On asymptotics of a tracer advected in a locally self-similar, correlated flow

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Abstract

In this paper we consider the motion of a tracer in a flow that is locally self-similar and whose correlations decay at infinity but at the rate that does not guarantee that the flow does not have "memory effect". We show that when the field is Gaussian the appropriately regularized scaling limit of the trajectory is a super-diffusive fractional Brownian motion. This complements our previous result contained in [6].

1 Introduction

The motion of a particle propagating in a random velocity field

$$\frac{d\mathbf{X}}{dt} = \mathbf{V}(t, \mathbf{X}),$$

is typically well approximated by a Brownian motion in the long time limit when the random field $\mathbf{V}(t, \mathbf{x})$ has sufficient mixing properties in time. The heuristic reason for this behavior is that the particle's speed at different times is weakly correlated and, loosely speaking, the central limit theorem implies that its position behaves as a Brownian motion. Another regime which leads to the "renewal" of the Lagrangian velocity of the particle is when the velocity is dominated by a large constant drift:

$$\frac{d\mathbf{X}(t)}{dt} = \mathbf{v} + \delta\mathbf{F}(\mathbf{X}(t)), \quad \mathbf{X}(0; \mathbf{x}) = 0. \tag{1.1}$$

Here $\mathbf{v} \neq 0$ is a constant drift and $\delta \ll 1$ is a small parameter. In that case, even if the random fluctuation $\mathbf{F}(\mathbf{x})$ is time-independent, the particle never returns to the regions it has previously visited, hence, if $\mathbf{F}(\mathbf{x})$ is sufficiently mixing in space, the particle "always sees new randomness". Then the fluctuation of its trajectory around the mean position $\mathbf{y}(t) = \mathbf{X}(t) - \mathbf{v}t$ converges to the Brownian motion. More precisely, if $\mathbf{F}(\mathbf{x})$ is a spatially homogeneous mixing field with the correlation matrix $R(\mathbf{x}) = [R_{ij}(\mathbf{x})]$ then the process

$$\mathbf{Y}_\delta(t) = \mathbf{X}\left(\frac{t}{\delta^2}\right) - \frac{\mathbf{v}t}{\delta^2},$$

converges as $\delta \to 0$ to the Brownian motion with the covariance matrix given by the Kubo-Taylor formula

$$D_{ij} = \frac{1}{2}\int_0^\infty (R_{ij}(\mathbf{v}t) + R_{ji}(\mathbf{v}t))\,dt, \quad i, j = 1, \ldots, d. \tag{1.2}$$

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This result was established by Kesten and Papanicolaou in [5] provided that $|R(\mathbf{x})| \le C/(1+|\mathbf{x}|^m)$ with a sufficiently large $m>0$ and some $C>0$. The strong decay of the correlation tensor ensures the aforementioned renewal of the particle Lagrangian velocity and also guarantees that the integral in the definition of the diffusivity tensor is finite, see (1.2).

We are interested in the situation when the correlation tensor decays slowly in space so that the diffusivity becomes infinite and the diffusive limit, at least as established in [5] needs not hold. To be more precise, suppose that the field $\mathbf{F}:\ \mathbb{R}^d\times\Omega\to\mathbb{R}^d$ is a random Gaussian field defined over a probability space $(\Omega,\mathcal{F},\mathbb{P})$ that is homogeneous and isotropic. We assume further that the power–energy spectrum of the field satisfies *the power law*, that is, its covariance matrix is of the form

$$R_{ij}(\mathbf{x}) = \int_{\mathbb{R}^d} \mathrm{e}^{i\mathbf{k}\cdot\mathbf{x}}\hat{R}_{ij}(\mathbf{k})d\mathbf{k}, \tag{1.3}$$

with the power-energy spectrum

$$\hat{R}_{ij}(\mathbf{k}) = \frac{\mathbf{1}_{[0,K]}(|\mathbf{k}|)}{|\mathbf{k}|^{2\alpha+d-2}}\,\Gamma_{ij}(\hat{\mathbf{k}}). \tag{1.4}$$

Here $0<K<+\infty$ is a large frequency cut-off, $\hat{\mathbf{k}} := \mathbf{k}/|\mathbf{k}|$ and $\Gamma_{ij}(\hat{\mathbf{k}}) = \delta_{ij} - \hat{k}_i\hat{k}_j$. To ensure integrability of the spectrum we assume $\alpha<1$. We have shown in our companion paper [6] that the behavior of the particle remains diffusive so long as the diffusivity tensor appearing in the Kubo formula (1.2) is finite, that is, when $\alpha<1/2$. In the present paper, we shall consider the case when $\alpha\in(1/2,1)$. Then (see e.g. (2.1) below) the correlation tensor of $\mathbf{F}(\mathbf{x})$ behaves as

$$R_{ij}(\mathbf{x}) \sim c_{ij}|\mathbf{x}|^{2\alpha-2}, \quad \text{for } |\mathbf{x}|\gg 1 \tag{1.5}$$

for some constants c_{ij}. We would like to understand the behavior of the trajectory of (1.1) for long times $t\sim\delta^{-2\beta}$ for an appropriate $\beta>0$. In the diffusive regime the Brownian motion limit is observed for $\beta=1$ but here the situation is different: it has been rigorously shown in [6] that for

$$\beta = \frac{1}{2\alpha} \tag{1.6}$$

and with $\mathbf{y}(t) = \mathbf{X}(t) - \mathbf{v}t$, we have

$$\lim_{\delta\to 0+}\mathbb{E}|\mathbf{y}(t/\delta^{2\beta(1+\rho)})|^2 = +\infty \qquad \text{and} \qquad \lim_{\delta\to 0+}\mathbb{E}|\mathbf{y}(t/\delta^{2\beta(1-\rho)})|^2 = 0, \tag{1.7}$$

where the first of the two limits is taken in the Cesaro-sense and $\mathbb{E}$ denotes the expectation with respect to the probability measure $\mathbb{P}$. This identifies the time-scale on which the process $\mathbf{y}(t)$ has a non-trivial behavior but leaves open the question on how exactly $\mathbf{y}(t)$ behaves on this time-scale.

Regarding the last question, we have also presented in [6] a formal argument that substantiated the claim that the one dimensional statistics of $\mathbf{y}(t/\delta^{2\beta})$ converge weakly in law to those of $\mathbf{D}^{1/2}(\mathbf{v})B^{(\alpha)}(t)$, where $B^{(\alpha)}(t)$ is a standard d–dimensional fractional Brownian motion with the Hurst exponent α. It is a Gaussian process with stationary increments and such that

$$\mathbb{E}[B_i^{(\alpha)}(t)B_j^{(\alpha)}(t)] = \delta_{ij}t^{2\alpha}, \quad \forall\, t\ge 0.$$

The matrix $\mathbf{D}(\mathbf{v}) = [D_{ij}(\mathbf{v})]$, where

$$D_{ij}(\mathbf{v}) = \frac{1}{2\alpha^2}\int \frac{e^{i\mathbf{k}\cdot\mathbf{v}}}{|\mathbf{k}|^{2\alpha+d-2}}\Gamma_{ij}(\hat{\mathbf{k}})d\mathbf{k}. \tag{1.8}$$

The purpose of the current paper is to present a rigorous argument that supports that claim in case we regularize the limiting procedure. More specifically, suppose that $u_0 \in \mathcal{S}(\mathbb{R}^d)$ is a smooth rapidly decaying function, $|\mathbf{v}| = 1$, $\mathbf{G}(t, \mathbf{x}) := -\mathbf{F}(\mathbf{x} + \mathbf{v}t)$ and $\mathbf{X}(t; \mathbf{x})$ is a trajectory of (1.1) starting at $\mathbf{X}(0; \mathbf{x}) = \mathbf{x}$. Let us set $\mathbf{y}_\delta(t, \mathbf{x}) = \mathbf{X}(t/\delta^{2\beta}; \mathbf{x}) - \mathbf{v}t/\delta^{2\beta}$. One can easily observe that $\mathbb{E}u_0(\mathbf{y}_\delta(t; \mathbf{x})) = \mathbb{E}u_\delta(t, \mathbf{x})$, where $u_\delta(t, \mathbf{x})$ is a solution of the advection equation

$$\begin{aligned} &\partial_t u_\delta(t, \mathbf{x}) + \delta^{1-2\beta}\mathbf{G}\left(\frac{t}{\delta^{2\beta}}, \mathbf{x}\right) \cdot \nabla_{\mathbf{x}} u_\delta(t, \mathbf{x}) = 0, \\ &u_\delta(0, \mathbf{x}) = u_0(\mathbf{x}). \end{aligned} \tag{1.9}$$

Here, as we have already mentioned, β is given by (1.6). For given $\mu, \kappa > 0$ consider $v_\delta(t, \mathbf{x})$, the mild solution of the regularized equation

$$\begin{aligned} &\partial_t v_\delta(t, \mathbf{x}) + \delta^{1-2\beta}\mathbf{G}\left(\frac{t}{\delta^{2\beta}}, \mathbf{x}\right) \cdot \nabla_{\mathbf{x}} v_\delta(t, \mathbf{x}) = \kappa[-(-\Delta)^\mu] v_\delta(t, \mathbf{x}), \\ &v_\delta(0, \mathbf{x}) = u_0(\mathbf{x}). \end{aligned} \tag{1.10}$$

The precise meaning of the mild solution to (1.10) is defined below in the remark preceding (2.12). Define also $v(t, \mathbf{x})$ as the respective solution to the initial value problem

$$\begin{aligned} \partial_t v(t, \mathbf{x}) &= \kappa[-(-\Delta)^\mu] v(t, \mathbf{x}), \\ v(0, \mathbf{x}) &= u_0(\mathbf{x}). \end{aligned} \tag{1.11}$$

Our principal result can be stated as follows.

Theorem 1.1 *Assume that the Gaussian field $\mathbf{F}(\cdot)$ satisfies the hypotheses made above, β is given by* (1.6) *and the functions $v_\delta(t, \mathbf{x})$ and $v(t, \mathbf{x})$ are the solutions of* (1.10) *and* (1.11)*, respectively. Then, for an arbitrary $\mu > 0$ satisfying $2\alpha - 1 - 1/\mu > 0$ we have*

$$\lim_{\delta \to 0+} \mathbb{E} v_\delta(t, \mathbf{x}) = \mathbb{E} v(t, \mathbf{x} + \mathbf{D}^{1/2}(\mathbf{v}) B^{(\alpha)}(t)), \tag{1.12}$$

where $B^{(\alpha)}(t)$ is a standard, d-dimensional fractional Brownian motion with the Hurst exponent α.

The proof of this theorem uses the perturbation series expansion of the solution to (1.10) obtained with the help of Green's function of equation (1.11), see Theorem 2.5 below. The expression for $\mathbb{E}v_\delta(t, \mathbf{x})$ can be rewritten in the form of an infinite series, see (2.16) below. Calculation of the limit appearing on the right hand side of (1.12) can be reduced therefore to the problem of computing the limit for each term of the expansion separately, see Section 3. The latter can be achieved using diagrammatic representations of the terms of the series. Section 4 contains the proof of Lemma 3.4 which plays the crucial role in substantiating the exchange of the limit with the infinite summation. Section 5 is devoted to the proof of the existence and uniqueness result concerning the solutions of equation (1.10).

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2 Preliminaries

2.1 The random field

The Spectral Theorem (see e.g., Theorem 2.1.2, p. 25 of [1]) implies that there exists a complex d–dimensional vector valued spectral measure $\hat{\mathbf{F}}(\cdot) = (\hat{F}_1(\cdot), \cdots, \hat{F}_d(\cdot))$ such that

$$\mathbf{F}(\mathbf{x}) = \int e^{i\mathbf{k}\cdot\mathbf{x}} \hat{\mathbf{F}}(d\mathbf{k}), \tag{2.1}$$

where

$$\mathbb{E}[\hat{F}_i(d\mathbf{k})\hat{F}_j(d\mathbf{k}')] = \delta(\mathbf{k}+\mathbf{k}')\hat{R}_{ij}(\mathbf{k})d\mathbf{k}\,d\mathbf{k}', \quad i,j=1,\ldots,d. \tag{2.2}$$

We note here that $\hat{\mathbf{F}}(\cdot)$ is Gaussian, if considered as a $2d$–dimensional real vector valued spectral measure. The covariance matrix of the field is given then by (1.3). Let us show that it decays as announced in (1.5).

Proposition 2.1 *For any $\mathbf{x}\in\mathbb{R}^d$ and $\mathbf{v}\in\mathbb{S}^{d-1}$ we have*

$$\lim_{s\to+\infty} s^{2(1-\alpha)}\mathbf{R}\left(\mathbf{x}+\mathbf{v}s\right) = \mathbf{D}(\mathbf{v}),$$

where $\mathbf{D}(\mathbf{v}) = [D_{ij}(\mathbf{v})]$ and $D_{ij}(\mathbf{v})$ is given by (1.8).

Proof. Using (1.3) and (1.4) we can write

$$\mathbf{R}\left(\mathbf{x}+\mathbf{v}s\right) = \int\limits_0^K \frac{D_{ij}(k,\hat{\xi})dk}{k^{2\alpha-1}},$$

where $\hat{\xi} := (\mathbf{v}+s^{-1}\mathbf{x})/|\mathbf{v}+s^{-1}\mathbf{x}|$ and

$$D_{ij}(k,\hat{\xi}) := \int\limits_{\mathbb{S}^{d-1}} \exp\left\{i\hat{\mathbf{k}}\cdot\hat{\xi}(k|\mathbf{x}+\mathbf{v}s|)\right\}\Gamma(\hat{\mathbf{k}})S(d\hat{\mathbf{k}}).$$

We change variables $k' := k|\mathbf{x}+\mathbf{v}s|$. Then,

$$R_{ij}\left(\mathbf{x}+\mathbf{v}s\right) = s^{2(\alpha-1)}|s^{-1}\mathbf{x}+\mathbf{v}|^{2(\alpha-1)} \int\limits_0^{K|\mathbf{x}+\mathbf{v}s|} \frac{D_{ij}(k,\hat{\xi})dk}{k^{2\alpha-1}}.$$

Using the Hecke-Funk theorem (see e.g. [4], p. 181) we can obtain a more explicit formula for the tensor $D_{ij}(k,\hat{\xi})$. First, note that for $i\neq j$ the expression k_ik_j is a harmonic polynomial, hence by the aforementioned theorem

$$D_{ij}(k,\hat{\xi}) := \omega_{d-1}\hat{\xi}_i\hat{\xi}_j\int\limits_{-1}^{1} \mathrm{e}^{ikt}P_{2,d}(t)(1-t^2)^{(d-3)/2}dt,$$

where $P_{n,d}(t)$ is the n-th degree Legendre polynomial in dimension d, see Section 6.3 of [4]. Here ω_{d-1} is the surface area of $\mathbb{S}^{d-1}$. When, on the other hand, $i=j$ note that

$$1-\hat{k}_i^2 = u_i(\hat{\mathbf{k}})+1-\frac{1}{d},$$

where $u_i(\mathbf{k}) := d^{-1}|\mathbf{k}|^2-k_i^2$ is a harmonic polynomial. Then,

$$D_{ii}(k,\hat{\xi}) := \omega_{d-1}\left(1-\frac{1}{d}\right)\hat{\xi}_i^2\int\limits_{-1}^{1}\mathrm{e}^{ikt}P_{2,d}(t)(1-t^2)^{(d-3)/2}dt + \frac{\omega_{d-1}}{d}\int\limits_{-1}^{1}\mathrm{e}^{ikt}(1-t^2)^{(d-3)/2}dt.$$

We use here the fact that $P_{0,d}(t) = \mathbf{1}$. The conclusion of the proposition follows now upon the passage to the limit $s\to+\infty$. $\Box$

We recall the following proposition from [7] (Theorem 3.2 in this reference).

Proposition 2.2 *Suppose that a Gaussian field* $\mathbf{F}(\mathbf{x})$ *satisfies the above assumptions. Then for any* $\gamma > 1$ *there exists a positive random variable* F *such that*

$$|\mathbf{F}(\mathbf{x};\omega)| \le F(\omega)(1+\log^+|\mathbf{x}|)^{\gamma/2}, \quad \forall\, \mathbf{x} \in \mathbb{R}^d.$$

In addition, there exists a constant $C > 0$ *such that*

$$\mathbb{P}[F \ge \lambda] \le Ce^{-\lambda^2/C}, \quad \forall\, \lambda > 0. \tag{2.3}$$

Proof. Define $G(\mathbf{x};\omega) := (1+\log^+|\mathbf{x}|)^{-\gamma/2}\mathbf{F}(\mathbf{x};\omega)$. Recall that for a given number $\varepsilon > 0$ the entropy number $N(\varepsilon)$ of the field $G(\mathbf{x})$ is the smallest number of balls in the pseudo-metric

$$\mathrm{d}(\mathbf{x};\mathbf{y}) := \sqrt{\mathbb{E}|G(\mathbf{x}) - G(\mathbf{y})|^2}, \quad \mathbf{x},\, \mathbf{y} \in \mathbb{R}^d$$

needed to cover $\mathbb{R}^d$. An elementary calculation shows that there exist constants $C_1, C_2 > 0$ such that

$$N(\varepsilon) \le C_1 \exp\{C_2 \varepsilon^{-2/\gamma}\} \quad \forall\, \varepsilon \in (0,1]. \tag{2.4}$$

This can be seen as follows. Choose $R := \exp\{\varepsilon^{-2/\gamma}\}$ and let $B_R := [|\mathbf{x}| \le R]$. The set $B_R^c := [|\mathbf{x}| \ge R]$ can be covered by a single d-ball of radius $\varepsilon > 0$. Since the function $(\mathbf{x},\mathbf{y}) \mapsto \mathrm{d}(\mathbf{x},\mathbf{y})$ is Hölder continuous with the exponent $1/2$ on $B_R \times B_R$ one can cover B_R by $C(R/\varepsilon^2)^d$ d-balls with radius ε, where $C > 0$ is a certain constant, and (2.4) follows. The conclusion of the proposition is a direct consequence of Theorem 5.4 p. 121 of [1]. $\square$

2.2 The Green's function for the fractional heat equation

Let us set

$$\hat{q}_0(t,\mathbf{k}) := \exp\left\{-\kappa|\mathbf{k}|^{2\mu}t\right\}. \tag{2.5}$$

Then the Green's function for the fractional heat equation (1.11) is

$$q_{\kappa,d}(t,\mathbf{x}) := \int\limits_{\mathbb{R}^d} e^{i\mathbf{k}\cdot\mathbf{x}}\hat{q}_\kappa(t,\mathbf{k})d\mathbf{k} = t^{-d/(2\mu)} q_0\left(\frac{|\mathbf{x}|}{t^{1/(2\mu)}}\right), \tag{2.6}$$

where

$$q_0(x) := \int\limits_{\mathbb{R}^d} e^{ix\mathbf{k}\cdot\mathbf{e}} e^{-\kappa|\mathbf{k}|^{2\mu}} d\mathbf{k}, \quad x \in \mathbb{R},$$

and $\mathbf{e}$ is an arbitrary unit vector: $|\mathbf{e}| = 1$ (rotational invariance implies that $q_0(x)$ does not depend on the choice of $\mathbf{e}$). We shall drop the indices κ, d when they are obvious from the context.

We will need the following properties of the kernel $q_{\kappa,d}(t,\mathbf{x})$ in the proof of the main theorem. They shall be proved in Section 5.

Proposition 2.3 *(i) There exists a constant* $C > 0$ *such that*

$$|q_0(x)| \le \frac{C}{(1+x^2)^{d/2+\mu}}, \quad \forall\, x \in \mathbb{R}. \tag{2.7}$$

(ii) We have

$$\nabla_{\mathbf{x}} q_{\kappa,d}(t,\mathbf{x}) = -2\pi\mathbf{x}\, q_{\kappa,d+2}(t,\mathbf{x}'), \quad \forall\, t > 0,\, \mathbf{x} \in \mathbb{R}^d, \tag{2.8}$$

where $\mathbf{x}' = (\mathbf{x},0,0) \in \mathbb{R}^{d+2}$.

The estimate (2.7) is not sharp when μ is a positive integer. Indeed, in that case $q_\kappa(x)$ is a Schwartz class function. As a direct consequence of (2.7) and (2.6) we conclude the following.

Corollary 2.4 *There exists a constant $C > 0$ such that*

$$|q_{\kappa,d}(t,\mathbf{x})| \le \frac{C}{t^{d/(2\mu)}}\left(1+\frac{|\mathbf{x}|^2}{t^{1/\mu}}\right)^{-d/2-\mu} \tag{2.9}$$

and

$$|\nabla_{\mathbf{x}} q_{\kappa,d}(t,\mathbf{x})| \le \frac{C|\mathbf{x}|}{t^{1/\mu}}|q_\kappa(t,\mathbf{x})| \tag{2.10}$$

for all $t > 0$, $\mathbf{x} \in \mathbb{R}^d$.

2.3 An infinite expansion for random mild solutions

Suppose that the initial data $u_0 \in \mathcal{S}(\mathbb{R}^d)$. One can the rewrite equation (1.10) in the mild form as follows

$$v_\delta(t,\mathbf{x}) = \int\limits_{\mathbb{R}^d} u_0(y) q_\kappa(t,\mathbf{x}-\mathbf{y})d\mathbf{y} - \delta^{1-2\beta}\int\limits_0^t\int\limits_{\mathbb{R}^d} q_\kappa(t-s_1,\mathbf{x}-\mathbf{y}_1)\mathbf{G}\left(\frac{s_1}{\delta^{2\beta}},\mathbf{y}_1\right)\cdot(\nabla v_\delta)(s_1,\mathbf{y}_1)ds_1 d\mathbf{y}_1. \tag{2.11}$$

In fact, thanks to the incompressibility of the field $\mathbf{F}(\mathbf{x})$, we can reformulate (1.10) further. Namely, we can define a mild solution to that equation, with the initial condition $u_0 \in \mathcal{S}(\mathbb{R}^d)$ as a function $v_\delta(t,\mathbf{x})$ that is continuous on $[0,+\infty)\times\mathbb{R}^d$, bounded on any $[0,T]\times\mathbb{R}^d$ for $T > 0$ and such that

$$v_\delta(t,\mathbf{x}) = \int\limits_{\mathbb{R}^d} u_0(y) q_\kappa(t,\mathbf{x}-\mathbf{y})d\mathbf{y} - \delta^{1-2\beta}\int\limits_0^t\int\limits_{\mathbb{R}^d} (\nabla q_\kappa)(t-s_1,\mathbf{x}-\mathbf{y}_1)\cdot\mathbf{G}\left(\frac{s_1}{\delta^{2\beta}},\mathbf{y}_1\right) v_\delta(s_1,\mathbf{y}_1)ds_1 d\mathbf{y}_1 \tag{2.12}$$

for all $t \ge 0$. The results of [7], see Theorem 3.2. p. 169, imply that $|\mathbf{F}(\mathbf{x})|$ can grow as $\log^{1/2}|\mathbf{x}|$ for $|\mathbf{x}| \gg 1$. Hence, in order to establish the existence and uniqueness result for mild solutions to (1.10) we have to deal with the issue of unbounded coefficients.

Let us set

$$q_0^{(\delta)}(t,\mathbf{x},s,\mathbf{y}) := q_\kappa(t-s,\mathbf{x}-\mathbf{y})$$

and for any $n \ge 1$

$$q_n^{(\delta)}(t,\mathbf{x},s,\mathbf{y}) := (-1)^n\delta^{(1-2\beta)n}\int\limits_{\Delta_n(t,s)}\!\!\dots\!\int\int\limits_{(\mathbb{R}^d)^n}\!\!\dots\!\int Q_\delta(t,\mathbf{x},s_1,\mathbf{y}_1)\dots Q_\delta(s_{n-1},\mathbf{y}_{n-1},s_n,\mathbf{y}_n)$$
$$\times\, q_\kappa(s_n-s_{n+1},\mathbf{y}_n-\mathbf{y}_{n+1})ds^{(n)}d\mathbf{y}^{(n)},$$

where $ds^{(n)} := ds_1\dots ds_n$, $d\mathbf{y}^{(n)} := d\mathbf{y}_1\dots d\mathbf{y}_n$, with the convention that $\mathbf{y}_{n+1} := \mathbf{y}$, $s_{n+1} := s$. The integration above is carried out over the simplex

$$\Delta_n(t,s) := [(s_1,\dots,s_n)\in\mathbb{R}^n : s \le s_n \le \dots \le s_1 \le t],$$

and the integral kernel is

$$Q_\delta(t,\mathbf{x},s,\mathbf{y}) := (\nabla q_\kappa)(t-s,\mathbf{x}-\mathbf{y})\cdot\mathbf{G}\left(\frac{s}{\delta^{2\beta}},\mathbf{y}\right). \tag{2.13}$$

We have then the following result.

Theorem 2.5 *For any fixed $\delta > 0$ the series $\sum_{n\geq 0} q_n^{(\delta)}(t,\mathbf{x},s,\mathbf{y})$ is uniformly convergent on compact subsets of $[(t,\mathbf{x},s,\mathbf{y}) : t > s, \mathbf{x} \neq \mathbf{y}]$ to a certain limit $r^{(\delta)}(t,\mathbf{x},s,\mathbf{y})$ for $\mathbb{P}$-a.s. ω. This function is a fundamental solution of* (1.10) *in the following sense:*
(i) for any $u_0 \in C_b(\mathbb{R}^d)$ the function

$$v_\delta(t,\mathbf{x}) := \int_{\mathbb{R}^d} r^{(\delta)}(t,\mathbf{x},0,\mathbf{y})u_0(\mathbf{y})d\mathbf{y} \tag{2.14}$$

is a mild solution to (1.10) *in the sense of* (2.12),
(ii) for any $\mathbf{x} \in \mathbb{R}^d$ we have

$$\lim_{t\to 0+} v_\delta(t,\mathbf{x}) = u_0(\mathbf{x}).$$

(iii) In addition, $v_\delta(t,\mathbf{x})$ is a unique mild solution of (1.10).
Moreover, for any $R > 0$ and $\mathbf{x} \in \mathbb{R}^d$ we have

$$\sum_{n=0}^{+\infty} \mathbb{E}\left[\sup_{|\mathbf{y}|\leq R} |q_n^{(\delta)}(t,\mathbf{x},0,\mathbf{y})|\right] < +\infty. \tag{2.15}$$

Remark 2.6 Observe that as an immediate consequence of the above proposition and the fact that the expectation of a product of an odd number of jointly Gaussian random variables vanishes we obtain that

$$\mathbb{E}v_\delta(t,\mathbf{x}) = \sum_{n=0}^{+\infty} \mathbb{E}\int q_{2n}^{(\delta)}(t,\mathbf{x},0,\mathbf{y})u_0(\mathbf{y})d\mathbf{y} \tag{2.16}$$

for any $u_0 \in C_c^\infty(\mathbb{R}^d)$.

We postpone the proof of the above theorem till Section 5.

3 Proof of Theorem 1.1

3.1 Outline of the proof

The proof of Theorem 1.1 is based on passing to the limit $\delta \downarrow 0$ in expression (2.16) for $\mathbb{E}v_\delta(t,\mathbf{x})$. For that we need the following proposition. We abbreviate $q_{2n}^{(\delta)}(t,\mathbf{x},\mathbf{y}) := q_{2n}^{(\delta)}(t,\mathbf{x},0,\mathbf{y})$

Proposition 3.1 *For an arbitrary $\varrho, \mu, t > 0$ there exists a constant $C > 0$ such that*

$$|\mathbb{E}q_{2n}^{(\delta)}(t,\mathbf{x},\mathbf{y})| \leq \frac{C^n\|u_0\|_\infty}{(n!)^{2\alpha-1-1/(2\mu)-\varrho}}, \quad \forall\, n \geq 1,\ \delta > 0,\ \mathbf{x},\ \mathbf{y} \in \mathbb{R}^d. \tag{3.1}$$

This proposition allows us to interchange the infinite summation over n and the limit $\delta \downarrow 0$, provided that $2\alpha - 1 - 1/(2\mu) > 0$, to claim that

$$\bar{u}(t,\mathbf{x}) := \lim_{\delta\downarrow 0} \mathbb{E}v_\delta(t,\mathbf{x}) = \sum_{n=0}^{+\infty} \lim_{\delta\downarrow 0} \mathbb{E}\int q_{2n}^{(\delta)}(t,\mathbf{x},\mathbf{y})u_0(\mathbf{y})d\mathbf{y}. \tag{3.2}$$

The next step is to identify the limit $\bar{u}(t,\mathbf{x})$ defined above as the right side of (1.12).
The limits of the individual terms in (3.2) are identified as follows.

Proposition 3.2 *We have, for each n:*

$$\lim_{\delta\downarrow 0}\mathbb{E}\int q_{2n}^{(\delta)}(t,\mathbf{x},\mathbf{y})u_0(\mathbf{y})d\mathbf{y}=\frac{1}{(2n)!}\mathbb{E}\left[\int q_\kappa(t,\mathbf{x}-\mathbf{y})\left(\mathbf{D}^{1/2}(\mathbf{v})B^{(\alpha)}(t)\cdot\nabla_{\mathbf{y}}\right)^{2n}u_0(\mathbf{y})d\mathbf{y}\right]. \tag{3.3}$$

Here $B^{(\alpha)}(t)$ is a d-dimensional, standard fractional Brownian motion with Hurst exponent α, defined in the statement of Theorem 1.1.

With the help of Propositions 3.1 and 3.2 the proof of Theorem 1.1 is short.

Proof of Theorem 1.1. We obtain from (3.2) and (3.3), that

$$\lim_{\delta\downarrow 0}\mathbb{E}v_\delta(t,\mathbf{x})=\sum_{n=0}^{+\infty}\bar I_{2n}(t,\mathbf{x}), \tag{3.4}$$

where $\bar I_{2n}(t,\mathbf{x})$, $n\ge 0$ are given by the right side of (3.3). To sum the series appearing on the right side of (3.4) we rewrite the expression for $\bar I_{2n}(t,\mathbf{x})$ using the Fourier transform of u_0. As a result we get

$$\bar I_{2n}(t,\mathbf{x})=\frac{(-1)^n}{(2\pi)^d(2n)!}\mathbb{E}\left[\int e^{i\mathbf{x}\cdot\mathbf{k}}\hat q_\kappa(t,\mathbf{k})\left(\mathbf{D}^{1/2}(\mathbf{v})B^{(\alpha)}(t),\mathbf{k}\right)^{2n}\hat u_0(\mathbf{k})d\mathbf{k}\right], \tag{3.5}$$

where $\hat q_\kappa(t,\mathbf{k})$ is given by (2.5). Calculating out the expectation of the Gaussian, using the formula $\mathbb{E}X^{2n}=(2n-1)!!(\mathbb{E}X^2)^n$ valid for any centered Gaussian random variable X, we obtain

$$\bar I_{2n}(t,\mathbf{x})=\frac{(-1)^n t^{2n\alpha}}{(2\pi)^d 2^n n!}\int e^{i\mathbf{x}\cdot\mathbf{k}}\hat q_\kappa(t,\mathbf{k})\left(\mathbf{D}(\mathbf{v})\mathbf{k},\mathbf{k}\right)^n\hat u_0(\mathbf{k})d\mathbf{k}, \tag{3.6}$$

We substitute this expression for $\bar I_{2n}$ into (3.4):

$$\begin{aligned}\lim_{\delta\downarrow 0}\mathbb{E}v_\delta(t,\mathbf{x})&=\frac{1}{(2\pi)^d}\int e^{i\mathbf{k}\cdot\mathbf{x}}\hat q_\kappa(t,\mathbf{k})\exp\left\{-\frac12(\mathbf{D}(\mathbf{v})\mathbf{k},\mathbf{k})t^{2\alpha}\right\}\hat u_0(\mathbf{k})d\mathbf{k}\\&=\mathbb{E}\left[\frac{1}{(2\pi)^d}\int\hat q_\kappa(t,\mathbf{k})\exp\left\{i\mathbf{k}\cdot\left(\mathbf{x}+\mathbf{D}^{1/2}(\mathbf{v})B^{(\alpha)}(t)\right)\right\}\hat u_0(\mathbf{k})d\mathbf{k}\right]\\&=\mathbb{E}\left[v(t,\mathbf{x}+\mathbf{D}^{1/2}(\mathbf{v})B^{(\alpha)}(t))\right],\end{aligned}$$

where $v(t,\mathbf{x})$ is the mild solution to (1.11) and the conclusion of Theorem 1.1 follows. □

The rest of the paper is devoted to the demonstrations of Propositions 3.1 and 3.2.

3.2 Proof of Proposition 3.2

Step 1. Passing to the limit $\delta\downarrow 0$. We will first compute the expectation of $q_{2n}^{(\delta)}(t,\mathbf{x},\mathbf{y})$ in terms of the correlation function $R(\mathbf{x})$ as a sum over the Feynman diagrams. We will then pass to the limit $\delta\downarrow 0$ with the help of the uniform integrability, see Lemma 3.3 below and Proposition 2.1. This leads to expressions (3.11)-(3.12) below for the limit of $\mathbb{E}q_{2n}^{(\delta)}(t,\mathbf{x},\mathbf{y})$.

Let $\Delta_n(t):=\Delta_n(t,0)$. With this notation we have

$$\begin{aligned}\mathbb{E}q_{2n}^{(\delta)}(t,\mathbf{x},\mathbf{y})=&\sum_{i_1,\ldots,i_{2n}=1}^{d}\delta^{2(1-2\beta)n}\int_{\Delta_{2n}(t)}\!\!\ldots\!\int\int_{(\mathbb{R}^d)^{2n}}\!\!\ldots\!\int\prod_{r=1}^{2n}\partial_{i_r}q_\kappa(s_{r-1}-s_r,\mathbf{y}_{r-1}-\mathbf{y}_r)\\&\times\mathbb{E}\left[\prod_{p=1}^{2n}G_{i_p}\left(\frac{s_p}{\delta^{2\beta}},\mathbf{y}_p\right)\right]q_\kappa(s_{2n},\mathbf{y}_{2n}-\mathbf{y}_{2n+1})ds^{(2n)}d\mathbf{y}^{(2n)}.\end{aligned} \tag{3.7}$$

Here, by convention, $\mathbf{y}_0 := \mathbf{x}$, $s_0 := t$ and $\mathbf{y}^{(2n+1)} = \mathbf{y}$. To compute the expectation of the product of Gaussians we shall use the Feynman diagrams. Suppose that we are given the set of vertices $\mathbb{Z}_{2n} := \{1, \ldots, 2n\}$. A graph $\mathcal{F}$ consisting of edges made of all the vertices belonging to $\mathbb{Z}_{2n}$ and such that no two edges have a common vertex is called *a Feynman diagram*. If p, q, with $p < q$, are the vertices of a given edge we write $(p, q) \in \mathcal{F}$. We denote by $\mathfrak{F}(n)$ the family of all Feynman diagrams made of the vertices of $\mathbb{Z}_{2n}$.

Using the above notation we can write that

$$\mathbb{E}q_{2n}^{(\delta)}(t, \mathbf{x}, \mathbf{y}) = \sum_{i_1,\ldots,i_{2n}=1}^{d} \delta^{2(1-2\beta)n} \int_{\Delta_{2n}(t)} \!\!\!\cdots\int \int_{(\mathbb{R}^d)^{2n}} \!\!\!\cdots\int \prod_{r=1}^{2n} \partial_{i_r} q_\kappa(s_{r-1} - s_r, \mathbf{y}_{r-1} - \mathbf{y}_r) \tag{3.8}$$

$$\times \mathbb{E}\left[\prod_{p=1}^{2n} G_{i_p}\left(\frac{s_p}{\delta^{2\beta}}, \mathbf{y}_p\right)\right] q_\kappa(s_{2n}, \mathbf{y}_{2n} - \mathbf{y}_{2n+1}) ds^{(2n)} d\mathbf{y}^{(2n)}$$

$$= \delta^{4(\alpha-1)\beta n} \sum_{i_1,\ldots,i_{2n}=1}^{d} \sum_{\mathcal{F}\in\mathfrak{F}(n)} \int_{\Delta_{2n}(t)} \!\!\!\cdots\int \int_{(\mathbb{R}^d)^{2n}} \!\!\!\cdots\int \prod_{r=1}^{2n} \partial_{i_r} q_\kappa(s_{r-1} - s_r, \mathbf{y}_{r-1} - \mathbf{y}_r)$$

$$\times \prod_{(p,q)\in\mathcal{F}} R_{i_p,i_q}\left(\mathbf{v}\frac{s_q - s_p}{\delta^{2\beta}} + \mathbf{y}_q - \mathbf{y}_p\right) q_\kappa(s_{2n}, \mathbf{y}_{2n} - \mathbf{y}_{2n+1}) ds^{(2n)} d\mathbf{y}^{(2n)}.$$

In the last equality we have used the fact that $4(1 - \alpha)\beta = 2(2\beta - 1)$. Proposition 2.1 implies that pointwise we have

$$\lim_{\delta\downarrow 0} \delta^{4(\alpha-1)\beta} R_{i_p,i_q}\left(\mathbf{v}\frac{s_q - s_p}{\delta^{2\beta}} + \mathbf{y}_q - \mathbf{y}_p\right) = |s_q - s_p|^{-2(1-\alpha)} D_{i_p i_q}(\mathbf{v}). \tag{3.9}$$

In order to use this pointwise convergence result inside the integral in (3.8) we need the following uniform integrability estimate.

Lemma 3.3 *For any n fixed and a given Feynman diagram $\mathcal{F} \in \mathfrak{F}(n)$ there exits $\nu > 0$ such that*

$$\limsup_{\delta\to 0+} \delta^{4(\alpha-1)\beta n} \left\{ \int_{\Delta_{2n}(t)} \!\!\!\cdots\int \int_{(\mathbb{R}^d)^{2n}} \!\!\!\cdots\int \left|\prod_{r=1}^{2n} \partial_{i_r} q_\kappa(s_{r-1} - s_r, \mathbf{y}_{r-1} - \mathbf{y}_r) q_\kappa(s_{2n}, \mathbf{y}_{2n} - \mathbf{y}_{2n+1})\right| \right.$$

$$\left. \times \left|\prod_{(p,q)\in\mathcal{F}} R_{i_p,i_q}\left(\mathbf{v}\frac{s_q - s_p}{\delta^{2\beta}} + \mathbf{y}_q - \mathbf{y}_p\right)\right|^{1+\nu} ds^{(2n)} d\mathbf{y}^{(2n)} \right\}^{1/(1+\nu)} < +\infty. \tag{3.10}$$

We shall postpone for a moment the proof of the lemma and use it first in order to calculate the limit $\lim_{\delta\downarrow 0} \mathbb{E}\int q_{2n}^{(\delta)}(t, \mathbf{x}, \mathbf{y}) u_0(\mathbf{y}) d\mathbf{y}$. Applying Lemma 3.3 and Proposition 2.1, as in (3.9), we obtain

$$\lim_{\delta\downarrow 0} \mathbb{E}\int q_{2n}^{(\delta)}(t, \mathbf{x}, \mathbf{y}) u_0(\mathbf{y}) d\mathbf{y} = \int \bar{q}_{2n}(t, \mathbf{x}, \mathbf{y}) u_0(\mathbf{y}) d\mathbf{y}, \tag{3.11}$$

where

$$\bar{q}_{2n}(t, \mathbf{x}, \mathbf{y}) = \sum_{i_1,\ldots,i_{2n}=1}^{d} \sum_{\mathcal{F}\in\mathfrak{F}(n)} \int_{\Delta_{2n}(t)} \!\!\!\cdots\int \int_{(\mathbb{R}^d)^{2n}} \!\!\!\cdots\int \prod_{r=1}^{2n} \partial_{i_r} q_\kappa(s_{r-1} - s_r, \mathbf{y}_{r-1} - \mathbf{y}_r) \tag{3.12}$$

$$\times q_\kappa(s_{2n}, \mathbf{y}_{2n} - \mathbf{y}_{2n+1}) \prod_{(p,q)\in\mathcal{F}} D_{i_p,i_q}(\mathbf{v})(s_q - s_p)^{-2(1-\alpha)} ds^{(2n)} d\mathbf{y}^{(2n)}.$$

Step 2. Identification of the limit in terms of a fractional Brownian motion. We will now transform the expression for $\bar{q}_{2n}(t,\mathbf{x},\mathbf{y})$, leading to (3.19) below and thus finish the proof of Proposition 3.2. After performing $2n$-times integration by parts we obtain

$$
\begin{aligned}
\int\limits_{\mathbb{R}^d} \bar{q}_{2n}(t,\mathbf{x},\mathbf{y})u_0(\mathbf{y})d\mathbf{y} &= \sum_{i_1,\ldots,i_{2n}=1}^{d} \int\limits_{\Delta_{2n}(t)}\!\!\!\ldots\!\!\int \int\limits_{(\mathbb{R}^d)^{2n+1}}\!\!\!\ldots\!\!\int \prod_{r=1}^{2n+1} q_\kappa(s_{r-1}-s_r,\mathbf{y}_{r-1}-\mathbf{y}_r) \\
&\times \prod_{(p,q)\in\mathcal{F}} [D_{i_p,i_q}(\mathbf{v})(s_q-s_p)^{-2(1-\alpha)}]\partial^{2n}_{i_1,\ldots,i_{2n}} u_0(\mathbf{y}_{2n+1}) ds^{(2n)} d\mathbf{y}^{(2n+1)} \\
&= \sum_{i_1,\ldots,i_{2n}=1}^{d} \int\limits_{\Delta_{2n}(t)}\!\!\!\ldots\!\!\int \int q_\kappa(t,\mathbf{x}-\mathbf{y}) \prod_{(p,q)\in\mathcal{F}} [D_{i_p,i_q}(\mathbf{v})(s_q-s_p)^{-2(1-\alpha)}]\partial^{2n}_{i_1,\ldots,i_{2n}} u_0(\mathbf{y}) ds^{(2n)} d\mathbf{y}.
\end{aligned}
$$

The last equality follows from the semi-group property of the kernels:

$$
\int q_\kappa(t,\mathbf{x}-\mathbf{z})q_\kappa(s,\mathbf{z}-\mathbf{y})d\mathbf{z} = q_\kappa(t+s,\mathbf{x}-\mathbf{y}), \quad \forall\, \mathbf{x},\, \mathbf{y}\in\mathbb{R}^d,\ t,s\ge 0.
$$

Note that the function

$$
f(s_1,\ldots,s_{2n}) := \sum_{i_1,\ldots,i_{2n}=1}^{d} \prod_{(p,q)\in\mathcal{F}} [D_{i_p,i_q}(\mathbf{v})|s_q-s_p|^{-2(1-\alpha)}]\partial^{2n}_{i_1,\ldots,i_{2n}} u_0(\mathbf{y}_{2n+1})
$$

is symmetric in all its arguments, that is, $f(s_1,\ldots,s_{2n}) = f(s_{\pi(1)},\ldots,s_{\pi(2n)})$ where π is an arbitrary permutation of $\{1,\ldots,2n\}$. Using this fact we can replace integration over the simplex by integration over a $2n$-dimensional cube $[0,t]^{2n}$ and rewrite the above integral as

$$
\begin{aligned}
\int\limits_{\mathbb{R}^d} \bar{q}_{2n}(t,\mathbf{x},\mathbf{y})u_0(\mathbf{y})d\mathbf{y} &= \frac{1}{(2n)!} \sum_{i_1,\ldots,i_{2n}=1}^{d} \int_0^t\!\!\ldots\!\int_0^t \int q_\kappa(t,\mathbf{x}-\mathbf{y}) \\
&\times \prod_{(p,q)\in\mathcal{F}} [D_{i_p,i_q}(\mathbf{v})|s_q-s_p|^{-2(1-\alpha)}]\partial^{2n}_{i_1,\ldots,i_{2n}} u_0(\mathbf{y}) ds^{(2n)} d\mathbf{y}. \qquad (3.13)
\end{aligned}
$$

We will now re-write the integral appearing above to bring about the expectation with respect to an appropriate fractional Brownian motion. Let $w(dk)$ be an $\mathbb{R}^d$-valued Gaussian noise with the covariance matrix

$$
\mathbb{E}[w_i(dk)w_j(dk')] = \delta_{ij}\delta(k+k')dkdk'.
$$

We have then

$$
\begin{aligned}
&\lim_{R\to+\infty} \mathbb{E}\left[\int_{-R}^{R} \frac{\mathrm{e}^{ik_1 s}}{|k_1|^{\alpha-1/2}} w_i(dk_1) \int_{-R}^{R} \frac{\mathrm{e}^{ik_2 r}}{|k_2|^{\alpha-1/2}} w_j(dk_2)\right] \qquad (3.14) \\
&= \delta_{ij} \int_{-\infty}^{\infty} \frac{\mathrm{e}^{ik_1(s-r)}}{|k_1|^{2\alpha-1}} dk_1 = c_\alpha^2 \delta_{ij}|s-r|^{-2(1-\alpha)},
\end{aligned}
$$

where $c_\alpha > 0$ is defined by

$$
c_\alpha^2 := \int_{-\infty}^{\infty} \frac{e^{ik}dk}{|k|^{2\alpha-1}} = 2\int_0^{+\infty} \frac{\cos k dk}{k^{2\alpha-1}} = \frac{\pi}{\Gamma(2\alpha-1)\sin(\pi\alpha)}. \qquad (3.15)
$$

The last equality follows, for instance, from formula 3), in paragraph [539] of [3]. We can now replace $|s_q - s_p|^{-2(1-\alpha)}$ in (3.13) using the above expression and Gaussianity of $w(dk)$, and obtain,

$$\begin{aligned}\int\limits_{\mathbb{R}^d} \bar{q}_{2n}(t,\mathbf{x},\mathbf{y})u_0(\mathbf{y})d\mathbf{y} &= \frac{c_\alpha^{-2n}}{(2n)!}\sum_{i_1,\ldots,i_{2n}=1}^{d}\int\limits_0^t\ldots\int\limits_0^t\int\limits_{\mathbb{R}^d} q_\kappa(t,\mathbf{x}-\mathbf{y}) \qquad (3.16)\\ &\times \lim_{R\to+\infty}\mathbb{E}\left\{\prod_{p=1}^{2n}\left[\int_{-R}^{R}\frac{\mathrm{e}^{ik_ps_p}\left[D^{1/2}(\mathbf{v})w(dk_p)\right]_{i_p}}{|k_p|^{\alpha-1/2}}\right]\right\}\partial^{2n}_{i_1,\ldots,i_{2n}}u_0(\mathbf{y})ds^{(2n)}d\mathbf{y}\\ &= \frac{c_\alpha^{-2n}}{(2n)!}\sum_{i_1,\ldots,i_{2n}=1}^{d}\lim_{R\to+\infty}\int\limits_0^t\ldots\int\limits_0^t\int\limits_{\mathbb{R}^d} q_\kappa(t,\mathbf{x}-\mathbf{y})\\ &\times \mathbb{E}\left\{\prod_{p=1}^{2n}\left[\int_{-R}^{R}\frac{\mathrm{e}^{ik_ps_p}\left[D^{1/2}(\mathbf{v})w(dk_p)\right]_{i_p}}{|k_p|^{\alpha-1/2}}\right]\right\}\partial^{2n}_{i_1,\ldots,i_{2n}}u_0(\mathbf{y})ds^{(2n)}d\mathbf{y}.\end{aligned}$$

The possibility of interchanging the limit with integration, used in the last equality, can be substantiated as follows. With the help of Feynman diagrams the problem of estimating the integrand can be reduced to the problem of estimating

$$\left|\prod_{(p,q)\in\mathcal{F}}\mathbb{E}\left[\int_{-R}^{R}\frac{\mathrm{e}^{ik_ps_p}\left[D^{1/2}(\mathbf{v})w(dk_p)\right]_{i_p}}{|k_p|^{\alpha-1/2}}\int_{-R}^{R}\frac{\mathrm{e}^{ik_qs_q}\left[D^{1/2}(\mathbf{v})w(dk_q)\right]_{i_q}}{|k_q|^{\alpha-1/2}}\right]\right|$$

for all diagrams $\mathcal{F}$. The above expression equals however

$$\left|\prod_{(p,q)\in\mathcal{F}}D_{i_pi_q}(\mathbf{v})\int_{-R}^{R}\frac{\mathrm{e}^{ik_p(s_p-s_q)}dk_p}{|k_p|^{2\alpha-1}}\right|\le C\prod_{(p,q)\in\mathcal{F}}\frac{1}{|s_p-s_q|^{2(1-\alpha)}}, \tag{3.17}$$

where $C>0$ can be chosen independently of $R>0$. The right hand side of (3.17) is weakly singular because $\alpha>1/2$ and the last equality in (3.16) follows from the dominated convergence theorem.

Note that

$$\lim_{R\to+\infty}\int_0^t\left(\int_{-R}^{R}\frac{\mathrm{e}^{ik_ps_p}D^{1/2}(\mathbf{v})w(dk_p)}{|k_p|^{\alpha-1/2}}\right)ds_p = c_\alpha^2 D^{1/2}(\mathbf{v})B^{(\alpha)}(t), \tag{3.18}$$

where

$$B^{(\alpha)}(t) := c_\alpha^{-2}\int\limits_{-\infty}^{+\infty}\frac{\mathrm{e}^{ik_pt}-1}{ik_p|k_p|^{\alpha-1/2}}w(dk_p)$$

is a d-dimensional, standard fractional Brownian motion with the Hurst exponent α. The limit appearing on the left hand side of (3.18) is understood in the $L^2(\mathbb{P})$ sense. We obtain therefore

$$\int\limits_{\mathbb{R}^d}\bar{q}_{2n}(t,\mathbf{x},\mathbf{y})u_0(\mathbf{y})d\mathbf{y} = \frac{1}{(2n)!}\mathbb{E}\left[\int q_\kappa(t,\mathbf{x}-\mathbf{y})\left(\mathbf{D}^{1/2}(\mathbf{v})B^{(\alpha)}(t)\cdot\nabla_{\mathbf{y}}\right)^{2n}u_0(\mathbf{y})d\mathbf{y}\right], \tag{3.19}$$

which is nothing but (3.3) and the proof of Proposition 3.2 is complete, except for the proof of Lemma 3.3, which we postpone until Section 3.4. □

3.3 Proof of Proposition 3.1

An auxiliary lemma. Let $q_+(x) := |xq(x)|$, $x \in \mathbb{R}$. The following lemma is crucial in both the forthcoming proof of Lemma 3.3, and the proof of Proposition 3.1 presented in this section.

Lemma 3.4 *Suppose that $t > 0$, $\alpha \in (1/2, 1)$, $\mu > 1$ and $\varrho \in (0, 2\alpha - 1 - 1/(2\mu))$ are fixed. Then, one can find $\nu > 0$ so that*

$$2\alpha - 1 - \frac{1}{2\mu} - 2\nu(1-\alpha) > 0 \tag{3.20}$$

and for any $\nu' \in [0, \nu]$ there exists $C > 0$ such that for all $n, k \geq 0$, $\delta \in (0,1)$ we have

$$\sup_{\mathbf{y}} \int_{\mathbb{R}^d} q_+(|\mathbf{z}|)d\mathbf{z} \left\{ \int_0^\tau s^{-1/(2\mu)}(\tau - s)^{n\epsilon + k\epsilon_1} \left[\delta^{2(2\beta-1)} + |\mathbf{v}s - \delta^{2\beta}s^{1/(2\mu)}\mathbf{z} + \mathbf{y}|^{2(1-\alpha)}\right]^{-(1+\nu')} ds \right\}$$
$$\leq \frac{C\tau^{(n+1)\epsilon + k\epsilon_1}}{(k+n+1)^{\epsilon - \varrho}}, \quad \forall \tau \in [0, t]. \tag{3.21}$$

Here $\epsilon := 2\alpha - 1 - 1/(2\mu) - 2\nu'(1-\alpha)$ and $\epsilon_1 := 1 - 1/(2\mu) > 0$.

We present the proof of the lemma in Section 4. Let us apply first the result in order to show Proposition 3.1. In this case we take $\nu' = 0$.

Reduction to one Feynman diagram. We now rewrite the expression for $E \int q_{2n}^{(\delta)}(t, \mathbf{x}, \mathbf{y})$ reducing the problem to a bound for one Feynman diagram. Using Proposition 2.3, Corollary 2.4 and the rules of computing the product of $2n$ Gaussian variables we can estimate the left hand side of (3.7) as follows

$$\left|\mathbb{E} \int q_{2n}^{(\delta)}(t, \mathbf{x}, \mathbf{y}) u_0(\mathbf{y}) d\mathbf{y}\right| \leq \delta^{(2-4\beta)n} \sum_{i_1, \ldots, i_{2n}=1}^{d} \sum_{\mathcal{F} \in \mathfrak{F}(n)} \int_{\Delta_{2n}(t)} \int_{(\mathbb{R}^d)^{2n+1}} |u_0(\mathbf{y}_{2n+1})| \prod_{r=1}^{2n} (s_{r-1} - s_r)^{-\frac{1}{2\mu}}$$
$$\times \left| \prod_{(p,q) \in \mathcal{F}} R_{i_p, i_q}\left(\mathbf{v}\frac{(s_q - s_p)}{\delta^{2\beta}} + \mathbf{y}_q - \mathbf{y}_p\right)\right| \prod_{l=1}^{2n+1} (s_{l-1} - s_l)^{-\frac{d}{2\mu}} q_+\left(\frac{|\mathbf{y}_{l-1} - \mathbf{y}_l|}{(s_{l-1} - s_l)^{\frac{1}{2\mu}}}\right) ds^{(2n)} d\mathbf{y}^{(2n+1)}.$$

Recall that

$$|R(\mathbf{x})| \leq \frac{C}{1 + |\mathbf{x}|^{2(1-\alpha)}} \tag{3.22}$$

for some $C > 0$ and all $\mathbf{x} \in \mathbb{R}^d$. Using the above and the relation $4(1-\alpha)\beta = 2(2\beta - 1)$ we can further write that

$$\left|\mathbb{E} \int q_{2n}^{(\delta)}(t, \mathbf{x}, \mathbf{y}) u_0(\mathbf{y}) d\mathbf{y}\right| \leq C^n \sum_{i_1, \ldots, i_{2n}=1}^{d} \sum_{\mathcal{F} \in \mathfrak{F}(n)} \int_{\Delta_{2n}(t)} \int_{(\mathbb{R}^d)^{2n+1}} \prod_{l=1}^{2n} (s_{l-1} - s_l)^{-1/(2\mu)}$$
$$\times \prod_{(p,q) \in \mathcal{F}} \left[\delta^{2(2\beta-1)} + |(s_q - s_p)\mathbf{v} + \delta^{2\beta}(\mathbf{y}_q - \mathbf{y}_p)|^{2(1-\alpha)}\right]^{-1}$$
$$\times \prod_{r=1}^{2n+1} (s_{r-1} - s_r)^{-\frac{d}{2\mu}} q_+\left(\frac{|\mathbf{y}_{r-1} - \mathbf{y}_r|}{(s_{r-1} - s_r)^{1/(2\mu)}}\right) |u_0(\mathbf{y}_{2n+1})| ds^{(2n)} d\mathbf{y}^{(2n+1)}.$$

Changing variables $\mathbf{z}_p := (s_{p-1}-s_p)^{-1/(2\mu)}(\mathbf{y}_{p-1}-\mathbf{y}_p)$, $p=1,\ldots,2n+1$, with the convention $s_0=t$, $s_{2n+1}=0$, we obtain

$$|\mathbb{E}\int q_{2n}^{(\delta)}(t,\mathbf{x},\mathbf{y})u_0(\mathbf{y})d\mathbf{y}| \le C^n\|u_0\|_\infty \int q_+(|\mathbf{z}_{2n+1}|)d\mathbf{z}_{2n+1}\sum_{\mathcal{F}\in\mathfrak{F}(n)} J(\mathcal{F}), \tag{3.23}$$

where

$$J(\mathcal{F}) := \underset{\Delta_{2n}(t)}{\int\ldots\int}\ \underset{(\mathbb{R}^d)^{2n+1}}{\int\ldots\int}\prod_{l=1}^{2n}\left[q_+(|\mathbf{z}_l|)(s_{l-1}-s_l)^{-1/(2\mu)}\right]$$
$$\times \prod_{(p,q)\in\mathcal{F}}\left[\delta^{2(2\beta-1)}+\left|(s_q-s_p)\mathbf{v}-\delta^{2\beta}\sum_{r=p+1}^{q}\mathbf{z}_r(s_{r-1}-s_r)^{1/(2\mu)}\right|^{2(1-\alpha)}\right]^{-1} ds^{(2n)}d\mathbf{z}^{(2n+1)}.$$

An estimate for $J(\mathcal{F})$. The key step in the proof of Proposition 3.1 is the following estimate for $J(\mathcal{F})$ for a given Feynman diagram $\mathcal{F}$.

Lemma 3.5 *For any $\varrho>0$ there exists a constant C independent of $\mathcal{F}$ and n such that*

$$J(\mathcal{F}) \le \frac{C^n t^{n(2\alpha-1/\mu)}}{[(2n)!]^{\epsilon_1}}\left[(2n-1)!!\right]^{2(1-\alpha)+\varrho} \tag{3.24}$$

End of proof of Proposition 3.1. Now, Proposition 3.1 is a simple consequence of Lemma 3.5. Combining (3.23) with (3.24) we obtain that the left hand side of (3.23) can be estimated by

$$\frac{C^n t^{n(\epsilon+1)}}{[(2n)!]^{\epsilon_1}}(2n-1)!!\left[(2n-1)!!\right]^{2(1-\alpha)+\varrho}\|u_0\|_\infty \le \frac{C^n t^{n(\epsilon+1)}}{(n!)^{2\alpha-1-1/\mu-\varrho}}\|u_0\|_\infty.$$

The last estimate follows from the fact that $(2n)!\sim(n!)^2$, $(2n-1)!!\sim n!$ for $n\gg 1$. $\Box$

Proof of Lemma 3.5. Step 1: removal of one edge. Let us fix a Feynman diagram $\mathcal{F}$ and suppose that $e:=(p_n,2n)\in\mathcal{F}$. We can estimate

$$J(\mathcal{F}) \le \underset{\Delta_{2n-1}(t)}{\int\ldots\int}\ \underset{(\mathbb{R}^d)^{2n-1}}{\int\ldots\int}\prod_{l=1}^{2n-1}\left[q_+(|\mathbf{z}_l|)(s_{l-1}-s_l)^{-1/(2\mu)}\right]Q(s_{2n-1}) \tag{3.25}$$
$$\times \prod_{(p,q)\in\mathcal{F}\setminus e}\left[\delta^{2(2\beta-1)}+\left|(s_q-s_p)\mathbf{v}-\delta^{2\beta}\sum_{r=p+1}^{q}\mathbf{z}_r(s_{r-1}-s_r)^{1/(2\mu)}\right|^{2(1-\alpha)}\right]^{-1} ds^{(2n-1)}d\mathbf{z}^{(2n-1)},$$

where

$$Q(s_{2n-1}) := \sup_{\mathbf{y}}\int_0^{s_{2n-1}}\int q_+(|\mathbf{z}|)s^{-1/(2\mu)}\left[\delta^{2\beta-1}+|\mathbf{v}s-\delta^{2\beta}\mathbf{z}s^{1/(2\mu)}+\mathbf{y}|^{2(1-\alpha)}\right]^{-1}dsd\mathbf{z} \le Cs_{2n-1}^{\epsilon},$$

by virtue of Lemma 3.4.

There are now two possibilities: either $p_n = 2n-1$, or not. In the first case, integrating out $\int q_+\left(|\mathbf{z}_{2n-1}|\right) d\mathbf{z}_{2n-1}$, we obtain the estimate

$$
\begin{aligned}
J(\mathcal{F}) \leq C^2 & \int \cdots \int_{\Delta_{2n-2}(t)} \int \cdots \int_{(\mathbb{R}^d)^{2(n-1)}} \prod_{p=1}^{2n-2} \left[q_+\left(|\mathbf{z}_p|\right) (s_{p-1} - s_p)^{-1/(2\mu)} \right] \\
& \times \prod_{(p,q)\in\mathcal{F}\setminus e} \left[\delta^{2(2\beta-1)} + \left| (s_q - s_p)\mathbf{v} - \delta^{2\beta} \sum_{r=p+1}^{q} \mathbf{z}_r (s_{r-1} - s_r)^{-1/(2\mu)} \right|^{2(1-\alpha)} \right]^{-1} \\
& \times \left[\int_0^{s_{2n-2}} s^{\epsilon} (s_{2n-2} - s)^{-1/(2\mu)} ds \right] ds^{(2n-2)} d\mathbf{z}^{(2n-2)}.
\end{aligned}
$$

Evaluating the last integral we obtain

$$
\begin{aligned}
J(\mathcal{F}) \leq C^2 B\left(\epsilon+1, \epsilon_1\right) & \int \cdots \int_{\Delta_{2n-2}(t)} \int \cdots \int_{(\mathbb{R}^d)^{2(n-1)}} \prod_{l=1}^{2n-2} \left[q_+\left(|\mathbf{z}_l|\right) (s_{l-1} - s_l)^{-1/(2\mu)} \right] \\
\times \prod_{(p,q)\in\mathcal{F}\setminus e} & \left[\delta^{2(2\beta-1)} + \left| (s_q - s_p)\mathbf{v} - \delta^{2\beta} \sum_{r=p+1}^{q} \mathbf{z}_r (s_{r-1} - s_r)^{-1/(2\mu)} \right|^{2(1-\alpha)} \right]^{-1} s_{2n-2}^{\epsilon+\epsilon_1} ds^{(2n-2)} d\mathbf{z}^{(2n-2)}.
\end{aligned}
\tag{3.26}
$$

Here $B(\cdot,\cdot)$ denotes Euler's beta function. When, on the other hand $p_n \neq 2n-1$ then $2n-1$ must be a right vertex of a certain edge $e' := (p_{n-1}, 2n-1)$. We can repeat now the same estimate that lead to (3.25) and obtain that

$$
\begin{aligned}
J(\mathcal{F}) \leq C & \int \cdots \int_{\Delta_{2n-1}(t)} \int \cdots \int_{(\mathbb{R}^d)^{2n-1}} \prod_{l=1}^{2n-1} \left[q_+\left(|\mathbf{z}_l|\right) (s_{l-1} - s_l)^{-1/(2\mu)} \right] s_{2n-1}^{\epsilon} \\
\times \prod_{(p,q)\in\mathcal{F}\setminus e} & \left[\delta^{2(2\beta-1)} + \left| (s_q - s_p)\mathbf{v} - \delta^{2\beta} \sum_{r=p+1}^{q} \mathbf{z}_r (s_{r-1} - s_r)^{1/(2\mu)} \right|^{2(1-\alpha)} \right]^{-1} ds^{(2n-1)} d\mathbf{z}^{(2n-1)}.
\end{aligned}
\tag{3.27}
$$

Step 2. Removing finitely many edges. We formulate the following result that generalizes (3.26) and (3.27).

Proposition 3.6 *Suppose that $\mathcal{F}$ is a Feynman diagram with the right vertices $2n+1-k_1, \ldots, 2n+1-k_n$, where $1 = k_1 < \ldots < k_n$. Then for any $\varrho > 0$ and $r = 1, \ldots, n$ there exists a constant $C > 0$ independent of n and $\mathcal{F}$ such that*

$$
\begin{aligned}
J(\mathcal{F}) \leq & \frac{C^{k_{r+1}-1}}{[(k_{r+1}-1)!]^{\epsilon_1}} \left(\prod_{l=1}^{r} k_l \right)^{2(1-\alpha)+\varrho} \\
\times & \int \cdots \int_{\Delta_{2n+1-k_{r+1}}(t)} \int \cdots \int_{(\mathbb{R}^d)^{2n+1-k_{r+1}}} \prod_{p=1}^{2n+1-k_{r+1}} \left[q_+\left(|\mathbf{z}_p|\right) (s_{p-1} - s_p)^{-1/(2\mu)} \right] s_{2n+1-k_{r+1}}^{r\epsilon + (k_{r+1}-r-1)\epsilon_1} \\
\times \prod_{(p,q)\in\mathcal{F}_r} & \left[\delta^{2(2\beta-1)} + \left| (s_q - s_p)\mathbf{v} - \delta^{2\beta} \sum_{r=p+1}^{q} \mathbf{z}_r (s_{r-1} - s_r)^{-1/(2\mu)} \right|^{2(1-\alpha)} \right]^{-1} ds^{(2n+1-k_{r+1})} d\mathbf{z}^{(2n+1-k_{r+1})}.
\end{aligned}
\tag{3.28}
$$

Here, by convention we denote $k_{n+1} := 2n+1$, $s_0 := t$ and $\mathcal{F}_r$ is the set of edges that remains after subtracting from $\mathcal{F}$ all the edges that have right vertices $2n+1-k_i$, $i = 1, \ldots, r$.

Proof. We establish the proposition by induction on r. The proof for $r = 1$ has already been carried out above in (3.26) and (3.27). Suppose that estimate (3.28) holds for a certain r and let $e := (p_r, 2n+1-k_{r+1}) \in \mathcal{F}_r$. We can bound the expression appearing on the right hand side of (3.28) by

$$\frac{C^{k_{r+1}-1}}{[(k_{r+1}-1)!]^{\epsilon_1}} \left(\prod_{l=1}^{r} k_l\right)^{2(1-\alpha)+\varrho} \int\limits_{\Delta_{2n-k_{r+1}}(t)} \!\!\cdots\!\! \int \int\limits_{(\mathbb{R}^d)^{2n-k_{r+1}}} \!\!\cdots\!\! \int \prod_{j=1}^{2n-k_{r+1}} \left[q_+(|\mathbf{z}_j|)(s_{j-1}-s_j)^{-1/(2\mu)}\right]$$
$$\times \prod_{(p,q)\in\mathcal{F}_r\setminus\{e\}} \left[\delta^{2(2\beta-1)} + \left|(s_q - s_p)\mathbf{v} - \delta^{2\beta} \sum_{r=p+1}^{q} \mathbf{z}_r (s_{r-1}-s_r)^{-1/(2\mu)}\right|^{2(1-\alpha)}\right]^{-1} \tag{3.29}$$
$$\times \tilde{Q}(s_{2n-k_{r+1}}) ds^{(2n+1-k_{r+1})} d\mathbf{z}^{(2n+2-k_{r+1})},$$

where

$$\tilde{Q}(s_{2n-k_{r+1}}) := \sup_{\mathbf{y}} \int q_+(|\mathbf{z}|) d\mathbf{z} \left\{ \int_0^{s_{2n-k_{r+1}}} s^{-1/(2\mu)} (s_{2n-k_{r+1}} - s)^{r\epsilon + (k_{r+1}-r-1)\epsilon_1}\right.$$
$$\left. \times \left[\delta^{2(2\beta-1)} + |\mathbf{v}s - \delta^{2\beta}\mathbf{z}s^{1/(2\mu)} + \mathbf{y}|^{2(1-\alpha)}\right]^{-1} ds \right\} \overset{\text{Lemma 3.4}}{\leq} \frac{C}{k_{r+1}^{\epsilon-\varrho}} s_{2n-k_{r+1}}^{(r+1)\epsilon+(k_{r+1}-r-1)\epsilon_1}.$$

The numbers from $2n - k_{r+1}$ to $2n + 2 - k_{r+2}$ are left vertices that are not represented in the graph $\mathcal{F}_{r+1} = \mathcal{F}_r \setminus \{e\}$. We can estimate therefore (3.29) by

$$\frac{C^{k_{r+1}-1}}{[(k_{r+1}-1)!]^{\epsilon_1} k_{r+1}^{\epsilon-\varrho}} \left(\prod_{l=1}^{r} k_l\right)^{2(1-\alpha)+\varrho} \prod_{j=1}^{k_{r+2}-k_{r+1}-1} B(\epsilon_1, (r+1)\epsilon + (k_{r+1}-r-1+j)\epsilon_1 + 1)$$
$$\int\limits_{\Delta_{2n+1-k_{r+2}}(t)} \!\!\cdots\!\! \int \int\limits_{(\mathbb{R}^d)^{2n+1-k_{r+2}}} \!\!\cdots\!\! \int \prod_{m=1}^{2n+1-k_{r+2}} \left[q_+(|\mathbf{z}_m|)(s_{m-1}-s_m)^{-1/(2\mu)}\right]$$
$$\times \prod_{(p,q)\in\mathcal{F}_{r+1}} \left[\delta^{2(2\beta-1)} + \left|(s_q - s_p)\mathbf{v} - \delta^{2\beta} \sum_{r=p+1}^{q} \mathbf{z}_r (s_{r-1}-s_r)^{-1/(2\mu)}\right|^{2(1-\alpha)}\right]^{-1} \tag{3.30}$$
$$\times s_{2n+1-k_{r+2}}^{(r+1)\epsilon+(k_{r+2}-r-2)\epsilon_1} ds^{(2n+1-k_{r+2})} d\mathbf{z}^{(2n+1-k_{r+2})}.$$

Using the well known formula $B(a,b) = \Gamma(a)\Gamma(b)\Gamma^{-1}(a+b)$ we can re-write the product of beta functions in (3.30) as

$$\prod_{j=1}^{k_{r+2}-k_{r+1}-1} \frac{[\Gamma(\epsilon_1)]^{k_{r+2}-k_{r+1}-1}\Gamma((r+1)\epsilon + (k_{r+1}-r)\epsilon_1 + 1)}{\Gamma((r+1)\epsilon + (k_{r+2}-r-1)\epsilon_1 + 1)}. \tag{3.31}$$

With the help of Stirling's formula

$$\Gamma(a) = \left(\frac{2\pi}{a}\right)^{1/2} \left(\frac{a}{e}\right)^a \exp\left\{\frac{\theta}{12a}\right\}$$

for any $a > 0$ and a suitably chosen $\theta \in (0,1)$ we conclude that for a fixed $b > 0$ there exists a constant $C > 0$ such that for all $x > 0$ and non-negative integer k we have

$$\frac{\Gamma(x+(k+1)b)}{\Gamma(x+kb)} \geq C(k+1)^b. \tag{3.32}$$

The above estimate allows us to bound (3.30) by

$$\begin{aligned} &\frac{C^{k_{r+2}}}{[(k_{r+1}-1)!]^{\epsilon_1} k_{r+1}^{\epsilon-\varrho}} \left(\prod_{l=1}^{r} k_l\right)^{2(1-\alpha)+\varrho} \left(\prod_{j=1}^{k_{r+2}-k_{r+1}-1} \frac{1}{k_{r+1}+j}\right)^{\epsilon_1} \\ &\int_{\Delta_{2n+1-k_{r+2}}(t)} \cdots \int \int_{(\mathbb{R}^d)^{2n+1-k_{r+2}}} \cdots \int \prod_{m=1}^{2n+1-k_{r+2}} \left[q_+\left(|\mathbf{z}_m|\right)(s_{m-1}-s_m)^{-1/(2\mu)}\right] \\ &\times \prod_{(p,q)\in\mathcal{F}_{r+1}} \left[\delta^{2(2\beta-1)} + \left|(s_q-s_p)\mathbf{v} - \delta^{2\beta}\sum_{r=p+1}^{q} \mathbf{z}_r (s_{r-1}-s_r)^{-1/(2\mu)}\right|^{2(1-\alpha)-1/(2\mu)}\right]^{-1} \\ &\times s_{2n+1-k_{r+2}}^{(r+1)\epsilon+(k_{r+2}-r-2)\epsilon_1} ds^{(2n+1-k_{r+2})} d\mathbf{z}^{(2n+1-k_{r+2})}. \end{aligned} \tag{3.33}$$

Equality $\epsilon_1 - \epsilon = 2(1-\alpha)$ concludes the induction argument. $\square$

In the particular case when $r = n$ we obtain the following.

Corollary 3.7 *For any $\varrho > 0$ there exists a constant C independent of $\mathcal{F}$ and n such that*

$$J(\mathcal{F}) \leq \frac{C^n t^{n(2\alpha-1/\mu)}}{[(2n)!]^{\epsilon_1}} \left(\prod_{p=1}^{n} k_p\right)^{2(1-\alpha)+\varrho} \tag{3.34}$$

All we need to finish the proof of Lemma 3.5 is the following.

Lemma 3.8 *For any $1 = k_1 < \ldots < k_n$ as in the statement of Proposition 3.6 we have*

$$n! \leq \prod_{p=1}^{n} k_p \leq (2n-1)!!. \tag{3.35}$$

Proof. The lower bound is obvious. Note that for each k_r there exists $2n \geq l_r > k_r$ such that $(2n+1-l_r, 2n+1-k_r) \in \mathcal{F}$. As a result we have $k_r + 1 \leq 2n$. Since the sequence k_r, $r = 1, \ldots, n$ is strictly increasing this implies that $k_r \leq 2r-1$ for all $r = 1, \ldots, n$. The upper bound is now obvious. $\square$

The proof of Lemma 3.5 is now complete. As a consequence Proposition 3.1 has also been shown except for the proof of Lemma 3.4.

3.4 The proof of Lemma 3.3

Another consequence of Lemma 3.4 is the proof of Lemma 3.3 which is the only remaining part in the proof of Proposition 3.2. Using (3.22) we conclude that (3.10) follows if we could prove that for a certain $\nu > 0$ we have

$$\begin{aligned} &\limsup_{\delta\downarrow 0} \int_{\Delta_{2n}(t)} \cdots \int \int_{(\mathbb{R}^d)^{2n+1}} \cdots \int \left|\prod_{r=1}^{2n} \partial_{i_r} q_\kappa(s_{r-1}-s_r, \mathbf{y}_{r-1}-\mathbf{y}_r) q_\kappa(s_{2n}, \mathbf{y}_{2n}-\mathbf{y}_{2n+1})\right| \\ &\times \prod_{(p,q)\in\mathcal{F}} \left[\delta^{2(2\beta-1)} + |(s_q-s_p)\mathbf{v} + \delta^{2\beta}(\mathbf{y}_q-\mathbf{y}_p)|^{2(1-\alpha)}\right]^{-(1+\nu)} ds^{(2n)} d\mathbf{y}^{(2n+1)} < +\infty. \end{aligned} \tag{3.36}$$

Proposition 2.3 and Corollary 2.4 allow us to estimate the expression under the limit in (3.36) by

$$C^n \int_{\Delta_{2n}(t)}\!\!\!\ldots\!\int \int_{(\mathbb{R}^d)^{2n+1}}\!\!\!\ldots\!\int \prod_{p=1}^{2n+1} (s_{p-1}-s_p)^{-d/(2\mu)} q_+\left(\frac{|\mathbf{y}_{p-1}-\mathbf{y}_p|}{(s_{p-1}-s_p)^{1/(2\mu)}}\right) \prod_{p=1}^{2n} (s_{p-1}-s_p)^{-1/(2\mu)}$$

$$\times \prod_{(p,q)\in\mathcal{F}} \left[\delta^{2(2\beta-1)} + |(s_q-s_p)\mathbf{v} + \delta^{2\beta}(\mathbf{y}_q-\mathbf{y}_p)|^{2(1-\alpha)}\right]^{-(1+\nu)} ds^{(2n)} d\mathbf{y}^{(2n+1)}. \tag{3.37}$$

Changing variables $\mathbf{z}_p := (s_{p-1}-s_p)^{-1/(2\mu)}(\mathbf{y}_{p-1}-\mathbf{y}_p)$, $p = 1,\ldots,2n+1$, where $s_{2n+1} = 0$, we can apply Lemma 3.4, this time with some $\nu > 0$, and proceed in the same way as it was done in Section 3.3. We shall obtain then that the expression in (3.37) can be estimated by the expression appearing on the right hand side of (3.34), which is independent of δ. The conclusion of the lemma then follows. □

4 The proof of Lemma 3.4

Let us recall that we need to show that

$$\sup_{\mathbf{y}} \int_{\mathbb{R}^d} q_+(|\mathbf{z}|)d\mathbf{z} \left\{ \int_0^\tau s^{-1/(2\mu)}(\tau-s)^{n\epsilon+k\epsilon_1} \left[\delta^{2(2\beta-1)} + |\mathbf{v}s - \delta^{2\beta}s^{1/(2\mu)}\mathbf{z} + \mathbf{y}|^{2(1-\alpha)}\right]^{-(1+\nu')} ds \right\}$$

$$\le \frac{C\tau^{(n+1)\epsilon+k\epsilon_1}}{(k+n+1)^{\epsilon-\varrho}}, \quad \forall \tau \in [0,t], \tag{4.1}$$

with $\epsilon := 2\alpha - 1 - 1/(2\mu) - 2\nu'(1-\alpha)$and $\epsilon_1 := 1 - 1/(2\mu) > 0$.

Denote by $I(\mathbf{y})$ the expression under the supremum on the left side of (4.1) and set

$$\gamma_0 := \frac{2(1-\alpha)}{\alpha(2\mu-1)}, \quad \gamma := \gamma_0 + \delta\gamma, \tag{4.2}$$

with $\delta\gamma > 0$. We split integration over $\mathbf{z}$ in (4.1) into the regions $[|\mathbf{z}| \ge \delta^{-\gamma}]$ and $[|\mathbf{z}| < \delta^{-\gamma}]$ and denote the respective integrals as $I_1(\mathbf{y})$ and $I_2(\mathbf{y})$ respectively. The first integral can be estimated by

$$I_1(\mathbf{y}) \le C\delta^{2(1-2\beta)(1+\nu')+\gamma(2\mu-1)} \int_0^\tau s^{-1/(2\mu)}(\tau-s)^{n\epsilon+k\epsilon_1} ds$$

$$= C\delta^{2(1-2\beta)(1+\nu')+\gamma(2\mu-1)} \tau^{n\epsilon+(k+1)\epsilon_1} B(\epsilon_1, n\epsilon+k\epsilon_1+1)$$

and the conclusion of the lemma for $I_1(\mathbf{y})$ follows (since $\epsilon < \epsilon_1$) upon an appropriate choice of $0 < \nu < \delta\gamma(\mu - 1/2)(1/\alpha - 1)^{-1}$.

Therefore, it remains to estimate the "inner" integral $I_2(\mathbf{y})$. However, since $|\mathbf{v}| = 1$, we can write

$$\sup_{\mathbf{y}} I_2(\mathbf{y}) \le \sup_{y\in\mathbb{R},\, |\sigma|\le\delta^{2\beta-\gamma}} \int_0^\tau s^{-1/(2\mu)}(\tau-s)^{n\epsilon+k\epsilon_1} \left[\delta^{2(2\beta-1)} + |f(s^{1/(2\mu)}) - y|^{2(1-\alpha)}\right]^{-(1+\nu')} ds, \tag{4.3}$$

where $f(\rho) := \rho^{2\mu} - \sigma\rho$. Let us denote, with a slight abuse of notation, the integral appearing in the above formula by I_2. We estimate it below by considering several possible cases for the parameters y and σ.

4.1 Parameter $\sigma \in (0, \delta^{2\beta-\gamma})$

The function f has a then a unique critical point at $\rho_0 = [\sigma/(2\mu)]^{1/(2\mu-1)}$ and two zeros at 0 and $\rho_* := \sigma^{1/(2\mu-1)} \in (\rho_0, \delta^{(2\beta-\gamma)/(2\mu-1)})$.

4.1.1 Parameter $y > 0$

In this case the equation $f(\rho) = y$ has a unique solution in $[0, +\infty)$, say at $\rho_1 > \rho_*$. Note that we can only consider the case when $\rho_1 \in [0, \tau^{1/(2\mu)}]$. If $\rho_1 > \tau^{1/(2\mu)}$ we have $y - f(\rho) > f(\tau^{1/(2\mu)}) - f(\rho)$ and this reduces to the former case.

We divide the integral appearing in (4.3) into integrals corresponding to the subintervals $[0, \rho_*^{2\mu}]$, $[\rho_*^{2\mu}, \tau]$ and denote them by I_{21}, I_{22} respectively.

To estimate I_{21} we make a change of variables $s := \rho^{2\mu}$, $ds = 2\mu\rho^{2\mu-1}d\rho$ and note that

$$I_{21} \leq 2\mu\delta^{2(1-2\beta)(1+\nu')} \int_0^{\rho_*} \rho^{2(\mu-1)}(\tau - \rho^{2\mu})^{n\epsilon+k\epsilon_1} d\rho \tag{4.4}$$

$$\leq 2\mu\delta^{2(1-2\beta)(1+\nu')}\rho_*^{4\mu(1-\alpha)(1+\nu')} \int_0^{\tau^{1/(2\mu)}} \rho^{2\mu[2\alpha-1-1/\mu-2\nu'(1-\alpha)]}(\tau - \rho^{2\mu})^{n\epsilon+k\epsilon_1} d\rho.$$

Remembering that $\rho_* < \delta^{(2\beta-\gamma)(2\mu-1)^{-1}}$ we can estimate $\delta^{2(1-2\beta)(1+\nu')}\rho_*^{4\mu(1-\alpha)}$ by δ^{ϵ_2}, where

$$\epsilon_2 := \left[(2\beta - \gamma)\frac{4\mu(1-\alpha)}{2\mu - 1} + 2(1 - 2\beta)\right](1 + \nu').$$

It is straightforward to verify (recall that $\beta = 1/(2\alpha)$) that for μ satisfying the assumptions of the lemma the exponent ϵ_2 corresponding to $\gamma = \gamma_0$ and $\nu' = 0$ is positive.

We can choose therefore appropriate $\nu, \delta\gamma > 0$ in such a way that ϵ_2 corresponding to those parameters also remains positive. Reverting to the s variable in the integral appearing on the utmost right hand side of (4.4) we obtain that

$$\sup_{y\in\mathbb{R},\, |\sigma|\leq\delta^{2\beta-\gamma}} I_{21} \leq \delta^{\epsilon_2} \int_0^{\tau} s^{\epsilon-1}(\tau - s)^{n\epsilon+k\epsilon_1} ds \leq \delta^{\epsilon_2}\tau^{(n+1)\epsilon+k\epsilon_1} B(\epsilon, n\epsilon + k\epsilon_1 + 1). \tag{4.5}$$

To estimate I_{22} we claim that

$$y - f(\rho) \geq \rho^{2\mu-1}(\rho_1 - \rho). \tag{4.6}$$

In order to verify (4.6) we note that it is equivalent to

$$\rho_1(\rho_1^{2\mu-1} - \rho^{2\mu-1}) \geq \sigma(\rho_1 - \rho), \tag{4.7}$$

which can be seen with an elementary inequality

$$[(m-1)a^{m-1} + b^{m-1}](b - a) < b^m - a^m \tag{4.8}$$

valid for any $b > a > 0$ and $m > 1$. It allows to estimate the left side of (4.7) from below by

$$\rho_1[(2\mu - 2)\rho^{2\mu-2} + \rho_1^{2\mu-2})](\rho_1 - \rho) \geq (2\mu - 1)\rho_*^{2\mu-1}(\rho_1 - \rho) = (2\mu - 1)\sigma(\rho_1 - \rho).$$

In the last equality we have used the fact that $\rho_*^{2\mu-1} = \sigma$ and (4.7) follows. Likewise, we have $f(\rho) - y \geq (\rho - \rho_1)\rho^{2\mu-1}$ for $\rho \in [\rho_1, \tau^{1/(2\mu)}]$. We have shown therefore that

$$|f(\rho) - y| \geq |\rho - \rho_1|\rho^{2\mu-1} \quad \text{for } \rho \in [\rho_*, \tau^{1/(2\mu)}].$$

Dropping the term $\delta^{2(2\beta-1)}$ in the denominator of I_{22} and using the above estimate we obtain, upon the substitution $\rho := s^{1/(2\mu)}$, that

$$I_{22} \leq C \int_{\rho_*}^{\tau^{1/(2\mu)}} \frac{\rho^{\epsilon_3}(\tau - \rho^{2\mu})^{n\epsilon+k\epsilon_1} d\rho}{(\rho_1 - \rho)^{2(1-\alpha)(1+\nu')}}, \tag{4.9}$$

where $\epsilon_3 := (2\mu-1)[2\alpha - 1 - 2\nu'(1-\alpha)] - 1$ and $C > 0$ is a certain constant. Suppose that $\varrho \in (0, \epsilon)$ is arbitrary. We let $1/p := 2(1-\alpha)(1+\nu') + \varrho > 1$ for a suitable choice of ν. Let also $1/q = 1 - 1/p$. Using Hölder's inequality we obtain that the last integral is less than, or equal to

$$C \left[\int_{\rho_*}^{\tau^{1/(2\mu)}} |\rho_1 - \rho|^{-2p(1-\alpha)(1+\nu')} d\rho \right]^{1/p} \left[\int_{\rho_*}^{\tau^{1/(2\mu)}} \rho^{q\epsilon_3}(\tau - \rho^{2\mu})^{q(n\epsilon+k\epsilon_1)} d\rho \right]^{1/q}.$$

Using substitution $\rho := s^{1/(2\mu)}$ in the second integral one can estimate the above expression by

$$C\tau^{[1/p - 2(1+\nu')(1-\alpha)]/(2\mu)} \times \tau^{n\epsilon+k\epsilon_1+\epsilon_3/(2\mu)+1/(2\mu q)} B^{1/q}((q\epsilon_3 + 1)/2\mu, q(n\epsilon + k\epsilon_1)) \tag{4.10}$$

Applying (3.32) we can further bound (4.10) by

$$\frac{C\tau^{2\alpha-1-2\nu'[1+(1-\alpha)/\mu]+n\epsilon+k\epsilon_1}}{(n+k+1)^{\epsilon_3/(2\mu)+1/(2q\mu)}}. \tag{4.11}$$

Taking into account that $\epsilon_3/(2\mu) + 1/(2q\mu) = \epsilon - \varrho/(2\mu)$ we obtain the required estimate

$$\sup_{y\in\mathbb{R},\, |\sigma|\leq\delta^{2\beta-\gamma}} I_{22} \leq \frac{C\tau^{(n+1)\epsilon+k\epsilon_1}}{(k+n+1)^{\epsilon-\varrho}}. \tag{4.12}$$

4.1.2 Parameter $y < 0$

The equation $f(\rho) = y$ can possibly have two solutions ρ_1, ρ_2 satisfying $0 \leq \rho_1 < \rho_0 < \rho_2 \leq \rho_*$. In fact we can assume that $y \in [f(\rho_0), 0]$, since otherwise we have $f(\rho) - y \geq f(\rho) - f(\rho_0)$ and the situation reduces to the case $y = f(\rho_0)$. We should consider two possibilities: either $\tau^{1/(2\mu)} < 2\rho_*$, or otherwise. In the first case we estimate I_2 precisely as in (4.4). In the second one we distinguish two regions of integration for I_2, namely $[0, 2\rho_*]$ and $[2\rho_*, \tau^{1/(2\mu)}]$. We denote the respective integrals by I_{21} and I_{22} correspondingly. The estimate for I_{21} is the same as in (4.4). To deal with I_{22} note that for $\rho \in [2\rho_*, \tau^{1/(2\mu)}]$ we have

$$f(\rho) - y \geq \rho^{2\mu} - \rho_2^{2\mu} - \sigma(\rho - \rho_2) = \rho^{2\mu} - \rho_2^{2\mu} - 2\mu\rho_0^{2\mu-1}(\rho - \rho_2).$$

Using inequality (4.8) we conclude easily that

$$\begin{aligned} f(\rho) - y &\geq (\rho - \rho_2)\left\{[(2\mu-1)\rho_2^{2\mu-1} + \rho^{2\mu-1}] - 2\mu\rho_0^{2\mu-1}\right\} \\ &\geq (\rho - \rho_2)(\rho - \rho_0)\rho^{2\mu-2} \geq (\rho - \rho_*)^2\rho^{2\mu-2} \geq \frac{\rho^{2\mu}}{4}. \end{aligned}$$

The last inequality follows from the fact that $\rho(\rho-\rho_*)^{-1} \le 1+\rho_*(\rho-\rho_*)^{-1} \le 2$ for $\rho \in [2\rho_*, \tau^{1/(2\mu)}]$. We, therefore, obtain

$$I_{22} \le 4 \int\limits_{(2\rho_*)^{2\mu}}^{\tau} (\tau - s)^{n\epsilon+k\epsilon_1} s^{\epsilon-1} ds \le \tau^{(n+1)\epsilon+k\epsilon_1} B(n\epsilon + k\epsilon_1 + 1, \epsilon), \tag{4.13}$$

and proceed then as in the passage from (4.10) to (4.12).

4.2 Parameter $\sigma \le 0$.

Then $f(s)$ is a strictly increasing function on $[0, +\infty)$ and hence only the case when $y > 0$ needs to be considered. Otherwise, we would have $f(s) - y > f(s) - f(0)$ and this reduces to the aforementioned situation. When $y > 0$ the estimations essentially repeat the case considered in Section 4.1.1. In fact they can be reduced only to the consideration of integral I_{22} considered there. As a result we again obtain (3.21).

5 The fundamental solutions to the random fractional advecion-heat equation

Here we establish the existence of fundamental solutions for (1.10) and prove Proposition 2.3 and Theorem 2.5. We shall assume that $\delta = 1$ and suppress it from the subsequent notation.

5.1 The proof of Proposition 2.3

Part (i) of Proposition 2.3 is proved in [2], Theorem 2.1, p. 263. To prove part (ii) note that using Funk-Hecke theorem, see p. 181 of [4], we can write

$$q_{\kappa,d}(t, \mathbf{x}) = \int\limits_{\mathbb{R}^d} \exp\{i\mathbf{x} \cdot \mathbf{k}\} e^{-t|\xi|^{2\mu}} \frac{d\mathbf{k}}{(2\pi)^d} = \frac{\omega_{d-2}}{(2\pi)^d} \int\limits_0^{+\infty} e^{-t\rho^{2\alpha}} \rho^{d-1} d\rho \left\{ \int\limits_{-1}^{1} \cos(u|\mathbf{x}|\rho)(1-u^2)^{(d-3)/2} du \right\}.$$

Hence, we have

$$\nabla_{\mathbf{x}} q_{\kappa,d}(t, \mathbf{x}) = -\frac{\omega_{d-2}\mathbf{x}}{(2\pi)^d|\mathbf{x}|} \int\limits_0^{+\infty} e^{-t\rho^{2\alpha}} \rho^d d\rho \left\{ \int\limits_{-1}^{1} u \sin(u|\mathbf{x}|\rho)(1-u^2)^{(d-3)/2} du \right\}. \tag{5.1}$$

Integrating by parts with respect to u we obtain that the right side of (5.1) equals

$$-\frac{\omega_{d-2}\mathbf{x}}{(2\pi)^d(d-1)} \int\limits_0^{+\infty} e^{-t\rho^{2\alpha}} \rho^{d+1} d\rho \left\{ \int\limits_{-1}^{1} \cos(u|\mathbf{x}|\rho)(1-u^2)^{(d-1)/2} du \right\} = -2\pi\mathbf{x} q_{\kappa,d+2}(t, \mathbf{x}'),$$

where $\mathbf{x}' = (\mathbf{x}, 0, 0) \in \mathbb{R}^{d+2}$. $\square$

5.2 Proof of Theorem 2.5

Let us fix $T > 0$ and $\gamma_0 > 1/2$. Using Proposition 2.2 we can easily conclude that there exists a random variable $F_T(\omega)$ that satisfies (2.3) and such that

$$|\mathbf{G}(\mathbf{v}s+\mathbf{y})| \le F_T(\omega)(1+\log^+|\mathbf{y}|)^{\gamma_0}$$

for all $(s, \mathbf{y}) \in [0, T] \times \mathbb{R}^d$. Theorem 2.5 is a consequence of the following lemma.

Lemma 5.1 *For any $T > 0$, $\mu' \in (1, \mu)$ and $\gamma_0 > 1/2$, there exists a deterministic constant $C_T > 0$ such that for $\mathbb{P}$-a.s. ω we have*

$$|q_n(t,\mathbf{x},s,\mathbf{y})| \le \frac{[C_T F_T(\omega)]^n (t-s)^{n(1-1/(2\mu))-d/(2\mu)}}{\Gamma(n(1-1/(2\mu)))} \times \frac{(1+\log^+|\mathbf{y}|)^{n\gamma_0}}{[1+|\mathbf{x}-\mathbf{y}|^2(t-s)^{-1/\mu}]^{d/2+n\mu'+\mu}} \tag{5.2}$$

for all $T \ge t > s \ge 0$, $\mathbf{x}, \mathbf{y} \in \mathbb{R}^d$, $n \ge 1$.

Statements (i) and (ii) of Theorem 2.5 then easily follow from (5.2). As for equality (2.15) a simple extremum consideration shows below that for any $\gamma > 0$ there exists a constant $C > 0$ such that for any $t > 0$, $\mathbf{x}, \mathbf{y} \in \mathbb{R}^d$ we have

$$\frac{(1+\log^+|\mathbf{x}|)^{\gamma_0}}{(1+|\mathbf{x}-\mathbf{y}|^2 t^{-1/\mu})^{\gamma}} \le C(t+1)\left(1+\log^+|\mathbf{y}|\right)^{\gamma_0}. \tag{5.3}$$

It suffices to verify (5.3) for $\gamma < \mu$ and $|\mathbf{x}| > 1$. Consider two cases: either $|\mathbf{x}-\mathbf{y}| < 10^{-2}|\mathbf{x}|$, or otherwise. In the first situation we have

$$|\mathbf{y}| \ge |\mathbf{x}| - |\mathbf{y}-\mathbf{x}| \ge \frac{99}{100}|\mathbf{x}|$$

so we can estimate the left hand side of (5.3) by by

$$(1+\log(100|\mathbf{y}|/99))^{\gamma_0} \tag{5.4}$$

and (5.3) follows. In the other case the left side of (5.3) is at most

$$\frac{(1+\log|\mathbf{x}|)^{\gamma_0}}{(1+10^{-4}|\mathbf{x}|^2 t^{-1/\mu})^{\gamma}} \le t^{\gamma/\mu}\frac{(1+\log|\mathbf{x}|)^{\gamma_0}}{|\mathbf{x}|^{2\gamma}} \le C(t+1) \tag{5.5}$$

for an appropriate $C > 0$.

From (5.2) and (5.3) we conclude therefore that

$$|q_n(t,\mathbf{x},s,\mathbf{y})| \le \frac{[C_T F_T(\omega)]^n (t-s)^{n(1-1/(2\mu))-d/(2\mu)}}{\Gamma(n(1-1/(2\mu)))} \times \frac{(1+\log^+|\mathbf{x}|)^{n\gamma_0}}{[1+|\mathbf{x}-\mathbf{y}|^2(t-s)^{-1/\mu}]^{d/2+\mu}} \tag{5.6}$$

To finish the proof of (2.15) we note that $\mathbb{E}F_T^n \sim \Gamma(n/2)$ so after integration in $\mathbf{y}$ the series appearing in that formula can be estimated from the above by

$$\sum_{n\ge 1} \frac{C_T^n \Gamma(n/2)(1+\log^+|\mathbf{x}|)^{n\gamma_0}}{\Gamma(n(1-1/(2\mu)))}\|u_0\|_\infty < +\infty.$$

It remains only to prove part (iii) of Theorem 2.5: uniqueness of the mild solution of (2.12). One can observe with the help of (2.8) that the mild solution $v(t, \mathbf{x})$ of (2.12) with vanishing initial data satisfies

$$|v(t,\mathbf{x})| \le C\int_0^t \int (t-s_1)^{-1/(2\mu)}\Phi(t,\mathbf{x},s_1,\mathbf{y}_1)|v(s_1,\mathbf{y}_1)|ds_1 d\mathbf{y}_1 \tag{5.7}$$

for some constant $C > 0$ and all $t \in [0, T]$ for some $T > 0$. Here we have defined

$$\Phi(t, \mathbf{x}, s, \mathbf{y}) := (t-s)^{-d/(2\mu)} \frac{(1 + \log^+ |\mathbf{y}|)^{\gamma_0}}{[1 + |\mathbf{x}-\mathbf{y}|^2 (t-s)^{-1/\mu}]^{d/2+\mu}}.$$

We would like to iterate (5.7) and use a Gronwall type argument. For that we need the following auxiliary lemma.

Lemma 5.2 *Suppose that $\mu, \gamma_1, \gamma_2 > 0$. Then, there exists a constant $C > 0$ depending only on γ_1, γ_2 and the dimension d such that for all $t > 0$ and $a, b < 1 + d/(2\mu)$ and $\mathbf{x} \in \mathbb{R}^d$ one has*

$$\int_0^t \int_{\mathbb{R}^d} (t-s)^{-a} s^{-b} \left[1 + \frac{|\mathbf{x}-\mathbf{y}|^2}{(t-s)^{1/\mu}}\right]^{-d/2-\gamma_1} \left[1 + \frac{|\mathbf{y}|^2}{s^{1/\mu}}\right]^{-d/2-\gamma_2} ds d\mathbf{y} \tag{5.8}$$

$$\leq CB\left(\frac{d}{2\mu} + 1 - a, \frac{d}{2\mu} + 1 - b\right) t^{1+d/(2\mu)-a-b} \left(1 + \frac{|\mathbf{x}|^2}{t^{1/\mu}}\right)^{-d/2-\gamma_1-\gamma_2}.$$

Here $B(\cdot, \cdot)$ is the Euler beta function.

We claim that, as consequence of (5.7) and Lemma 5.2, for any $\mu' \in (1, \mu)$ there exists $C > 0$ such that for all $n \geq 1$ we have

$$|v(t, \mathbf{x})| \leq \frac{C^n}{\Gamma(n(1-1/(2\mu))} \int_0^t \int \frac{(t-s_1)^{n(1-1/(2\mu))-1-d/(2\mu)} (1 + \log^+ |\mathbf{y}_1|)^{n\gamma_0}}{[1 + |\mathbf{x}-\mathbf{y}_1|^2/(t-s_1)^{-1/\mu}]^{d/2+n\mu'}} |v(s_1, \mathbf{y}_1)| ds_1 d\mathbf{y}_1. \tag{5.9}$$

The proof of (5.9) is done by induction with respect to n. For $n = 1$ it is just a consequence of (5.7). Suppose that (5.9) holds for a certain positive integer n. Substituting the right side of (5.9) into (5.7) in place of $v(s_1, \mathbf{y}_1)$, we obtain

$$|v(t, \mathbf{x})| \leq \frac{C^n}{\Gamma(n(1-1/(2\mu))} \int_0^t \int (t-s_1)^{-(d+1)/(2\mu)} \frac{(1 + \log^+ |\mathbf{y}_1|)^{\gamma_0}}{[1 + |\mathbf{x}-\mathbf{y}_1|^2 (t-s_1)^{-1/\mu}]^{d/2+\mu}} \tag{5.10}$$

$$\times \int_0^{s_1} \int \frac{(s_1-s)^{n(1-1/(2\mu))-1-d/(2\mu)} (1 + \log^+ |\mathbf{y}|)^{n\gamma_0}}{(1 + |\mathbf{y}_1-\mathbf{y}|^2/(s_1-s)^{-1/\mu})^{d/2+n\mu'}} |v(s, \mathbf{y})| ds d\mathbf{y} ds_1 d\mathbf{y}_1.$$

Let us set

$$\gamma := \min\{\mu - \mu', \mu'/2\}. \tag{5.11}$$

Then, with the help of (5.3) we obtain

$$\frac{(1 + \log^+ |\mathbf{y}_1|)^{\gamma_0}}{(1 + |\mathbf{y}_1-\mathbf{y}|^2/(s_1-s)^{-1/\mu})^{d/2+\gamma}} \leq C(T+1)(1 + \log^+ |\mathbf{y}|)^{\gamma_0}.$$

Thus, we can re-write (5.10) as

$$|v(t, \mathbf{x})| \leq \frac{C^n}{\Gamma(n(1-1/(2\mu))} \int_0^t \int |v(s, \mathbf{y})| (1 + \log^+ |\mathbf{y}|)^{(n+1)\gamma_0} \int_s^t \int \frac{(t-s_1)^{-(d+1)/(2\mu)}}{[1 + |\mathbf{x}-\mathbf{y}_1|^2 (t-s_1)^{-1/\mu}]^{d/2+\mu}}$$

$$\times \frac{(s_1-s)^{n(1-1/(2\mu))-1-d/(2\mu)}}{[1 + |\mathbf{y}_1-\mathbf{y}|^2/(s_1-s)^{-1/\mu}]^{d/2+n\mu'-\gamma}} ds_1 d\mathbf{y}_1 ds d\mathbf{y}. \tag{5.12}$$

Lemma 5.2 allows to estimate the $(s_1, \mathbf{y}_1)$-integral above as

$$\int_s^t \int \frac{(t-s_1)^{-(d+1)/(2\mu)}}{[1+|\mathbf{x}-\mathbf{y}_1|^2(t-s_1)^{-1/\mu}]^{d/2+\mu}} \frac{(s_1-s)^{n(1-1/(2\mu))-1-d/(2\mu)}(1+\log^+|\mathbf{y}|)^{(n+1)\gamma_0}}{[1+|\mathbf{y}_1-\mathbf{y}|^2/(s_1-s)^{-1/\mu}]^{d/2+n\mu'-\gamma}} ds_1 d\mathbf{y}_1$$

$$\leq CB\left(1-\frac{1}{2\mu}, n\left(1-\frac{1}{2\mu}\right)\right)(t-s)^{(n+1)(1-1/(2\mu))-1-d/(2\mu)}\left[1+\frac{|\mathbf{x}-\mathbf{y}|^2}{(t-s)^{1/\mu}}\right]^{-\frac{d}{2}-(n+1)\mu'} \tag{5.13}$$

as $\mu - \gamma > \mu'$. Using (5.13) in (5.12), together with $B(x,y) = \Gamma(x)\Gamma(y)/\Gamma(x+y)$ leads to (5.9). As a result of that estimate we conclude that

$$|v(t,\mathbf{x})| \leq \frac{C^n(1+\log^+|\mathbf{x}|)^{n\gamma_0}}{\Gamma(n(1-1/(2\mu)))}\|v\|_\infty \int_0^t \int \frac{(t-s_1)^{n(1-1/(2\mu))-1-d/(2\mu)}}{[1+|\mathbf{x}-\mathbf{y}_1|^2/(t-s_1)^{-1/\mu}]^{d/2+n(\mu'-\gamma)}} ds_1 d\mathbf{y}_1.$$

for $0 \leq t \leq T$ and all $n \geq 1$, with a constant C which is independent of n , As $\|v\|_\infty < +\infty$ because v is a mild solution, the above estimate in turn, after passing to the limit $n \to +\infty$, implies that $v(t,\mathbf{x}) \equiv 0$. Therefore, the mild solution of (2.12) is unique. This finishes the proof of Theorem 2.5 except for the proof of Lemmas 5.1 and 5.2. □

5.3 Proof of Lemma 5.2

Using identity

$$\int_0^{+\infty} t^{\beta-1}e^{-bt}dt = \frac{\Gamma(\beta)}{b^\beta} \tag{5.14}$$

that holds for all $\beta, b > 0$ we can rewrite the left hand side of (5.8) as being equal to

$$\frac{1}{\Gamma(d/2+\gamma_1)\Gamma(d/2+\gamma_2)} \int_0^t \int_{\mathbb{R}^d} \int_0^{+\infty} \int_0^{+\infty} (t-s)^{-a}s^{-b}\rho_1^{d/2+\gamma_1-1}\rho_2^{d/2+\gamma_2-1}e^{-(\rho_1+\rho_2)} \tag{5.15}$$
$$\times \exp\left\{-\frac{\rho_1}{(t-s)^{1/\mu}}|\mathbf{x}-\mathbf{y}|^2 - \frac{\rho_2}{s^{1/\mu}}|\mathbf{y}|^2\right\} ds d\mathbf{y} d\rho_1 d\rho_2.$$

Performing first the integration over $\mathbf{y}$ variable and using the well known formula for the convolution of Gaussian densities we obtain that

$$\int_{\mathbb{R}^d} \exp\left\{-\frac{\rho_1}{(t-s)^{1/\mu}}|\mathbf{x}-\mathbf{y}|^2 - \frac{\rho_2}{s^{1/\mu}}|\mathbf{y}|^2\right\} d\mathbf{y} \tag{5.16}$$
$$= \pi^{d/2}\left[\frac{(t-s)^{1/\mu}}{\rho_1} \times \frac{s^{1/\mu}}{\rho_2}\right]^{d/2}\left[\frac{(t-s)^{1/\mu}}{\rho_1} + \frac{s^{1/\mu}}{\rho_2}\right]^{-d/2} \exp\left\{-\left[\frac{(t-s)^{1/\mu}}{\rho_1} + \frac{s^{1/\mu}}{\rho_2}\right]^{-1}|\mathbf{x}|^2\right\}.$$

We perform the integration with respect to ρ_1, ρ_2 variables using the polar coordinates simultaneously rescaling the size of the interval over s to the unit one, namely substitute $\rho_1 = r\cos\phi$, $\rho_2 = r\sin\phi$ and $s := s/t$. We find then that the respective integral of the expression in (5.15) equals

$$\frac{\pi^{d/2}t^{d/(2\mu)-a-b+1}}{\Gamma(d/2+\gamma_1)\Gamma(d/2+\gamma_2)} \int_0^1 \int_0^{+\infty} \int_0^{\pi/2} (1-s)^{d/(2\mu)-a}s^{d/(2\mu)-b}f^{d/2}(s,\phi)r^{d+\gamma_1+\gamma_2-1}$$
$$\times \exp\left\{-r\left[(\cos\phi+\sin\phi) + f(s,\phi)\frac{|\mathbf{x}|^2}{t^{1/\mu}}\right]\right\} \cos^{d/2+\gamma_1-1}\phi \sin^{d/2+\gamma_2-1}\phi ds dr d\phi,$$

where

$$f(s,\phi) := \left[\frac{(1-s)^{1/\mu}}{\cos\phi} + \frac{s^{1/\mu}}{\sin\phi}\right]^{-1} = \frac{\sin\phi\cos\phi}{(1-s)^{1/\mu}\sin\phi + s^{1/\mu}\cos\phi}.$$

Using the fact that $\cos\phi + \sin\phi \geq \sqrt{2}$ and integrating out the r variable with the help of (5.14) we can estimate the above expression by

$$\pi^{d/2} t^{d/(2\mu)-a-b+1} B(d/2+\gamma_1, d/2+\gamma_2) \int_0^1 \int_0^{\pi/2} (1-s)^{d/(2\mu)-a} s^{d/(2\mu)-b} \cos^{d/2+\gamma_1-1}\phi \sin^{d/2+\gamma_2-1}\phi$$

$$\times \left[\sqrt{2} + f(s,\phi)\frac{|\mathbf{x}|^2}{t^{1/\mu}}\right]^{-(d/2+\gamma_1+\gamma_2)} f^{d/2}(s,\phi) ds d\phi$$

$$\leq \pi^{d/2} t^{d/(2\mu)-a-b+1} B(d/2+\gamma_1, d/2+\gamma_2) \left[\sqrt{2}c(\mu) + \frac{|\mathbf{x}|^2}{t^{1/\mu}}\right]^{-(d/2+\gamma_1+\gamma_2)} \tag{5.17}$$

$$\times \int_0^1 (1-s)^{d/(2\mu)-a} s^{d/(2\mu)-b} ds \times \sup_{s'\in(0,1)} \int_0^{\pi/2} \cos^{d/2+\gamma_1-1}\phi \sin^{d/2+\gamma_2-1}\phi f^{-\gamma_1-\gamma_2}(s',\phi) d\phi$$

where $c(\mu)$ is the infimum of $f^{-1}(s,\phi)$ over $s \in (0,1)$, $\phi \in (0,\pi/2)$ and the supremum over s' is finite thanks to the fact that $f^{-1}(s',\phi) \leq 2(\cos\phi\sin\phi)^{-1}$. The conclusion of the lemma then clearly follows. □

5.4 Proof of Lemma 5.1

Let us define the function

$$\Psi(t,\mathbf{x}) := t^{-d/(2\mu)} \frac{1}{(1+|\mathbf{x}|^2 t^{-1/\mu})^{d/2+\mu}}$$

and let γ be given by (5.11). Using Lemma 5.2 and expression (5.3) we obtain that for any $\gamma_0 > 1/2$ and $\mu > 1$ there exists a constant $C > 0$ such that for $0 \leq s < t \leq T$, $\mathbf{x}, \mathbf{y} \in \mathbb{R}^d$

$$\int_s^t \int (t-s_1)^{-1/(2\mu)} \Phi(t,\mathbf{x},s_1,\mathbf{y}_1) \Psi(s_1-s,\mathbf{y}_1-\mathbf{y}) ds_1 d\mathbf{y}_1 = \int_s^t \int (t-s_1)^{-1/(2\mu)-d/(2\mu)} (s_1-s)^{-d/(2\mu)}$$

$$\times \frac{(1+\log^+|\mathbf{y}_1|)^{\gamma_0}}{[1+|\mathbf{x}-\mathbf{y}_1|^2(t-s_1)^{-1/\mu}]^{d/2+\mu}} \times \frac{1}{[1+|\mathbf{y}_1-\mathbf{y}|^2(s_1-s)^{-1/\mu}]^{d/2+\mu}} ds_1 d\mathbf{y}_1 \tag{5.18}$$

$$\stackrel{(5.3)}{\leq} C \int_s^t \int (t-s_1)^{-1/(2\mu)-d/(2\mu)} (s_1-s)^{-d/(2\mu)} \times \frac{1}{[1+|\mathbf{x}-\mathbf{y}_1|^2(t-s_1)^{-1/\mu}]^{d/2+\mu}}$$

$$\times \frac{(1+\log^+|\mathbf{y}|)^{\gamma_0}}{[1+|\mathbf{y}_1-\mathbf{y}|^2(s_1-s)^{-1/\mu}]^{d/2+\mu'}} ds_1 d\mathbf{y}_1 \stackrel{(5.8)}{\leq} \frac{C(t-s)^{1-1/(2\mu)-d/(2\mu)}(1+\log^+|\mathbf{y}|)^{\gamma_0}}{[1+|\mathbf{x}-\mathbf{y}|^2(t-s)^{-1/\mu}]^{d/2+\mu+\mu'}}.$$

In fact, we can generalize the above estimate to obtain.

Proposition 5.3 *For any $T > 0$, $\mu' \in (1,\mu)$ and $\gamma_0 > 1/2$ there exists a constant $C_T > 0$ such that for all $n \geq 1$*

$$\int_{\Delta_n(t,s)} \!\!\cdots\!\int \int_{(\mathbb{R}^d)^n} \!\!\cdots\!\int \prod_{k=1}^n \left[(s_{k-1}-s_k)^{-1/(2\mu)} \Phi(s_{k-1},\mathbf{y}_{k-1},s_k,\mathbf{y}_k)\right] \Psi(s_n-s,\mathbf{y}_n-\mathbf{y}) ds^{(n)} d\mathbf{y}^{(n)}$$

$$\leq \frac{C_T^n (t-s)^{n(1-1/(2\mu))-d/(2\mu)}}{\Gamma(n(1-1/(2\mu)))} \times \frac{(1+\log^+|\mathbf{y}|)^{n\gamma_0}}{[1+|\mathbf{x}-\mathbf{y}|^2(t-s)^{-1/\mu}]^{d/2+n\mu'+\mu}}, \tag{5.19}$$

with the convention that $s_0 = t$ *and* $\mathbf{y}_0 = \mathbf{x}$.

Proof. We show this result by induction in n. The result for $n = 1$ is just (5.18) with an appropriate choice of the constant $C > 0$. Suppose that it holds for a certain n. We can estimate the left side of expression (5.19) written for $n + 1$ by

$$\frac{C_T^n}{\Gamma(n(1-1/(2\mu)))} \int_s^t \int_{\mathbb{R}^d} (t-s_1)^{-1/(2\mu)} \Phi(t,\mathbf{x},s_1,\mathbf{y}_1)(s_1-s)^{n(1-1/(2\mu))-d/(2\mu)} \tag{5.20}$$

$$\times \frac{(1+\log^+|\mathbf{y}|)^{n\gamma_0} ds_1 d\mathbf{y}_1}{[1+|\mathbf{y}_1-\mathbf{y}|^2(s_1-s)^{-1/\mu}]^{d/2+n\mu'+\mu}}$$

$$\stackrel{(5.3)}{\leq} \frac{C_T^n C(t+1)}{\Gamma(n(1-1/(2\mu)))} \int_s^t \int_{\mathbb{R}^d} (t-s_1)^{-(d+1)/(2\mu)} (s_1-s)^{n(1-1/(2\mu))-d/(2\mu)}$$

$$\times \frac{1}{[1+|\mathbf{x}-\mathbf{y}_1|^2(t-s_1)^{-1/\mu}]^{d/2+\mu'}} \times \frac{(1+\log^+|\mathbf{y}|)^{(n+1)\gamma_0}}{[1+|\mathbf{y}_1-\mathbf{y}|^2(s_1-s)^{-1/\mu}]^{d/2+n\mu'+\mu}} ds_1 d\mathbf{y}_1.$$

Using Lemma 5.2 we can estimate the right side of (5.20) by

$$\frac{C_T^n C(t+1)(t-s)^{(n+1)(1-1/(2\mu))-d/(2\mu)}}{\Gamma(n(1-1/(2\mu)))} \times \frac{B(1-1/(2\mu), n(1-1/(2\mu)))(1+\log^+|\mathbf{y}|)^{(n+1)\gamma_0}}{[1+|\mathbf{x}-\mathbf{y}|^2(t-s)^{-1/\mu}]^{d/2+(n+1)\mu'+\mu}}.$$

Again, the induction argument can be concluded from the identity $B(\alpha,\beta) = \Gamma(\alpha)\Gamma(\beta)\Gamma^{-1}(\alpha+\beta)$ for $\alpha, \beta > 0$. □

We can now finish the proof of Lemma 5.1: it follows immediately from Proposition 5.3 and expression (2.13). □

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