{4(t+T)} + \frac{(x+B)^2}{8(t+T)^2}\frac{\sqrt{\theta}}{16} + \frac{3}{(t+T)(x+B)} \\ &- \frac{2}{(x+B)^2} + \frac{2}{x^2} - \frac{4C}{xt} + \frac{4\log x}{x^3} - \frac{4\log C}{x^3} - \frac{C}{x^3} - \frac{C\log x}{x^4}. \end{aligned} \tag{4.6.31}$$

After decreasing δ_0 (which, by (4.6.26), increases the lower bound for x), we find

$$\frac{\overline{z}_t - \frac{3}{2(t+T)}(\overline{z}_x - \overline{z}) - \overline{z}_{xx} - \overline{z}R(\tilde{u})}{\overline{z}} \ge \frac{\sqrt{\theta}}{4(t+T)} + \frac{(x+B)^2}{8(t+T)^2}\frac{\sqrt{\theta}}{16} + \frac{3}{(t+T)(x+B)} - \frac{4C}{xt} + \frac{2\log x}{x^3}.$$

There is only one negative term above. Applying Young's inequality with $p = 3/2$ and $q = 3$ yields

$$-\frac{4C}{xt} \ge -\frac{\sqrt{\theta}}{4(t+T)} - C\left(\left(\frac{\sqrt{\theta}}{(t+T)}\right)^{-\frac{2}{3}}\frac{1}{xt}\right)^3 = -\frac{\sqrt{\theta}}{4(t+T)} - C\frac{(t+T)^3}{T}\frac{1}{x^3t^3} \ge -\frac{\sqrt{\theta}}{4(t+T)} - C\frac{T^2}{x^3}. \tag{4.6.32}$$

Hence, we have

$$\frac{\overline{z}_t - \frac{3}{2(t+T)}(\overline{z}_x - \overline{z}) - \overline{z}_{xx} - \overline{z}R(\tilde{u})}{\overline{z}} \ge \frac{(x+B)^2}{8(t+T)^2}\frac{\sqrt{\theta}}{16} + \frac{3}{2(t+T)}\frac{2}{x+B} - C\frac{T^2}{x^3} + \frac{\log x}{2x^3}. \tag{4.6.33}$$

which is positive after further decreasing δ_0 (which, by (4.6.26), increases x). This concludes the proof of (v) and, thus, the proof of the lemma. $\square$

4.7 Proofs of the bounds on η and Q

4.7.1 Concavity of η: Proposition 4.3.2

We make two observations. First, arguing as in Lemma 4.4.4, it is easy to check that, for any f, its traveling wave profile function η satisfies

$$\eta^2(u)\eta''(u) \to 0, \quad \text{as } u \to 1^-. \tag{4.7.1}$$ {{mar2325}}

Second, Proposition 4.3.2 follows from the following more general result.

Lemma 4.7.1. *Assume that (4.1.1)-(4.1.3) hold. Suppose that either:*

(i) (pulled case) the asymptotics (4.3.5) hold and $f'' \le 0$ on $(0,1)$;

(ii) (pulled case) the asymptotics (4.3.5) hold and there is $u_0 \in [0,1]$ such that $f'' \ge 0$ on $(0,u_0)$ and $f'' \le 0$ on $(u_0,1)$;

(iii) (pushed and pushmi-pullyu cases) there is $\chi \geq 1$ and A satisfying $A(0) = A'(0) = 0$ and $A(1) = 1$ such that

$$f(u) = (u - A(u))(1 + \chi A'(u)) \quad \text{and} \quad A'', A''' \geq 0. \tag{4.7.2}$$ {{e.c062505}}

If $\chi = 1$, the condition $A''' \geq 0$ is not necessary.

Then $\eta'' \leq 0$ and $Q \leq 1$.

Proof in cases (i) and (ii). First, note that, case (i) is really the subcase of (ii) where $u_0 = 0$. Hence, we only consider case (ii). Let us also recall that $f'(0) = 1$, according to assumption (4.1.3). If the asymptotics (4.3.5) holds and $c \geq 2$ is the speed of the wave, then, by linearization as $x \to +\infty$, it is easy to see that λ_0 must be a double root of the equation

$$c\lambda = \lambda^2 + 1. \tag{4.7.3}$$

It follows that $c = c_* = 2$ and $\lambda_0 = 1$.

Observe that it is thus enough to show that $\eta'' \leq 0$. Indeed,

$$Q = \eta'(2 - \eta') + \eta''\eta \leq 1 + \eta''\eta \leq 1. \tag{4.7.4}$$

By Lemma 4.4.4(iii), there exists $u_1 > 0$ so that

$$\eta''(u) \leq 0, \quad \text{for all } 0 < u < u_1. \tag{4.7.5}$$ {{mar2702}}

Thus, the following is well-defined and positive:

$$\bar{u} = \sup\{\tilde{u} \in (0,1) : \ \eta''(u) \leq 0 \text{ on } (0, \tilde{u}]\}. \tag{4.7.6}$$ {{e.bar_u}}

Our goal is to prove that $\bar{u} = 1$.

Suppose, for the sake of a contradiction, that $\bar{u} < 1$. Writing (4.3.26) as

$$2 - \eta' = \frac{f(u)}{\eta(u)}, \tag{4.7.7}$$ {{e.eta_ode}}

we find $\eta^2\eta'' = \eta' f - f'\eta$ and, hence,

$$(\eta^2\eta'')' = (\eta' f - f'\eta)' = \eta'' f - \eta f''. \tag{4.7.8}$$ {{e.c031902}}

It follows that, at $\bar{u}$, we have

$$0 \leq (\eta^2\eta'')'(\bar{u}) = \eta''(\bar{u})f(\bar{u}) - \eta(\bar{u})f''(\bar{u}) = -\eta(\bar{u})f''(\bar{u}). \tag{4.7.9}$$ {{e.c031901}}

The first inequality follows from the fact that $\eta^2\eta''$ crosses zero at $\bar{u}$ due to (4.7.6). As $\eta(\bar{u}) > 0$, it follows that $f''(\bar{u}) \leq 0$, which in turn implies that

$$\bar{u} \geq u_0. \tag{4.7.10}$$ {{mar2627}}

We deduce that

$$f''(u) \leq 0 \quad \text{for all } u \geq \bar{u}. \tag{4.7.11}$$ {{mar2706}}

We now claim that $\eta'' > 0$ on $(\bar{u}, 1)$. The definition (4.7.6) of $\bar{u}$ implies that if $\bar{u} < 1$ then for every $\varepsilon > 0$ sufficiently small, there is $u_\varepsilon \in (\bar{u}, \bar{u} + \varepsilon)$ such that $\eta''(u_\varepsilon) > 0$. Suppose that there is $\bar{v}_\varepsilon \in (u_\varepsilon, 1)$ such that $\eta''(u) > 0$ for $u \in (u_\varepsilon, \bar{v}_\varepsilon)$ and $\eta''(\bar{v}_\varepsilon) = 0$. Then, integrating (4.7.8) gives

$$0 > -\eta^2(u_\varepsilon)\eta''(u_\varepsilon) = \int_{u_\varepsilon}^{v_\varepsilon} \left(\eta'' f - \eta f''\right) du > 0, \tag{4.7.12}$$ {{mar2624}}

which is a contradiction. The second inequality in (4.7.12) follows from the fact that, on the domain on integration, $\eta, f, \eta'' > 0$ and $f'' \leq 0$. We conclude that $\eta''(u) > 0$ for $u \in (u_\varepsilon, 1)$. By the arbitrariness of $\varepsilon > 0$, it follows that $\eta'' > 0$ on $(\bar{u}, 1)$, as claimed.

Finally, we conclude by obtaining a contradiction at $u = 1$. Going back to (4.7.8) and recalling (4.7.10), we deduce that

$$(\eta^2\eta'')' = \eta'' f - \eta f'' > 0, \quad \text{for } \bar{u} < u < 1. \tag{4.7.13}$$

{{mar2628}}

Recall that $\eta''(\bar{u}) = 0$, by construction. As a consequence, we obtain, for any $u > \bar{u}$,

$$\eta^2(u)\eta''(u) = \eta^2(\bar{u})\eta''(\bar{u}) + \int_{\bar{u}}^{u} \left(\eta'' f - \eta f''\right) du = \int_{\bar{u}}^{u} \left(\eta'' f - \eta f''\right) du > 0. \tag{4.7.14}$$

{{mar2628bis

Taking the limit $u \nearrow 1$ and using (4.7.1), we obtain

$$0 = \lim_{u \nearrow 1} \eta^2(u)\eta''(u) = \int_{\bar{u}}^{1} \left(\eta'' f - \eta f''\right) du > 0. \tag{4.7.15}$$

Here, the last inequality follows from (4.7.13). This contradiction shows that it is impossible that $\bar{u} < 1$. It follows that $\bar{u} = 1$ and $\eta''(u) < 0$ for all $u \in (0, 1)$. This concludes the proof. □

Proof in case (iii). Here, we have the explicit form of η due to [2, Proposition A.2]:

$$\eta(u) = \sqrt{\chi}(u - A(u)) \quad \text{ and } \quad c_* = \sqrt{\chi} + \frac{1}{\sqrt{\chi}}. \tag{4.7.16}$$

{{e.c060501}

It is immediate that $\eta'' \leq 0$; hence, we need only show that $Q \leq 1$. A direct computation yields

$$Q = 1 + (\chi - 1)A' - \chi|A'|^2 - \chi(u - A)A'' \leq 1 + \chi(A' - |A'|^2 - (u - A)A''). \tag{4.7.17}$$

The second inequality follows from the convexity of A and the fact that $A'(0) = 0$, which imply that $A' \geq 0$. It is, hence, enough to show that

$$A' - |A'|^2 - (u - A)A'' \leq 0. \tag{4.7.18}$$

{{jun2106}}

Note that, at $u = 0$, the expression above vanishes On the other hand,

$$(A' - |A'|^2 - (u - A)A'')' = -A'A'' - (u - A)A''' \leq 0, \tag{4.7.19}$$

{{jun2202}}

since $u - A, A', A'', A''' \geq 0$. We conclude that $Q \leq 1$. This completes the proof.

Finally, we consider the last statement for $\chi = 1$. We have already observed that $\eta'' \leq 0$. We conclude by noting that, from (4.7.16), $c_* = 2$ and then arguing as in the second paragraph of the proof for cases (i) and (ii). □

4.7.2 Refined bounds on Q: proof of Lemma 4.4.6

First, we note that the bounds in (4.4.8) follow from Lemma 4.7.1. Second, the bounds (4.4.11) follow directly from Lemma 4.4.4.

We now address the bounds in (4.4.10) for the remainder of the proof. We first investigate the first alternative in (4.4.10). In the case $\chi \geq 1$, the proof of Lemma 4.7.1 clearly shows that if $A'', A''' < 0$, then Q is bounded away from 1 on compact subsets of $(0, 1]$. This is exactly the first alternative in (4.4.10) for the case $\chi \geq 1$.

When $0 \leq \chi < 1$, the first inequality in (4.4.10) is deduced using only the concavity of η (4.4.8) and the asymptotics Lemma 4.4.4. Indeed, these imply that $\eta'(u) \leq \eta'(\delta_0) < 1$ for all $u \in (\delta_0, 1)$. Hence,

$$\begin{aligned} R(u) = 1 - Q(u) &= 1 - \eta'(u)(2 - \eta'(u)) - \eta(u)\eta''(u) \geq 1 - \eta'(u)(2 - \eta'(u)) \\ &> 1 - \eta'((\delta_0)(2 - \eta'(\delta_0)) > 0. \end{aligned} \tag{4.7.20}$$

{{e.c062702}

This yields the first alternative in (4.4.10) in the pulled case.

We now investigate the second alternative in (4.4.10). Notice that

$$Q(1) = \eta'(1)(c_* - \eta'(1)) + \eta(1)\eta''(1) = -\lambda_1(c_* + \lambda_1) = f'(1) < 0. \tag{4.7.21}$$

{{e.c060701}

The second equality above follows from (4.3.9), while the third is due to (4.3.10). The inequality uses the particular form of f. This concludes the proof. $\Box$

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